Automorphism groups of Joyce twistor spaces

Akira Fujiki

To the memory of Professor Masaki Maruyama

Abstract We determine the automorphism groups of torus invariant self-dual structures defined by Joyce on the connected sum of copies of the complex projective plane. We determine, actually, the automorphism groups of the associated twistor spaces by using the results of our previous work. When the self-dual structures of Joyce and LeBrun coincide, our results recover the recent results of Honda and Viaclovsky on the automorphism groups of LeBrun's self-dual structures.

1. Introduction

For a positive integer m let $M = m\boldsymbol{P}^2$ denote the connected sum of m copies of complex projective plane $\boldsymbol{P}^2$. The two families of self-dual structures on M constructed, respectively, by LeBrun [7] and by Joyce [6] are the most typical examples of explicit constructions of such structures. More precisely, for each m there exists a semifree smooth S^1-action on M, which is up to diffeomorphisms unique if $m \geq 3$. LeBrun [7] then explicitly constructed a smooth and connected family of S^1-invariant self-dual structures on M. On the other hand, there exists for each m a finite number of smooth actions up to diffeomorphisms of the real two torus $K := S^1 \times S^1$ on M. For each such K-action Joyce [6] constructed explicitly a smooth and connected family of K-invariant self-dual structures on M with real $(m-1)$-dimensional parameters.

Recently Honda and Viaclovsky [5] have determined the whole group of conformal automorphisms for the self-dual manifolds by LeBrun above. Inspired by their work, in this note we shall determine the structure of the whole automorphism group of self-dual structures by Joyce.

Our main results are described as follows. We assume throughout the paper that $m \geq 2$ since otherwise everything is well known. Let $M = m\boldsymbol{P}^2$ with a fixed K-action as above, and let $[g]$ be any K-invariant self-dual structure on M by Joyce. Let $\mathrm{Conf}(M, [g])$ denote the group of conformal automorphisms of the self-dual manifold $(M, [g])$, and let $\mathrm{Conf}_0(M, [g])$ be its identity component.

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Let Z be the associated twistor space with twistor fibration $t: Z \to M$ and the real structure σ. Let $\mathrm{Aut}(Z, t, \sigma)$ denote the group of biholomorphic automorphisms of Z which commute with σ and which preserve the twistor fibration. Let $\mathrm{Aut}_0(Z, t, \sigma)$ be its identity component. By the Penrose correspondence we have a natural isomorphism $t_*: \mathrm{Aut}(Z, t, \sigma) \cong \mathrm{Conf}(M, [g])$ (cf. [10, Proposition 2.1]). In this paper we shall determine the structure of $\mathrm{Aut}(Z, t, \sigma)$, depending heavily on the results on the structure of the twistor space Z in our previous paper [3].

In this introduction, however, we state our results in terms of $\mathrm{Conf}(M, [g])$, leaving the description in terms of $\mathrm{Aut}(Z, t, \sigma)$ for subsequent sections.

Let $N := M/K$ be the quotient of M by K with the quotient map $q: M \to N$. By construction in [6], N is naturally conformally identified with the closed unit disc $\{|z| \leq 1\}$ in the complex plane $\boldsymbol{C} = \boldsymbol{C}(z)$. On the other hand, as a smooth orbit space of M by K, N is endowed with natural *weights* along its boundary as defined in [9, Section 4.1], based on the data of the stabilizer groups of various K-orbits on M. Let $\mathrm{Conf}(N, w)$ be the subgroup of the conformal automorphism group of the interior $\mathrm{Int}\, N$ consisting of all elements which preserve the weights w of N. (Note that any conformal automorphism of $\mathrm{Int}\, N$ is a hyperbolic isometry and extends smoothly up to the boundary.)

Now since $m \geq 2$ the identity component $\mathrm{Conf}_0(M, [g])$ coincides with K by Poon [12], and hence K is normalized by $\mathrm{Conf}(M, [g])$. Thus any element in $\mathrm{Conf}(M, [g])$ induces a diffeomorphism of N in $\mathrm{Conf}(N, w)$. Then we prove the following.

(1) The induced homomorphism $q_*: \mathrm{Conf}(M, [g]) \to \mathrm{Conf}(N, w)$ is surjective.

(2) The kernel $\mathrm{Conf}_1(M, [g])$ of q_* is isomorphic to the semidirect product $\boldsymbol{Z}_2 \ltimes K$.

In particular we have the following exact sequence of groups:

$$1 \to \boldsymbol{Z}_2 \ltimes K \to \mathrm{Conf}\big(M, [g]\big) \to \mathrm{Conf}(N, w) \to 1.$$

(3) For some point $x \in M$ $\mathrm{Conf}(M, [g])$ has a natural semidirect product structure

$$\mathrm{Conf}\big(M, [g]\big) \cong \mathrm{Conf}\big(M, [g], x\big) \ltimes K,$$

which is determined up to conjugations by elements of K, where $\mathrm{Conf}(M, [g], x)$ is the subgroup of $\mathrm{Conf}(M, [g])$ consisting of elements which fix x. In particular $\mathrm{Conf}(M, [g], x)$ is isomorphic to the quotient group $\overline{\mathrm{Conf}}(M, [g]) := \mathrm{Conf}(M, [g])/K$ and thus is a central extension of $\mathrm{Conf}(N, w)$ by $\boldsymbol{Z}_2$.

(4) Elements of $\mathrm{Conf}_1(M, [g]) \setminus K$ are involutions which are mutually K-conjugate to each other. The fixed point set F of any such element τ is diffeomorphic to a connected sum of m copies of the real projective plane. For any fixed point x of K-action on M, τ is realized on $M - x$ by an antiholomorphic involution with respect to its natural (complex algebraic) toric surface structure.

(5) $\overline{\mathrm{Conf}}(M, [g])$ is finite and is isomorphic to a subgroup of either of the dihedral groups D_4 or D_6 containing their center. Moreover, given such a subgroup

H, if $\nu(m)$ is the number of diffeomorphism classes of K-actions on $M = m\boldsymbol{P}^2$ such that there exists a K-invariant self-dual structure with $\overline{\mathrm{Conf}}(M,[g]) \cong H$, we have $\limsup_{m\to\infty} \nu(m) = \infty$.

(6) It is known that for each m there exists a unique K-action on M for which some S^1-subgroup induces a semifree action, and in this case, the conformal structures of Joyce coincide with a subfamily of those of LeBrun with respect to this S^1-action. For this special family our results basically recover the results of [5] in our Joyce context, including the important special case $m = 2$ of Poon's self-dual structure [11]. (Such a Joyce twistor space will be *of type LJ* or of LeBrun–Joyce type in what follows.)

The paper is arranged as follows. Any Joyce twistor space admits a distinguished pencil P of K-invariant symmetric toric surfaces, and we study the structure of the group $A := \mathrm{Aut}(Z, t, \sigma)$ through its action on P. In Sections 2 and 3 we recall from [3] the basic facts on symmetric toric surfaces and the pencil P, respectively. In Section 2 we also study the structure of the automorphism group of symmetric toric surfaces rather in detail, which will be used in Section 6. The pencil P is preserved by A and σ. Let $A' := \mathrm{Aut}(Z/P, t, \sigma)$ be the normal subgroup of A consisting of elements which induce the identity on P. Then in Section 4 we determine the structure of A' as a natural semidirect product $A' = \boldsymbol{Z}_2 \ltimes K$. Next in Section 5 we identify the quotient group A/A'. Namely, we show by using some of the main results of [3] that the natural image of the homomorphism $A \to \mathrm{Aut}(P, \sigma)$ coincides with an explicitly determined subgroup $\bar{A} := \mathrm{Aut}(P, \{a_i\}, w)$. In Section 6 we determine the possible structure of quotient group A/A', and hence of $B := A/K$, by using the results of Section 2. As for B we have a natural semidirect product decomposition $A = B \ltimes K$ up to K-conjugation. This will be shown in Section 7, again by using the results of [3]. In Section 8 we give a geometric correspondence of the results obtained so far in terms of twistor spaces to the results in terms of self-dual structures as stated above. Finally in Section 9 we describe the involutions in (4) above in terms of the elementary surfaces in Z and the real structure σ of Z.

2. Symmetric toric surfaces and the associated weighted graphs

2.1.

A Joyce twistor space Z admits a canonical pencil of symmetric toric surfaces, which plays the central role for the study of geometry of Z in general. In this section we give some preliminary and elementary results on symmetric toric surfaces which will be used later and mainly in Section 6.

Let $G = \boldsymbol{C}^* \times \boldsymbol{C}^*$ be the two-dimensional algebraic torus with maximal compact subgroup $K := S^1 \times S^1$. In this paper for simplicity by a *toric surface* we shall mean a projective smooth surface with an effective G-action. Let S be any toric surface and U the unique open G-orbit. Then the complement C of U is a cycle of rational curves $C = C_1 + \cdots + C_d$, $C_i \cong \boldsymbol{P}^1$; that is, two successive curves (in the cyclic sense) intersect transversally at a unique point, and there are no other intersections among two irreducible components of C. Moreover, C is an anticanonical divisor on S and is called the *anticanonical cycle* of S.

To S one associates a weighted circular graph $\Gamma = \Gamma(S)$ by associating to each C_i a vertex with weight $b_i = C_i^2$, the self-intersection number of C_i, and by connecting two vertices by an edge if the corresponding irreducible components intersect. We denote by $\mathrm{Aut}\,\Gamma$ the group of automorphisms of Γ *preserving the weights.* $\mathrm{Aut}\,\Gamma$ is naturally a subgroup of dihedral group D_d of order $2d$.

We call S *symmetric* if $d = 2k$ is even with $k \geq 2$ and $b_i = b_{k+i}, 1 \leq i \leq k$. The condition is equivalent to the existence of an *admissible involution* τ; namely, τ is a holomorphic involution of S such that $\tau(C_i) = C_{k+i}, 1 \leq i \leq k$, which implies that $\tau g \tau^{-1} = g^{-1}$ for any $g \in G$ (see [3, Proposition 2.4]). The involution τ is determined up to multiplication by an element of G. The group $\mathrm{Aut}\,S$ of biholomorphic automorphisms of S contains a distinguished subgroup $\mathrm{Aut}_1 S := G \cup \tau G$, where τG consists of admissible involutions. We also note that the second Betti number of S is given by $2(k-1)$.

2.2.

We call an antiholomorphic involution σ of S *admissible* if $\sigma g \sigma^{-1} = g^*$ as an element of $\mathrm{Aut}\,S$, where $g \to g^*$ is the unique antiholomorphic involution of the group G with fixed point set K. For later use we note the following.

LEMMA 2.1

Let S be a toric surface with open orbit U. We also assume that S is symmetric in (1) and (2) below. Then we have the following.

(1) *For any points $x, y \in U$ (not necessarily distinct) there exists a unique admissible involution τ such that $\tau(x) = y$.*

(2) *If σ is an admissible antiholomorphic involution, $\tau_1 := \sigma\tau\sigma$ is again an admissible involution.*

(3) *If an antiholomorphic involution σ of S satisfies $\sigma g \sigma^{-1} = g^{*-1}$, it is unique up to conjugation by elements of K.*

Proof

(1) *Existence.* Start with any admissible involution τ' with $\tau' x = z$. Take an element $g \in G$ with $gz = y$. Then the admissible involution $\tau := g\tau'$ satisfies $\tau(x) = y$.

Uniqueness. If $g\tau$, $g \in G$, maps x to y as well as τ, we have $(g\tau)(x) = g(y) \neq y$ except when g is the unit element e.

(2) Clearly τ_1 is a holomorphic involution. Moreover, since $\sigma(C_i) = C_{i+k}$ (cf. the proof of [3, Lemma 2.6]), we have $\tau_1(C_i) = C_{i+k}$. The assertion follows.

(3) We proceed as in the proof of [3, Lemma 2.7]. Let σ and σ' satisfy the conditions of (3). Note that g^{*-1} is the complex conjugation in the standard coordinates $G = \boldsymbol{C}^*(z) \times \boldsymbol{C}^*(w)$. Then we conclude that both σ and σ', and hence $h := \sigma^{-1}\sigma'$ also, leave invariant each irreducible component of the anticanonical

cycle; we further get that $h\bar{h} = 1$ or equivalently $h \in K$. Then if we take an element k with $k^2 = h$ in K, we have $\sigma' = k^{-1}\sigma k$. □

2.3.

We continue to assume that S is symmetric. Suppose further that $k \geq 3$, or equivalently, $S \ncong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Then G is identified with the identity component $\mathrm{Aut}_0\, S$ of $\mathrm{Aut}\, S$, and $\mathrm{Aut}\, S$ thus normalizes G. This implies that for any $s \in \mathrm{Aut}\, S$, $s(C)$ is G-invariant and therefore never intersects with the open orbit U. Hence C is preserved by $\mathrm{Aut}\, S$. We have thus the induced homomorphism $u : \mathrm{Aut}\, S \to \mathrm{Aut}\, \Gamma$.

LEMMA 2.2

$\mathrm{Aut}\, S$ is isomorphic to a semidirect product $\mathrm{Aut}\, \Gamma \ltimes G$ in such a way that the quotient map $\mathrm{Aut}\, \Gamma \ltimes G \to \mathrm{Aut}\, \Gamma$ is identified with u above.

Proof

S is expressed as a torus embedding $S = T_N\mathrm{emb}(\Delta)$ for the lattice $N \cong \boldsymbol{Z}^2$ and a fan Δ in $N_{\boldsymbol{R}}$ (cf. [8, Theorem 1.5]). Then the group $\mathrm{Aut}(N, \Delta)$ of automorphisms of the fan is naturally identified with a subgroup of $\mathrm{Aut}\, S$ (see [8, Theorem 1.13]) so that $\mathrm{Aut}\, S$ becomes a semidirect product $\mathrm{Aut}(N, \Delta) \ltimes G$. On the other hand, in our two-dimensional case the fan $\Delta = (N, \Delta)$ determines and is determined by the associated weighted graph in such a way that we have a natural isomorphism $\mathrm{Aut}(N, \Delta) \cong \mathrm{Aut}\, \Gamma$ (cf. [8, Corollary 1.29]). Hence we get a semidirect product decomposition $\mathrm{Aut}\, \Gamma \ltimes G \to \mathrm{Aut}\, \Gamma$ as in the lemma with the quotient map identified with u. □

In what follows we identify $\mathrm{Aut}\, \Gamma$ as a subgroup of $\mathrm{Aut}\, S$ with respect to any fixed torus embedding as above. (The results below are independent of the choice of such an identification.)

2.4.

Since $\mathrm{Aut}\, S$ normalizes G we have the induced homomorphism $v : \mathrm{Aut}\, \Gamma \to \mathrm{Aut}\, G \cong \mathrm{GL}_2(\boldsymbol{Z})$. This is injective since this action is naturally identified with the action on the homology group of the open G-orbit U via the induced action of $\mathrm{Aut}\, S$ on U. By the classification of the finite subgroups of $\mathrm{GL}_2(\boldsymbol{Z})$ we see that $\mathrm{Aut}\, \Gamma$ is a subgroup of either of the dihedral groups D_4 or D_6, and hence is isomorphic to either of D_{2l}, or is a cyclic group of order $2l$, where $l = 1, 2, 3$, where D_2 denotes the four group $\boldsymbol{Z}_2^2 := \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$.

Any admissible involution τ of S defines an element of the center of $\mathrm{Aut}\, \Gamma$ mapping v_i to v_{k+i}. The quotient of Γ by the induced action of τ gives rise to another circular weighted graph $\bar{\Gamma}$ whose weight is $\{b_1, \ldots, b_k\}$. We call $\bar{\Gamma}$ and its weight the reduced weighted graph of S and the reduced weight of Γ, respectively . We have $\mathrm{Aut}\, \bar{\Gamma} = \mathrm{Aut}\, \Gamma / \langle \tau \rangle$, and it is naturally a subgroup of either of $D_l, l = 2, 3$. The original weighted graph Γ is recovered from $\bar{\Gamma}$ by taking the unique unramified double covering. Note that $\bar{\Gamma}$ is in general not a weighted graph of any toric surface.

Summarizing the above discussion we obtain the following lemma. (Also, in the case $k = 2$, the statements are directly verified.)

LEMMA 2.3
For any $k \geq 2$ we have the following.

(1) $\operatorname{Aut}\Gamma$ *is either cyclic of order* $2, 4,$ *or* 6 *or is isomorphic to one of the groups* $D_{2l}, l = 1, 2, 3$.

(2) $\operatorname{Aut}\bar{\Gamma}$ *is either trivial, cyclic of order* $2, 3,$ *or isomorphic to either of the groups* $D_l, l = 2, 3$.

REMARK 2.1
Let $\bar{H}$ be any of the nontrivial groups appearing in Lemma 2.3(2). Then a subgroup of $\mathrm{PGL}_2(\boldsymbol{R})$ which is isomorphic to $\bar{H}$ is unique up to conjugacy unless it is of cyclic of order two. In the latter case there are exactly two subgroups as above up to conjugation, one in $\mathrm{PSL}_2(\boldsymbol{R})$ and the other not.

2.5.
We shall see more in detail how these groups appear in $\operatorname{Aut}\Gamma$ depending on individual Γ. It is also convenient to set $m = k - 2 \geq 0$ in what follows. Let S be a symmetric toric surface. The possible reduced weights for $\Gamma = \Gamma(S)$ for $k = m + 2 \leq 6$ and the corresponding automorphism groups $\operatorname{Aut}\Gamma$ are easily seen to be given in Table 1 (see [3, (2.7)]).

For $m = 0, 1, 2, 3$ the surface and the associated graph are unique and will be denoted by S_m and Γ_m, respectively. The case $m = 0$ or $= 1$ is particularly important as we see below. If $m = 0$ (resp., 1), we have $\operatorname{Aut}\Gamma_m \cong D_{2k}, k = m + 2$, and it contains a unique cyclic group of order $2k$, and two (resp., three) subgroups which are isomorphic to D_2. In the latter case the three subgroups in $\operatorname{Aut}\Gamma_1$ are conjugate to each other, while the two subgroups in $\operatorname{Aut}\Gamma_0$ are not, since one, say, D_2^I, acts transitively on the nodes of C, while the other, say, D_2^{II}, has two orbits each consisting of two nodes. In the latter case if we blow up any two nodes in the same orbit in S_0 we get S_1, and D_2^{II} is realized as one of the three subgroups of $\operatorname{Aut}\Gamma_1$ above.

Table 1

m	k	$(-b_1, \ldots, -b_k)$	$\operatorname{Aut}\Gamma$
0	2	$(0, 0)$	D_4
1	3	$(1, 1, 1)$	D_6
2	4	$(1, 2, 1, 2)$	D_4
3	5	$(2, 1, 3, 1, 2)$	D_2
		$(1, 2, 3, 1, 2, 3)$	$\boldsymbol{Z}_4$
4	6	$(1, 3, 1, 3, 1, 3)$	D_6
		$(2, 1, 4, 1, 2, 2)$	D_2

REMARK 2.2

For each k there exists a unique symmetric toric surface whose reduced weight sequence up to sign is of the form $(2,\ldots,2,1,m,1,2,\ldots,2)$ (up to cyclic permutations). In this case S, or the associated Γ, is called *of type LJ* (LeBrun–Joyce type). S_m and Γ_m are of type LJ for $m=1,2,3$. For a surface of type LJ with $m\geq 3$, $\operatorname{Aut}\Gamma$ is obviously isomorphic to $D_2\cong \boldsymbol{Z}_2^2$. The case $m=2$ is special in the sense that Γ_2 is considered to be of type LJ in two ways since there are two $-m=-2$ in the reduced weights.

2.6.

Now for a toric surface S and a subgroup H of $\operatorname{Aut}S$ an H*-admissible blowup* of S is a blowup $f:\tilde{S}\to S$ of S with center an H-orbit with respect to the induced H-action on the set of nodes of C. In this case the H-action lifts naturally to $\tilde{S}$.

Conversely, for S and H as above an H*-admissible blowdown* of S is a contraction $f:S\to\bar{S}$ of mutually disjoint irreducible components of C with self-intersection number -1 which form an H-orbit with respect to the induced H-action on the set of such (-1)-curves of C. In this case the H-action descends naturally to $\bar{S}$.

Then we call in general a birational morphism $d:S\to\bar{S}$ of toric surfaces with H-action on $\bar{S}$ (resp., S) H*-admissible* if it is a finite succession of H-admissible blowups (resp., blowdowns) with respect to the successively induced H-action. We consider the identity also H-admissible by convention. In either case we can eventually speak of the natural H-actions on both S and $\bar{S}$ making them d-equivariant.

PROPOSITION 2.4

Let S be a symmetric toric surface with associated weighted graph Γ with $m=m(S)\geq 0$. Let H be a subgroup of $\operatorname{Aut}S$ which is mapped injectively into $\operatorname{Aut}\Gamma$ by $u:\operatorname{Aut}S\to\operatorname{Aut}\Gamma$ and which contains an admissible involution τ. Assume that $H\neq\langle\tau\rangle$. Then there exists a birational morphism $d:S\to S_n$ for $n=0$ or $=1$ such that the H-action descends uniquely to S_n making d H-admissible, where $n=0$ if H is isomorphic to D_4, D_2, or $\boldsymbol{Z}_4$ and $n=1$ if H is isomorphic to D_6 or $\boldsymbol{Z}_6$.

REMARK 2.3

For each H as above the surfaces $S_n, n=0,1$, with H-action can be thought of as the unique minimal model for symmetric toric surfaces with H-action. In the case $H=D_2$ two types of H-surfaces should be distinguished according to whether H is of type D_2^I or D_2^{II} on the minimal model S_0. The case $H=\langle\tau\rangle$ is treated in [3, Section 2] where the minimal model is S_0.

Proof

We show the proposition by descending induction on m. First we treat the cases $m=0$ and 1.

When $m=0$, the possibilities for H are $D_6, \boldsymbol{Z}_6$, and D_2 (with two types D_2^I and D_2^{II}). In this case we have only to take h to be the identity of $S=\bar{S}=S_0$. In the case $m=1$ we have to consider the cases $H=D_6, \boldsymbol{Z}_6$, and D_2. For the first two cases we have only to take h again to be the identity of S_1. In the case of D_2 the set of (-1)-curves in C is divided into two H-orbits consisting two and four elements, respectively. We then define h to be the blowdown of the pair of curves in the first orbits to S_0 to obtain a D_2-action of D_2^{II}-type there.

Suppose now that $m \geq 2$. Note that by using admissible H-blowups, one shows easily by induction on m that for any symmetric toric surface S no two irreducible components of C of self-intersection number -1 intersect.

Suppose that the result is true for any symmetric toric surface S' with smaller $m(S')$. Take any irreducible component E of C with $E^2=-1$ in S. Then consider the H-orbit B of E with respect to the action of H on the set of (-1)-curves in C. By the above remark the curves belonging to B do not intersect each other. Thus we may contract these curves to obtain another symmetric toric surface S' with smaller m, which is again symmetric since H contains an admissible involution. Moreover, the original H-action descends to S'. Then by the induction hypothesis we easily derived the desired conclusion. □

2.7.

We shall give more detailed descriptions of the individual cases. Namely, any symmetric toric surface S admitting a holomorphic H-action as in Proposition 2.4 is obtained in the following way.

Case $H=D_{2l}, l=2$ (resp., 3). Start from $S_m, m=0$ (resp., $=1$). The induced action of H on the set N_m of nodes on C is transitive with each stabilizer group isomorphic to $\boldsymbol{Z}_2$. Blow up S_m with center the unique H-orbit in N_m, and obtain another surface $S_{[1]}$ with $4l$ nodes on the anticanonical cycle $C_{[1]}$, on which the lifted H-action is simply transitive. Then blow up $S_{[1]}$ further with center the set of these $4l$ nodes, and obtain a surface $S_{[2]}$ with H-action and with $8l$ nodes on $C_{[2]}$. The H-action on the set $N_{[2]}$ of nodes on $C_{[2]}$ is now free and has two orbits. Take any of the H-orbits, and blow up $S_{[2]}$ H-admissibly with center this orbit, and obtain a new surface $S_{[3]}$ with H-action. In this way after r times blowups one obtains a surface $S_{[r]}$ with $4rl$ nodes in C and r H-orbits among them. The resulting toric surfaces are all symmetric since τ is in H.

The case $H=D_2^I$ is similar. In this case the action on N_0 is already simply transitive so that after r times H-admissible blowups we obtain $4(r+1)$ nodes with $r+1$ orbits. Similar are the cases of $H=\boldsymbol{Z}_{2l}, l=2,3$, with $2(r+1)l$ nodes after r times blowups.

The case $H=D_2^{II}$ is slightly different. We should distinguish two types of H-admissible birational morphisms which lead to the H-actions with different orbit structures. First blow up one of the two H-orbits in N_0 consisting of two nodes. The resulting D_2-action on S_1 still has two orbits, say, A and B, consisting of two and four nodes, respectively.

Type $D_{2,A}^{II}$. We blow up S_1 with center A and get the two H-orbits, on each of which H-action is simply transitive. Continuing the blowups with center any of the orbits successively, after r-times the blowup starting from S_0, we get $4r$ nodes on the resulting surface with D_2-action if $r > 1$.

Type $D_{2,B}^{II}$. We blow up S_1 with center B and then perform the successive blowups with any of the orbits with four nodes as centers, leaving the orbit A untouched. Then we get $4r+2$ nodes after r-times the blowups starting from S_0.

Table 2 gives the values of the invariant $k = m + 2$ of a symmetric toric surface S with H-action obtained after r-times successive H-admissible blowups of S_n, where $n = 0$ or 1 depending on H as in Proposition 2.4. Recall that $2k$ is the number of nodes of the anticanonical cycle C. In Table 2 we have

Table 2

H	D_{2l}	D_2^I	$D_{2,A}^{II}$	$D_{2,B}^{II}$	Z_{2l}
$k = m+2$	$2lr$	$2(r+1)$	$2r$	$2r+1$	$l(r+1)$

$l = 2, 3$ for D_{2l} and Z_{2l}, and in the $D_{2,A}^{II}$-case we have the exceptional case: $k = 3$ if $r = 1$.

2.8.

For later purposes we formulate the result in terms of the automorphism group $\operatorname{Aut}\Gamma$ of the weighted graph Γ. The proof is immediate from the above description.

PROPOSITION 2.5

Let H be any subgroup of D_4 or D_6 containing the center. Let k be an integer with $k \geq 4$. Then there exists a symmetric toric surface S with $k(S) = k$ such that H is isomorphic to a subgroup of $\operatorname{Aut}\Gamma(S)$ if and only if one of the following is true:

(a) *$H = D_{2l}$ or $\boldsymbol{Z}_{2l}$ for $l = 2, 3$, and k is divisible by half of the order of H; and*

(b) *$H \cong D_2 := \boldsymbol{Z}_2^2$.*

REMARK 2.4

(1) Let H be as in Proposition 2.5. Let $n(k)$ be the number of isomorphism classes of symmetric toric surfaces S with $k(S) = k$ and with $H \subseteq \operatorname{Aut}\Gamma(S)$. Then from the above argument it is clear that $\limsup_{k\to\infty} n(k) = \infty$.

(2) We have the parallel statement for the subgroups of $\operatorname{Aut}\bar{\Gamma}(S)$ by considering $H/\langle\tau\rangle$ instead of H in the above proposition.

3. Joyce self-dual structures and the associated twistor spaces

3.1.

For any integer $m \geq 2$ let $M = m\boldsymbol{P}^2$ be the connected sum of m copies of the complex projective plane $\boldsymbol{P}^2$. We fix an effective smooth action of the real two-torus $K = S^1 \times S^1$ on M. Then by Joyce [6] there exists a smooth family of K-invariant self-dual conformal structures $[g]$ on M depending on real $(m-1)$-dimensional parameters.

For any such self-dual structure $(M, [g])$ we have the associated twistor space Z which is a complex manifold of dimension three with a smooth *twistor fibration* $t : Z \to M$ with each fiber a complex submanifold isomorphic to the complex projective line $\boldsymbol{P}^1$, called a *twistor line*. Furthermore Z admits a real structure σ which is a fixed-point free antiholomorphic involution preserving each twistor line. The smooth action of K on M lifts to a biholomorphic action on Z and then extends to a biholomorphic action of its complexfication $G := \boldsymbol{C}^* \times \boldsymbol{C}^*$.

3.2.

We shall summarize the basic structure of Z according to [3]. Recall the natural isomorphism $t^* : \mathrm{Conf}(M, [g]) \cong \mathrm{Aut}(Z, t, \sigma)$ mentioned in the introduction. Under our assumption that $m \geq 2$ we have the natural identifications $K = \mathrm{Conf}_0(M, [g])$ and $G = \mathrm{Aut}_0 Z$, where $\mathrm{Aut}_0 Z$ is the identity component of the biholomorphic automorphism group $\mathrm{Aut}\, Z$.

Let K^{-1} be the anticanonical line bundle on Z. It admits a unique square root $K^{-1/2}$. Denote by $F := |K^{-1/2}|^G$ the linear subsystem of the complete linear system $|K^{-1/2}|$ consisting of all G-invariant members. (Unless Z is of LeBrun–Joyce type, we have $|K^{-1/2}|^G = |K^{-1/2}|$ and F is an invariant of Z alone; see [3, Proposition 6.14].) F is a pencil, and its base locus is reduced and consists of a cycle of rational curves

$$C = C_1^+ + \cdots + C_k^+ + C_1^- + \cdots + C_k^- \tag{1}$$

consisting of $2k$ smooth rational curves $C_i^{\pm}, 1 \leq i \leq k$, where $k := m + 2$.

Let P denote the parameter space of F which is thus a smooth rational curve. F determines a meromorphic map $\varphi : Z \to P$ which fits into the commutative diagram

$$\begin{array}{l} \hat{Z} \ \xrightarrow{\mu}\ Z \\ f \downarrow \swarrow \varphi \\ P \end{array} \tag{2}$$

where $\hat{Z} \subseteq P \times Z$ is the graph of the meromorphic map $\varphi = f\mu^{-1}$ and μ and f are induced by the natural projections on the factors. More intrinsically one may also think of P as a connected component of the set of G-fixed points on the space $\mathrm{Div}Z$ of effective divisors on Z, and the diagram denotes the universal family of G-invariant effective divisors on Z where the fiber $S_a := f^{-1}(a), a \in P$, is naturally identified with its image in Z. (In what follows S_a will denote the fiber over a of f^{-1} and the corresponding surface in Z interchangeably.) By construction S_a has the induced G-action with respect to which S_a becomes naturally a toric surface.

3.3.

The real structure σ on Z induces a natural action, denoted by the same letter, on the diagram (2), where the action on P is nontrivial with a nonempty fixed point set R. It is sometimes convenient to fix an identification:

$$P = \boldsymbol{P}^1 = \{z \in \boldsymbol{C}\} \cup \{\infty\}, \qquad R = \{|z| = 1\}, \qquad \sigma(z) = 1/\bar{z}. \tag{3}$$

There exist exactly k points $a_1, \dots, a_k, k = m+2$, cyclically distributed on R in this order such that $S_i := S_{a_i}$ are precisely the singular members of the system F or, equivalently, the singular fibers of f. The smooth members $S = S_a, a \neq a_i$, are mutually isomorphic symmetric toric surfaces, $S_a, a \in P$; all contain the cycle C, and for $a \neq a_i$, C coincides with the anticanonical cycle of S_a (see [3, Proposition 6.5]). The intersection numbers $(C_i^{\pm})^2$ of $C_i^{\pm}$ in S_a are independent of $a \neq a_i$, and we have $(C_i^+)^2 = (C_i^-)^2 \leq -1$, which will be denoted by b_i with $b_i \leq -1$. Also, Z is of type LJ if and only if so is $S_a, a \neq a_i$ (see [3, Proposition 6.14]).

Let Γ denote the weighted circular graph associated to the toric surface (S_a, C). Then it is up to (weighted) isomorphisms independent of the choice of $a \neq a_i$. Thus $\Gamma = \Gamma(C)$ is an invariant of Z. (It actually depends only on the underlying K-action on M alone.)

3.4.

For each i, the singular fiber S_i is a union of mutually σ-conjugate and biholomorphic toric surfaces $S_i^{\pm}$; their intersection is transversal and the intersection is a twistor line L_i [3, Proposition 6.12]. $L_i, 1 \leq i \leq k$, are unique G-invariant twistor lines on Z, and each of them admits exactly two fixed points of the G-action. We can arrange the numbering of $C_i^{\pm}$ so that these two points are precisely $z_i^{\pm} := C_i^{\pm} \cap C_{i+1}^{\pm}$, which are σ-conjugate to each other. The three curves $L_i, C_i^{\pm}$, and $C_{i+1}^{\pm}$ intersect transversally at $z_i^{\pm}$, and L_i intersects with no other irreducible components of C. Further, by [3, Proposition 6.12], $S_i^{\pm}$ is isomorphic to the toric surface $S_i^{\pm}(\rho)$ given in [3, (2.10)]. Therefore, as explained there its anticanonical cycle of $S_i^{\pm}$ is given by

$$C_1^{\pm} + \cdots + C_i^{\pm} + L_i + C_{i+1}^{\pm} + \cdots + C_k^{\pm}, \tag{4}$$

where the intersection numbers of the irreducible components are given in this order by

$$b_1, \dots, b_{i-1}, b_i + 1, 1, b_{i+1} + 1, b_{i+2}, \dots, b_k. \tag{5}$$

Noting that the isomorphism class of the union S_i is determined by those of the irreducible components $S_i^{\pm}$, we deduce immediately the following.

LEMMA 3.1

The isomorphism classes of the singular fibers $S_i, 1 \leq i \leq k$, are completely determined by the reduced weighted graph $\bar{\Gamma}$ above (and vice versa).

3.5.

Since $K^{-1/2}$ is intrinsic to Z, the action of $\operatorname{Aut} Z$ lifts to a biholomorphic action on the line bundle $K^{-1/2}$; since G is normalized by $\operatorname{Aut} Z$, this action further induces a natural biholomorphic action of $\operatorname{Aut} Z$ on the commutative diagram (2). Or more intrinsically, the natural action of $\operatorname{Aut} Z$ on the space $\operatorname{Div} Z$ of effective divisors on Z leaves invariant $P(\subseteq \operatorname{Div} Z)$ and hence induces a natural action on the universal family of divisors restricted to P. In particular $\operatorname{Aut} Z$ leaves invariant the base locus $C \subseteq Z$ of F or equivalently of φ. Moreover, since the action of $\operatorname{Aut} Z$ maps a smooth fiber to another smooth fiber isomorphically, it induces an automorphism of the associated weighted dual graph. This gives rise to a homomorphism

$$h : \operatorname{Aut} Z \to \operatorname{Aut} \Gamma \tag{6}$$

which factors through the quotient $\operatorname{Aut} Z / \operatorname{Aut}_0 Z$ as $\operatorname{Aut}_0 Z \cong G$ acts trivially on Γ. Similarly since σ commutes with K-action and hence normalizes G-action, it also has the induced antiholomorphic action on the diagram (2) which commutes with the induced action of $\operatorname{Aut}(Z, t, \sigma)$.

4. Structure of the relative automorphism group

4.1.

We now start our study of the automorphism group $\operatorname{Aut}(Z, t, \sigma)$ for a fixed Joyce twistor space (Z, t, σ). In this section we mainly study the automorphisms which preserve each fiber S_a. Recall that we are assuming that $m \geq 2$.

LEMMA 4.1

Any automorphism of Z lifts to a unique automorphism of $\hat{Z}$. Conversely, any automorphism $\hat{g}$ of $\hat{Z}$ descends to a unique automorphism of Z.

Proof

The first assertion is already explained via the interpretation of the diagram (2) in terms of the subfamily of the universal family of divisors on Z. One may also note that μ is the blowup with center C with exceptional divisor $E := \mu^{-1}(C) \cong C \times P$ (see [3, Proposition 6.5, 1]), and C is invariant by $\operatorname{Aut} Z$ as the invariant part of the base locus of the fundamental system $|K^{-1/2}|^G$, where we use the assumption $m \geq 2$ so that $G(= \operatorname{Aut}_0 Z)$ is normalized by $\operatorname{Aut} Z$.

For the second assertion we first note that it is true for the elements of the identity component $\operatorname{Aut}_0 \hat{Z}$ (cf. [13]). In particular we have $\operatorname{Aut}_0 \hat{Z} \cong \operatorname{Aut}_0 Z \cong G$. Thus $\operatorname{Aut} \hat{Z}$ also normalizes G. Thus it maps fibers of f to other fibers, thus inducing a natural homomorphism $\beta : \operatorname{Aut} \hat{Z} \to \operatorname{Aut} P$, and interchanges the irreducible components $E_s = C_s \times P$ of E, $1 \leq s \leq 2k$, where $C_s = C_i^+$ for $1 \leq i \leq k$, and $= C_i^-$ for $k+1 \leq i \leq 2k$.

Now take any element $\hat{g} \in \operatorname{Aut} \hat{Z}$. Assume that $\hat{g}$ maps $C_s \times P$ to $C_t \times P$ which takes the form $\hat{g}(d, p) \to (h(d, p), \bar{g}(p))$, where $\bar{g}$ is the image of $\hat{g}$ by β. If we fix p, the map $d \to h(d, p) \in C_t$ is a holomorphic map from C_s to C_t, and hence it defines a holomorphic map H from P to the space of holomorphic maps $\mathrm{Hol}(C_s, C_t)$ from C_s to C_t. Since $\mathrm{Hol}(C_s, C_t)$ is isomorphic as a complex manifold to $\mathrm{PGL}_2(\boldsymbol{C})$ and is affine, the map H must be constant; that is, $h = h(d)$ is independent of p. Hence $\hat{g}$ descends to an automorphism of Z. □

4.2.

By identifying $\mathrm{Aut}\, Z$ and $\mathrm{Aut}\, \hat{Z}$ by the above lemma we now get the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Aut}(Z/P, t, \sigma) & \longrightarrow & \mathrm{Aut}(Z, t, \sigma) & \xrightarrow{\beta} & \mathrm{Aut}(P, \sigma) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{Aut}(Z/P) & \longrightarrow & \mathrm{Aut}\, Z & \xrightarrow{\beta} & \mathrm{Aut}\, P \end{array}$$

Here $\mathrm{Aut}(Z/P)$ is the subgroup of $\mathrm{Aut}\, Z$ consisting of elements which preserve each fiber of f, or each member S_a of F, according as they are considered to be automorphisms of $\hat{Z}$ or Z, the vertical arrows are inclusions, with each term of the first line denoting the subgroups consisting of elements which commute with σ and t, and β is the natural homomorphism defined above.

We note that sometimes the role of t in $\mathrm{Aut}(Z, t, \sigma)$ is superfluous.

LEMMA 4.2

If an element g of $\mathrm{Aut}\, Z$ commutes with σ and leaves invariant a twistor line, it is contained in $\mathrm{Aut}(Z, t, \sigma)$.

Proof

Consider the induced action of g on the smooth part D_0 of the Douady space of Z. Then σ induces an antiholomorphic involution of D_0 whose fixed point set F is a smooth submanifold of D_0. By assumption g permutes the connected components of F and fixes a point of M in D_0, which implies that g preserves M and hence that g maps any twistor line to another twistor line. Namely, $g \in \mathrm{Aut}(Z, t, \sigma)$. □

4.3.

We now consider the natural homomorphism $h : \mathrm{Aut}\, Z \to \mathrm{Aut}\, \Gamma$ in (6) and the induced one $\bar{h} := \pi h : \mathrm{Aut}\, Z \to \mathrm{Aut}\, \bar{\Gamma}$, where $\pi : \mathrm{Aut}\, \Gamma \to \mathrm{Aut}\, \bar{\Gamma}$ is the natural projection with kernel isomorphic to $\boldsymbol{Z}_2$. We identify the kernels of $\bar{h}$ and h as follows.

LEMMA 4.3

(1) *We have* $\ker \bar{h} = \mathrm{Aut}(Z/P)$.
(2) *We have* $\ker h = G = \mathrm{Aut}_0\, Z$.
(3) *Any element α of* $\mathrm{Aut}\, Z$ *which centralizes G is an element of G.*

Proof

(1) Let g be an element of $\ker \bar{h}$. Then either g preserve all $C_i^{\pm}$ or g interchanges $C_i^{\pm}$ for all i. On the other hand, since g normalizes G, g maps G-orbits to G-orbits. In particular it permutes $L_i, 1 \le i \le k$, among themselves. However, since g either preserves all $z_i^{\pm}$ or interchanges $z_i^{\pm}$ for all i, and $z_i^{\pm}$ lies both on L_i but not on any of $L_j, j \ne i$, g must preserve each L_i. This implies that g must preserve each singular fiber S_i, which in turn implies that the induced element $\bar{g} \in \operatorname{Aut} P$ preserves each a_i. Since $k = m+2 \ge 3$, $\bar{g}$ must be the identity. Thus $g \in \operatorname{Aut}(Z/P)$.

Conversely, take any element g of $\operatorname{Aut}(Z/P)$. Then since g preserves each singular fiber S_i, it preserves each G-invariant twistor line L_i, which is the singular locus of S_i. On the other hand, $C_i^{\pm}$ are the unique curves among $C_j^{\pm}, 1 \le j \le k$, which intersects both L_i and L_{i-1}. Therefore g maps $C_i^{\pm}$ to either of $C_i^{\pm}$. This implies that g is in $\ker \bar{h}$.

(2) Let g be an element of $\ker h \subseteq \operatorname{Aut}(Z/P)$. It suffices to show that $g \in G$. By assumption g must preserve all the irreducible components $C_i^{\pm}$. Therefore for any $a \in P, a \ne a_i$, the induced element $g_a \in \operatorname{Aut} S_a$ belongs to the identity component $\operatorname{Aut}_0 S_a \cong G$ (see [3, Lemma 2.1]). Fix a general point $o \in P$. Take an element $g' \in G$ such that $g_o = g'_o$ on S_o. Then taking $g'^{-1} g$ instead of g it suffices to show that g is the identity, assuming that $g_o = e_o$, the unit element. In particular $g_o \mid C_i^{\pm} \times o$ is the identity $\mathrm{id}_i^{\pm}$ of $C_i^{\pm}$. Let $E = \mu^{-1}(C)$ with irreducible components $E_i^{\pm} \cong C_i^{\pm} \times P$ (see [3, Proposition 6.5]). Then g preserves each E_i. Since $g \in \operatorname{Aut}(Z/P)$, $g \mid E_i$ defines a holomorphic map $P \to \operatorname{Aut} C_i^{\pm} \cong \mathrm{PGL}_2 \boldsymbol{C}$, $a \to g_{a,i}^{\pm}$, which must be a constant map as before. Thus $g_{a,i}^{\pm} = g_{o,i}^{\pm} = \mathrm{id}_{o,i}^{\pm}$ for any $a \in P$, that is, as an automorphism of Z $g \mid C_i$ is the identity for each i. This implies that g_a is the identity on S_a for any a and hence so is g on Z.

(3) We first show that $\alpha \in \ker \bar{h}$. Otherwise, α maps $C_i^{\pm}$ to $C_j^{\pm}$ for some i and j with $j \ne i$. Take any element $g \in G$ which pointwise fixes $C_i^{\pm}$ and does not fix any general point on $C_j^{\pm}$. Then for any general point z of $C_i^{\pm}$, $g\alpha(z) = \alpha g(z) = \alpha(z)$. This contradicts the assumption that α does not fix a general point of $C_j^{\pm}$. Hence $\alpha \in \ker \bar{h}$. Then by (1), α preserves each fiber S_a of f. For a general point $s \in S_a$ take an element α of G such that $\alpha g(s) = s$. Since αg also centralizes G, this implies that αg fixes any point of the open orbit of S_a. Since a is arbitrary, we conclude that αg is the identity and $\alpha \in G$. $\square$

In particular, restricting h to $\operatorname{Aut}(Z, t, \sigma)$ (and still denoting it by the same letter h) $\bar{h}$ factors through $\operatorname{Im} \beta$ via $\bar{h}_\beta$ in the following commutative diagram of exact sequences:

$$
\begin{tikzcd}
1 \arrow[r] & \operatorname{Aut}(Z/P, t, \sigma) \arrow[r] \arrow[d, "h'"'] & \operatorname{Aut}(Z, t, \sigma) \arrow[r, "\beta"] \arrow[d, "h"'] & \operatorname{Im} \beta \arrow[r] \arrow[d, "\bar{h}_\beta"'] & 1 \\
1 \arrow[r] & \boldsymbol{Z}_2 \arrow[r] & \operatorname{Aut} \Gamma \arrow[r, "\pi"] & \operatorname{Aut} \bar{\Gamma} \arrow[r] & 1
\end{tikzcd}
\tag{7}
$$

where the kernels of h and $h' := h \mid \mathrm{Aut}(Z/P,t,\sigma)$ both coincide with $K = \mathrm{Aut}_0(Z,t,\sigma) = \mathrm{Aut}_0(Z/P,t,\sigma), \mathrm{Aut}_0$ denoting the identity component.

4.4.

The main purpose of this section is to prove the following.

PROPOSITION 4.4

There exists an involution τ in $\mathrm{Aut}(Z/P)$ such that on any smooth fiber S_a its restriction τ_a belongs to $\mathrm{Aut}_1 S_a$ (cf. Section 2). Moreover, for any $g \in G$ we have $\tau g \tau^{-1} = g^{-1}$ in $\mathrm{Aut}(Z/P)$, and τ can always be taken to commute with σ so that $\tau \in \mathrm{Aut}(Z/P,t,\sigma)$.

By the map h' any τ as in the theorem gives a nonzero element of $\boldsymbol{Z}_2$. Together with the diagram chase in (7) we get the following.

COROLLARY 4.5

In (7) h' is surjective and $\bar{h}_\beta$ is injective. Also we have $\mathrm{Aut}(Z/P,t,\sigma) = \ker \bar{h}$ in $\mathrm{Aut}(Z,t,\sigma)$.

Proof of Proposition 4.4

We fix a general twistor line $L \subseteq Z$. In particular L is not left fixed by any $g \in G$ other than e. Then for all $a \in P$, $L \cap S_a$ consists of points x_a and y_a, where the case $x_a = y_a$ occurs exactly at two points a and $\sigma(a)$ with $a \notin R$ (cf. [3, Proposition 6.15, 1]). Moreover, if $a \neq a_i$, x_a and y_a belong to the open orbit U_a, while for $a = a_i$, x_a and y_a belong to different components $S_i^{\pm}$ of S_i, say, $x_a \in S_i^+$ and $y_a \in S_i^-$.

Now by Lemma 2.1(1) for each $a \neq a_i$ we have a unique $\tau = \tau_a \in \mathrm{Aut}_1 S_a$ such that $\tau(x_a) = y_a$, which is easily seen to depend holomorphically on a.

We show that the map $a \to \tau_a$ extends holomorphically across a_i for each i so that we obtain a holomorphic family of fiberwise holomorphic automorphisms for f. This clearly gives an automorphism τ of desired type apart from the compatibility with t and σ.

First, by [3, Proposition 7.15] in a neighborhood V of a_i in P we can find a holomorphic family of relative involutions $a \to \tau'_a \in \mathrm{Aut}\, S_a$ such that for $a \neq a_i$ we have $\tau'_a \in \mathrm{Aut}_1 S_a$, and τ'_{a_i} interchanges the two components $S_i^{\pm}$,

Suppose that $\tau'_a(x_a) = z_a$. In particular $z_{a_i} \in S_i^-$. Then by a standard argument we show that there exists a unique holomorphic map $V \to G$, $a \to g_a$, such that $g_a(z_a) = y_a$ even for $a = a_i$ since z_{a_i}, y_{a_i} are in the open orbit of S_i^-. (Note that G acts on the open orbits U_a, $a \neq a_i$, and $U_{a_i}^-$ simply transitively and $\bigcup_{a \neq a_i} U_a \cup U_{a_i}^-$ is open in $f^{-1}(V)$.) Hence if we replace τ'_a by $g_a \tau'_a$ we get a family of holomorphic involutions which coincides with τ_a for each a except for $a = a_i$. This shows that τ_a is extendible across a_i as asserted. Recall that by Lemma 4.3 the resulting involution τ on $\hat{Z}$ descends to Z. Further, the formula $\tau g \tau^{-1} = g^{-1}$ follows since it holds on general fibers.

It remains to show the compatibility of τ with t and σ. Since τ preserves the twistor line L chosen at the beginning of the proof, by Lemma 4.2 it is compatible with t if it is compatible with σ. So we show the latter. Note that x_a and y_a are interchanged by σ when $a \in R$ since L and S_a are both σ-invariant and σ is fixed-point free. Thus when $a \in R$, if $\tau(x_a) = y_a$, we have $\sigma\tau\sigma(x_a) = y_a$. Since $\sigma g \sigma^{-1} = g^*$ for $g \in G$, by Lemma 2.1(1), (2) we have $\sigma\tau_a\sigma = \tau_a$ for $a \in R$. Since $\sigma\tau\sigma$ and τ are holomorphic, the equality then holds true on the minimal complex subspace of Z containing $f^{-1}(R)$, that is, on Z. □

We call any involution τ as in the proposition also a *(real) admissible* involution. In view of (7) we also get the following.

COROLLARY 4.6

We have $\mathrm{Aut}(Z/P, t, \sigma) = K \cup \tau K$ *for any admissible involution* τ*, where* τK *consists of admissible involutions. Two admissible involutions are conjugate to each other by an element of* K*.*

5. Identification of the quotient group

5.1.

In this section we determine the quotient group $\mathrm{Aut}(Z, t, \sigma)/\mathrm{Aut}(Z/P, t, \sigma)$ identifying it with the image of $\beta : \mathrm{Aut}(Z, t, \sigma) \to \mathrm{Aut}(P, \sigma)$ in (7). We proceed as follows. The diagram (2) can be considered as the canonical model for the meromorphic quotient of Z by the complexfication G of K, with P as the quotient. We then put *weights* on this quotient, in analogy with the *weights* on the K-quotient N of M via Orlik–Raymond invariants as in the introduction.

Let $a_i, 1 \le i \le k$, be the points of R corresponding to the singular fibers S_i of f distributed cyclically on R. Denote by (a_{i-1}, a_i), $1 \le i \le k$, $a_0 = a_k$, the open arc from a_{i-1} to a_i on R. We base our definition of weights on the following facts, essentially proved in [3].

LEMMA 5.1

Let a be any point on the arc (a_{i-1}, a_i). Then the set T of twistor lines contained in the smooth toric surface S_a is nonempty, and twistor lines in T are mutually K-equivalent. Each of these twistor lines intersects with both of $C_i^{\pm}$ but with none of $C_j^{\pm}$, $j \ne i$.

Proof

The proof is essentially given in the proof of [3, Proposition 6.16]. In fact, the nonemptyness of T is shown in the “Surjectivity” part of the proof, and the K-homogeneity of T is in the “Injectivity” part (cf. also the remark following that proposition). The latter implies that S_a admits a G- and σ-equivariant fiber space structure $\nu_i : S_a \to C_i$ onto a smooth rational curve C_i with G- and σ-actions such that (1) $C_i^{\pm}$ is mapped isomorphically onto C_i, (2) the twistor lines in T coincide with the fibers of points of the σ-real part $C_{i,R}$, (3) $C_{i,R}$ is a single

K-orbit, and (4) all the other $C_j^{\pm}$ are mapped to points of $C_i \setminus C_{i,R}$. From this the remaining assertions follow (see also [3, Lemmas 2.5, 2.6]). $\square$

5.2.

In view of Lemma 5.1, we may naturally attach to each open arc (a_{i-1}, a_i) the self-intersection number b_i of $C_i^{\pm}$ considered as the *weight* of (a_{i-1}, a_i). Let $\mathrm{Aut}(P, \{a_i\}, w) = \mathrm{Aut}(P, \sigma, \{a_i\}, w)$ be the subgroup of $\mathrm{Aut}(P, \sigma)$ consisting of elements which preserve the set $\{a_i\}$ on R and leave the corresponding weights unchanged. (Since a_i are real points and $k \geq 4$ we may omit σ in the notation.)

LEMMA 5.2

The image by β of $\mathrm{Aut}(Z, t, \sigma)$ is contained in the subgroup $\mathrm{Aut}(P, \{a_i\}, w)$ above.

Proof

For any $g \in \mathrm{Aut}(Z, t, \sigma)$ let $\bar{g} = \beta(g)$. Suppose that for $a \in (a_{i-1}, a_i)$ we have $\bar{g}(a) = a' \in (a_{j-1}, a_j)$. Then we have to show that $b_i = b_j$. We have $g(S_a) = S_{a'}$, and if L is a twistor line on S_a, $g(L)$ is a twistor line on $S_{a'}$. Since L intersects with $C_i^{\pm}$ by Lemma 5.1, $g(L)$ intersects with $g(C_i^{\pm})$. Then again by Lemma 5.1 we must have $g(C_i^{\pm}) = C_j^{\pm}$ or $= C_j^{\mp}$. Therefore $b_i = (C_i^{\pm})^2 = (C_j^{\pm})^2 = b_j$ as desired. $\square$

We associate to the pointed space $(R, \{a_i\})$ the (dual) weighted graph whose vertices u_i correspond to the arcs (a_{i-1}, a_i) with weights b_i, $1 \leq i \leq k$, and whose edges correspond to the points a_i which connect the vertices u_i and u_{i+1}. Let $\Gamma(R, \{a_i\})$ be the resulting circular weighted graph. Then the following lemma follows easily from Lemma 5.1 and the definitions.

LEMMA 5.3

$\Gamma(R, \{a_i\})$ is naturally identified with the reduced weighted graph $\bar{\Gamma}$ as constructed from Γ in (2.4), and hence we have a natural injective homomorphism $\bar{\eta} : \mathrm{Aut}(P, \{a_i\}, w) \to \mathrm{Aut}\,\bar{\Gamma}$ such that $\bar{\eta}\beta = \bar{h}$.

5.3.

Now the main purpose of this section is to show the following.

PROPOSITION 5.4

The image of $\beta : \mathrm{Aut}(Z, t, \sigma) \to \mathrm{Aut}(P, \sigma)$ coincides with the subgroup $\mathrm{Aut}(P, \{a_i\}, w)$.

The commutative diagram (7) may thus be written as

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Aut}(Z/P, t, \sigma) & \longrightarrow & \mathrm{Aut}(Z, t, \sigma) & \xrightarrow{\beta} & \mathrm{Aut}\big(P, \{a_i\}, w\big) & \longrightarrow & 1 \\
 & & \Big\downarrow h' & & \Big\downarrow h & & \Big\downarrow \bar{\eta} & & \\
1 & \longrightarrow & \boldsymbol{Z}_2 & \longrightarrow & \mathrm{Aut}\,\Gamma & \xrightarrow{\pi} & \mathrm{Aut}\,\bar{\Gamma} & \longrightarrow & 1
\end{array}
\tag{8}
$$

where the kernels of h and h' both coincide with K, h' is surjective, and $\bar{\eta}$ is injective. The main point of proof of the proposition is contained in the following lemma which is one of the main results, Theorem 8.1, of [3].

LEMMA 5.5
For $n=1,2$ let Z_n be two Joyce twistor spaces over $m\boldsymbol{P}^2$ (coming from the same K-action on $M=m\boldsymbol{P}^2$), and let $f_n:\hat{Z}_n\to P$ be the associated fiber space as in the diagram (2). Suppose that the set of points $a_1,\dots,a_k$, $k:=m+2$, on the real part R of P corresponding to the singular fibers of f_1 and f_2 coincide. If the singular fibers of f_1 and f_2 over a_i are isomorphic for each i, there exists an isomorphism $\hat{j}:\hat{Z}_1\to\hat{Z}_2$ over P, which descends to an isomorphism $j:Z_1\to Z_2$ of twistor spaces, which commutes with the real structures σ_n of Z_n, and which maps twistor lines to twistor lines.

An important remark here is that the second assumption can be rephrased as follows.

LEMMA 5.6
Let $Z_n, f_n:\hat{Z}_n\to P$, $n=1,2$, be as in Lemma 5.5 with common critical values $a_i\in P$, $1\le i\le k$. Then singular fibers of f_n for each a_i are mutually isomorphic if the corresponding circular weighted graphs $\bar{\Gamma}_n=\Gamma(R,\{a_i\})_n$ defined above coincide for $n=1$ and 2.

Proof
Note that since $k=m+2\ge 4$, the real structure σ on P, and hence its real part R, is independent of $n=1,2$. Hence the last condition makes sense. Now in view of our definition of weights the assertion follows from Lemma 3.1. $\square$

Proof of Proposition 5.4
Let $\bar{g}\in\mathrm{Aut}(P,\{a_i\},w)$ be any element. Take the pull-back $\hat{Z}':=\bar{g}^*\hat{Z}:=\hat{Z}\times_P P$ of $f:\hat{Z}\to P$ by the isomorphism $\bar{g}:(P,\sigma)\to(P,\sigma)$. The natural projections to the factors give two natural morphisms $\hat{g}:\hat{Z}'\to\hat{Z}$ and $f':\hat{Z}'\to P$ such that $f\hat{g}=f'\bar{g}$, where $\hat{g}$ is isomorphic. Moreover, there exist natural G- and σ-actions on $\hat{Z}'$, which makes $\hat{g}$ both G- and σ-equivariant, where G acts trivially on P. Since $\hat{g}$ is an isomorphism of complex spaces, we may blow down $E':=\hat{g}^{-1}(E)\cong E$ to a cycle C' of rational curves in a complex manifold Z' such that the isomorphism $\hat{g}$ descends to a G- and σ-equivariant isomorphism of complex manifolds $\tilde{g}:Z'\to Z$.

Then $\tilde{g}$ induces an isomorphism of the Douady spaces $\tilde{g}_D:D'\to D$ of Z' and Z, commuting with the induced real structures σ' and σ on D. Let M' be the real subspace of D' which is mapped by $\tilde{g}_D$ diffeomorphically and K-equivariantly onto M, considered as a subspace of D parameterizing the twistor lines on Z. Then the twistor fibration $t:Z\to M$ induces a smooth fibraiton $t':Z'\to M'$ such that $(t':Z'\to M',\sigma)$ becomes a twistor space with K-action which is isomorphic to the original one. Then (Z', f') gives the meromorphic G-quotient diagram for Z' corresponding to (2).

Now we compare two fiber space f and f' over the same base space P. Since $\bar{g} \in \mathrm{Aut}(P, \{a_i\}, w)$, the singular fibers of f and f' are precisely the same over the points a_i, and since $\bar{g}$ is weight preserving, by Lemma 5.6 the corresponding singular fibers are mutually isomorphic. Thus by Lemma 5.5 there exists an isomorphism $j : Z \to Z'$ over (the identity of) P of complex manifolds which commutes with the real structures and which maps twistor lines to twistor lines. Then the composite map $\tilde{g}j : Z \to Z$ gives an automorphism of Z which lifts the given automorphism $\bar{g}$ of P commuting with real structure σ and mapping twistor lines to twistor lines. Namely, $\tilde{g}j$ gives the desired lift to $\mathrm{Aut}(Z, t, \sigma)$ of $\bar{g}$. □

5.4.

We summarize what we have obtained in the following theorem.

THEOREM 5.7

Let Z be a Joyce twistor space associated with a smooth K-action on $M = m\boldsymbol{P}^2$ with $m \geq 2$. Then there exists a natural surjective homomorphism β : $\mathrm{Aut}(Z, t, \sigma) \to \mathrm{Aut}(P, \{a_i\}, w)$ whose kernel $\mathrm{Aut}(Z/P, t, \sigma)$ is isomorphic to $\boldsymbol{Z}_2 \ltimes K$.

The structure of $\mathrm{Aut}(P, \{a_i\}, w)$ will be determined in the next section.

6. Structure of the quotient groups

6.1.

In this section we shall study the structure of the quotient groups

$$B(Z, t, \sigma) := \mathrm{Aut}(Z, t, \sigma)/K \qquad \text{and} \qquad B(Z/P, t, \sigma) := \mathrm{Aut}(Z/P, t, \sigma)/K$$

which, by (8), fit into the following commutative diagram of exact sequences giving the central extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B(Z/P, t, \sigma) & \longrightarrow & B(Z, t, \sigma) & \xrightarrow{\bar{\beta}} & \mathrm{Aut}(P, \{a_i\}, w) & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle \eta'} & & \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle \bar{\eta}} & & \\ 1 & \longrightarrow & \boldsymbol{Z}_2 & \longrightarrow & \mathrm{Aut}\,\Gamma & \xrightarrow{\pi} & \mathrm{Aut}\,\bar{\Gamma} & \longrightarrow & 1 \end{array} \tag{9}$$

where η' is isomorphic, and η and $\bar{\eta}$ are injective. Moreover, the generator of $B(Z/P, t, \sigma)$ is given by the image of any admissible involution $\tau \in \mathrm{Aut}(Z, t, \sigma)$.

The possible structure of $\mathrm{Aut}\,\Gamma$ is given in Proposition 2.5. As noted in Remark 2.4(2), this gives the structure of $\mathrm{Aut}\,\bar{\Gamma}$ as follows. (See Table 2 for the values of k, which are also the number of vertices of $\bar{\Gamma}$. We assume as before that $k \geq 4$.)

LEMMA 6.1

Suppose that $\operatorname{Aut}\bar{\Gamma}$ *is nontrivial. Then it is isomorphic to one of the following groups: the dihedral group* D_l *of order* $2l$, $l = 1, 2, 3$, *or a cyclic group* $\boldsymbol{Z}_l$ *of order* l, *for* $l = 2, 3$, *where by convention* D_1 *is the group of order two generated by a reflection of* $\bar{\Gamma}$. *The number* k *is divisible by* $2l$ *for* D_l, *except for the case* D_1 *with* k *odd, and by* l *for* $\boldsymbol{Z}_l$.

6.2.

$\operatorname{Aut}\bar{\Gamma}$ depends only on m and w, while the structure of $\operatorname{Aut}(P, \{a_i\}, w)$ depends on the analytic isomorphism class of the point sequence a_i, $1 \le i \le k$, on R. Thus the injection $\bar{\eta}$ is in general not surjective, but for the given invariants m and w it can always be made surjective by a suitable choice of a twistor space.

PROPOSITION 6.2

For the given K*-action on* $M = m\boldsymbol{P}^2$, $m \ge 2$, *among the associated Joyce twistor spaces there always exist those for which the above injective homomorphism* $\bar{\eta}$: $\operatorname{Aut}(P, \{a_i\}, w) \to \operatorname{Aut}\bar{\Gamma}$ *is isomorphic.*

Proof

We shall give the proof only for the case $\operatorname{Aut}\bar{\Gamma} \cong D_l$, $l = 1, 2, 3$ since the cyclic case can be similarly treated. By Lemma 6.1 for $l = 2, 3$ we may write $k = 2rl$ for some integer r, while for $l = 1$, k can take an arbitrary value. The description after the proof of Proposition 2.4 shows that $\operatorname{Aut}\bar{\Gamma}$ acts freely on the set of edges of $\bar{\Gamma}$ unless $l = 1$ and k is odd. Here by Lemma 5.3 the edges correspond to the points a_i on R. Since $m \ge 2$, we have $r \ge 3$.

Let $\eta = e^{\pi\sqrt{-1}/l}$. We divide R into $2l$ arcs $A_i := (\eta^i, \eta^{i+1})$, $0 \le i \le 2l - 1$. Then with respect to the realization of D_l in $\operatorname{Aut}(P, \sigma)$ (which is unique up to conjugacy by Remark 2.1.) as the subgroup $\langle \rho, \phi \rangle$ generated by $\rho(z) = e^{2\pi\sqrt{-1}/l} z$ and $\phi(z) = 1/z$, D_l acts simply transitively on the set of arcs A_i, and the induced homomorphism $\operatorname{Aut}(P, \{a_i\}, w) \to \operatorname{Aut}\bar{\Gamma}$ is naturally identified with $\bar{\eta}$.

We first exclude the case of D_1 with k odd. Then we may write $k = 2rl$ for any $l = 1, 2, 3$. We take mutually distinct points $d_1, \ldots, d_r$ arbitrarily in A_0 and define $a_1, \ldots, a_k$ to be its images by elements of D_l, arranged counterclockwise in this order on R. Then clearly D_l preserves the set $\{a_1, \ldots, a_k\}$, and D_l is realized as a subgroup of $\operatorname{Aut}(P, \{a_i\}, w)$ which is mapped surjectively onto $\operatorname{Aut}\bar{\Gamma}$. For the case D_1 with odd k, say, $k = 2r + 1$, we have only to put $a_k = 1$ and take $a_1, \ldots, a_{r+1}$ arbitrarily in A_0 and $a_{r+1+i} = -a_i$ for $i = 1, \ldots, r+1$ and can proceed as above.

It remains to note that by combining [3, Theorem 9.1] (cf. also Section 8 below) and the construction of Joyce [6] it follows that any sequence $\{a_i\}$ on R can be realized by some Joyce twistor space with the given w, that is, with the given smooth K-action on $M = m\boldsymbol{P}^2$. $\square$

REMARK 6.1

Let H be any subgroup of D_4 or D_6 containing the center. Let $\nu(m)$ be the number of diffeomorphism classes of the K-action on $m\boldsymbol{P}^2$ such that there exists a Joyce twistor space Z over $m\boldsymbol{P}^2$ associated with this K-action and that H is isomorphic to a subgroup of $B(Z/P,t,\sigma)$. Then from Proposition 6.2 and Remark 2.4 we have $\limsup_{m\to\infty}\nu(m)=\infty$.

EXAMPLE 6.1

Suppose that Γ is of type LJ with k even (resp., odd) and $k>4$. Then $\mathrm{Aut}\,\bar\Gamma\cong D_1\cong\boldsymbol{Z}_2$, and $\bar\eta:\mathrm{Aut}(P,\{a_i\},w)\to\mathrm{Aut}\,\bar\Gamma$ is isomorphic if and only if the sequence $\{a_i\}$ is equivalent under the action of $\mathrm{Aut}(P,\sigma)$ to a sequence distributed symmetrically with respect to the diameter of $D=\{|z|\le 1\}$ passing through the middle point of a_k and a_1 (resp., through the point a_1).

6.3.

If $k=4$, Γ is unique and is of type LJ. However, since the associated reduced weight sequence is $(1,2,1,2)$ in this case, we may think of Γ as of LJ type in two ways, depending on which of the 2's corresponds to the maximal component. In this case the self-dual structures and the associated twistor spaces are exactly those constructed by Poon [11], and a stronger result holds true. We have $\mathrm{Aut}\,\bar\Gamma\cong D_2=\boldsymbol{Z}_2^2$.

PROPOSITION 6.3

Suppose that $k=4$, that is, $m=2$. Then for any *Joyce twistor space Z the homomorphism $\bar\eta:\mathrm{Aut}(P,\{a_i\},w)\to\mathrm{Aut}\,\bar\Gamma$ is always isomorphic independently of the choice of the four points a_i.*

Proof

When the sequence of points is of the form

$$(a_1,a_2,a_3,a_4)=(e^{i\theta},-e^{-i\theta},-e^{i\theta},e^{-i\theta}),\quad 0<\theta<\pi/2,\tag{10}$$

$\bar\eta$ is isomorphic by the proof of Proposition 6.2. The cross ratio of any general sequence (a_1,a_2,a_3,a_4) we consider has value in the interval $(1,\infty)$, and for a sequence of the form (10) we compute its value to be $1/\sin^2\theta>1$. We conclude from this that any sequence of four points arranged cyclically on R is $\mathrm{Aut}(P,\sigma)$-equivalent to some sequence of four points in (10). The proposition follows. $\square$

Together with the general results obtained so far this implies the following corollary, which gives the structure of $\mathrm{Aut}(Z,t,\sigma)$ for the twistor spaces of Poon [11] from the viewpoint of Joyce twistor spaces (see Theorem 7.1 below for the semidirect product structure). The result was originally obtained in [5] in terms of LeBrun's construction of self-dual metrics.

COROLLARY 6.4

When $m = 2$, we always have $\mathrm{Aut}(Z, t, \sigma) \cong D_4 \ltimes K$.

REMARK 6.2

The above formula for the values of cross ratios shows that no two sequences in (10) are equivalent under $\mathrm{Aut}(P, \sigma)$. Hence together with [3, Theorem 8.1] we see that θ, $0 < \theta < \pi/2$, gives a moduli parameter for the Poon–Joyce twistor spaces.

7. Semidirect product structures

7.1.

In this section we shall show that there exist natural splittings of the defining exact sequence for $B(Z, t, \sigma)$:

$$1 \longrightarrow K \longrightarrow \mathrm{Aut}(Z, t, \sigma) \xrightarrow{\delta} B(Z, t, \sigma) \longrightarrow 1. \tag{11}$$

Recall that $B(Z, t, \sigma)$ is isomorphic to one of the subgroups of D_4 or D_6 containing their center $C(D) \cong \boldsymbol{Z}_2$. Then by (9), $\mathrm{Aut}(P, \{a_i\}, w)$ is naturally isomorphic to $B(Z, t, \sigma)/C(D)$ and hence is isomorphic to some subgroup of D_l, $l = 2$ or 3. Thus it always preserves at least one set $\{a, \sigma(a)\}$, $a \notin R$, of a mutually σ-conjugate pair of points on P. (Such a set is unique unless $\mathrm{Aut}(P, \{a_i\}, w)$ is of order two and is generated by a reflection. With respect to the realization of the group as in the proof of Proposition 6.2 one may take $a = 0$ and $\sigma(a) = \infty$.) Then there exists a twistor line L in Z with $L \cap C = \emptyset$ such that its proper transform in $\hat{Z}$ (denoted by the same letter) is a ramified double covering over P branched exactly over a and $\sigma(a)$ with respect to the natural projection $f \mid L : L \to P$. Moreover, such twistor lines are mapped to each other by the K-action on Z (see [3, (9.2)] for the details).

7.2.

Now let $\mathrm{Aut}(Z, t, \sigma, L)$ be the subgroup of elements of $\mathrm{Aut}(Z, t, \sigma)$ which preserve L. By the above remark these groups for various L are conjugate to each other by some element of K.

THEOREM 7.1

The restricted homomorphism $\delta_L : \mathrm{Aut}(Z, t, \sigma, L) \to B(Z, t, \sigma)$ *is isomorphic so that we have a natural semidirect product decomposition* $\mathrm{Aut}(Z, t, \sigma) = \mathrm{Aut}(Z, t, \sigma, L) \ltimes K$. *In particular the extension (11) splits naturally. Moreover, the splitting is unique up to conjugations of elements of K when a and L vary.*

We need a refined version of Lemma 5.5.

LEMMA 7.2

Let the notation and assumptions be as in Lemma 5.5. Suppose that we have twistor lines L_n, $n = 1, 2$, on Z_n whose proper transforms in $\hat{Z}_n$, denoted by the same letters, is a double covering over P with the common branch points a

and $\sigma(a)$ in P with $a \notin R$. Then we can take an isomorphism $j : Z_1 \to Z_2$ as in Lemma 5.5 such that j maps L_1 onto L_2.

In fact, the proof of Lemma 5.5, that is, [3, Theorem 8.1], was done by proving the above statement.

7.3.

Proof of Theorem 7.1

For the first statement it clearly suffices to show that the homomorphism δ_L is isomorphic. First we show that $\beta_L : \mathrm{Aut}(Z,t,\sigma,L) \to \mathrm{Aut}(P,\{a_i\},\sigma)$ is surjective. Take any element $\bar{g} \in \mathrm{Aut}(P,\{a_i\},\sigma)$. By Proposition 6.2, $\bar{g}$ lifts to an element $\tilde{g}j$ of $\mathrm{Aut}(Z,t,\sigma)$ in the notation of the proof of that proposition. In particular $\tilde{g}$ is an isomorphism of twistor spaces $Z' \to Z$ in the notation there.

Since $\bar{g}$ preserves the set $\{a,\sigma(a)\}$, the branch points of the double covering $L' := \tilde{g}^{-1}(L) \to P$ are also given by a and $\sigma(a)$. Hence by Lemma 7.2 we may take an isomorphism j in such a way that $j(L) = L'$. Then L is preserved by $\tilde{g}j$; that is, $\tilde{g}j \in \mathrm{Aut}(Z,t,\sigma,L)$. Thus β_L is surjective.

Next we note that the proof of Proposition 4.4 shows that there exists a real admissible involution $\tau \in \mathrm{Aut}(Z,t,\sigma,L)$, whose image gives the generator of the kernel of $\bar{\beta} : B(Z,t,\sigma) \to \mathrm{Aut}(P,\{a_i\},\sigma)$. This implies that δ_L is surjective.

We prove the injectivity. We have $\ker \delta_L = K \cap \mathrm{Aut}(Z,t,\sigma,L)$. Thus if $k \in \ker \delta_L$, k preserves S_a and L, and hence $S_a \cap L$, which is a single point. But K acts freely outside C on S_a, so k must be the identity. Hence δ_L is injective. This proves the first statement.

The uniqueness up to K-conjugacy follows from the uniqueness of the set $\{a,\sigma(a)\}$ and the mentioned K-equivalence of L unless $\mathrm{Aut}(P,\{a_i\},\sigma)$ is of order two and is generated by reflection. On the other hand, in the latter case, in the previous realization of the group we can take a to be any point on the real line inside the unit disc. But we can show that up to K-conjugacy the resulting groups are still independent of the choice of a as above. The details will be left to the reader. (One may also use Table 3.) $\square$

In the LeBrun–Joyce case the splittings are obtained explicitly from the LeBrun viewpoint in [5].

7.4.

The group extension of $B = B(Z,t,\sigma)$ by K is classified by the second cohomology group $H^2(B,K)$, and when the extension splits, the conjugacy classes by elements of K of the splittings are classified by the first cohomology group $H^1(B,K)$ (cf. [4], [14], [1]). Here K is considered to be a B-module via the natural embedding $B \subseteq \mathrm{GL}_2(Z) \cong \mathrm{Aut}\, K$. We can directly compute the cohomology groups for various B as follows, where we use [14, VII, Proposition 5] for the dihedral groups. Although H^2 does not vanish for $B = \boldsymbol{Z}_4$ or $\boldsymbol{Z}_2$, one checks easily that the extension class of (11) gives the zero element by using Corollary 4.6.

Table 3

B	D_6	D_4	D_2	$\boldsymbol{Z}_6$	$\boldsymbol{Z}_4$	$\boldsymbol{Z}_2$
$H^2(B,K)$	0	$\boldsymbol{Z}_2$	0	0	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2^2$
$H^1(B,K)$	0	$\boldsymbol{Z}_2$	0	0	0	0

8. Automorphism groups of twistor spaces and self-dual structures

8.1.

In this section we shall relate the results obtained so far on the automorphism group of a Joyce twistor space to the corresponding results on the conformal automorphisms group $\mathrm{Conf}(M,[g])$ stated in the introduction. Recall the isomorphism $t^*: \mathrm{Conf}(M,[g]) \cong \mathrm{Aut}(Z,t,\sigma)$ and the homomorphisms $\beta(=f_*): \mathrm{Aut}(Z,t,\sigma) \to \mathrm{Aut}(P,\{a_i\},w)$ and $q_*: \mathrm{Conf}(M,[g]) \to \mathrm{Conf}(N,\{y_i\},w)$. Here $\mathrm{Conf}(N,\{y_i\},w) = \mathrm{Conf}(N,w)$ in the notation of the introduction, where y_i, $1 \le i \le k$, are the images of the fixed points of K on M distributed cyclically on the boundary $bN(\cong S^1)$ of N.

The weight w in the sense of [9, 4,1] associates to each arc $A_i := (y_{i-1}, y_i)$ on bN, $1 \le i \le k$ with $y_k = y_0$, a pair $\chi_i = (m_i, n_i)$ of coprime integers defined up to sign which describes the common stabilizer group K_i of any points of the inverse image $q^{-1}(A_i)$ in M as $K_i = \{(s,t) \in K = S^1 \times S^1; s^{m_i} t^{n_i} = 1\} \cong S^1$.

Thus an element of the group $\mathrm{Conf}(N,\{y_i\},w)$ is a conformal diffeomorphism of N which preserves the set $\{y_i\}$ and for which $\chi_i = \chi_j$ if it maps A_i to A_j. We show that $t^*: \mathrm{Conf}(M,[g]) \cong \mathrm{Aut}(Z,t,\sigma)$ descends to a natural isomorphism $\mathrm{Conf}(N,\{y_i\},w) \cong \mathrm{Aut}(P,\{a_i\},w)$ with respect to q_* and β.

8.2.

For this purpose we use the homeomorphism $\psi: N \to D$, constructed in [3, Theorem 9.1], which gives a conformal automorphism of the interiors, where $D = \{z \in P; |z| \le 1\}$ as in (3). (It is actually more intrinsic to consider the target space D as the quotient $P/\langle\sigma\rangle$ so that D inherits just the conformal structure from P forgetting the complex structure. In this way D and $\sigma(D)$ play a symmetric role.) In any case we get a natural isomorphism $u: \mathrm{Aut}(P,\sigma) \cong \mathrm{Conf}(D)$ by defining $u(g) = g \mid D$ if $g(D) = D$, and $u(g) = \sigma g \mid D$ if $g(D) = \sigma(D)$. Note that $u(g) = g$ on R since σ is the identity on R. Therefore u induces an isomorphism $\mathrm{Aut}(P,\{a_i\},w) \xrightarrow{\sim} \mathrm{Conf}(D,\{a_i\},w)$, where the latter is defined in the obvious way noting that w depends only on the pair $(R,\{a_i\})$. Namely, w associates to each arc (a_{i-1},a_i) the common self-intersection number b_i of $C_i^{\pm}$.

PROPOSITION 8.1

Let $\psi: N \to D$ be as above. Then ψ induces an isomorphism $\psi_*: \mathrm{Conf}(N,\{y_i\}, w) \to \mathrm{Aut}(P,\{a_i\},w)$ such that $\beta t^* = \psi_* q_*$.

Proof

By the remark preceding the proposition we have only to prove the proposition with $\mathrm{Aut}(P,\{a_i\},w)$ replaced by $\mathrm{Conf}(D,\{a_i\},w)$. Recall first that ψ gives a conformal isomorphism of the interiors of N and D and maps y_i to a_i for all i. Thus ψ induces an isomorphism $\psi_{*0} : \mathrm{Conf}(N, \{y_i\}) \to \mathrm{Conf}(D, \{a_i\})$. We have to check that ψ_* also preserves weights.

For this and the other purposes we next recall a more detailed definition of ψ. For any point y of N, $Z_y := (qt)^{-1}(y)$ consists of a union of twistor lines which are mutually K-equivalent. If $y \in bN$, Z_y is contained in a unique member S_a, $a \in R$, of F, and we define $\psi(y) = a$. In particular, the image y_i of a fixed point on M is sent to $a_i \in P$, and the arc (y_{i-1}, y_i) is mapped to the arc (a_{i-1}, a_i) for all i.

If $y \in \mathrm{Int}\, N$, there exists a unique pair $\{S_a, S_{\sigma(a)}\}$ of mutually σ-conjugate members of F with $a \in D$ such that any twistor line L in Z_y is tangent to both S_a and $S_{\sigma(a)}$. Then we define $\psi(y) = a$.

The curves $C_i^{\pm}$ are mapped to the closed arc $[y_{i-1}, y_i]$ on bN, and the corresponding one-parameter subgroups $\pm\rho_i$ are given dually by $\pm(-n_i, m_i)$, $1 \le i \le k$, with a suitable choice of the sign. Moreover, the sequence (b_i) of common self-intersection numbers of $C_i^{\pm}$ in S_a, $a \ne a_i$, determines and is determined by the sequence $(+\rho_1, \dots, +\rho_k, -\rho_1, \dots, -\rho_k)$ associated to (1) (cf. [8, Corollary 1.29]). Since the weight b_i is attached to the arc (a_{i-1}, a_i), this implies that if an element u of $\mathrm{Conf}(N, \{y_i\})$ preserves the sequence of weights on $(bN, \{y_i\})$, then $\psi_*(u)$ preserves the weights on $(D, \{a_i\})$.

Finally, from the geometric description of the map ψ above we verify easily the desired commutativity $\beta t^* = \psi_* q_*$. $\square$

8.3.

By this proposition we can translate what we have obtained in terms of twistor space into the results on the automorphism groups of the associated self-dual structures. Let d be any fixed point of $\mathrm{Conf}(N, \{y_i\}, w)$ in the interior of N, and take any point $x \in q^{-1}(d)$. Denote by $\mathrm{Conf}(M, [g], x)$ the subgroup of $\mathrm{Conf}(M, [g])$ consisting of elements which fix x.

THEOREM 8.2

Let $(M, [g])$ be a K-invariant self-dual structure of Joyce on $M = m\boldsymbol{P}^2$, $m \ge 2$, with a fixed K-action as above. Let $q : M \to N := M/K$ be the quotient map with N identified with hyperbolic plane with boundary at infinity bN. Then q induces a natural surjective homomorphism $q_* : \mathrm{Conf}(M, [g]) \to \mathrm{Conf}(N, \{y_i\}, w)$ whose kernel is isomorphic to $\boldsymbol{Z}_2 \ltimes K$, where $y_i \in bN$ and w are as explained above. Furthermore, $\mathrm{Conf}(M, [g])$ has the natural semidirect product decomposition $\mathrm{Conf}(M, [g]) = \mathrm{Conf}(M, [g], x) \ltimes K$. Such a decomposition is, up to conjugation by elements of K, independent of the choice of points d and x as above.

9. Properties of admissible involutions

9.1.

We shall give geometric properties of a real admissible involution τ in $\mathrm{Aut}(Z/P, t, \sigma)$ constructed in Proposition 4.4 in terms of the associated self-dual manifold

$(M, [g])$. In the LeBrun–Joyce case they coincide with the involutions studied in [5]. Let ι be the element of $\mathrm{Conf}(M, [g])$ corresponding to τ with respect to the isomorphism $t_* : \mathrm{Aut}(Z, t, \sigma) \cong \mathrm{Conf}(M, [g])$; in particular we have $t\tau = \iota t$ for the twistor fibration $t : Z \to M$. We give an antiholomorphic description of the action of ι on M.

9.2.

Fix any i, $1 \le i \le k$. Consider the induced action of τ on the singular fiber S_i. The twistor fibration $t : Z \to M$ induces by restriction the map $t_i^{\pm} : S_i^{\pm} \to M$ which is just a contraction of L_i to a point, say, p_i, on M and gives a diffeomorphism $S_i^{\pm} - L_i \to M - \{p_i\}$.

Now since σ and τ both interchange $S_i^{\pm}, \sigma\tau$ is an antiholomorphic involution which leaves invariant each $S_i^{\pm}$. Let $\iota^{\pm}$ be the restriction of $\sigma\tau$ on $S_i^{\pm}$. Since σ leaves invariant each fiber of t, $t_i^{\pm}$ is $(\iota^{\pm}, \iota)$-equivariant, that is, ι is essentially described by the action of $\iota^{\pm}$ on $S_i^{\pm}$. Note that both τ and σ leave invariant L_i, and hence p_i is a fixed point of ι.

PROPOSITION 9.1

The involution ι preserves each K-orbit on M and induces the identity on the quotient $N = M/K$. The set of fixed points M^{ι} of ι on M is diffeomorphic to a connected sum $m\boldsymbol{RP}^2$ of m copies of the real projective plane $\boldsymbol{RP}^2$. The action of ι on the second integral homology group of M is given by the multiplication by -1.

Proof

First note that ι satisfies $\iota g \iota = g^{*-1}$ since $\tau g \tau = g^{-1}$ and $\sigma g \sigma = g^*$. In particular $\iota k \iota^{-1} = k^{-1}$, and the first assertion follows.

Now any smooth toric surface is covered by coordinate planes $\boldsymbol{C}^2(z_i, w_i)$ whose transition functions are monomial maps $(z_j, w_j) \to (z_i, w_i) = (z_j^a w_j^b, z_j^c w_j^d)$ for some integers a, b, c, d with $ad - bc = \pm 1$. Thus complex conjugation on each such coordinate plane patches together and gives an antiholomorphic involution κ satisfying $\kappa g \kappa^{-1} = g^{*-1}$. Thus by Lemma 2.1(3) for the proof we may assume that $\iota^{\pm} = \kappa$ on $S_i^{\pm}$ is of this form. In fact, the coordinate open subsets are in this case given by the complements of the anticanonical cycle with any of two of its adjacent irreducible components deleted. Then the fixed point set $(S_i^{\pm})^{\iota^{\pm}}$ is the real part of the toric surface $S_i^{\pm}$ with respect to the standard real structure as above.

Any irreducible components $B_{i\alpha}^{\pm}$ of anticanonical cycle $B_i^{\pm}$ of $S_i^{\pm}$ are preserved by $\iota^{\pm}$, and the corresponding fixed point set $(B_{i\alpha}^{\pm})^{\iota^{\pm}}$ is a circle. Since it reverses the orientation of each $B_{i\alpha}^{\pm}$, it follows that the induced action on the second homology group, generated by the classes of $B_{i\alpha}^{\pm}$, is by multiplication by -1.

In our case $(S_i^{\pm}, B_i^{\pm})$ is obtained from a toric complex projective plane $(\boldsymbol{P}^2, l)$ by m times successive blowups, where l consists of three lines l_0, l_1, l_2 and the centers of the blowing up are all mapped to the single point $p := l_1 \cap l_2$. Note that $L_i \subseteq B_i^{\pm}$ is the isomorphic proper transform of l_0 in $S_i^{\pm}$. We put the standard complex conjugation ι_0 as above on $\boldsymbol{P}^2$, which leaves invariant all the l_i and fixes p. Then we may assume that ι_0 lifts successively to the involution $\iota^{\pm}$ on $S_i^{\pm}$. The blowing up process is smoothly identified with the connected sum $m\bar{P}^2$ of m copies of a complex projective plane with orientation reversed and, when restricted to the real part, that is, the fixed point set of the involution, with connected sum of $\boldsymbol{RP}^2$ with $m\boldsymbol{RP}^2$.

As M is obtained from $S_i^{\pm}$ by contracting L_i to a point or, equivalently, by dropping the first factor $\boldsymbol{P}^2$ and $\boldsymbol{RP}^2$ from $S_i^{\pm}$. The assertions of the proposition follow. □

REMARK 9.1

(1) The subgroup B of commuters of ι in K is isomorphic to $\boldsymbol{Z}_2^2$ and has the induced action on M^ι making it a small cover over the quotient N for the quasi-toric manifold M in the sense of [2]. When we consider the B-action on $S_i^{\pm}$, a natural fundamental domain is familiar in toric geometry as $T_N \operatorname{emb}(\Delta)_{\geq 0}$ if $S_i^{\pm}$ is given as $T_N \operatorname{emb}(\Delta)$ (cf. [8]).

(2) Involutions ι above are conjugate to each other by elements of K. This family of involutions was first considered in the LeBrun–Joyce case in [5, Theorem 3.13] from a LeBrun viewpoint and denoted there by Φ_3. In this case there exists a unique irreducible component $B_{i\beta}^{\pm}$ of $B_i^{\pm}$ which is pointwise fixed by LeBrun's S^1-action. If $\bar{H}^3$ is the hyperbolic 3-space with boundary which is the quotient of M by this S^1-action, the natural image of $B_i^{\pm} \setminus B_{i\beta}^{\pm}$ in $\bar{H}^3$, which is independent of i, is precisely the geodesic passing through all the images of the isolated fixed points of the S^1-action. The image of $(S_i^{\pm})^{\iota^{\pm}}$ then gives a hemisphere containing the geodesic.

(3) There are other cases where an involution on M is described in terms of a certain complex conjugation on a toric surface as for ι above. They occur when $\operatorname{Aut}(P, \{a_i\}, w)$ contains a subgroup $\bar{H}$ isomorphic to $D_1 \cong \boldsymbol{Z}_2$ with k odd. In this case some of the points a_i, say, a_1, are fixed by the generator $\bar{\alpha}$ of $\bar{H}$. Then any of its lift α to an involution on $\hat{Z}$ interchanges the two irreducible components $S_1^{\pm}$ of the fiber over a_1, and the composition $\sigma\alpha$ gives an antiholomorphic involution $\iota_1^{\pm}$ which can be explicitly described and which describes our involution ι_1 on M after blowing down the twistor line L_1 in $S_1^{\pm}$. The fixed-point set in this case is a real projective plane.

Typical examples arise in the LeBrun–Joyce case (cf. Example 6.1). In this case an explicit description of ι_1 is obtained in [5] and is denoted by Φ_2 there.

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka, 560-0043, Japan; fujiki@math.sci.osaka-u.ac.jp
Current: Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606, Japan; fujiki@kurims.kyoto-u.ac.jp