Perverse coherent sheaves and Fourier–Mukai transforms on surfaces, I

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Abstract We study perverse coherent sheaves on the resolution of rational double points. As examples, we consider rational double points on 2-dimensional moduli spaces of stable sheaves on $K3$ and elliptic surfaces. Then we show that perverse coherent sheaves appear in the theory of Fourier–Mukai transforms. As an application, we generalize the Fourier–Mukai duality for $K3$ surfaces to our situation.

Contents

0. Introduction

Let $\pi : X \to Y$ be a birational map such that $\dim \pi^{-1}(y) \leq 1, y \in Y$. Bridgeland [Br3] introduced the abelian category ${}^p\operatorname{Per}(X/Y)(\subset \mathbf{D}(X))$ of perverse coherent sheaves in order to show that flops of smooth 3-folds preserve the derived categories of coherent sheaves. By using the moduli of perverse coherent sheaves on X, Bridgeland constructed the flop $X' \to Y$ of $X \to Y$. Then the Fourier–Mukai transform by the universal family induces an equivalence $\mathbf{D}(X) \cong \mathbf{D}(X')$. In [VB], Van den Bergh showed that ${}^p\operatorname{Per}(X/Y)$ is Morita equivalent to the category $\operatorname{Coh}_{\mathcal{A}}(Y)$ of $\mathcal{A}$-modules on Y and gave a different proof of Bridgeland's result, where $\mathcal{A}$ is a sheaf of (noncommutative) algebras over Y. Although the main examples of the birational contraction are small contractions of 3-folds, 2-dimensional cases seem to be still interesting. In [NY1], [NY2], and [NY3], Nakajima and Yoshioka studied perverse coherent sheaves for the blowup $X \to Y$ of a smooth surface Y at a point. In this case, by analyzing wall-crossing phenomena, we related the moduli of stable perverse coherent sheaves to the moduli of usual stable sheaves. The next example is the minimal resolution of a rational double point. Let G be a finite subgroup of $\mathrm{SU}(2)$ acting on $\mathbb{C}^2$, and set

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$Y := \mathbb{C}^2/G$. Let $\pi : X \to Y$ be the resolution of Y. Then the relation between the perverse coherent sheaves and the usual coherent sheaves on X was discussed by Nakajima. Their moduli spaces are constructed as Nakajima's quiver varieties in [N1], and their differences are described by the wall-crossing phenomena in [N2]. Toda [T] also treated special cases. In this and a subsequent paper (see [Y7]), we are interested in the global case. Thus we consider the minimal resolution $\pi : X \to Y$ of a normal projective surface Y with rational double points as singularities.

The main example of global case comes from Fourier–Mukai transforms on $K3$ and elliptic surfaces. Let (X, H) be a pair of a $K3$ surface X and an ample divisor H on X. We take a locally free sheaf G on X. Replacing the usual Hilbert polynomial $\chi(E(nH))$ of E by $\chi(G^\vee \otimes E(nH))$, we have a notion of G-twisted semistability and the coarse moduli space $\overline{M}_H^G(v)$, where $v \in H^*(X, \mathbb{Z})$ is the Mukai vector of G-twisted semistable sheaves. Since the G-twisted semistability depends only on $c_1(G)/\operatorname{rk} G$, we may write $\overline{M}_H^G(v) = \overline{M}_H^w(v)$, where w is the Mukai vector of G. As in the usual Gieseker–Maruyama semistability, it is a refinement of the slope semistability due to Mumford and Takemoto. Assume that $\overline{M}_H^w(v)$ contains a w-twisted stable sheaf and $\dim \overline{M}_H^w(v) = 2$. For the moduli space $\overline{M}_H^w(v)$, we can associate a natural $\mathbb{Q}$-divisor $\widehat{H}$ which also appears in the theory of Donaldson invariants. This $\widehat{H}$ is nef and big and defines a morphism whose image is contained in the differential geometric compactification (i.e., the Uhlenbeck compactification) of the moduli of slope stable vector bundles. For $w = v$, $\widehat{H}$ is ample and $\overline{M}_H^v(v)$ is a normal $K3$ surface (see [OY, Propositions 1.3, 2.16]). Thus $\overline{M}_H^v(v)$ is a natural object if we focus on the divisor $\widehat{H}$.

If H is a general polarization, then $Y' := \overline{M}_H^v(v)$ consists of v-twisted stable sheaves and is a smooth $K3$ surface. Moreover, if there is a universal family $\mathcal{E}$ on $X \times Y'$, then we have a Fourier–Mukai transform $\Phi_{X \to Y'}^{\mathcal{E}^\vee} : \mathbf{D}(X) \to \mathbf{D}(Y')$ (see [Br2], [O]). Even if there is no universal family, we still have a universal family $\mathcal{E}$ as a twisted sheaf and get a Fourier–Mukai transform $\Phi_{X \to Y'}^{\mathcal{E}^\vee} : \mathbf{D}(X) \to \mathbf{D}^\alpha(Y')$, where α is a representative of a torsion element $[\alpha] \in H^2(Y', \mathcal{O}_{Y'}^\times)$ and $\mathbf{D}^\alpha(Y')$ is the bounded derived category of the category of coherent α-twisted sheaves $\operatorname{Coh}^\alpha(Y')$. By choosing a locally free twisted sheaf G on Y', the Morita equivalence $\operatorname{Coh}^\alpha(Y') \to \operatorname{Coh}_{\mathcal{A}}(Y')$ induces an equivalence $\mathbf{D}(X) \to \mathbf{D}_{\mathcal{A}}(Y')$, where $\mathcal{A} = G^\vee \otimes G$ is a sheaf of $\mathcal{O}_{Y'}$-algebras. We would like to generalize these kinds of equivalences to the case where Y' has singularities. In this case, we shall construct a sheaf of $\mathcal{O}_{Y'}$-algebras $\mathcal{A}$ and get an equivalence $\mathbf{D}(X) \to \mathbf{D}_{\mathcal{A}}(Y')$. Let us briefly explain our construction of the equivalence. We take a minimal resolution $X' \to Y'$. For a sufficiently small $\xi \in \operatorname{NS}(X) \otimes \mathbb{Q}$, we set $w = ve^\xi$. Then there is a projective morphism $\pi' : \overline{M}_H^w(v) \to \overline{M}_H^v(v)$ which gives a minimal resolution of Y'. So we set $X' := \overline{M}_H^w(v)$. Then we have a Fourier–Mukai transform $\Phi_{X \to X'}^{\mathcal{E}^\vee} : \mathbf{D}(X) \to \mathbf{D}^\alpha(X')$. For a suitable locally free α-twisted sheaf G, $\mathcal{A} := \pi'_*(G^\vee \otimes G)$ is a sheaf of $\mathcal{O}_{Y'}$-algebras, and we have an equivalence $\mathbf{D}^\alpha(X') \to \mathbf{D}_{\mathcal{A}}(Y')$ via the Morita equivalence ($E \mapsto \mathbf{R}\pi'_*(E \otimes G^\vee)$). In this way, we have an equivalence (see [Y7, Proposition 2.3.6])

$$\mathbf{D}(X) \to \mathbf{D}^\alpha(X') \to \mathbf{D}_{\mathcal{A}}(Y'). \tag{0.1}$$

In our previous papers [Y1] and [Y2], we studied the relation of Gieseker–Maruyama stability and the Fourier–Mukai transforms if Y' is smooth, that is, $X' \to Y'$ is isomorphic. In these papers, we assumed that the universal family $\mathcal{E}$ on $X \times X'$ satisfies that $\mathcal{E}_{|\{x\} \times X'}$ is stable for all $x \in X$ and X is the moduli space of stable sheaves on X'. As in the abelian variety and its dual, these properties mean that we can regard X' as the dual of X. We call these properties the *Fourier–Mukai duality* [Mu1]. A nontrivial example of Fourier–Mukai duality was first constructed by Bartocci, Bruzzo, and Hernández Ruipérez [BBH]. For a general member (X, H) of the moduli space of polarized $K3$ surfaces, Mukai [Mu3], Orlov [O] and Bridgeland [Br2] showed the Fourier–Mukai duality. Moreover, a recent paper by Huybrechts [H] showed the Fourier–Mukai duality if $\overline{M}_H^v(v)$ consists of slope stable vector bundles. This was achieved in his study of Bridgeland's work [Br4] on the stability conditions for $K3$ surfaces. Bridgeland's stability condition $(\mathfrak{A}, Z)$ consists of an abelian subcategory $\mathfrak{A}$ of $\mathbf{D}(X)$ and a stability function $Z : \mathbf{D}(X) \to \mathbb{C}$ satisfying some properties. As examples, Bridgeland constructed stability conditions $(\mathfrak{A}_\beta, Z_{(\beta,\omega)})$ associated to $(\beta, \omega) \in \mathrm{NS}(X)_\mathbb{Q} \times \mathrm{Amp}(X)_\mathbb{R}$, where $\mathfrak{A}_\beta$ is independent of ω and $Z_{(\beta,\omega)}(\bullet) = \langle e^{\beta+\sqrt{-1}\omega}, v(\bullet)\rangle$. Bridgeland characterized these kinds of stability conditions. Then Huybrechts realized that the Fourier–Mukai transform induces an equivalence

$$\Phi_{X \to X'}^{\mathcal{E}^\vee[1]} : \mathfrak{A}_\beta \to \mathfrak{A}_{\beta'}, \tag{0.2}$$

where $\beta = c_1(\mathcal{E}_{|X \times \{x'\}})/\operatorname{rk}\mathcal{E}_{|X \times \{x'\}}$ and $\beta' = -c_1(\mathcal{E}_{|\{x\} \times X'})/\operatorname{rk}\mathcal{E}_{|\{x\} \times X'}$ (see [H, Proposition 4.2]). Combining this equivalence with a classification of irreducible objects of $\mathfrak{A}_\beta$ (in his terminology, irreducible objects mean minimal objects; see [H, Theorem 0.2]), the Fourier–Mukai duality is easily deduced. Then inspired by [H], we showed the stability of $\Phi_{X \to X'}^{\mathcal{E}^\vee}(E(nH))$ $(E \in M_H(u))$ for sufficiently large n depending on H, v, u (see [Y5, Theorem 1.7]) if $\overline{M}_H^v(v)$ consists of slope stable vector bundles. Actually we first gave more direct proofs of the Fourier–Mukai duality and the equivalence (0.2) under the same conditions as in [H]. Then by using (0.2), we got the above asymptotic result on the stability of $\Phi_{X \to X'}^{\mathcal{E}^\vee}(E(nH))$. We would like to remark that results in [Y1] and [Y2] can be easily derived by using (0.2) or its variant, although we did not state them in [Y5] (cf. [H, Section 6]).

In this paper, we establish the Fourier–Mukai duality for X and X' without any assumption on X', and generalize all results in [Y5] to our situation (see [Y7, Theorem 2.5.9, Proposition 2.7.2]). In particular, if Y' is singular, then we show that X is a moduli space of stable perverse coherent sheaves with respect to $\widehat{H}$. Let (X, H) be a pair of a smooth $K3$ surface X and H a nef and big divisor on X which gives a birational contraction $\pi : X \to Y$ to the normal $K3$ surface Y. For the Fourier–Mukai duality, the roles of X and X' are the same. This means that it is natural to formulate the Fourier–Mukai duality as a relation between $\pi : X \to Y$ and $\pi' : X' \to Y'$. Thus we also consider the Fourier–Mukai transforms associated to the moduli spaces of perverse coherent sheaves on X.

For an elliptic surface $X \to C$ with a section and reducible singular fibers, the Weierstrass model $Y \to C$ is a normal surface whose singularities are rational double points. Hence the category of perverse coherent sheaves associated to $X \to Y$ naturally appears. For the elliptic surface $X \to C$, there are many stable sheaves E with $E \otimes K_X \cong E$. Let Y' be the moduli space of semistable sheaves containing E. If Y' is smooth, then the universal family induces a Fourier–Mukai transform $\mathbf{D}(X) \to \mathbf{D}^{\alpha}(Y')$. In general, Y' is singular. For example, let $Y' := \overline{M}_H(0, f, 0)$ be the moduli space of semistable 1-dimensional sheaves E with $c_1(E) = f$ and $\chi(E) = 0$, where H is an ample divisor on X. Then Y' is a compactified relative Picard scheme $\overline{\operatorname{Pic}}^0_{X/C} \to C$, and it is the Weierstrass model of an elliptic surface $X' \to C$. Moreover, X' is constructed as a moduli space $\overline{M}_H^G(0, f, 0)$ of G-twisted semistable sheaves, where $G \in K(X)_{\mathbb{Q}}$. Then we have an equivalence (0.1). Thus we can show similar results to those for a $K3$ surface. In particular, we can formulate the Fourier–Mukai duality by using perverse coherent sheaves and study the preservation of Gieseker semistability under Fourier–Mukai transforms.

Let G be a finite group acting on a projective surface X. Assume that K_X is the pullback of a line bundle on $Y := X/G$. Then the McKay correspondence (see [VB]) implies that $\operatorname{Coh}_G(X)$ is equivalent to ${}^{-1}\operatorname{Per}(X'/Y)$, where $X' \to Y$ is the minimal resolution of Y. The equivalence is given by a Fourier–Mukai transform associated to a moduli space of stable G-sheaves of dimension zero. If X is a $K3$ surface or an abelian surface, then we have many 2-dimensional moduli spaces of stable G-sheaves. We also treat the Fourier–Mukai transform induced by the moduli of G-sheaves.

Let us explain the content of the first half part. In Section 1, we consider an abelian subcategory $\mathcal{C}$ of $\mathbf{D}(X)$ which is Morita equivalent to $\operatorname{Coh}_{\mathcal{A}}(Y)$, where $\pi : X \to Y$ is a birational contraction from a smooth variety X and $\mathcal{A}$ is a sheaf of (noncommutative) algebras over Y. We call an object of $\mathcal{C}$ a perverse coherent sheaf. Since ${}^{-1}\operatorname{Per}(X/Y)$ is Morita equivalent to $\operatorname{Coh}_{\mathcal{A}}(Y)$ for an algebra $\mathcal{A}$ (see [VB]), our definition is compatible with Bridgeland's definition. We also study irreducible objects and local projective generators of $\mathcal{C}$. As examples, we shall give generalizations of ${}^{p}\operatorname{Per}(X/Y)$, $p = -1, 0$. We next explain families of perverse coherent sheaves and the relative version of Morita equivalence. Then we can use Simpson's [S] moduli spaces of stable $\mathcal{A}$-modules to construct the moduli spaces of stable perverse coherent sheaves. Since Simpson's stability is not good enough for the zero-dimensionional objects, we also introduce a refinement of the stability and construct the moduli space, which is close to King's [K] stability. Finally we explain how to modify our arguments in Section 1 to be applicable to the category of twisted sheaves. This is necessary for applications of Fourier–Mukai transforms, as we have explained in this introduction.

In Section 2, we study perverse coherent sheaves on the resolution of rational double points. We first introduce two kind of categories $\operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$ and $\operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)^*$ associated to a sequence of line bundles on the exceptional curves of the resolution of rational singularities and show that they are the category of perverse coherent sheaves in the sense in Section 1. They are generalizations of ${}^{-1}\operatorname{Per}(X/Y)$ and ${}^{0}\operatorname{Per}(X/Y)$, respectively.

We next study the moduli of zero-dimensional objects on the resolution of rational double points. We introduce the wall and the chamber structure and study the Fourier–Mukai transforms induced by the moduli spaces. Under a suitable stability condition for $\mathbb{C}_x$, $x \in X$, we show that the category of perverse coherent sheaves is equivalent to ${}^{-1}\operatorname{Per}(X/Y)$ (cf. Proposition 2.3.27). We also construct local projective generators under suitable conditions.

Examples of the categories of perverse coherent sheaves and the relation with the Fourier–Mukai transforms will be treated in the second part (see [Y7]).

NOTATION

(i) For a scheme X, $\operatorname{Coh}(X)$ denotes the category of coherent sheaves on X and $\mathbf{D}(X)$ denotes the bounded derived category of $\operatorname{Coh}(X)$. We denote the Grothendieck group of X by $K(X)$.

(ii) Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras on a scheme X which is coherent as an $\mathcal{O}_X$-module. Let $\operatorname{Coh}_{\mathcal{A}}(X)$ be the category of coherent $\mathcal{A}$-modules on X, and let $\mathbf{D}_{\mathcal{A}}(X)$ be the bounded derived category of $\operatorname{Coh}_{\mathcal{A}}(X)$.

(iii) Assume that X is a smooth projective variety. Let E be an object of $\mathbf{D}(X)$. Let $E^\vee := \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ denote the dual of E. We denote the rank of E by $\operatorname{rk} E$.

(iv) G-**twisted semistability**. Let X be a smooth projective variety, and let L be an ample divisor on X. For $G \in K(X)$, $\operatorname{rk} G > 0$, and a coherent sheaf E on X, we define $a_i(E)$ by

$$\chi(G, E \otimes L^{\otimes n}) = \sum_i a_i(E) \binom{n+i}{i}. \tag{0.3}$$

A d-dimensional coherent sheaf E is G-twisted semistable with respect to L if

$$\chi(G^\vee \otimes F \otimes L^{\otimes n}) \le \frac{a_d(F)}{a_d(E)} \chi(G^\vee \otimes E \otimes L^{\otimes n}), \quad n \gg 0, \tag{0.4}$$

for all subsheaves F of E. If E is 1-dimensional, then the condition is

$$\chi(G, F) \le \frac{(c_1(L), \operatorname{ch}_{\dim X - 1}(F))}{(c_1(L), \operatorname{ch}_{\dim X - 1}(E))} \chi(G, E) \tag{0.5}$$

for all subsheaves F of E. In particular if $\chi(G, E) = 0$, then the condition is

$$\chi(G, F) \le 0 \quad \text{for all subsheaves } F \text{ of } E. \tag{0.6}$$

Thus the condition does not depend on the choice of L. If $G = \mathcal{O}_X$, then G-twisted semistability is the usual semistability of Gieseker, Maruyama, and Simpson.

(v) **Integral functor**. For two schemes X, Y and an object $\mathcal{E} \in \mathbf{D}(X \times Y)$, $\Phi^{\mathcal{E}}_{X \to Y} : \mathbf{D}(X) \to \mathbf{D}(Y)$ is the integral functor

$$\Phi^{\mathcal{E}}_{X \to Y}(E) := \mathbf{R}p_{Y*}\big(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_X^*(E)\big), \quad E \in \mathbf{D}(X), \tag{0.7}$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are projections. If $\Phi^{\mathcal{E}}_{X \to Y}$ is an equivalence, it is said to be the *Fourier–Mukai transform*.

(vi) $\mathbf{D}(X)_{\mathrm{op}}$ denotes the opposite category of $\mathbf{D}(X)$. We have a functor

$$\begin{aligned} D_X : \mathbf{D}(X) &\to \mathbf{D}(X)_{\mathrm{op}}, \\ E &\mapsto E^{\vee}. \end{aligned}$$

(vii) Assume that X is a smooth projective surface.

(a) We set $H^{\mathrm{ev}}(X,\mathbb{Z}) := \bigoplus_{i=0}^{2} H^{2i}(X,\mathbb{Z})$. In order to describe the element x of $H^{\mathrm{ev}}(X,\mathbb{Z})$, we use two kinds of expressions: $x = (x_0, x_1, x_2) = x_0 + x_1 + x_2 \varrho_X$, where $x_0 \in \mathbb{Z}, x_1 \in H^2(X,\mathbb{Z}), x_2 \in \mathbb{Z}$, and $\int_X \varrho_X = 1$. For $x = (x_0, x_1, x_2)$, we set $\operatorname{rk} x := x_0$ and $c_1(x) = x_1$.

(b) We define a homomorphism

$$\begin{aligned} \gamma : K(X) &\to \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}, \\ E &\mapsto \big(\operatorname{rk} E, c_1(E), \chi(E)\big) \end{aligned} \tag{0.8}$$

and set $K(X)_{\mathrm{top}} := K(X)/\ker\gamma$. We denote $E \mod \ker\gamma$ by $\tau(E)$. $K(X)_{\mathrm{top}}$ has a bilinear form $\chi(\ ,\ )$.

(c) **Mukai lattice** [Mu2]. We define a lattice structure $\langle\ ,\ \rangle$ on $H^{\mathrm{ev}}(X,\mathbb{Z})$ by

$$\begin{aligned} \langle x, y\rangle &:= -\int_X x^{\vee} \cup y \\ &= (x_1, y_1) - (x_0 y_2 + x_2 y_0), \end{aligned} \tag{0.9}$$

where $x = (x_0, x_1, x_2)$ (resp., $y = (y_0, y_1, y_2)$) and $x^{\vee} = (x_0, -x_1, x_2)$. It is now called the *Mukai lattice*. The Mukai lattice has a weight 2 Hodge structure such that the (p,q)-part is $\bigoplus_i H^{p+i,q+i}(X)$. We set

$$\begin{aligned} H^{\mathrm{ev}}(X,\mathbb{Z})_{\mathrm{alg}} &= H^{1,1}\big(H^{\mathrm{ev}}(X,\mathbb{C})\big) \cap H^{\mathrm{ev}}(X,\mathbb{Z}) \\ &\cong \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}. \end{aligned} \tag{0.10}$$

Let E be an object of $\mathbf{D}(X)$. If X is a $K3$ surface or $\operatorname{rk} E = 0$, we define the *Mukai vector* of E as

$$v(E) := \mathrm{rk}(E) + c_1(E) + \big(\chi(E) - \mathrm{rk}(E)\big)\varrho_X \in H^{ev}(X,\mathbb{Z}). \tag{0.11}$$

Then for $E, F \in \mathbf{D}(X)$ such that the Mukai vectors are well defined, we have

$$\chi(E,F) = -\big\langle v(E), v(F)\big\rangle. \tag{0.12}$$

1. Perverse coherent sheaves and their moduli spaces

1.1. Tilting and Morita equivalence

Let X be a smooth projective variety, and let $\pi : X \to Y$ be a birational map. Let $\mathcal{O}_Y(1)$ be an ample line bundle on Y, and let $\mathcal{O}_X(1) := \pi^*(\mathcal{O}_Y(1))$. In Sections 1.1 and 1.2, we impose the following assumption.

ASSUMPTION 1.1.1

(1) Take $\dim \pi^{-1}(y) \le 1$ for all $y \in Y$, and set

$$Y_\pi := \{ y \in Y \mid \dim \pi^{-1}(y) = 1 \}. \tag{1.1}$$

(2) We have $\mathbf{R}\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$; that is, $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ and $R^1\pi_*(\mathcal{O}_X) = 0$.

More precisely, we impose Assumption 1.1.1(1) from Definition 1.1.10. Lemma 1.1.11(2) and Proposition 1.1.13 will explain that Assumption 1.1.1(2) is a reasonable assumption. Then we impose Assumption 1.1.1(2) after Proposition 1.1.13.

REMARK 1.1.2

Since $\pi : X \to Y$ is birational, $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ means that Y is normal.

We are interested in the following type of abelian categories.

DEFINITION 1.1.3

(1) A subcategory $\mathcal{C}$ of $\mathbf{D}(X)$ is a *category of perverse coherent sheaves* if the following conditions are satisfied.
   (i) $\mathcal{C}$ is the heart of a bounded t-structure of $\mathbf{D}(X)$.
   (ii) There is an object $G \in \mathcal{C}$ such that
      (a) $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E) \in \mathrm{Coh}(Y)$ for all $E \in \mathcal{C}$ and
      (b) $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E) = 0$, $E \in \mathcal{C}$ if and only if $E = 0$.
(2) We say G is a *a local projective generator* of $\mathcal{C}$ if it satisfies (a) and (b).
(3) A *perverse coherent sheaf* E is an object of $\mathcal{C}$.
(4) For $E \in \mathbf{D}(X)$, ${}^pH^i(E) \in \mathcal{C}$ denotes the ith cohomology object of E with respect to the t-structure.

By these properties, we get

$$\mathcal{C} = \{ E \in \mathbf{D}(X) \mid \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E) \in \mathrm{Coh}(Y) \}. \tag{1.2}$$

Indeed for $E \in \mathbf{D}(X)$, (a) implies $H^i(\mathbf{R}\pi_*(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, {}^pH^j(E)))) = 0$ for $i \neq 0$. Hence the spectral sequence

$$\begin{aligned} E_2^{i,j} &= H^i\big(\mathbf{R}\pi_*\big(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(G, {}^pH^j(E)\big)\big)\big) \\ &\implies E_\infty^{i+j} = H^{i+j}\big(\mathbf{R}\pi_*\big(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E)\big)\big) \end{aligned} \tag{1.3}$$

degenerates, and (b) implies (1.2).

REMARK 1.1.4

(1) $\mathcal{C}$ in Definition 1.1.3 is not determined by π, unlike in [Br4] and [VB], and does depend on G.

(2) Our definition of a local projective generator is in the global nature of $\mathrm{Coh}(X)$. So it is different from the one in [VB]. Under Assumptions 1.1.1 and

1.1.6, we can show that $G_{|\pi^{-1}(U)}$ is a local projective generator of a local category

$$\{E \in \mathbf{D}(\pi^{-1}(U)) \mid \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_{\pi^{-1}(U)}}(G_{|\pi^{-1}(U)}, E) \in \mathrm{Coh}(U)\} \tag{1.4}$$

in Corollary 1.1.18, where U is an open subset of Y. This is the link of two notions of local projective generators.

As we shall see in Section 1.4, the existence of G in Definition 1.1.3 or the Morita equivalence (Proposition 1.1.7 below) which follows from the existence of G is essential for the construction of moduli spaces of stable objects. This is our motivation to require a local projective generator G in Definition 1.1.3. Then it is desirable to know what kind of categories G has in Definition 1.1.3. We shall discuss this problem in Section 1.1.2.

The following is an easy consequence of the properties (a) and (b) of G. For the sake of convenience, we give a proof.

LEMMA 1.1.5

Let G be a local projective generator of $\mathcal{C}$.

(1) *For $E \in \mathcal{C}$, there is a locally free sheaf V on Y and a surjective morphism*

$$\phi : \pi^*(V) \otimes G \to E \tag{1.5}$$

in $\mathcal{C}$. In particular, we have a resolution

$$\cdots \to \pi^*(V_{-1}) \otimes G \to \pi^*(V_0) \otimes G \to E \to 0 \tag{1.6}$$

of E such that V_i, $i \leq 0$, are locally free sheaves on Y.

(2) *Let $G' \in \mathcal{C}$ be a local projective object of $\mathcal{C}$: $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G', E) \in \mathrm{Coh}(Y)$ for all $E \in \mathcal{C}$. If G is a locally free sheaf, then so is G'.*

Proof

(1) By property (a) of G (see Definition 1.1.3), we can take a morphism $\varphi : V \to \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E)$ in $\mathbf{D}(Y)$ such that $V \to H^0(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E))$ is surjective in $\mathrm{Coh}(Y)$. Since

$$\begin{aligned} &\mathrm{Hom}\big(\mathbf{L}\pi^*\big(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E)\big) \otimes G, E\big) \\ &\quad = \mathrm{Hom}\big(\mathbf{L}\pi^*\big(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E)\big), \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E)\big) \\ &\quad = \mathrm{Hom}\big(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E), \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E)\big), \end{aligned} \tag{1.7}$$

we have a morphism $\phi : \pi^*(V) \otimes G \to E$ such that the induced morphism

$$V \to \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(G, \pi^*(V) \otimes G\big) \to \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, E)$$

is φ. Then $\operatorname{coker}\phi \in \mathcal{C}$ satisfies $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G, \operatorname{coker}\phi) = 0$. By our assumption on G, $\operatorname{coker}\phi = 0$. Thus ϕ is surjective in $\mathcal{C}$.

(2) We take a surjective homomorphism (1.5) for G'. Let U be an affine open subset of Y. We note that

$$\begin{aligned}(1.8)\qquad &\operatorname{Hom}\bigl(G'_{|\pi^{-1}(U)}, \ker\phi_{|\pi^{-1}(U)}[1]\bigr)\\ &= H^1\bigl(U, \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G'_{|\pi^{-1}(U)}, \ker\phi_{|\pi^{-1}(U)})\bigr) = 0.\end{aligned}$$

Hence

$$(1.9)\qquad \operatorname{Hom}\bigl(G'_{|\pi^{-1}(U)}, \pi^*(V)\otimes G_{|\pi^{-1}(U)}\bigr) \to \operatorname{Hom}(G'_{|\pi^{-1}(U)}, G'_{|\pi^{-1}(U)})$$

is surjective. Therefore $G'_{|\pi^{-1}(U)}$ is a direct summand of $\pi^*(V)\otimes G_{|\pi^{-1}(U)}$. □

ASSUMPTION 1.1.6

From now on, we add the following equivalent conditions for the definition of our category of perverse coherent sheaves $\mathcal{C}$ in Definition 1.1.3.

(1) There is a local projective generator which is a locally free sheaf.
(2) Every local projective generator is a locally free sheaf.

By this assumption, a local projective generator G satisfies $R^i\pi_*(G^\vee\otimes G) = 0$ for $i > 0$ by Definition 1.1.3(a).

PROPOSITION 1.1.7 ([VB, LEMMA 3.2, COROLLARY 3.2.8])

For a local projective generator G of $\mathcal{C}$, we set $\mathcal{A} := \pi_*(G^\vee\otimes G)$. Then we have an equivalence

$$(1.10)\qquad \begin{aligned}&\mathcal{C}\to \operatorname{Coh}_{\mathcal{A}}(Y),\\ &E \mapsto \mathbf{R}\pi_*(G^\vee\otimes E)\end{aligned}$$

whose inverse is $F \mapsto \pi^{-1}(F)\overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G$. Moreover, this equivalence induces an equivalence $\mathbf{D}(X)\to\mathbf{D}_{\mathcal{A}}(Y)$.

For the convenience of the reader, let us briefly explain the correspondence (1.10). For $F\in\operatorname{Coh}_{\mathcal{A}}(Y)$, we have a surjective morphism $H^0(Y, F(n))\otimes\mathcal{A}(-n)\to F$, $n\gg 0$. Hence we have a resolution $V^\bullet\to F$ by locally free $\mathcal{A}$-modules V^i. If $V^i_{|U}\cong \mathcal{A}_U^{\oplus n}$ on an open subset of Y, then $(\pi^{-1}(V^i)\otimes_{\pi^{-1}(\mathcal{A})} G)_{|\pi^{-1}(U)}\cong G^{\oplus n}_{|\pi^{-1}(U)}$. Thus $\pi^{-1}(F)\overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G$ is isomorphic to $\pi^{-1}(V^\bullet)\otimes_{\pi^{-1}(\mathcal{A})} G$. Then $\pi^{-1}(V^\bullet)\otimes_{\pi^{-1}(\mathcal{A})} G\in\mathcal{C}$ follows from the next lemma.

LEMMA 1.1.8

(1) *For a morphism $V\xrightarrow{\psi} W$ of locally free $\mathcal{A}$-modules on Y, we have a morphism $\pi^{-1}(V)\otimes_{\pi^{-1}(\mathcal{A})} G\xrightarrow{\psi'}\pi^{-1}(W)\otimes_{\pi^{-1}(\mathcal{A})} G$. Then $\mathbf{R}\pi_*(G^\vee\otimes\ker\psi') = \ker\psi$ and $\mathbf{R}\pi_*(G^\vee\otimes\operatorname{im}\psi') = \operatorname{im}\psi$.*

(2) *Let $U\xrightarrow{\phi} V\xrightarrow{\psi} W$ be an exact sequence of locally free $\mathcal{A}$-modules on Y. Then $\pi^{-1}(U)\otimes_{\pi^{-1}(\mathcal{A})} G\xrightarrow{\phi'}\pi^{-1}(V)\otimes_{\pi^{-1}(\mathcal{A})} G\xrightarrow{\psi'}\pi^{-1}(W)\otimes_{\pi^{-1}(\mathcal{A})} G$ is exact in $\mathcal{C}$.*

Proof

(1) We have exact sequences in $\mathcal{C}$,

$$\begin{aligned} &0 \to \operatorname{im}\psi' \to \pi^{-1}(W) \otimes_{\pi^{-1}(\mathcal{A})} G \to \operatorname{coker}\psi' \to 0, \\ &0 \to \ker\psi' \to \pi^{-1}(V) \otimes_{\pi^{-1}(\mathcal{A})} G \to \operatorname{im}\psi' \to 0. \end{aligned} \tag{1.11}$$

Applying $\mathbf{R}\pi_*(G^\vee \otimes \bullet)$ to these sequences, we have exact sequences

$$\begin{aligned} &0 \to \mathbf{R}\pi_*(G^\vee \otimes \operatorname{im}\psi') \to W \to \mathbf{R}\pi_*(G^\vee \otimes \operatorname{coker}\psi') \to 0, \\ &0 \to \mathbf{R}\pi_*(G^\vee \otimes \ker\psi') \to V \to \mathbf{R}\pi_*(G^\vee \otimes \operatorname{im}\psi') \to 0 \end{aligned} \tag{1.12}$$

by Definition 1.1.3(a). Thus claim (1) holds.

(2) We have an exact sequence

$$\begin{aligned} &0 \to \mathbf{R}\pi_*(G^\vee \otimes \operatorname{im}\phi') \\ &\quad \to \mathbf{R}\pi_*(G^\vee \otimes \ker\psi') \to \mathbf{R}\pi_*(G^\vee \otimes \ker\psi'/\operatorname{im}\phi') \to 0. \end{aligned} \tag{1.13}$$

Hence (2) follows from (1) and Definition 1.1.3(b). $\square$

Finally, by using Lemma 1.1.5 and the construction of $\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G$, the equivalence (1.10) follows.

REMARK 1.1.9

(1) For $E^\bullet \in \mathbf{D}(X)$, there is a bounded complex $E_1^\bullet$ such that $E^\bullet \cong E_1^\bullet$ in $\mathbf{D}(X)$ and $E_1^i \in \operatorname{Coh}(X) \cap \mathcal{C}$ (see the proof of Lemma 1.3.6 below).

(2) By taking a local projective resolution $\pi^*(V_\bullet) \otimes G$ of $E \in \mathcal{C}$ in Lemma 1.1.5, we have

$$\begin{aligned} &\operatorname{Hom}_{\mathbf{D}(X)}\bigl(E, F[q]\bigr) \\ &\quad \cong \operatorname{Hom}_{\mathbf{D}(X)}\bigl(\pi^*(V_\bullet) \otimes G, F[q]\bigr) \\ &\quad \cong \operatorname{Hom}_{\mathbf{D}(Y)}\bigl(V_\bullet, \mathbf{R}\pi_*(G^\vee \otimes F)[q]\bigr) \\ &\quad \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}\bigl(V_\bullet \otimes \mathcal{A}, \mathbf{R}\pi_*(G^\vee \otimes F)[q]\bigr) \\ &\quad \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}\bigl(\mathbf{R}\pi_*\bigl(G^\vee \otimes \bigl(\pi^*(V_\bullet) \otimes G\bigr)\bigr), \mathbf{R}\pi_*(G^\vee \otimes F)[q]\bigr) \\ &\quad \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}\bigl(\mathbf{R}\pi_*(G^\vee \otimes E), \mathbf{R}\pi_*(G^\vee \otimes F)[q]\bigr), \end{aligned} \tag{1.14}$$

where we put suffixes $\mathbf{D}(X), \mathbf{D}(Y), \mathbf{D}(\mathcal{A})$ for Hom in order to clarify the categories. In particular, we have an isomorphism of the space of morphisms

$$\operatorname{Hom}(E, F) \cong \operatorname{Hom}_{\mathcal{A}}\bigl(\mathbf{R}\pi_*(G^\vee \otimes E), \mathbf{R}\pi_*(G^\vee \otimes F)\bigr) \tag{1.15}$$

for $E, F \in \mathcal{C}$.

(3) We can also explain (1.15) as follows. We take a local projective presentation $\pi^*(W_{-1}) \otimes G \xrightarrow{\phi} \pi^*(W_0) \otimes G \to F \to 0$ of $F \in \mathcal{C}$. For $F' \in \mathcal{C}$ and a locally free sheaf V on Y, Serre's vanishing theorem says that $\operatorname{Hom}(\pi^*(V(-n)) \otimes$

$G, F'[q]) = 0$ for $q \neq 0$ and $n \gg 0$. Hence we can take a local projective presentation $\pi^*(V_{-1}) \otimes G \xrightarrow{\psi} \pi^*(V_0) \otimes G \to E \to 0$ of $E \in \mathcal{C}$ such that

$$(1.16) \qquad \begin{aligned} \operatorname{Hom}\bigl(\pi^*(V_i) \otimes G, \pi^*(W_j) \otimes G[q]\bigr) &= \operatorname{Hom}\bigl(\pi^*(V_i) \otimes G, \operatorname{im}\phi[q]\bigr) \\ &= \operatorname{Hom}\bigl(\pi^*(V_i) \otimes G, F[q]\bigr) = 0 \end{aligned}$$

for $q \neq 0$. Then $\operatorname{Hom}(E, F)$ is the zeroth cohomology group of the complex $\operatorname{Hom}(\pi^*(V_\bullet) \otimes G, \pi^*(W_\bullet) \otimes G)$, which is isomorphic to $\operatorname{Hom}_{\mathcal{A}}(V_\bullet \otimes \mathcal{A}, W_\bullet \otimes \mathcal{A})$. Since $V_\bullet \otimes \mathcal{A}$, $W_\bullet \otimes \mathcal{A}$ give locally free presentations of $\mathbf{R}\pi_*(G^\vee \otimes E)$, $\mathbf{R}\pi_*(G^\vee \otimes F)$ in $\operatorname{Coh}_{\mathcal{A}}(Y)$ and similar properties to (1.16) hold, we see that $\mathbf{R}\pi_*(G^\vee \otimes \bullet)$ induces (1.15).

As we explained, we assume Assumption 1.1.1(1) from now on.

DEFINITION 1.1.10

For a locally free sheaf G on X, we set

$$(1.17) \qquad \begin{aligned} T(G) &:= \bigl\{E \in \operatorname{Coh}(X) \bigm| R^1\pi_*(G^\vee \otimes E) = 0\bigr\}, \\ S(G) &:= \bigl\{E \in \operatorname{Coh}(X) \bigm| \pi_*(G^\vee \otimes E) = 0\bigr\}, \\ S_0(G) &:= \bigl\{E \in \operatorname{Coh}(X) \bigm| \mathbf{R}\pi_*(G^\vee \otimes E) = 0\bigr\} = T(G) \cap S(G). \end{aligned}$$

If $(T(G), S(G))$ is a torsion pair of $\operatorname{Coh}(X)$, then

$$(1.18) \qquad \begin{aligned} \mathcal{C}(G) := \bigl\{E \in \mathbf{D}(X) \bigm| H^{-1}(E) \in S(G), H^0(E) \in T(G), H^i(E) = 0, \\ i \neq -1, 0\bigr\} \end{aligned}$$

denotes the tilted category.

LEMMA 1.1.11

Let G be a locally free sheaf on X.

(1) *$(T(G), S(G))$ is a torsion pair of $\operatorname{Coh}(X)$ such that $G \in T(G)$ if and only if $R^1\pi_*(G^\vee \otimes G) = 0$ and $S_0(G) = 0$.*
(2) *Assume that $(T(G), S(G))$ is a torsion pair such that $G \in T(G)$. Then the following assertions hold:*
    (a) *$R^1\pi_*(\mathcal{O}_X) = 0$;*
    (b) *G is a local projective generator of $\mathcal{C}(G)$;*
    (c) *if (T, S) is a torsion pair of $\operatorname{Coh}(X)$ such that $G \in T$ and $S(G) \cap T = 0$, then $(T, S) = (T(G), S(G))$.*

Proof

(1) The only if part is obvious. So we only prove the if part. For $E \in \operatorname{Coh}(X)$, let $\phi : \pi^*(\pi_*(G^\vee \otimes E)) \otimes G \to E$ be the evaluation map. Then we see that $\pi_*(G^\vee \otimes \operatorname{coker}\phi) = 0$, $R^1\pi_*(G^\vee \otimes \operatorname{im}\phi) = 0$, and $R^1\pi_*(G^\vee \otimes E) \cong R^1\pi_*(G^\vee \otimes \operatorname{coker}\phi)$.

Hence we have a desired decomposition

$$0 \to E_1 \to E \to E_2 \to 0, \tag{1.19}$$

where $E_1 := \operatorname{im}\phi \in T(G)$ and $E_2 := \operatorname{coker}\phi \in S(G)$.

(2) (a) Since the trace map $G^\vee \otimes G \to \mathcal{O}_X$ is surjective, we have a surjective homomorphism $R^1\pi_*(G^\vee \otimes G) \to R^1\pi_*(\mathcal{O}_X)$. Then (1) implies the claim.

(b) For $E \in \mathcal{C}(G)$, we have an exact sequence

$$0 \to R^1\pi_*\big(G^\vee \otimes H^{-1}(E)\big) \to \mathbf{R}\pi_*(G^\vee \otimes E) \to \pi_*\big(G^\vee \otimes H^0(E)\big) \to 0. \tag{1.20}$$

Hence $\mathbf{R}\pi_*(G^\vee \otimes E) \in \operatorname{Coh}(Y)$ and $\mathbf{R}\pi_*(G^\vee \otimes E) = 0$ if and only if $R^1\pi_*(G^\vee \otimes H^{-1}(E)) = \pi_*(G^\vee \otimes H^0(E)) = 0$, which is equivalent to $H^{-1}(E), H^0(E) \in S_0(G) = 0$. Therefore G is a local projective generator of $\mathcal{C}(G)$.

(c) We first prove that $T(G) \subset T$. For an object $E \in T(G)$, (b) implies that there is a surjective morphism $\phi : \pi^*(V) \otimes G \to E$ in $\mathcal{C}(G)$, where V is a locally free sheaf on Y. Since ϕ is surjective in $\operatorname{Coh}(X)$ and $G \in T$, $E \in T$. Since $S(G) \cap T = 0$, we get $S(G) \subset S$. Therefore $(T, S) = (T(G), S(G))$. □

By the proof of Lemma 1.1.11, we get the following.

COROLLARY 1.1.12

Let G be as in Lemma 1.1.11, and suppose that $(T(G), S(G))$ is a torsion pair with $G \in T(G)$. Let E be a coherent sheaf on X, and let $\phi : \pi^*(\pi_*(G^\vee \otimes E)) \otimes G \to E$ be the evaluation map. Then $E_1 := \operatorname{im}\phi \in T(G)$ and $E_2 := \operatorname{coker}\phi \in S(G)$. Thus we have a decomposition of E,

$$0 \to \operatorname{im}\phi \to E \to \operatorname{coker}\phi \to 0, \tag{1.21}$$

with respect to the torsion pair $(T(G), S(G))$.

PROPOSITION 1.1.13

Let $\mathcal{C}$ be a category of perverse coherent sheaves, and let G be a local projective generator. Then $(T(G), S(G))$ is a torsion pair of $\operatorname{Coh}(X)$ whose tilting is $\mathcal{C}$.

Proof

We first note that G is a locally free sheaf by Assumption 1.1.6. Since $G \in \mathcal{C}$, we have $\mathbf{R}\pi_*(G^\vee \otimes G) \in \operatorname{Coh}(Y)$. By the definition of a local projective generator, we have $S_0(G) = 0$. By Lemma 1.1.11, $(T(G), S(G))$ is a torsion pair. Since $S(G)[1], T(G) \subset \mathcal{C}$, we get $\mathcal{C}(G) \subset \mathcal{C}$. Conversely for $E \in \mathcal{C}$, we have a spectral sequence

$$E_2^{p,q} = R^p\pi_*\big(G^\vee \otimes H^q(E)\big) \Longrightarrow E_\infty^{p+q} = R^{p+q}\pi_*(G^\vee \otimes E). \tag{1.22}$$

Since $\pi^{-1}(y) \leq 1$ for all $y \in Y$, this spectral sequence degenerates. Hence we have $\mathbf{R}\pi_*(G^\vee \otimes H^q(E)) = 0$ for $q \neq -1, 0$, $\pi_*(G^\vee \otimes H^{-1}(E)) = 0$ and $R^1\pi_*(G^\vee \otimes H^0(E)) = 0$. Therefore $E \in \mathcal{C}(G)$. □

From now on, we impose Assumption 1.1.1(2), which is reasonable by Proposition 1.1.13 and Lemma 1.1.11(2).

LEMMA 1.1.14

Let $\mathcal{C}$ be a category of perverse coherent sheaves, and let G be a local projective generator of $\mathcal{C}$. Then

(1) *$(T(G^\vee), S(G^\vee))$ is a torsion pair, and $G^\vee$ is a local projective generator of $\mathcal{C}(G^\vee)$;*

(2) *if E is a local projective object of $\mathcal{C}$, that is, $R^1\pi_*(E^\vee \otimes F) = 0$ for all $F \in \mathcal{C}$, then $E^\vee$ is a local projective object of $\mathcal{C}(G^\vee)$;*

(3) *$(T(G^\vee), S(G^\vee))$ is independent of the choice of G;*

(4) *for $E \in \mathbf{D}(X)$, $\mathbf{R}\pi_*(G^\vee \otimes E)$ is a zero-dimensional sheaf on Y if and only if $\mathbf{R}\pi_*(G \otimes D_X(E)(K_X)[n])$ is a zero-dimensional sheaf on Y, where $n = \dim X$.*

By (3), we denote $\mathcal{C}(G^\vee)$ by $\mathcal{C}^D$.

Proof

(1) Since $R^1\pi_*(G^\vee \otimes G) = 0$, $G^\vee \in T(G^\vee)$. We show that $S_0(G^\vee) = 0$. Assume that $\mathbf{R}\pi_*(G \otimes E) = 0$ for a coherent sheaf E on X. Since

$$\begin{aligned} H^i\big(Y, \mathbf{R}\pi_*(G \otimes E)(-k)\big) &= H^i\big(X, G \otimes E(-k)\big) \\ &= H^{n-i}\big(X, G^\vee \otimes D_X(E)(K_X) \otimes \mathcal{O}_X(k)\big)^\vee \\ &= H^{n-i}\big(Y, \mathbf{R}\pi_*\big(G^\vee \otimes D_X(E)(K_X)\big)(k)\big)^\vee \end{aligned} \tag{1.23}$$

for all $k \in \mathbb{Z}$ and $H^j(Y, H^{n-i}(\mathbf{R}\pi_*(G^\vee \otimes D_X(E)(K_X)))(k)) = 0$ for $k \gg 0$ and $j \neq 0$, we get $H^{n-i}(Y, \mathbf{R}\pi_*(G^\vee \otimes D_X(E)(K_X))(k)) = H^0(Y, H^{n-i}(\mathbf{R}\pi_*(G^\vee \otimes D_X(E)(K_X)))(k)) = 0$ for $k \gg 0$. Therefore $\mathbf{R}\pi_*(G^\vee \otimes D_X(E)(K_X)) = 0$. Since $\dim \pi^{-1}(y) \le 1$ for all $y \in Y$, we see that $\mathbf{R}\pi_*(G^\vee \otimes H^i(D_X(E)(K_X))) = \mathbf{R}\pi_*(H^i(G^\vee \otimes D(E)(K_X))) = 0$ (see the proof of Proposition 1.1.13). Since G is a local projective generator of $\mathcal{C}(G) = \mathcal{C}$, $H^i(D_X(E)(K_X)) = 0$ for all i. Therefore $D_X(E)(K_X) = 0$, which implies that $E = 0$.

(2) We note that E is a locally free sheaf on X by Lemma 1.1.5(2). By $G \in \mathcal{C}$, we have $R^1\pi_*(E^\vee \otimes G) = 0$, which implies that $E^\vee \in T(G^\vee)$. By Corollary 1.1.12, there is a surjection $G^\vee \otimes \pi^*(W) \to E^\vee$, where W is a locally free sheaf on Y. Then there is an inclusion $E \hookrightarrow G \otimes \pi^*(W^\vee)$. Hence $\pi_*(E \otimes F) = 0$ for $F \in \operatorname{Coh}(X)$ with $F \in S(G^\vee)$. Since there is a surjection $G \otimes \pi^*(V) \to E$, $R^1\pi_*(E \otimes F) = 0$ for $F \in T(G^\vee)$.

(3) Let G' be a local projective generator of $\mathcal{C}$. Then (1) implies that $(T(G'^\vee), S(G'^\vee))$ is also a torsion pair. By (2), $G^\vee$ is a local projective object of $\mathcal{C}(G'^\vee)$. In particular, $G^\vee \in T(G'^\vee)$. Then we have $T(G^\vee) \subset T(G'^\vee)$ by Corollary 1.1.12. In the same way, we also have $T(G'^\vee) \subset T(G^\vee)$. Therefore the claim holds.

(4) Assuming that $\mathbf{R}\pi_*(G^\vee \otimes E)$ is a zero-dimensional sheaf on Y, we shall show that $\mathbf{R}\pi_*(G \otimes D_X(E)(K_X)[n])$ is a zero-dimensional sheaf on Y. The argument is similar to that for (1). By our assumption,

$$\operatorname{Hom}\bigl(G, E(-m)[k]\bigr) = \begin{cases} H^0(Y, \mathbf{R}\pi_*(G^\vee \otimes E)), & k=0, \\ 0, & k \neq 0. \end{cases} \tag{1.24}$$

By the Serre duality, we have

$$\operatorname{Hom}\bigl(G, E(-m)[k]\bigr) = \operatorname{Hom}\bigl(G^\vee, \bigl(D_X(E)(K_X)[n]\bigr)(m)[-k]\bigr)^\vee. \tag{1.25}$$

For $m \gg 0$, we have

$$\begin{aligned} &\operatorname{Hom}\bigl(G^\vee, \bigl(D_X(E)(K_X)[n]\bigr)(m)[-k]\bigr) \\ &\quad = H^0\bigl(Y, H^{-k}\bigl(\mathbf{R}\pi_*\bigl(G \otimes \bigl(D_X(E)(K_X)[n]\bigr)\bigr)\bigr)(m)\bigr). \end{aligned}$$

Hence $H^{-k}(\mathbf{R}\pi_*(G \otimes (D_X(E)(K_X)[n]))) = 0$ for $k \neq 0$ and $H^0(\mathbf{R}\pi_*(G \otimes (D_X(E)(K_X)[n])))$ is a zero-dimensional sheaf. □

We characterize $S_0(G)$ in terms of the Gieseker semistability of a 1-dimensional sheaf (see the definition in (0.6)).

LEMMA 1.1.15

Let G be a locally free sheaf on X such that $R^1\pi_*(G^\vee \otimes G) = 0$. Let E be a 1-dimensional sheaf on a fiber of π such that $\chi(G, E) = 0$. Then $\mathbf{R}\pi_*(G^\vee \otimes E) = 0$ if and only if E is a G-twisted semistable sheaf on X.

Proof

By the proof of Lemma 1.1.11(1), we can take a decomposition

$$0 \to E_1 \to E \to E_2 \to 0 \tag{1.26}$$

such that $\mathbf{R}\pi_*(G^\vee \otimes E_1) = \pi_*(G^\vee \otimes E)$ and $\mathbf{R}\pi_*(G^\vee \otimes E_2) = R^1\pi_*(G^\vee \otimes E)[-1]$. Then $\chi(G, E_1) \geq 0 \geq \chi(G, E_2)$. Hence if E is G-twisted semistable, then $\pi_*(G^\vee \otimes E_1) = \pi_*(G^\vee \otimes E) = 0$, which also implies that $R^1\pi_*(G^\vee \otimes E) = 0$. Conversely if $\pi_*(G^\vee \otimes E) = R^1\pi_*(G^\vee \otimes E) = 0$, then $\pi_*(G^\vee \otimes E') = 0$ for any subsheaf E' of E. Hence E is G-twisted semistable. □

COROLLARY 1.1.16

Assume that $\pi: X \to Y$ is the minimal resolution of a rational double point. Let G be a locally free sheaf on X. Then $(T(G), S(G))$ is a torsion pair with $G \in T(G)$ if and only if

(i) $R^1\pi_*(G^\vee \otimes G) = 0$ *and*

(ii) *there is no G-twisted stable sheaf E such that* $\operatorname{rk} E = 0$, $\chi(G^\vee \otimes E) = 0$, $(c_1(E), c_1(\mathcal{O}_X(1))) = 0$, *and* $(c_1(E)^2) = -2$.

Moreover, (ii) is equivalent to $\operatorname{rk} G \nmid (c_1(G), D)$ for D with $(D, c_1(\mathcal{O}_X(1))) = 0$ and $(D^2) = -2$.

Proof

Let E be a 1-dimensional G-twisted stable sheaf on X. Then E is a sheaf on the exceptional locus if and only if $(c_1(E), c_1(\mathcal{O}_X(1))) = 0$. Under this assumption, we have $\chi(E,E) = -(c_1(E)^2) > 0$. Since $E \otimes K_X \cong E$, we see that $\chi(E,E) \leq 2$. Hence $(c_1(E)^2) = -2$. By Lemma 1.1.15, we get the first part of our claim. Since $\chi(G,E) = -(c_1(G), c_1(E)) + \operatorname{rk} G \chi(E)$, we also get the second claim by [Y6, Proposition 4.6]. $\square$

1.1.1. Irreducible objects of $\mathcal{C}$

LEMMA 1.1.17

Let G be a locally free sheaf on X such that $\mathbf{R}\pi_*(G^\vee \otimes F) \neq 0$ for all nonzero coherent sheaves F on a fiber of π. Then for a coherent sheaf E on X, $\pi_*(G^\vee \otimes E) = 0$ implies that $R^1\pi_*(G^\vee \otimes E_{|\pi^{-1}(y)}) \neq 0$ for all $y \in \pi(\operatorname{Supp}(E))$.

Proof

Assume that $R^1\pi_*(G^\vee \otimes E_{|\pi^{-1}(y)}) = 0$. By Lemma 1.1.24 below, $R^1\pi_*(G^\vee \otimes E) = 0$ in a neighborhood of y. Thus $\mathbf{R}\pi_*(G^\vee \otimes E) = 0$ in a neighborhood of y. Then $\mathbf{R}\pi_*(G^\vee \otimes E \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi^*(\mathbb{C}_y)) = \mathbf{R}\pi_*(G^\vee \otimes E) \overset{\mathbf{L}}{\otimes} \mathbb{C}_y = 0$. Since the spectral sequence

$$\begin{aligned} E_2^{pq} &= R^p\pi_*\big(H^q\big(G^\vee \otimes E \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi^*(\mathbb{C}_y)\big)\big) \\ &\Longrightarrow E_\infty^{p+q} = H^{p+q}\big(\mathbf{R}\pi_*\big(G^\vee \otimes E \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi^*(\mathbb{C}_y)\big)\big) \end{aligned} \tag{1.27}$$

degenerates, $R^p\pi_*(G^\vee \otimes E \otimes \pi^*(\mathbb{C}_y)) = 0$. By our assumption on G, we have $E_{|\pi^{-1}(y)} = 0$, which is a contradiction. $\square$

COROLLARY 1.1.18

Let G be a locally free sheaf on X such that $S_0(G) = 0$. For an open subset of Y, we extend Definition 1.1.10 to $G_{|\pi^{-1}(U)} \in \operatorname{Coh}(\pi^{-1}(U))$. Then $S_0(G_{|\pi^{-1}(U)}) = 0$. In particular, $G_{|\pi^{-1}(U)}$ is a local projective generator of $\mathcal{C}(G_{|\pi^{-1}(U)})$.

Proof

We first note that Lemma 1.1.17 holds for the morphism $\pi' : \pi^{-1}(U) \to U$, since the projectivity of Y is not used in its proof. Since $S_0(G) = 0$, $\mathbf{R}\pi'_*(G^\vee \otimes F) \neq 0$ for all nonzero coherent sheaves F on a fiber of π'. For $E \in \operatorname{Coh}(\pi^{-1}(U))$ with $\pi'_*(G^\vee \otimes E) = 0$, we have $R^1\pi'_*(G^\vee \otimes E_{|\pi^{-1}(y)}) \neq 0$ for all $y \in \pi'(\operatorname{Supp}(E))$. Therefore $S_0(G_{|\pi^{-1}(U)}) = 0$. $\square$

DEFINITION 1.1.19

Let G be a local projective generator of a category of perverse coherent sheaves $\mathcal{C}$.

(1) An object $E \in \mathcal{C}$ is *zero-dimensional*, if $\mathbf{R}\pi_*(G^\vee \otimes E)$ is zero-dimensional as an object of $\operatorname{Coh}(Y)$.

(2) An object $E \in \mathcal{C}$ is *irreducible*, if E does not have a proper subobject except zero.

(3) For a zero-dimensional object $E \in \mathcal{C}$, we take a filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{1.28}$$

such that F_i/F_{i-1} are irreducible objects of $\mathcal{C}$. Then $\bigoplus_i F_i/F_{i-1}$ is the *Jordan–Hölder decomposition* of E. As is well known, the Jordan–Hölder decomposition is unique, though (1.28) is not unique.

REMARK 1.1.20

In Section 1.4, we shall define the dimension of E generally. According to the definition of the stability in Definition 1.4.1, we also have the following.

(1) A zero-dimensional object E is G-twisted semistable, and a G-twisted stable object corresponds to an irreducible object.

(2) The Jordan–Hölder decomposition of E is nothing but the standard representative of the S-equivalence class of E.

LEMMA 1.1.21

Let G be as in Lemma 1.1.11, and suppose that $(T(G), S(G))$ is a torsion pair with $G \in T(G)$.

(1) For $y \in Y_\pi$, let $\pi^{-1}(y)_{\mathrm{red}}$ be the reduced subscheme of $\pi^{-1}(y)$. Then $\pi^{-1}(y)_{\mathrm{red}}$ is a tree of smooth rational curves.

(2) We have $\mathbb{C}_x \in \mathcal{C}(G)$ for all $x \in X$.

(3) For $\mathbb{C}_x, x \in \pi^{-1}(y)$, the Jordan–Hölder decomposition depends only on $y = \pi(x)$.

(4) Let $\bigoplus_{j=0}^{s_y} \mathbf{I}_{yj}^{\oplus a_{yj}}$ be the Jordan–Hölder decomposition of $\mathbb{C}_x$ ($y = \pi(x) \in Y_\pi$). Then the irreducible objects of $\mathcal{C}(G)$ are

$$\mathbb{C}_x \quad \bigl(x \in X \setminus \pi^{-1}(Y_\pi)\bigr), \qquad \mathbf{I}_{yj} \quad (y \in Y_\pi, 0 \le j \le s_y). \tag{1.29}$$

In particular, if $\mathbf{R}\pi_*(G^\vee \otimes E)$ is a zero-dimensional $\mathcal{A}$-module, then E is generated by (1.29).

Proof

For (2) we note that $\mathbf{R}\pi_*(G^\vee \otimes \mathbb{C}_x) = \pi_*(G^\vee \otimes \mathbb{C}_x)$. Hence $\mathbb{C}_x \in \mathcal{C}(G)$. For (1) and (3) we have a surjective map

$$R^1\pi_*(\mathcal{O}_X) \to R^1\pi_*(\mathcal{O}_{\pi^{-1}(y)_{\mathrm{red}}}). \tag{1.30}$$

Since $R^1\pi_*(\mathcal{O}_X) = 0$ by Assumption 1.1.1, we get

$$H^1\bigl(\pi^{-1}(y)_{\mathrm{red}}, \mathcal{O}_{\pi^{-1}(y)_{\mathrm{red}}}\bigr) = H^0\bigl(Y, R^1\pi_*(\mathcal{O}_{\pi^{-1}(y)_{\mathrm{red}}})\bigr) = 0.$$

Then we see that $\pi^{-1}(y)_{\mathrm{red}}$ is a tree of smooth rational curves. Let C_{yj}, $j = 1, \dots, t_y$, be the irreducible component of $\pi^{-1}(y)_{\mathrm{red}}$. Since the restriction map $R^1\pi_*(G^\vee \otimes G) \to R^1\pi_*(G^\vee \otimes G_{|C_{yj}})$ is surjective, $R^1\pi_*(G^\vee \otimes G_{|C_{yj}}) = 0$. Thus we can write $G_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}(d_{yj})^{\oplus r_{yj}} \oplus \mathcal{O}_{C_{yj}}(d_{yj} + 1)^{\oplus r'_{yj}}$. Since $R^1\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(d_{yj})) = 0$ and $\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(d_{yj} - 1)) = 0$, $\mathcal{O}_{C_{yj}}(d_{yj}), \mathcal{O}_{C_{yj}}(d_{yj} - 1)[1] \in$

$\mathcal{C}(G)$. For $x \in C_{yj}$, we have an exact sequence in $\mathcal{C}_G$,

$$0 \to \mathcal{O}_{C_{yj}}(d_{yj}) \to \mathbb{C}_x \to \mathcal{O}_{C_{yj}}(d_{yj}-1)[1] \to 0. \tag{1.31}$$

Hence the Jordan–Hölder decomposition of $\mathbb{C}_x$ is constant on C_{yj}. Since $\pi^{-1}(y)$ is connected, the Jordan–Hölder decomposition of $\mathbb{C}_x$ is determined by y.

To see (4) let E be an irreducible object of $\mathcal{C}(G)$. Then we have

(i) $E = F[1]$, $F \in \mathrm{Coh}(X)$, or
(ii) $E \in \mathrm{Coh}(X)$.

In the first case, since $F \in S(G)$, we have $\pi_*(G^\vee \otimes F) = 0$. By Lemma 1.1.17, we have $R^1\pi_*(G^\vee \otimes F_{|\pi^{-1}(y)}) \neq 0$ for $y \in \pi(\mathrm{Supp}(F))$, which implies that there is a quotient $F_{|\pi^{-1}(y)} \to F'$ such that $0 \neq F' \in S(G)$ for $y \in \pi(\mathrm{Supp}(F))$. Then we have a nontrivial morphism $F[1] \to F'[1]$, which should be injective in $\mathcal{C}(G)$. Therefore $\pi(\mathrm{Supp}(F))$ is a point. In the second case, we also see that $\pi(\mathrm{Supp}(E))$ is a point. Therefore $\mathbf{R}\pi_*(G^\vee \otimes E)$ is a zero-dimensional sheaf.

(i) If $E = F[1]$, then since $\pi_*(G^\vee \otimes F) = 0$, F is purely 1-dimensional. Then $\mathrm{Hom}(\mathbb{C}_x, F[1]) = \mathrm{Hom}(D(F)[n-1], D(\mathbb{C}_x)[n]) \neq 0$ for $x \in \mathrm{Supp}(F)$, where $n = \dim X$. Hence we have a nontrivial morphism $\mathbf{I}_{yj} \to E$, $y \in \pi(\mathrm{Supp}(F)) \cap Y_\pi$, which is an isomorphism.

(ii) If $E \in \mathrm{Coh}(X)$, then $\mathrm{Hom}(E, \mathbb{C}_x) \neq 0$ for $x \in \mathrm{Supp}(E)$, which also implies that $E \cong \mathbf{I}_{yj}$ for $\mathrm{Supp}(E) \subset \pi^{-1}(y)$ or $E \cong \mathbb{C}_x$ for $\mathrm{Supp}(E) \subset X \setminus \pi^{-1}(Y_\pi)$. □

REMARK 1.1.22

Since $\pi_*(G^\vee \otimes \mathbb{C}_x)$ is a coherent sheaf on the reduced point $\{y\}$, the multiplication $\pi^*(t) : \mathbf{I}_{yj} \to \mathbf{I}_{yj}$, $t \in I_y$ is zero. Thus $H^i(\mathbf{I}_{yj})$ are coherent sheaves on the scheme $\pi^{-1}(y)$.

LEMMA 1.1.23

Let $\mathbf{I}_{yj} \in \mathcal{C}(G)$ be irreducible objects in Lemma 1.1.21. Let E be a coherent sheaf such that $\pi(\mathrm{Supp}(E)) = \{y\} \subset Y_\pi$.

(1) *For $E \in T(G)$, there is a filtration*

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{1.32}$$

such that for every F_k/F_{k-1}, there is $\mathbf{I}_{yj} \in T(G)$ and a surjective homomorphism $\mathbf{I}_{yj} \to F_k/F_{k-1}$ in $\mathrm{Coh}(X)$.

(2) *For $E \in S(G)$, there is a filtration*

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{1.33}$$

such that, for every F_k/F_{k-1}, there is $\mathbf{I}_{yj}[-1] \in S(G)$ and an injective homomorphism $F_k/F_{k-1} \to \mathbf{I}_{yj}[-1]$ in $\mathrm{Coh}(X)$.

Proof
(1) Since $E \in T(G)$, E contains $\mathbf{I}_{yj}$ in $\mathcal{C}(G)$. Let F be the quotient in $\mathcal{C}(G)$. Then we have an exact sequence

$$0 \to H^{-1}(\mathbf{I}_{yj}) \to 0 \to H^{-1}(F) \to H^0(\mathbf{I}_{yj}) \to E \to H^0(F) \to 0. \tag{1.34}$$

Hence $\mathbf{I}_{yj} \in T(G)$ and $H^0(F) \in T(G)$. We set $F_1 := \mathrm{im}(\mathbf{I}_{yj} \to E)$ in $\mathrm{Coh}(X)$. Since $E/F_1 \in T(G)$ and $\mathrm{Supp}(E/F_1) \subset \pi^{-1}(y)$, by the induction on the support of E, we get the claim.

(2) Since $E \in S(G)$, there is a quotient $E[1] \to \mathbf{I}_{yj}$ in $\mathcal{C}(G)$. Let F be the kernel in $\mathcal{C}(G)$. Then we have an exact sequence

$$0 \to H^{-1}(F) \to E \to H^{-1}(\mathbf{I}_{yj}) \to H^0(F) \to 0 \to H^0(\mathbf{I}_{yj}) \to 0. \tag{1.35}$$

Hence $\mathbf{I}_{yj}[-1] \in S(G)$ and $H^{-1}(F) \in S(G)$. We set $E' := \mathrm{im}(E \to H^{-1}(\mathbf{I}_{yj}))$ in $\mathrm{Coh}(X)$. Then E' is a subsheaf of $\mathbf{I}_{yj}[-1]$, and E is an extension of E' by $H^{-1}(F) \in S(G)$. Since $\mathrm{Supp}(H^{-1}(F)) \subset \pi^{-1}(y)$, by the induction on the support of E, we get the claim. $\square$

LEMMA 1.1.24
(1) *The natural homomorphism* $\pi^*(\pi_*(I_{\pi^{-1}(y)})) \to I_{\pi^{-1}(y)}$ *is surjective. In particular,* $\mathrm{Hom}(I_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1)) = 0$ *for all* j.

(2) *We have* $\mathrm{Ext}^1(\mathcal{O}_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1)) = 0$ *for all* j. *In particular,*

$$H^1\big(X, \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{O}_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1)\big)\big) = H^0\big(X, \mathcal{E}xt^1_{\mathcal{O}_X}\big(\mathcal{O}_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1)\big)\big) = 0.$$

(3) *For a coherent sheaf* E *on* X, $R^1\pi_*(E) = 0$ *at* y *if and only if* $R^1\pi_*(E_{|\pi^{-1}(y)}) = 0$.

Proof
Since $I_{\pi^{-1}(y)} = \mathrm{im}(\pi^*(I_y) \to \mathcal{O}_X)$, (1) holds. (2) Since $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_{C_{yj}}(-1)[k]) = 0$ for all j and k, the first claim follows from the exact sequence

$$0 \to I_{\pi^{-1}(y)} \to \mathcal{O}_X \to \mathcal{O}_{\pi^{-1}(y)} \to 0. \tag{1.36}$$

Since $H^2(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1))) = 0$, the second claim follows from the local-global spectral sequence.

(3) The proof is similar to [Is1]. Assume that $R^1\pi_*(E_{|\pi^{-1}(y)}) = 0$. We take a locally free sheaf V on Y such that $V \to I_y$ is surjective. Then (1) implies that $\pi^*(V) \to I_{\pi^{-1}(y)}$ is surjective. Hence we have a surjective homomorphism $\pi^*(V^{\otimes n}) \otimes \mathcal{O}_{\pi^{-1}(y)} \to I^n_{\pi^{-1}(y)}/I^{n+1}_{\pi^{-1}(y)}$. Then we see that $R^1\pi_*(E \otimes \mathcal{O}_X/I^n_{\pi^{-1}(y)}) = 0$. By the theorem of formal functions, we get the claim. $\square$

LEMMA 1.1.25
Let $\mathbf{I}_{yj} \in \mathcal{C}(G)$ *be irreducible objects in Lemma 1.1.21. Let* E *be a coherent sheaf on* X. *If* $\mathrm{Hom}(E, \mathbf{I}_{yj}[-1]) = 0$ *for all* $\mathbf{I}_{yj}[-1] \in S(G)$, *then* $E \in T(G)$.

Proof
We note that $\operatorname{Hom}(E_{|\pi^{-1}(y)}, \mathbf{I}_{yj}[-1]) = 0$ for all $\mathbf{I}_{yj}[-1] \in S(G)$. By Lemma 1.1.23(2), $E_{|\pi^{-1}(y)} \in T(G)$. Then $R^1\pi_*(G^\vee \otimes E_{|\pi^{-1}(y)}) = 0$. By Lemma 1.1.24, $R^1\pi_*(G^\vee \otimes E) = 0$ in a neighborhood of y. Since y is any point of Y_π, $R^1\pi_*(G^\vee \otimes E) = 0$, which implies that $E \in T(G)$. $\square$

PROPOSITION 1.1.26
Assume that $\#Y_\pi < \infty$. Let $\mathbf{I}_{yj} \in \mathcal{C}(G)$ be irreducible objects in Lemma 1.1.21.

(1) *We set*

$$\Sigma := \left\{\mathbf{I}_{yj}[-1] \,\middle|\, y \in Y_\pi, j = 0, \dots, s_y\right\} \cap \operatorname{Coh}(X),$$
$$\mathcal{T} := \left\{E \in \operatorname{Coh}(X) \,\middle|\, \operatorname{Hom}(E, c) = 0, c \in \Sigma\right\}, \tag{1.37}$$
$$\mathcal{S} := \left\{E \in \operatorname{Coh}(X) \,\middle|\, E \text{ is a successive extension of subsheaves of } c \in \Sigma\right\}.$$

Then $(\mathcal{T}, \mathcal{S})$ is a torsion pair of $\operatorname{Coh}(X)$ *whose tilting is $\mathcal{C}(G)$. In particular, $\mathcal{C}(G)$ is characterized by Σ.*

(2) *For the category $\mathcal{C}^D$ in Lemma 1.1.14 for $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{C}(G)^D$ is characterized by*

$$\begin{aligned} \Sigma^D &:= \left\{\big(D_X(\mathbf{I}_{yj}) \otimes K_X[n]\big)[-1] \,\middle|\, y \in Y_\pi, j = 0, \dots, s_y\right\} \cap \operatorname{Coh}(X) \\ &= D_X\big(\{\mathbf{I}_{yj} | y \in Y_\pi, j = 0, \dots, s_y\} \cap \operatorname{Coh}(X)\big) \otimes K_X[n-1], \end{aligned} \tag{1.38}$$

where $n = \dim X$.

Proof
(1) For $E \in \operatorname{Coh}(X)$, we consider $\phi : G \otimes \pi^*(\pi_*(G^\vee \otimes E)) \to E$. We set $E_1 := \operatorname{im}\phi$ and $E_2 := \operatorname{coker}\phi$. Since $\operatorname{Hom}(G, \mathbf{I}_{yj}[-1]) = 0$ for all $\mathbf{I}_{yj}$, $G \in \mathcal{T}$. Hence $E_1 \in \mathcal{T}$. We shall show that $E_2 \in \mathcal{S}$. By Corollary 1.1.12, $E_1 \in T(G), E_2 \in S(G)$. Since $\operatorname{Supp}(E_2) \subset \pi^{-1}(Y_\pi)$, Lemma 1.1.23(2) implies that $E_2 \in \mathcal{S}$. Therefore $(\mathcal{T}, \mathcal{S})$ is a torsion pair of $\operatorname{Coh}(X)$. We also see that $(\mathcal{T}, \mathcal{S}) = (T(G), S(G))$. Thus (1) holds.

(2) By Lemma 1.1.14(4), $D_X(\mathbf{I}_{yj}) \otimes K_X[n]$ are the irreducible objects of $\mathcal{C}(G)^D$. Hence the claim follows from (1). $\square$

1.1.2. Local projective generators of $\mathcal{C}$
We shall give a criterion for a two-term complex of coherent sheaves to be a local projective generator of a category of perverse coherent sheaves. Since the existence of a local projective generator is the most essential part of our theory, we also discuss a certain torsion pair (see Definition 1.1.28(2)) to define a category of perverse coherent sheaves.

DEFINITION 1.1.27
Let $\mathcal{C}$ be an abelian subcategory of $\mathbf{D}(X)$. For $y \in Y$, we set

$$\mathcal{C}_y := \left\{E \in \mathcal{C} \,\middle|\, \pi\big(\operatorname{Supp}\big(H^i(E)\big)\big) = \{y\}, i \in \mathbb{Z}\right\}. \tag{1.39}$$

DEFINITION 1.1.28

In Section 1.1.2, let (T,S) be a torsion pair of $\mathrm{Coh}(X)$ such that the tilted category $\mathcal{C}$ satisfies one of the following conditions.

(1) There is a local projective generator $G \in T$ of $\mathcal{C}$; that is, $\mathcal{C}$ is a category of perverse coherent sheaves, or

(2) $\mathcal{C}$ satisfies the following conditions:

 (a) $\#Y_\pi < \infty$ and every object of $\mathcal{C}_y$, $y \in Y$ is of finite length;

 (b) $\pi(\mathrm{Supp}(E)) \subset Y_\pi$ for $E \in S$.

The condition $\#Y_\pi < \infty$ is a technical condition. Other conditions of (2) are satisfied for a category of perverse coherent sheaves.

DEFINITION 1.1.29

Assume that $F \subset E$ in $\mathcal{C}$ and $E \in \mathcal{C}_y$, $y \in Y$ implies $F \in \mathcal{C}_y$. Then $\mathbf{I}_{yj}, j \in J_y = \{0,1,\dots,s_y\}$ denote the irreducible objects of $\mathcal{C}_y$.

If $y \in Y \setminus Y_\pi$, then $s_y = 0$ and $E_{y0} = \mathbb{C}_x$ $(\pi(x) = y)$.

LEMMA 1.1.30

Let C_{yj} $(j = 1,\dots,t_y)$ be the irreducible components of $\pi^{-1}(y)_{\mathrm{red}}$, $y \in Y_\pi$. Assume that $\mathcal{C}$ satisfies Definition 1.1.28(2). Then the following assertions hold:

 (1) *$\mathbb{C}_x \in \mathcal{C}$ for all $x \in X$;*

 (2) *let L be a line bundle on C_{yj}; then $L \in T$ or $L \in S$; moreover, there is $n \in \mathbb{Z}$ such that $\mathcal{O}_{C_{yj}}(n) \in S$ and $\mathcal{O}_{C_{yj}}(n+1) \in T$;*

 (3) *the claims of Lemma 1.1.21, Lemma 1.1.23, and Lemma 1.1.25 hold.*

Proof

We first show that the assumption of Definition 1.1.29 holds. Let F be a subobject of E and $E \in \mathcal{C}_y$. Then we have a morphism $H^{-1}(E/F) \to H^0(F)$. Suppose that $H^{-1}(E/F)_{|U} \neq 0$ for the open set $U := X \setminus \pi^{-1}(y)$. Since $E \in \mathcal{C}_y$, we have an isomorphism $H^{-1}(E/F)_{|U} \to H^0(F)_{|U}$. Since $\mathrm{Supp}(H^{-1}(E/F)) \subset \pi^{-1}(Y_\pi)$ by (b), we have a decomposition $H^{-1}(E/F) \cong \bigoplus_{y' \in \pi_*(\mathrm{Supp}(H^{-1}(E/F)))} V_{y'}$, where $\pi_*(\mathrm{Supp}(V_{y'})) = \{y'\}$. In particular, we can regard $H^{-1}(E/F)_{|U}$ as a subsheaf of $H^{-1}(E/F)$. Then we have a nonzero homomorphism $H^0(F) \to H^0(F)_{|U} \to H^{-1}(E/F)_{|U} \hookrightarrow H^{-1}(E/F)$. Since (T,S) is a torsion pair, $H^0(F) \to H^{-1}(E/F)$ is a zero map. This is a contradiction. Therefore $\mathrm{Supp}(H^{-1}(E/F)), \mathrm{Supp}(H^0(F)) \subset \pi^{-1}(y)$, which implies the claim.

By Definition 1.1.27(2), irreducible objects are $E = \mathbb{C}_x, x \in X \setminus \pi^{-1}(Y_\pi)$, or irreducible objects of $\mathcal{C}_y, y \in Y_\pi$. For a point $x \in \pi^{-1}(Y_\pi)$, assume that $\mathbb{C}_x \notin T$. Since $\mathbb{C}_x$ is an irreducible object of $\mathrm{Coh}(X)$ and (T,S) is a torsion pair, $\mathbb{C}_x \in S$. We take a curve C_{yj} with $x \in C_{yj}$. Then for any line bundle L on C_{yj}, $\mathrm{Hom}(L, \mathbb{C}_x) \cong \mathbb{C}$ implies that $L \in S$ for all line bundles L on C_{yj}. Indeed let L_T be the subsheaf of L such that $L_T \in T$ and $L/L_T \in S$. Then $\mathbb{C}_x \in S$ implies that $\mathrm{Hom}(L_T, \mathbb{C}_x) = 0$. Hence $L_T = 0$. Since $L(-nx) \in S$ for all positive integers n, $L[1] \in \mathcal{C}_y$ is not of finite length. Therefore $\mathbb{C}_x \in T$ for $x \in \pi^{-1}(Y_\pi)$. Thus (1) holds.

(2) Let L be a line bundle on C_{yj}. Then we have a decomposition

$$0 \to L_1 \to L \to L_2 \to 0 \tag{1.40}$$

such that $L_1 \in T$ and $L_2 \in S$. We note that (1) implies that every torsion $\mathcal{O}_{C_{yj}}$-module belongs to T. If $L_1 \neq 0$, then L_2 is a torsion $\mathcal{O}_{C_{yj}}$-module, which implies that $L_2 = 0$ and $L \in T$. If $L_1 = 0$, then $L = L_2 \in S$. Thus the first claim holds. Assume that $\mathcal{O}_{C_{yj}} \in T$. If $\mathcal{O}_{C_{yj}}(-n) \in T$ for all $n > 0$, then $\mathcal{O}_{C_{yj}}$ is not of finite length. Hence there is a positive integer n such that $\mathcal{O}_{C_{yj}}(-n) \in S$ and $\mathcal{O}_{C_{yj}}(-n+1) \in T$. We next assume that $\mathcal{O}_{C_{yj}} \in S$. If $\mathcal{O}_{C_{yj}}(n) \in S$ for all positive integers n, then the exact sequence in $\mathcal{C}_y$,

$$0 \to \mathcal{O}_{C_{yj}}(n)/\mathcal{O}_{C_{yj}} \to \mathcal{O}_{C_{yj}}[1] \to \mathcal{O}_{C_{yj}}(n)[1] \to 0, \tag{1.41}$$

implies that $\mathcal{O}_{C_{yj}}[1] \in \mathcal{C}_y$ is not of finite length. Therefore there is a positive integer n such that $\mathcal{O}_{C_{yj}}(n-1) \in S$ and $\mathcal{O}_{C_{yj}}(n) \in T$.

(3) By (1), we get Lemma 1.1.21(2). We also get Lemma 1.1.21(3) from its proof and (2). The other claims of Lemmas 1.1.21 and 1.1.23 are obvious. For $0 \neq E \in S$, (i) and Lemma 1.1.23 imply that there is a coherent sheaf $\mathbf{I}_{yj}[-1] \in S$ such that $\mathrm{Hom}(E, \mathbf{I}_{yj}[-1]) \neq 0$. Hence Lemma 1.1.25 also holds. □

We shall give a criterion (Proposition 1.1.33) for a two-term complex to be a local projective generator of $\mathcal{C}$. Since objects in $\mathcal{C}$ are two-term complexes, our criterion is applicable to these objects.

LEMMA 1.1.31

Let E be an object of $\mathbf{D}(X)$ such that $H^i(E) = 0$ for $i \neq -1, 0$. If $\mathrm{Ext}^1(E, \mathbb{C}_x) = 0$, then E is a free sheaf in a neighborhood of x.

Proof

Since E fits in the exact triangle

$$\tau^{\leq -1}(E) \to E \to \tau^{\geq 0}(E) \to \big(\tau^{\leq -1}(E)\big)[1], \tag{1.42}$$

we have an exact sequence

$$\begin{aligned} &0 \to \mathcal{E}xt^1_{\mathcal{O}_X}\big(H^0(E), \mathbb{C}_x\big) \\ &\quad \to \mathcal{E}xt^1_{\mathcal{O}_X}(E, \mathbb{C}_x) \to \mathcal{H}om_{\mathcal{O}_X}\big(H^{-1}(E), \mathbb{C}_x\big) \to \mathcal{E}xt^2_{\mathcal{O}_X}\big(H^0(E), \mathbb{C}_x\big). \end{aligned} \tag{1.43}$$

Since $\mathrm{Ext}^1(E, \mathbb{C}_x) = H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(E, \mathbb{C}_x))$, $\mathcal{E}xt^1_{\mathcal{O}_X}(E, \mathbb{C}_x) = 0$. Then $\mathcal{E}xt^1_{\mathcal{O}_X}(H^0(E), \mathbb{C}_x) = 0$, which implies that $H^0(E)$ is a free sheaf in a neighborhood of x. Then $\mathcal{E}xt^i_{\mathcal{O}_X}(H^0(E), \mathbb{C}_x) = 0$ for $i > 0$. Hence $\mathcal{H}om_{\mathcal{O}_X}(H^{-1}(E), \mathbb{C}_x) = 0$. Therefore $H^{-1}(E) = 0$ in a neighborhood of x. □

LEMMA 1.1.32

Let G_1 be a locally free sheaf of rank r on X such that

$$\text{(a)} \quad \operatorname{Hom}\big(G_1, \mathbf{I}_{yj}[p]\big) = 0, \quad p \neq 0, \qquad \text{(b)} \quad \chi(G_1, \mathbf{I}_{yj}) > 0 \tag{1.44}$$

for all y, j.

(1) *If $0 \neq E \in S$, then $\pi_*(G_1^\vee \otimes E) = 0$ and $R^1\pi_*(G_1^\vee \otimes E) \neq 0$.*
(2) *If $R^1\pi_*(G_1^\vee \otimes E) = 0$, then $E \in T$.*
(3) *If $0 \neq E \in T$ and $\operatorname{Supp}(E) \subset \pi^{-1}(y)$, then $\pi_*(G_1^\vee \otimes E) \neq 0$ and $R^1\pi_*(G_1^\vee \otimes E) = 0$. In particular, $\chi(G_1, E) > 0$.*

Proof

(1) We note that $G_1 \in T$ by Lemma 1.1.25. We first treat the case where $\mathcal{C}$ is the category of perverse coherent sheaves. We consider the homomorphism $\pi^*(\pi_*(G_1^\vee \otimes E)) \otimes G_1 \to E$. Then $\operatorname{im}\phi \in T \cap S = 0$. Since $\pi_*(G_1^\vee \otimes \operatorname{im}\phi) = \pi_*(G_1^\vee \otimes E)$, we get $\pi_*(G_1^\vee \otimes E) = 0$. Let $F \neq 0$ be a coherent sheaf on a fiber, and take the decomposition

$$0 \to F_1 \to F \to F_2 \to 0 \tag{1.45}$$

with $F_1 \in T, F_2 \in S$. Since $F_1, F_2[1] \in \mathcal{C}$, the condition $\chi(G_1, \mathbf{I}_{yj}) > 0$ implies that $\chi(G_1, F_1) > 0$ or $\chi(G_1, F_2) < 0$, which imply that $\pi_*(G_1^\vee \otimes F_1) \neq 0$ or $R^1\pi_*(G_1^\vee \otimes F_2) \neq 0$. Since $\pi_*(G_1^\vee \otimes F_1)$ is a subsheaf of $\pi_*(G_1^\vee \otimes F)$ and $R^1\pi_*(G_1^\vee \otimes F_2)$ is a quotient of $R^1\pi_*(G_1^\vee \otimes F)$, we get $\mathbf{R}\pi_*(G_1^\vee \otimes F) \neq 0$. Then we can apply Lemma 1.1.17 to E and get $R^1\pi_*(G_1^\vee \otimes E_{|\pi^{-1}(y)}) \neq 0$ for $y \in \pi(\operatorname{Supp}(E))$. Since $R^1\pi_*(G_1^\vee \otimes E) \to R^1\pi_*(G_1^\vee \otimes E_{|\pi^{-1}(y)}) \neq 0$ is surjective, we get the claim.

We next assume that $\#Y_\pi < \infty$. Then $E[1]$ is generated by $\mathbf{I}_{yj}$. Hence (1.44) implies that $\chi(G_1, E[1]) > 0$ and $\mathbf{R}\pi_*(G_1^\vee \otimes E[1]) \in \operatorname{Coh}(Y)$. Hence $R^1\pi_*(G_1^\vee \otimes E) \neq 0$ and $\pi_*(G_1^\vee \otimes E) = 0$.

(2) For $E \in \operatorname{Coh}(X)$, we take a decomposition

$$0 \to E_1 \to E \to E_2 \to 0 \tag{1.46}$$

such that $E_1 \in T$ and $E_2 \in S$. If $R^1\pi_*(G_1^\vee \otimes E) = 0$, then (1) implies that $E_2 = 0$.

(3) By Lemma 1.1.23, we may assume that E is a quotient of $\mathbf{I}_{yj}$, $\mathbf{I}_{yj} \in T$ in $\operatorname{Coh}(X)$. Since $\mathbf{I}_{yj}$ is irreducible, $\phi : \mathbf{I}_{yj} \to E$ is injective in $\mathcal{C}$. We set $F := \ker(\mathbf{I}_{yj} \to E)$ in $\operatorname{Coh}(X)$. Then $F \in S$ and $F[1]$ is the cokernel of ϕ in $\mathcal{C}$. Hence $\pi_*(G_1^\vee \otimes F) = 0$ by (1). By our assumption, $\pi_*(G_1^\vee \otimes \mathbf{I}_{yj}) \neq 0$, $\mathbf{I}_{yj} \in T$, and $R^1\pi_*(G_1^\vee \otimes \mathbf{I}_{yj}) = 0$. Therefore our claim holds. □

PROPOSITION 1.1.33

Let G_1 be an object of $\mathbf{D}(X)$ such that $H^i(G_1) = 0$ for $i \neq -1, 0$ and satisfies

$$\text{(a)} \quad \operatorname{Hom}\big(G_1, \mathbf{I}_{yj}[p]\big) = 0, \quad p \neq 0, \qquad \text{(b)} \quad \chi(G_1, \mathbf{I}_{yj}) > 0 \tag{1.47}$$

for all $y \in Y$ and $j = 0, 1, \ldots, s_y$.

(1) *G_1 is a locally free sheaf on X.*

(2) *We have* $R^1\pi_*(G_1^\vee \otimes G_1) = 0$.

(3) *For* $E \in \mathrm{Coh}(X)$, $E \in T$ *if and only if* $R^1\pi_*(G_1^\vee \otimes E) = 0$, *and* $E \in S$ *if and only if* $\pi_*(G_1^\vee \otimes E) = 0$.

(4) G_1 *is a local projective generator of* $\mathcal{C}$.

Proof

(1) The claim follows from Lemma 1.1.31 and (a).

(2) It is sufficient to prove that $R^1\pi_*(G_1^\vee \otimes G_{1|\pi^{-1}(y)}) = 0$ for all $y \in Y_\pi$. By Lemma 1.1.25, $G_1 \in T$. Since $\mathrm{Supp}(G_{1|\pi^{-1}(y)}) = \pi^{-1}(y)$ and $G_{1|\pi^{-1}(y)} \in T$, Lemma 1.1.23(1) implies that $G_{1|\pi^{-1}(y)} \in T$ is a successive extension of quotients of $\mathbf{I}_{yj} \in T$. Hence it is sufficient to prove $R^1\pi_*(G_1^\vee \otimes Q) = 0$ for all quotients Q of $\mathbf{I}_{yj} \in T$. By our assumption on G_1, we have $R^1\pi_*(G_1^\vee \otimes \mathbf{I}_{yj}) = 0$ for $\mathbf{I}_{yj} \in T$. Therefore the claim holds.

(3) By Lemma 1.1.32(2), we get

$$T(G_1) \cap S(G_1) \subset T \cap S(G_1) = \left\{E \in T \;\middle|\; \pi_*(G_1^\vee \otimes E) = 0\right\}. \tag{1.48}$$

If $T \cap S(G_1) = 0$, then Lemma 1.1.11(1) implies that G_1 is a local projective generator of $\mathcal{C}(G_1)$. Since $G_1 \in T$ by (2), Lemma 1.1.11(3) also implies that $\mathcal{C} = \mathcal{C}(G_1)$. Therefore we shall prove that $T \cap S(G_1) = 0$. Assume that $E \in T$ satisfies $\pi_*(G_1^\vee \otimes E) = 0$. We first prove that $R^1\pi_*(G_1^\vee \otimes E) = 0$. By Lemma 1.1.24, it is sufficient to prove $R^1\pi_*(G_1^\vee \otimes E_{|\pi^{-1}(y)}) = 0$ for all $y \in Y$. This follows from Lemma 1.1.32(3). Hence $\mathbf{R}\pi_*(G_1^\vee \otimes E) = 0$. Then we see that $\mathbf{R}\pi_*(G_1^\vee \otimes E_{|\pi^{-1}(y)}) = 0$ for all $y \in Y$ by the proof of Lemma 1.1.17. Since $E_{|\pi^{-1}(y)} \in T$, Lemma 1.1.32(3) implies that $E_{|\pi^{-1}(y)} = 0$ for all $y \in Y$. Therefore $E = 0$.

(4) This is a consequence of (3) and Lemma 1.1.11(2). □

REMARK 1.1.34

According to (4), $\mathcal{C}$ satisfying Definition 1.1.28(2) is a category of perverse coherent sheaves, if there is G_1 in Proposition 1.1.33.

REMARK 1.1.35

In the condition (1.47), assume that G_1 satisfies $\mathrm{Hom}(G_1, \mathbf{I}_{yj}[p]) = 0$ $(p = \pm 1)$ only. Then we also see that G_1 is locally free by Lemma 1.1.31. Since $\dim \pi^{-1}(y) \leq 1$ for all $y \in Y$, we have $\mathrm{Hom}(G_1, \mathbf{I}_{yj}[p]) = 0$ for $p \neq 0$. Thus condition (a) follows. Then the proofs of Lemma 1.1.32 and Proposition 1.1.33 imply that $R^1\pi_*(G_1^\vee \otimes G_1) = 0$ and $\mathbf{R}\pi_*(G_1^\vee \otimes F) \in \mathrm{Coh}(Y)$ for $F \in \mathcal{C}$.

The following claim shows that $R^1\pi_*(G_1^\vee \otimes G_1) = 0$ is a fairly strong condition. Since $R^1\pi_*(G_1^\vee \otimes G_1) = 0$ is an open condition, it also says that a small deformation of a local projective generator is also a local projective generator.

LEMMA 1.1.36

Let G_1 be a locally free sheaf of rank r on X such that

$$\chi(G_1, \mathbf{I}_{yj}) > 0. \tag{1.49}$$

Then $\operatorname{Hom}(G_1, \mathbf{I}_{yj}[k]) = 0, k \neq 0$, *if and only if* $R^1\pi_*(G_1^\vee \otimes G_1) = 0$.

Proof

The only if part was already proved in Proposition 1.1.33. Assume that $R^1\pi_*(G_1^\vee \otimes G_1) = 0$. We first prove that $G_1 \in T$. Assume that $G_1 \notin T$. Then there is a surjective homomorphism $G_1 \to E$ in $\operatorname{Coh}(X)$ such that $E \in S$. If $\mathcal{C}$ has a local projective generator G, then $\pi_*(G^\vee \otimes E) = 0$. By Lemma 1.1.17, we have $R^1\pi_*(G^\vee \otimes E_{|\pi^{-1}(y)}) \neq 0$ for a point $y \in Y$. Hence we may assume that $\operatorname{Supp}(E) \subset \pi^{-1}(y)$. In the second case, since $\#Y_\pi < \infty$, we may also assume that $\operatorname{Supp}(E) \subset \pi^{-1}(y)$. Then $E[1]$ is generated by $\mathbf{I}_{yj}$, $0 \leq j \leq s_y$. By our assumption, $\chi(G_1, E[1]) > 0$. Hence $\operatorname{Ext}^1(G_1, E) \neq 0$, which implies that $R^1\pi_*(G_1^\vee \otimes G_1) \neq 0$. Therefore $G_1 \in T$. For $\mathbf{I}_{yj} \in T$, we consider the homomorphism $\phi : \pi^*(\pi_*(G_1^\vee \otimes \mathbf{I}_{yj})) \otimes G_1 \to \mathbf{I}_{yj}$. Since $\mathbf{I}_{yj}$ is an irreducible object, ϕ is surjective in $\mathcal{C}$, which implies that ϕ is surjective in $\operatorname{Coh}(X)$. Hence $\operatorname{Ext}^1(G_1, \mathbf{I}_{yj}) = 0$. Since $\dim \pi^{-1}(y) \leq 1$, we also get $\operatorname{Ext}^k(G_1, \mathbf{I}_{yj}) = 0$ for $k \geq 2$. Therefore $\operatorname{Hom}(G_1, \mathbf{I}_{yj}[k]) = 0$ for $k \neq 0$. For $\mathbf{I}_{yj} \in S[1]$, $\dim \pi^{-1}(y) \leq 1$ and the locally freeness of G_1 imply that $\operatorname{Ext}^1(G_1, \mathbf{I}_{yj}) = 0$. Since $G_1 \in T$, we also get $\operatorname{Hom}(G_1, \mathbf{I}_{yj}[-1]) = 0$ for all irreducible objects of $\mathcal{C}$. □

1.2. Examples of perverse coherent sheaves

Let $\pi : X \to Y$ be a birational map in Section 1.1. Let G be a locally free sheaf on X such that $R^1\pi_*(G^\vee \otimes G) = 0$, that is, $G \in T(G)$. We set $\mathcal{A} := \pi_*(G^\vee \otimes G)$ as before. Let F be a coherent $\mathcal{A}$-module on Y. Then $\mathbf{R}\pi_*((\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G) \otimes G^\vee) \cong F$ as an $\mathcal{A}$-module. By using the spectral sequence, we see that

$$R^p\pi_*\big(G^\vee \otimes H^q\big(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G\big)\big) = 0, \quad p + q \neq 0, \tag{1.50}$$

and we have an exact sequence

$$\begin{aligned} 0 &\to R^1\pi_*\big(G^\vee \otimes H^{-1}\big(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G\big)\big) \\ &\to F \overset{\lambda}{\to} \pi_*\big(G^\vee \otimes H^0\big(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G\big)\big) \to 0. \end{aligned} \tag{1.51}$$

We set

$$\pi^{-1}(F) \otimes_{\pi^{-1}(\mathcal{A})} G := H^0\big(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G\big) \in \operatorname{Coh}(X). \tag{1.52}$$

If $S_0(G) \neq 0$, then obviously $(T(G), S(G))$ is not a torsion pair of $\operatorname{Coh}(X)$ (cf. Lemma 1.1.11). We shall construct torsion pairs associated to $(T(G), S(G))$. We set

$$\begin{aligned} S &:= S(G), \\ T &:= \big\{E \in T(G) \,\big|\, \operatorname{Hom}(E, c) = 0 \text{ for } c \in S_0(G)\big\}. \end{aligned} \tag{1.53}$$

REMARK 1.2.1

We have $G \in T$. Indeed for $c \in S_0(G)$, $\operatorname{Hom}(G,c) = H^0(Y, \pi_*(G^\vee \otimes c)) = 0$.

LEMMA 1.2.2

For $E \in \operatorname{Coh}(X)$, let $\phi : \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G \to E$ be the evaluation map.

(1) *We have $\mathbf{R}\pi_*(G^\vee \otimes \ker\phi) = 0$, $\pi_*(G^\vee \otimes \operatorname{coker}\phi) = 0$, and $R^1\pi_*(G^\vee \otimes E) \cong R^1\pi_*(G^\vee \otimes \operatorname{coker}\phi)$.*

(2) *(T,S) is a torsion pair of $\operatorname{Coh}(X)$, and the decomposition of E is given by*

$$0 \to \operatorname{im}\phi \to E \to \operatorname{coker}\phi \to 0, \tag{1.54}$$

$\operatorname{im}\phi \in T$, $\operatorname{coker}\phi \in S$.

Proof

(1) For the morphisms

$$\begin{aligned} &\lambda : \pi_*(G^\vee \otimes E) \longrightarrow \pi_*\big(G^\vee \otimes \pi^{-1}\big(\pi_*(G^\vee \otimes E)\big) \otimes_{\pi^{-1}(\mathcal{A})} G\big), \\ &\pi_*(1_{G^\vee} \otimes \phi) : \pi_*\big(G^\vee \otimes \pi^{-1}\big(\pi_*(G^\vee \otimes E)\big) \otimes_{\pi^{-1}(\mathcal{A})} G\big) \longrightarrow \pi_*(G^\vee \otimes E), \end{aligned} \tag{1.55}$$

the composition

$$\begin{aligned} \pi_*(G^\vee \otimes E) \quad &\xrightarrow{\lambda} \quad \pi_*\big(G^\vee \otimes \pi^{-1}\big(\pi_*(G^\vee \otimes E)\big) \otimes_{\pi^{-1}(\mathcal{A})} G\big) \\ &\xrightarrow{\pi_*(1_{G^\vee}\otimes\phi)} \pi_*(G^\vee \otimes E) \end{aligned} \tag{1.56}$$

is the identity. By (1.51), λ and $\pi_*(1_{G^\vee} \otimes \phi)$ are isomorphic. Hence we get $\operatorname{im}\pi_*(1_{G^\vee} \otimes \phi) = \pi_*(G^\vee \otimes \operatorname{im}\phi) = \pi_*(G^\vee \otimes E)$. Since $R^1\pi_*(G^\vee \otimes \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G) = 0$, we get $\mathbf{R}\pi_*(G^\vee \otimes \ker\phi) = 0$. Since $R^1\pi_*(G^\vee \otimes \operatorname{im}\phi) = 0$, we also get the remaining claims.

(2) We shall prove that $\operatorname{im}\phi \in T$. If $\operatorname{im}\phi \notin T$, then there is a homomorphism $\psi : \operatorname{im}\phi \to F$ such that $F \in S$. Replacing F by $\operatorname{im}\psi$, we may assume that ψ is surjective. Since $\psi \circ \phi$ is surjective, $\operatorname{Hom}(G,F) \neq 0$, which is a contradiction. Therefore $\operatorname{im}\phi \in T$. Obviously we have $S \cap T = \{0\}$. Therefore (T,S) is a torsion pair. $\square$

DEFINITION 1.2.3

Let $\mathcal{C}(G)$ denote the tilting of $\operatorname{Coh}(X)$ with respect to the torsion pair (T,S) above.

This definition is a generalization of Definition 1.1.10. In the sense of Definition 1.1.3, $\mathcal{C}(G)$ is the category of perverse coherent sheaves. Indeed, we have the following.

LEMMA 1.2.4 ([VB, PROPOSITION 3.2.5])

The category $\mathcal{C}(G)$ has a local projective generator.

Proof
Let $\mathcal{O}_X(D)$ be a very ample line bundle on X such that $\pi^*(\pi_*(G^\vee \otimes G(D))) \otimes G \to G(D)$ is surjective. We set $L := G(D)$. We take a locally free resolution $0 \to L_{-1} \to L_0 \to L \to 0$ such that $R^1\pi_*(L_0^\vee \otimes G) = 0$. Then

$$\mathbf{R}\pi_*(L^\vee \otimes G)[1] = \operatorname{Cone}\bigl(\pi_*(L_0^\vee \otimes G) \to \pi_*(L_{-1}^\vee \otimes G)\bigr). \tag{1.57}$$

We take a surjective homomorphism $V \to \pi_*(L_{-1}^\vee \otimes G)$ from a locally free sheaf V on Y. Then we have a morphism $\pi^*(V) \otimes L \to \mathbf{L}\pi^*(\mathbf{R}\pi_*(L^\vee \otimes G))[1] \otimes L \to G[1]$, which induces a surjective homomorphism $V \to R^1\pi_*(L^\vee \otimes G)$. Hence we have a morphism

$$L \to G[1] \otimes \pi^*(V)^\vee \tag{1.58}$$

such that the induced homomorphism

$$V \to \pi_*\bigl(\mathcal{H}om_{\mathcal{O}_X}\left(G[1], G[1]\right)\bigr) \otimes V \to R^1\pi_*(L^\vee \otimes G) \tag{1.59}$$

is surjective. We set $E := \operatorname{Cone}(L \to G[1] \otimes \pi^*(V)^\vee)[-1]$. Then E is a locally free sheaf on X, and $\phi : \pi^*(\pi_*(G^\vee \otimes E)) \otimes G \to E$ is surjective by our choice of L. By (1.59) and our assumption, we have $R^1\pi_*(E^\vee \otimes G) = 0$. For $F \in T(G)$, we consider the evaluation map $\varphi : \pi^*(\pi_*(G^\vee \otimes F)) \otimes G \to F$. The proof of Lemma 1.1.11(1) implies that $\operatorname{coker}\varphi \in S_0(G)$. By the definition of T, $\operatorname{coker}\varphi = 0$. Thus φ is surjective. Hence $R^1\pi_*(E^\vee \otimes F) = 0$ for $F \in T(G)$.

For $F \in S(G)$, the surjectivity of ϕ implies that $\pi_*(E^\vee \otimes F) = 0$. If $F \notin S_0(G)$, then $R^1\pi_*(G^\vee \otimes F) \neq 0$, which implies that $R^1\pi_*(E^\vee \otimes F) \neq 0$. Assume that $F \in S_0(G)$. Then since $\mathbf{R}\pi_*(G^\vee \otimes F) = 0$ for $F \in S_0(G)$, we have $R^1\pi_*(E^\vee \otimes F) \cong R^1\pi_*(L^\vee \otimes F)$. Assume that $R^1\pi_*(L^\vee \otimes F) = 0$ and $F \neq 0$. We take a point $y \in \pi(\operatorname{Supp}(F))$. Since $\mathcal{O}_X(D)$ is very ample, we can take a smooth divisor $C \in |\mathcal{O}_X(D)|$ such that $\pi^{-1}(y) \cap C$ consists of finitely many points. We may assume that $C \cap \operatorname{Supp}(F_{|\pi^{-1}(y)}) \neq \emptyset$. Then we have an exact sequence

$$0 \to L^\vee \to G^\vee \to G^\vee_{|C} \to 0.$$

Since $C \to Y$ is generically finite, it is finite over an open neighborhood U of y. Since $\operatorname{Supp}(F) \cap \pi^{-1}(U) \cap C \neq \emptyset$, we have $G^\vee \otimes F_{|\pi^{-1}(U)\cap C} \neq 0$. Hence $\pi_*(G^\vee \otimes F_{|C}) \neq 0$. On the other hand, our assumptions imply that $\mathbf{R}\pi_*(F \overset{\mathbf{L}}{\otimes} \mathcal{O}_C \otimes G^\vee) = 0$. Since the spectral sequence

$$E_2^{pq} = R^p\pi_*\bigl(H^q(F \overset{\mathbf{L}}{\otimes} \mathcal{O}_C \otimes G^\vee)\bigr) \Rightarrow E_\infty^{p+q} = H^{p+q}\bigl(\mathbf{R}\pi_*(F \overset{\mathbf{L}}{\otimes} \mathcal{O}_C \otimes G^\vee)\bigr) \tag{1.60}$$

degenerates, we have $\pi_*(F \otimes \mathcal{O}_C \otimes G^\vee) = 0$, which is a contradiction. Hence $R^1\pi_*(L^\vee \otimes F) \neq 0$ for all nonzero $F \in S_0(G)$. Then $G_1 := G \oplus E$ satisfies

$$\begin{aligned} &\pi_*(G_1^\vee \otimes F) \neq 0, \qquad R^1\pi_*(G_1^\vee \otimes F) = 0, \qquad 0 \neq F \in T(G),\\ &\pi_*(G_1^\vee \otimes F) = 0, \qquad R^1\pi_*(G_1^\vee \otimes F) \neq 0, \qquad 0 \neq F \in S(G). \end{aligned} \tag{1.61}$$

Therefore G_1 is a local projective generator of $\mathcal{C}(G)$. $\square$

We also define another torsion pair associated to $(T(G), S(G))$:

$$
(1.62)\qquad
\begin{aligned}
S^* &:= \{E \in S(G) \mid \operatorname{Hom}(c, E) = 0 \text{ for } c \in S_0(G)\},\\
T^* &:= T(G).
\end{aligned}
$$

LEMMA 1.2.5

(T^*, S^*) is a torsion pair of $\operatorname{Coh}(X)$, and the tilted category has a local projective generator. We denote the category by $\mathcal{C}(G)^*$.

Proof

We set

$$
(1.63)\qquad
\begin{aligned}
S_1 &:= S(G^\vee),\\
T_1 &:= \{E \in T(G^\vee) \mid \operatorname{Hom}(E, c) = 0 \text{ for } c \in S_0(G^\vee)\}.
\end{aligned}
$$

Then (T_1, S_1) is a torsion pair of $\operatorname{Coh}(X)$, and Lemma 1.2.4 implies that the tilted category $\mathcal{C}(G^\vee)$ has a local projective generator $G^\vee \oplus E_1$, where E_1 is a locally free sheaf on X such that $\phi : \pi^*(\pi_*(G \otimes E_1)) \otimes G^\vee \to E_1$ is surjective and $R^1\pi_*(G^\vee \otimes E_1^\vee) = 0$. By Lemma 1.1.14, (T_1^D, S_1^D) is a torsion pair of $\operatorname{Coh}(X)$. We prove that $\mathcal{C}(G)^* = \mathcal{C}(G^\vee)^D$ by showing that $(T_1^D, S_1^D) = (T^*, S^*)$. By the surjectivity of ϕ, we have

$$
(1.64)\qquad T_1^D = \{E \in \operatorname{Coh}(X) \mid R^1\pi_*(G^\vee \otimes E) = R^1\pi_*(E_1 \otimes E) = 0\} = T^*.
$$

For a coherent sheaf E with $\pi_*(G^\vee \otimes E) = 0$, we consider $\psi : \pi^*(\pi_*(E_1 \otimes E)) \otimes E_1^\vee \to E$. Then $\operatorname{im}\psi \in T_1^D = T^*$ and $\operatorname{coker}\psi \in S_1^D$. Since $\pi_*(G^\vee \otimes \operatorname{im}\psi) = 0$, $\operatorname{im}\psi \in S_0(G)$. Therefore if $E \in S^*$, then $\operatorname{im}\psi = 0$, which means that $E \in S_1^D$. Conversely if $E \in S_1^D$, then $S_0(G) \subset T_1^D$ implies that $E \in S^*$. Therefore $(T_1^D, S_1^D) = (T^*, S^*)$. □

LEMMA 1.2.6

We set $S_{0y} := \{E \in S_0(G) \mid \pi(\operatorname{Supp}(E)) = \{y\}\}$. Then $S_{0y}[1]$ is generated by $\{\mathbf{I}_{yj} \mid \mathbf{I}_{yj} \in S_0(G)[1]\}$, where $\mathcal{C} = \mathcal{C}(G)$.

Proof

For an exact sequence

$$
(1.65)\qquad 0 \to E_1 \to E \to E_2 \to 0
$$

in $\mathcal{C}$, we have an exact sequence

$$
(1.66)\qquad 0 \to \mathbf{R}\pi_*(G^\vee \otimes E_1) \to \mathbf{R}\pi_*(G^\vee \otimes E) \to \mathbf{R}\pi_*(G^\vee \otimes E_2) \to 0
$$

in $\operatorname{Coh}(Y)$. If $E \in S_0(G)[1]$, then $\mathbf{R}\pi_*(G^\vee \otimes E_1) = \mathbf{R}\pi_*(G^\vee \otimes E_2) = 0$. Then $\mathbf{R}\pi_*(G^\vee \otimes H^{-1}(E_1)) = \mathbf{R}\pi_*(G^\vee \otimes H^{-1}(E_2)) = 0$ and $\mathbf{R}\pi_*(G^\vee \otimes H^0(E_1)) = \mathbf{R}\pi_*(G^\vee \otimes H^0(E_2)) = 0$. By the definition of T, $H^0(E_1) = H^0(E_2) = 0$. Hence $E_1, E_2 \in S_0(G)[1]$. Therefore the claim holds. □

By the construction of $\mathcal{C}(G)$ and $\mathcal{C}(G)^*$, we have the following.

PROPOSITION 1.2.7

We set $\mathcal{A} := \pi_*(G^\vee \otimes G)$. *Then we have morphisms*

$$\begin{aligned} \mathcal{C}(G) &\to \mathrm{Coh}_{\mathcal{A}}(Y), \\ E &\mapsto \mathbf{R}\pi_*(G^\vee \otimes E) \end{aligned} \tag{1.67}$$

and

$$\begin{aligned} \mathcal{C}(G)^* &\to \mathrm{Coh}_{\mathcal{A}}(Y), \\ E &\mapsto \mathbf{R}\pi_*(G^\vee \otimes E). \end{aligned} \tag{1.68}$$

Let $\tau^{\geq -1} : \mathbf{D}(X) \to \mathbf{D}(X)$ be the truncation morphism such that $H^p(\tau^{\geq -1}(E)) = 0$ for $p < -1$ and $H^p(\tau^{\geq -1}(E)) = H^p(E)$ for $p \geq -1$. By (1.50), we have

$$\begin{aligned} &H^q\big(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G\big) \in S_0(G), \quad q \neq -1, 0, \\ &\Sigma(F) := \tau^{\geq -1}\big(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G\big) \in \mathcal{C}(G). \end{aligned} \tag{1.69}$$

Thus we have a morphism $\Sigma : \mathrm{Coh}_{\mathcal{A}}(Y) \to \mathcal{C}(G)$ such that $\mathbf{R}\pi_*(G^\vee \otimes \Sigma(F)) = F$ for $F \in \mathrm{Coh}_{\mathcal{A}}(Y)$.

REMARK 1.2.8

We have a morphism $g : \Sigma(\mathbf{R}\pi_*(G^\vee \otimes E)) \to E$. It is not an isomorphism unless G is a local projective generator of $\mathcal{C}(G)$. For $E \in T$, Lemma 1.2.2 implies that g is injective and $\operatorname{coker} g \in S_0(G)[1]$.

1.2.1. ${}^p\,\mathrm{Per}(X/Y)$, $p = -1, 0$, *and their generalizations*

We give examples such that $S_0(G) \neq \{0\}$. For $y \in Y_\pi$, we set $Z_y := \pi^{-1}(y)$ and C_{yj}, $j = 1, \dots, t_y$, the irreducible components of Z_y. As we shall see later, we have $t_y = s_y$. By Assumption 1.1.1 and Lemma 1.1.21, C_{yj} are smooth rational curves and $\mathcal{O}_X \in T(\mathcal{O}_X)$. Then $S_0(\mathcal{O}_X)$ contains $\mathcal{O}_{C_{yj}}(-1)$, $y \in Y_\pi$, and $\mathcal{C}(\mathcal{O}_X)$ is nothing but the category ${}^{-1}\,\mathrm{Per}(X/Y)$ defined by Bridgeland. We also have $\mathcal{C}(\mathcal{O}_X)^* = \mathcal{C}(\mathcal{O}_X^\vee)^D = {}^0\,\mathrm{Per}(X/Y)$. We shall study $\mathcal{C}(G)$ such that $S_0(G)$ contains line bundles on C_{yj}, $y \in Y_\pi$. For this purpose, we first prepare some properties of $S_0(\mathcal{O}_X)$ and $\mathcal{C}(\mathcal{O}_X)$.

The following lemma shows that we do not need to specify the ample divisor for the ($\mathcal{O}_X$-twisted) semistability of E with $\chi(E) = 1$.

LEMMA 1.2.9

Let E be a 1-dimensional sheaf such that $\mathrm{Supp}(E) \subset Z_y$ *and* $\chi(E) = 1$. *Then the ($\mathcal{O}_X$-twisted) semistability of E is independent of the choice of an ample line bundle L on X.*

Proof

By (0.4), E is ($\mathcal{O}_X$-twisted) semistable if and only if $\chi(F) \leq 0$ for all proper subsheaves F of E. Hence the semistability is independent of L. $\square$

LEMMA 1.2.10

(1) *Let E be a semistable 1-dimensional sheaf such that* $\mathrm{Supp}(E) \subset Z_y$ *and* $\chi(E) = 1$. *Then there is a curve $D \subset Z_y$, and $E \cong \mathcal{O}_D$. Conversely, if $\mathcal{O}_D$ is 1-dimensional, $\chi(\mathcal{O}_D) = 1$, and $\pi(D) = \{y\}$, then $\mathcal{O}_D$ is stable. In particular, D is a subscheme of Z_y.*

(2) *$\mathcal{O}_{Z_y}$ is stable.*

Proof

(1) Since $\chi(E) = 1$, $\pi_*(E) \neq 0$. Since $\pi_*(E)$ is zero-dimensional, we have a homomorphism $\mathbb{C}_y \to \pi_*(E)$. Then we have a homomorphism $\phi : \mathcal{O}_{Z_y} = \pi^*(\mathbb{C}_y) \to E$. We denote the image by $\mathcal{O}_D$. Since $R^1\pi_*(\mathcal{O}_X) = 0$, we have $H^1(X, \mathcal{O}_D) = 0$. Hence $\chi(\mathcal{O}_D) \geq 1$. Since E is semistable, ϕ must be surjective.

Conversely, we assume that $\mathcal{O}_D$ satisfies $\chi(\mathcal{O}_D) = 1$. For a quotient $\mathcal{O}_D \to \mathcal{O}_C$, $H^1(X, \mathcal{O}_C) = 0$ implies that $\chi(\mathcal{O}_C) \geq 1$, which implies that $\mathcal{O}_D$ is stable.

(2) We have an exact sequence

$$0 \to I_{Z_y} \to \mathcal{O}_X \to \pi^*(\mathbb{C}_y) \to 0.$$

Since $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ and $R^1\pi_*(I_{Z_y}) = 0$ (see Lemma 1.1.24), we have a surjective homomorphism $\mathcal{O}_Y \to \pi_*(\pi^*(\mathbb{C}_y))$. Hence we get an isomorphsim $\mathbb{C}_y \to \pi_*(\pi^*(\mathbb{C}_y))$. Therefore $\chi(\mathcal{O}_{Z_y}) = 1$. By (1), $\mathcal{O}_{Z_y}$ is stable. □

LEMMA 1.2.11

(1) *Let E be a stable purely 1-dimensional sheaf such that* $\pi(\mathrm{Supp}(E)) = \{y\}$ *and $\chi(E) = 0$. Then $E \cong \mathcal{O}_{C_{yj}}(-1)$.*

(2) *Let E be a 1-dimensional sheaf such that $\mathbf{R}\pi_*(E) = 0$. Then E is a semistable 1-dimensional sheaf with $\chi(E) = 0$. In particular, E is a successive extension of $\mathcal{O}_{C_{yj}}(-1)$, $y \in Y$, $1 \leq j \leq t_y$.*

Proof

(1) We set $n := \dim X$. We take a point $x \in \mathrm{Supp}(E)$. Then $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, E) = \mathbb{C}_x \overset{\mathbf{L}}{\otimes} E[-n+1]$. Since E is purely 1-dimensional, $\mathrm{depth}_{\mathcal{O}_{X,x}} E_x = 1$. Hence the projective dimension of E at x is $n-1$. Then $\mathcal{T}or^{\mathcal{O}_X}_{n-1}(\mathbb{C}_x, E) = H^0(\mathbb{C}_x \overset{\mathbf{L}}{\otimes} E[-n+1]) \neq 0$. Since $\mathrm{Ext}^1(\mathbb{C}_x, E) = H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, E)) \neq 0$, we can take a nontrivial extension

$$0 \to E \to F \to \mathbb{C}_x \to 0. \tag{1.70}$$

If F is not semistable, then since $\chi(F) = 1$, there is a quotient $F \to F'$ of F such that F' is a stable sheaf with $\chi(F') \leq 0$. Then $E \to F'$ is an isomorphism, which is a contradiction. By Lemma 1.2.10, $F = \mathcal{O}_D$. We take an integral curve $C \subset D$ containing x. Since $\mathcal{O}_D \to \mathbb{C}_x$ factors through $\mathcal{O}_C$, we have a surjective homomorphism $E \to \mathcal{O}_C(-1)$. By the stability of E, $E \cong \mathcal{O}_C(-1)$.

(2) Let F be a subsheaf of E. Then we have $\pi_*(F) = 0$, which implies that $\chi(F) \leq 0$. Therefore E is semistable. □

We shall slightly generalize ${}^{-1}\operatorname{Per}(X/Y)$. Let G be a locally free sheaf on X.

ASSUMPTION 1.2.12

There are line bundles $\mathcal{O}_{C_{yj}}(b_{yj})$ on C_{yj} such that $\mathbf{R}\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj})) = 0$ for all $y \in Y_\pi$ and $j = 1, 2, \ldots, t_y$.

LEMMA 1.2.13

(1) *Let E be a locally free sheaf of rank r on X such that $E_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}^{\oplus r}$. Then E is the pullback of a locally free sheaf on Y.*

(2) *We have $G^\vee \otimes G \cong \pi^*(\pi_*(G^\vee \otimes G))$. In particular, $R^1\pi_*(G^\vee \otimes G) = 0$.*

Proof

(1) We consider the map $\phi : H^0(E_{|Z_y}) \otimes \mathcal{O}_{Z_y} \to E_{|Z_y}$. For any point $x \in Z_y$, we have an exact sequence

$$0 \to F_x \to \mathcal{O}_{Z_y} \to \mathbb{C}_x \to 0 \tag{1.71}$$

such that $\mathbf{R}\pi_*(F_x) = 0$. By Lemma 1.2.11(2) and our assumption, we have $\mathbf{R}\pi_*(E \otimes F_x) = 0$. Hence $H^0(E_{|Z_y}) \to H^0(E_{|\{x\}})$ is isomorphic and $H^1(E_{|Z_y}) = 0$. Therefore ϕ is a surjective homomorphism of locally free sheaves of the same rank, which implies that ϕ is an isomorphism. By $R^1\pi_*(E) = 0$ (see Lemma 1.1.24(3)) and the surjectivity of $\pi^*(\pi_*(I_{Z_y})) \to I_{Z_y}$, $R^1\pi_*(E \otimes I_{Z_y}) = 0$. Hence $\pi_*(E) \to \pi_*(E_{|Z_y})$ is surjective. Then we can take a homomorphism $\mathcal{O}_U^{\oplus r} \to \pi_*(E)_{|U}$ in a neighborhood of y such that $\mathcal{O}_U^{\oplus r} \to \pi_*(E_{|Z_y})$ is surjective. Then we have a homomorphism $\pi^*(\mathcal{O}_U^{\oplus r}) \to E_{|\pi^{-1}(U)}$ which is surjective on Z_y. Since π is proper, replacing U by a small neighborhood of y, we have an isomorphism $\pi^*(\mathcal{O}_U^{\oplus r}) \to E_{|\pi^{-1}(U)}$. Therefore E is the pullback of a locally free sheaf on Y.

(2) Since $G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj})$ is a locally free sheaf on C_{yj} with $\mathbf{R}\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj})) = 0$, we have $G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj}) \cong \mathcal{O}_{C_{yj}}(-1)^{\oplus \operatorname{rk} G}$. Hence $G_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}(1)^{\oplus \operatorname{rk} G} \otimes \mathcal{O}_{C_{yj}}(b_{yj})$. Hence $G^\vee \otimes G_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}^{\oplus (\operatorname{rk} G)^2}$. By (1), we get the first claim. Then Assumption 1.1.1 implies $R^1\pi_*(G^\vee \otimes G) = 0$. $\square$

LEMMA 1.2.14

For $E \in \operatorname{Coh}(X)$, we have

$$\pi^{-1}\big(\pi_*(G^\vee \otimes E)\big) \otimes_{\pi^{-1}(\mathcal{A})} G \otimes_{\mathcal{O}_X} G^\vee \cong \pi^*\pi_*(G^\vee \otimes E). \tag{1.72}$$

Proof

By Lemma 1.2.13, we get

$$\begin{aligned}
&\pi^{-1}\big(\pi_*(G^\vee \otimes E)\big) \otimes_{\pi^{-1}(\mathcal{A})} G \otimes_{\mathcal{O}_X} G^\vee \\
&\cong \pi^{-1}\big(\pi_*(G^\vee \otimes E)\big) \otimes_{\pi^{-1}(\mathcal{A})} \pi^{-1}\big(\pi_*(G \otimes_{\mathcal{O}_X} G^\vee)\big) \otimes_{\pi^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \\
&\cong \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \\
&= \pi^*\big(\pi_*(G^\vee \otimes E)\big).
\end{aligned} \tag{1.73}$$

Therefore the claims hold. $\square$

LEMMA 1.2.15

Let $y \in Y_\pi$. Then the $\mathcal{A}$-module $\pi_*(G^\vee \otimes \mathbb{C}_x)$ does not depend on the choice of $x \in \pi^{-1}(y)$. We set

$$A_y := \pi^{-1}\big(\pi_*(G^\vee \otimes \mathbb{C}_x)\big) \otimes_{\pi^{-1}(\mathcal{A})} G, \quad x \in Z_y. \tag{1.74}$$

Proof

For the exact sequence

$$0 \to \mathcal{O}_{C_{yj}}(b_{yj}) \to \mathcal{O}_{C_{yj}}(b_{yj}+1) \to \mathbb{C}_x \to 0, \tag{1.75}$$

we have $\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj}+1)) \cong \pi_*(G^\vee \otimes \mathbb{C}_x)$. Hence $\pi_*(G^\vee \otimes \mathbb{C}_x)$ does not depend on the choice of $x \in Z_y$. □

LEMMA 1.2.16

(1) *A_y is a unique line bundle on Z_y such that $A_{y|C_{yj}} \cong \mathcal{O}_{C_{yj}}(b_{yj}+1)$ for $j = 1, 2, \dots, t_y$.*

(2) *We have $G^\vee \otimes A_y \cong \mathcal{O}_{Z_y}^{\oplus \operatorname{rk} G}$.*

Proof

By Lemma 1.2.14, $G^\vee \otimes A_y \cong \pi^*(\pi_*(G^\vee \otimes \mathbb{C}_x)) \cong \mathcal{O}_{Z_y}^{\oplus \operatorname{rk} G}$. Thus (2) holds. Since $G_{|Z_y}$ is a locally free sheaf on Z_y, A_y is a line bundle on Z_y. Then $A_y^{\otimes \operatorname{rk} G} \cong \det G_{|Z_y}$. Since the restriction map $\operatorname{Pic}(Z_y) \to \prod_j \operatorname{Pic}(C_{yj})$ is bijective and $\operatorname{Pic}(C_{yj}) \cong \mathbb{Z}$, $G_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}(b_{yj}+1)^{\oplus \operatorname{rk} G}$ implies claim (1). □

LEMMA 1.2.17

For a coherent sheaf E with $\operatorname{Supp}(E) \subset Z_y$, $\chi(G, E) \in \mathbb{Z} \operatorname{rk} G$.

Proof

We note that $K(Z_y)$ is generated by $\mathcal{O}_{C_{yj}}(b_{yj})$ and $\mathbb{C}_x$. For E with $\operatorname{Supp}(E) \subset Z_y$, we have a filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$ such that $F_i/F_{i-1} \in \operatorname{Coh}(Z_y)$. Hence the claim follows from $\chi(G, \mathcal{O}_{C_{yj}}(b_{yj})) = 0$ and $\chi(G, \mathbb{C}_x) = \operatorname{rk} G$. □

Thanks to Lemma 1.2.17, we see that the G-twisted semistability of E with $\chi(G, E) = \operatorname{rk} G$ is independent of the choice of an ample line bundle L and is equivalent to the G-twisted stability (see the proof of Lemma 1.2.9).

LEMMA 1.2.18

(1) *Let E be a G-twisted, semistable 1-dimensional sheaf such that $\operatorname{Supp}(E) \subset Z_y$ and $\chi(G, E) = \operatorname{rk} G$. Then there is a subscheme C of Z_y such that $\chi(\mathcal{O}_C) = 1$ and $E \cong A_y \otimes \mathcal{O}_C$. Conversely, for a subscheme C of Z_y such that $\mathcal{O}_C$ is 1-dimensional, $\chi(\mathcal{O}_C) = 1$, $E = A_y \otimes \mathcal{O}_C$ is a G-twisted stable sheaf with $\chi(G, E) = \operatorname{rk} G$, and $\pi(\operatorname{Supp}(E)) = \{y\}$.*

(2) *A_y is G-twisted stable.*

Proof

(1) We choose an exact sequence

$$0 \to K \to E \to \mathbb{C}_x \to 0. \tag{1.76}$$

Since E is a G-twisted semistable 1-dimensional sheaf with $\chi(G,E) = \operatorname{rk} G$, K is a G-twisted semistable sheaf with $\chi(G,K) = 0$. If $\pi_*(G^\vee \otimes K) \neq 0$, then we have a nonzero homomorphism $\phi : \pi^{-1}(\pi_*(G^\vee \otimes K)) \otimes_{\pi^{-1}(\mathcal{A})} G \to K$ such that $\pi_*(G^\vee \otimes \operatorname{im}\phi) = \pi_*(G^\vee \otimes K)$. Since $R^1\pi_*(G^\vee \otimes \operatorname{im}\phi) = 0$, $\chi(G, \operatorname{im}\phi) > 0$, which is a contradiction. Therefore $\pi_*(G^\vee \otimes K) = 0$. Hence $\xi : \pi_*(G^\vee \otimes E) \to \pi_*(G^\vee \otimes \mathbb{C}_x)$ is injective. Since $\dim H^0(Y, \pi_*(G^\vee \otimes E)) \geq \chi(G,E) = \operatorname{rk} G$, ξ is an isomorphism. Then we have a homomorphism $\psi : A_y \to E$. Since $\pi_*(G^\vee \otimes \operatorname{im}\psi) = \pi_*(G^\vee \otimes E)$ and $R^1\pi_*(G^\vee \otimes \operatorname{im}\psi) = 0$, we get $\operatorname{im}\psi = E$. Since $E \otimes A_y^D$, $A_y^D := \mathcal{H}om_{\mathcal{O}_X}(A_y, \mathcal{O}_{Z_y})$ is a quotient of $\mathcal{O}_{Z_y}$, there is a subscheme C of Z_y such that $E \otimes A_y^D \cong \mathcal{O}_C$. Since $\chi(G,E) = \chi(G, A_y \otimes \mathcal{O}_C) = \chi(\mathcal{O}_C^{\oplus \operatorname{rk} G})$, we have $\chi(\mathcal{O}_C) = 1$.

Conversely, for $E \otimes A_y^\vee \cong \mathcal{O}_C$ such that $\mathcal{O}_C$ is 1-dimensional, $C \subset Z_y$, and $\chi(\mathcal{O}_C) = 1$, we consider a quotient $E \to F$. Then $F = A_y \otimes \mathcal{O}_D$, $D \subset C$. Since $R^1\pi_*(G^\vee \otimes F) = 0$ and $G^\vee \otimes A_y \otimes \mathcal{O}_D \cong \mathcal{O}_D^{\oplus \operatorname{rk} G}$, we get $\chi(G,F) \geq \operatorname{rk} G$. From this fact, we first see that E is purely 1-dimensional, and then we see that it is G-twisted stable.

(2) This follows from (1) and $\chi(\mathcal{O}_{Z_y}) = 1$. □

LEMMA 1.2.19

Let E be a G-twisted stable purely 1-dimensional sheaf such that $\pi(\operatorname{Supp}(E)) = \{y\}$ and $\chi(G,E) = 0$. Then $E \cong A_y \otimes \mathcal{O}_{C_{yj}}(-1) \cong \mathcal{O}_{C_{yj}}(b_{yj})$.

Proof

We set $n := \dim X$. We take a point $x \in \operatorname{Supp}(E)$. Then $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, E) = \mathbb{C}_x \overset{\mathbf{L}}{\otimes} E[-n+1]$. Since E is purely 1-dimensional, $\operatorname{depth}_{\mathcal{O}_{X,x}} E_x = 1$. Hence the projective dimension of E at x is $n-1$. Then $\mathcal{T}or_{n-1}^{\mathcal{O}_X}(\mathbb{C}_x, E) = H^0(\mathbb{C}_x \overset{\mathbf{L}}{\otimes} E[-n+1]) \neq 0$. Since $\operatorname{Ext}^1(\mathbb{C}_x, E) = H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, E)) \neq 0$, we can take a nontrivial extension

$$0 \to E \to F \to \mathbb{C}_x \to 0. \tag{1.77}$$

If F is not G-twisted semistable, then since $\chi(G,F) = \operatorname{rk} G$, there is a quotient $F \to F'$ of F such that F' is a G-twisted stable sheaf with $\chi(G,F') \leq 0$. Then $E \to F'$ is an isomorphism, which is a contradiction. By Lemma 1.2.18, F is a quotient of A_y. Thus we may write $F = A_y \otimes \mathcal{O}_D$, where D is a subscheme of Z_y. We take an integral curve $C \subset D$ containing x. Since $\mathcal{O}_D \to \mathbb{C}_x$ factor through $\mathcal{O}_C$, we have a surjective homomorphism $E \to A_y \otimes \mathcal{O}_C(-1)$. By the stability of E, $E \cong A_y \otimes \mathcal{O}_C(-1)$. □

LEMMA 1.2.20

Let E be a 1*-dimensional sheaf such that* $\chi(G,E)=0$ *and* $\pi(\mathrm{Supp}(E))=\{y\}$*. Then the following conditions are equivalent:*

(1) $\mathbf{R}\pi_*(G^\vee\otimes E)=0$.
(2) *E is a G-twisted semistable* 1*-dimensional sheaf with* $\pi(\mathrm{Supp}(E))=\{y\}$*.*
(3) *E is a successive extension of* $A_y\otimes\mathcal{O}_{C_{yj}}(-1)$, $1\le j\le t_y$*.*

Proof

Lemma 1.1.15 gives the equivalence of (1) and (2). The equivalence of (2) and (3) follows from Lemma 1.2.19. □

LEMMA 1.2.21

Let E be a 1*-dimensional sheaf such that* $\pi_*(G^\vee\otimes E)=0$*. Then there is a homomorphism* $E\to A_y\otimes\mathcal{O}_{C_{yj}}(-1)$*. In particular, E is generated by subsheaves of* $A_y\otimes\mathcal{O}_{C_{yj}}(-1)$, $y\in Y_\pi$, $1\le j\le t_y$*.*

Proof

Since $\pi(\mathrm{Supp}(E))$ is zero-dimensional, we have a decomposition $E=\bigoplus_i E_i$, $\mathrm{Supp}(E_i)\cap\mathrm{Supp}(E_j)=\emptyset$, $i\ne j$. So we may assume that $\pi(\mathrm{Supp}(E))$ is a point. We note that $\chi(G,E)\le 0$. If $\chi(G,E)=0$, then $\chi(R^1\pi_*(G^\vee\otimes E))=0$. Since $\dim E=1$ and $\pi_*(G^\vee\otimes E)=0$, we get $\dim\pi(\mathrm{Supp}(E))=0$. Then we have $R^1\pi_*(G^\vee\otimes E)=0$. Hence the claim follows from Lemma 1.2.20. We assume that $\chi(G,E)<0$. We set $n:=\dim X$. Let

$$0\subset F_1\subset F_2\subset\cdots\subset F_s=E \tag{1.78}$$

be a filtration such that $E_i:=F_i/F_{i-1}$, $1\le i\le s$, are G-twisted stable and $\chi(G,E_i)/(\mathrm{ch}_{n-1}(E_i),L)\le\chi(G,E_{i-1})/(\mathrm{ch}_{n-1}(E_{i-1}),L)$, where L is an ample divisor on X. Since $\pi_*(G^\vee\otimes E)=0$ for any G-twisted stable 1-dimensional sheaf E on a fiber with $\chi(G,E)\le 0$, replacing E by a G-twisted stable sheaf E_s, we may assume that E is G-twisted stable. We take a nontrivial extension

$$0\to E\to F\to\mathbb{C}_x\to 0. \tag{1.79}$$

Then F is purely 1-dimensional, and $\chi(G,F)=\chi(G,E)+\mathrm{rk}\,G\le 0$ by Lemma 1.2.17. Assume that there is a quotient $F\to F'$ of F such that F' is a G-twisted stable sheaf with $\chi(G,F')/(\mathrm{ch}_{n-1}(F'),L)<\chi(G,F)/(\mathrm{ch}_{n-1}(F),L)\le 0$. Then $\phi:E\to F'$ is surjective over $X\setminus\{x\}$. Hence $\chi(G,F')/(\mathrm{ch}_{n-1}(F'),L)\ge\chi(G,\mathrm{im}\,\phi)/(\mathrm{ch}_{n-1}(\mathrm{im}\,\phi),L)\ge\chi(G,E)/(\mathrm{ch}_{n-1}(E),L)$. Since $(\mathrm{ch}_{n-1}(F'),L)\le(\mathrm{ch}_{n-1}(F),L)=(\mathrm{ch}_{n-1}(E),L)$, we get $\chi(G,F')\ge\chi(G,E)(\mathrm{ch}_{n-1}(F'),L)/(\mathrm{ch}_{n-1}(E),L)\ge\chi(G,E)$. If $\chi(G,F')=\chi(G,E)$, then ϕ is an isomorphism. Since the extension is nontrivial, this is a contradiction. Therefore F is G-twisted semistable or $\chi(G,F')>\chi(G,E)$. Thus we get a homomorphism $\psi:E\to E'$ such that E' is a stable sheaf with $\chi(G,E)<\chi(G,E')<0$ and ψ is surjective in codimension $n-1$. By the induction on $\chi(G,E)$, we get the claim. □

LEMMA 1.2.22

For a point $y \in Y_\pi$, let E be a 1-dimensional sheaf on X satisfying the following two conditions:

(i) $\operatorname{Hom}(E, A_y \otimes \mathcal{O}_{C_{yj}}(-1)) = \operatorname{Ext}^1(E, A_y \otimes \mathcal{O}_{C_{yj}}(-1)) = 0$ *for all* j*;*
(ii) *There is an exact sequence*

$$0 \to F \to E \to \mathbb{C}_x \to 0 \tag{1.80}$$

such that F is a G-twisted semistable 1-dimensional sheaf with $\pi(\operatorname{Supp}(F)) = \{y\}$, $\chi(G, F) = 0$, and $x \in Z_y$.

Then $E \cong A_y$. Conversely, $E := A_y$ satisfies (i) and (ii).

Proof

We first prove that A_y satisfies (i) and (ii). For the exact sequence

$$0 \to F' \to A_y \to \mathbb{C}_x \to 0, \tag{1.81}$$

we have $\mathbf{R}\pi_*(G, F') = 0$. Hence (ii) holds by Lemma 1.2.20; (i) follows from Lemma 1.1.24. Conversely we assume that E satisfies (i) and (ii). By (ii), $\pi_*(G^\vee \otimes E) \cong \pi_*(G^\vee \otimes \mathbb{C}_x)$ and $R^1\pi_*(G^\vee \otimes E) = 0$. By (i), Lemma 1.2.2, and Lemma 1.2.20, $\pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G \to E$ is surjective. Hence we have an exact sequence

$$0 \to F' \to A_y \to E \to 0, \tag{1.82}$$

where F' is a G-twisted semistable 1-dimensional sheaf with $\chi(G, F') = 0$. Since $\operatorname{Ext}^1(E, A_y \otimes \mathcal{O}_{C_{yj}}(-1)) = 0$ for all j, $A_y \cong E \oplus F'$, which implies that $A_y \cong E$. □

PROPOSITION 1.2.23 ([VB, PROPOSITION 3.5.7])

(1) A_y *and* $A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1]$ $(j = 1, \ldots, t_y)$ *are irreducible objects of* $\mathcal{C}(G)$*;*
(2) $\mathbb{C}_x$, $\pi(x) = y \in Y_\pi$, *is generated by irreducible objects in (1).*

Proof

(1) Assume that there is an exact sequence in $\mathcal{C}(G)$:

$$0 \to E_1 \to A_y \to E_2 \to 0. \tag{1.83}$$

Since $H^{-1}(E_1) = 0$, $E_1 \in T$ and $\pi_*(G^\vee \otimes E_1) \cong \pi_*(G^\vee \otimes A_y) = \mathbb{C}_y^{\oplus \operatorname{rk} G}$. Hence we have a nonzero morphism $A_y \to E_1$. Since $\operatorname{Hom}(A_y, A_y) \cong \mathbb{C}$, $E_1 \cong A_y$ and $E_2 = 0$. For $A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1]$, assume that there is an exact sequence in $\mathcal{C}(G)$:

$$0 \to E_1 \to A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1] \to E_2 \to 0. \tag{1.84}$$

Since $H^0(E_2) = 0$, we have $E_2[-1] \in S$. Then Lemma 1.2.21 implies that we have a nonzero morphism $E_2 \to A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1]$. Since $\operatorname{Hom}(A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1], A_y \otimes$

$\mathcal{O}_{C_{yj}}(-1)[1]) = \mathbb{C}$, we get $E_1 = 0$. Therefore $A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1]$ is irreducible; (2) is obvious by Lemma 1.2.22. □

By Proposition 1.2.23 and Lemma 1.1.21, we have the following.

COROLLARY 1.2.24

We set

$$\mathbf{I}_{yj} := \begin{cases} \mathbb{C}_x, & \pi(x) = y \notin Y_\pi, j = 0, \\ A_y, & y \in Y_\pi, j = 0, \\ A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1], & y \in Y_\pi, j = 1, \dots, s_y. \end{cases} \tag{1.85}$$

Then they are the irreducible objects of $\mathcal{C}(G)$, *and* s_y *is equal to the number of* 1*-dimensional irreducible components of* $\pi^{-1}(y)$; *that is,* $s_y = 0$ *for* $y \in Y \setminus Y_\pi$ *and* $s_y = t_y$ *for* $y \in Y_\pi$.

We give a characterization of $T = T(G)$.

PROPOSITION 1.2.25

(1) *For* $E \in \mathrm{Coh}(X)$, *the following are equivalent:*

  (a) $E \in T(G)$;
  (b) $\mathrm{Hom}(E, A_y \otimes \mathcal{O}_{C_{yj}}(-1)) = 0$ *for all* y, j;
  (c) $\phi : \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G \to E$ *is surjective.*

(2) *If (c) holds, then* $\ker \phi \in S_0(G)$.

Proof

(1) This is a consequence of Lemmas 1.2.2 and 1.1.25.
(2) The claim follows from Lemma 1.2.2. □

We note that $G \otimes \mathcal{H}om_{\mathcal{O}_{Z_y}}(A_y, \mathcal{O}_{Z_y}) \cong \mathcal{O}_{Z_y}^{\oplus \mathrm{rk}\, G}$. Then we have $\mathcal{H}om_{\mathcal{O}_{Z_y}}(A_y, \mathcal{O}_{Z_y}) \cong \pi^{-1}(\pi_*(G \otimes \mathbb{C}_x)) \otimes_{\pi^{-1}(\mathcal{A})} G^\vee$. We set

$$\mathbf{I}^*_{yj} := \begin{cases} \mathbb{C}_x, & \pi(x) = y \notin Y_\pi, j = 0, \\ A_y \otimes \omega_{Z_y}[1], & y \in Y_\pi, j = 0, \\ A_y \otimes \mathcal{O}_{C_{yj}}(-1), & y \in Y_\pi, j = 1, \dots, s_y, \end{cases} \tag{1.86}$$

where s_y is the number of 1-dimensional irreducible components of $\pi^{-1}(y)$ as above. Then we also have the following.

PROPOSITION 1.2.26 ([VB, PROPOSITION 3.5.8])

We have the following:

(1) $\mathbf{I}^*_{yj}$, $j = 0, \dots, s_y$, *are irreducible objects of* $\mathcal{C}(G)^* = \mathcal{C}(G^\vee)^D$;

(2) $\mathbb{C}_x$, $\pi(x) = y \in Y_\pi$, *is generated by* $\mathbf{I}^*_{yj}$. *In particular, irreducible objects of* $\mathcal{C}(G)^*$ *are*

$$\mathbf{I}^*_{yj}, \quad y \in Y, j = 0, 1, \dots, s_y. \tag{1.87}$$

LEMMA 1.2.27

For a point $y \in Y_\pi$, *let* E *be a* 1*-dimensional sheaf on* X *satisfying the following two conditions:*

(i) $\mathrm{Hom}(A_y \otimes \mathcal{O}_{C_{yj}}(-1), E) = \mathrm{Ext}^1(A_y \otimes \mathcal{O}_{C_{yj}}(-1), E) = 0$ *for all* j*;*
(ii) *there is an exact sequence*

$$0 \to E \to F \to \mathbb{C}_x \to 0 \tag{1.88}$$

such that F *is a* G*-twisted semistable* 1*-dimensional sheaf with* $\pi(\mathrm{Supp}(F)) = \{y\}$, $\chi(G, F) = 0$ *and* $x \in Z_y$.

Then $E \cong A_y \otimes \omega_{Z_y}$.

Proof

We set $n := \dim X$. For a purely 1-dimensional sheaf E on X, $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, K_X[n-1]) \in \mathrm{Coh}(X)$ and $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, K_X[n-1]) = \mathcal{H}om_{\mathcal{O}_C}(E, \omega_C)$ if E is a locally free sheaf on a curve without embedded primes. Hence the claim follows from Lemma 1.2.22. □

1.3. Families of perverse coherent sheaves

We shall explain families of complexes which correspond to families of $\mathcal{A}$-modules via Morita equivalence. Let $f : X \to S$ and $g : Y \to S$ be flat families of projective varieties parameterized by a scheme S, and let $\pi : X \to Y$ be an S-morphism. Let $\mathcal{O}_Y(1)$ be a relatively ample line bundle over $Y \to S$. From Section 1.3 to Section 1.6, we assume the following.

ASSUMPTION 1.3.1

(i) The morphism $f : X \to S$ is a smooth morphism; $X \to S$ is a smooth family.

(ii) There is a locally free sheaf G on X such that $G_s := G_{|f^{-1}(s)}$, $s \in S$, are local projective generators of a family of abelian categories $\mathcal{C}_s \subset \mathbf{D}(X_s)$.

(iii) We have $\dim \pi^{-1}(y) \leq 1$ for all $y \in Y$; that is, π satisfies Assumption 1.1.1.

Then $\mathcal{C}_s$ is a tilting of $\mathrm{Coh}(X_s)$.

REMARK 1.3.2

Assumptions (i), (ii), and (iii) imply that

(iv) $R^1\pi_*(G^\vee \otimes G) = 0$;
(v) we have

$$\{E \in \mathrm{Coh}(X) \mid \mathbf{R}\pi_*(G^\vee \otimes E) = 0\} = 0. \tag{1.89}$$

Thus G defines a tilting $\mathcal{C}$ of $\mathrm{Coh}(X)$.

Indeed if $E \in \mathrm{Coh}(X)$ satisfies $\mathbf{R}\pi_*(G^\vee \otimes E) = 0$, then the projection formula implies that $\mathbf{R}\pi_*(G^\vee \otimes E \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathbb{C}_s)) = \mathbf{R}\pi_*(G^\vee \otimes E) \overset{\mathbf{L}}{\otimes} \mathbf{L}g^*(\mathbb{C}_s) = 0$ for all $s \in S$. Then $\mathbf{R}\pi_*(G^\vee \otimes H^p(E \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathbb{C}_s))) = 0$ for all p and $s \in S$. By (ii), $H^p(E \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathbb{C}_s)) = 0$ for all p and $s \in S$. Therefore (v) holds; (iv) is obvious. Conversely if (i), (iii), (iv), and (v) hold, then (ii) holds. So we may replace (ii) by (iv) and (v).

REMARK 1.3.3

We do not require the birationality of π. If π is finite and f is smooth, then conditions (ii) and (iii) hold.

For a morphism $T \to S$, we set $X_T := X \times_S T$, $Y_T := Y \times_S T$, and $\pi_T := \pi \times \mathrm{id}_T$.

DEFINITION 1.3.4

(1) A family of objects in $\mathcal{C}_s, s \in S$, means a bounded complex $F^\bullet$ of coherent sheaves on X such that F^i are flat over S and $F^\bullet_s \in \mathcal{C}_s$ for all $s \in S$.

(2) A family of local projective generators is a locally free sheaf G on X such that $G_s := G_{|f^{-1}(s)}$, $s \in S$, are local projective generators of a family of abelian categories $\mathcal{C}_s$.

REMARK 1.3.5

If $F^\bullet_s \in \mathrm{Coh}(X_s)$ for all $s \in S$, then $F^\bullet$ is isomorphic to a coherent sheaf on X which is flat over S.

LEMMA 1.3.6

For a family $F^\bullet$ of objects in $\mathcal{C}_s$, $s \in S$, there is a complex $\widetilde{F}^\bullet$ such that

(i) $\widetilde{F}^i_s \in \mathcal{C}_s$, $s \in S$,
(ii) $\widetilde{F}^i$ *are flat over* S, *and*
(iii) $F^\bullet \cong \widetilde{F}^\bullet$.

Proof

We set $d := \dim X_s, s \in S$. For the bounded complex $F^\bullet$, we take a locally free resolution of $\mathcal{O}_X$,

$$0 \to V_{-d} \to \cdots \to V_{-1} \to V_0 \to \mathcal{O}_X \to 0 \tag{1.90}$$

such that $R^k\pi_*((G^\vee \otimes V_{-i}^\vee \otimes F^j)_s) = 0$, $k > 0$, for $0 \le i \le d-1$ and all j. Since $X \to Y$ is projective, we can take such a resolution. Then $R^k\pi_*((G^\vee \otimes V_{-d}^\vee \otimes F^j)_s) = 0$, $k > 0$, for all j. Therefore we have an isomorphism $F^\bullet \cong V_\bullet^\vee \otimes F^\bullet$ such that $(V_\bullet^\vee \otimes F^\bullet)^i$ are S-flat and $(V_\bullet^\vee \otimes F^\bullet)^i_s = \bigoplus_{p+q=i} V_{-p}^\vee \otimes F^q_s \in \mathcal{C}_s$ for all $s \in S$. $\square$

PROPOSITION 1.3.7

(1) *Let* $F^\bullet$ *be a family of objects in* $\mathcal{C}_s$*,* $s \in S$*. Then we get*

$$F^\bullet \cong \operatorname{Cone}(E_1 \to E_2), \tag{1.91}$$

where $E_i \in \operatorname{Coh}(X)$ *are flat over* S *and* $(E_i)_s \in \mathcal{C}_s$*,* $s \in S$*.*

(2) *Let* $F^\bullet$ *be a family of objects in* $\mathcal{C}_s$*,* $s \in S$*. Then we have a complex*

$$G(-n_1) \otimes f^*(U_1) \to G(-n_2) \otimes f^*(U_2) \to F^\bullet \to 0 \tag{1.92}$$

whose restriction to $s \in S$ *is exact in* $\mathcal{C}_s$*, where* U_1, U_2 *are locally free sheaves on* S*.*

(3) *Let* F *be an* $\mathcal{A}$*-module flat over* S*. Then we can attach a family* E *of objects in* $\mathcal{C}_s$*,* $s \in S$*, such that* $\mathbf{R}\pi_*(G^\vee \otimes E) = F$*. The correspondence is functorial, and* E *is unique in* $\mathbf{D}(X)$*. We denote* E *by* $\pi^{-1}(F) \otimes_{\pi^{-1}(\mathcal{A})} G$*.*

Proof

(1) We may assume that Lemma 1.3.6(i)–(iii) hold for $F^\bullet$. We take a sufficiently large n with $\operatorname{Hom}_f(G(-n), F^j[i]) = 0$, $i > 0$, for all j. Then $W^j := \operatorname{Hom}_f(G(-n), F^j)$ are locally free sheaves. Let $W^\bullet := \mathbf{R}\operatorname{Hom}_f(G(-n), F^\bullet)$ be the complex defined by $W^j, j \in \mathbb{Z}$. Then we have a morphism $G(-n) \otimes f^*(W^\bullet) \to F^\bullet$. Since $F_s^\bullet \in \mathcal{C}_s$, $s \in S$, $\operatorname{Hom}(G_s(-n), F_s^\bullet[i]) = 0$ for $i \neq 0$ and all $s \in S$. Then the base-change theorem implies that $U := \operatorname{Hom}_f(G(-n), F^\bullet)$ is a locally free sheaf on S and $\operatorname{Hom}_f(G(-n), F^\bullet)_s \cong \operatorname{Hom}(G(-n)_s, F_s^\bullet)$. Hence $G(-n) \otimes f^*(W^\bullet) \cong G(-n) \otimes f^*(U)$, which defines a family of morphisms

$$G(-n) \otimes f^*(U) \to F^\bullet. \tag{1.93}$$

Since $F_s^\bullet \in \mathcal{C}_s$ for all $s \in S$, $\mathbf{R}\pi_*(G^\vee \otimes F^\bullet)$ is a coherent sheaf on Y which is flat over S, and $g^* g_*(\pi_*(G^\vee \otimes F^\bullet)(n)) \to \pi_*(G^\vee \otimes F^\bullet)(n)$ is surjective in $\operatorname{Coh}(Y)$ for $n \gg 0$. Since $W^\bullet \cong g_*(\pi_*(G^\vee \otimes F^\bullet)(n))$, the homomorphism

$$\pi_*(G^\vee \otimes G)(-n) \otimes g^*(U) \to \pi_*(G^\vee \otimes F^\bullet) \tag{1.94}$$

in $\operatorname{Coh}(Y)$ is surjective for $n \gg 0$. Thus we have a family of exact sequences

$$0 \to E^\bullet \to G(-n) \otimes f^*(U) \to F^\bullet \to 0 \tag{1.95}$$

in $\mathcal{C}_s$, $s \in S$. Since $G \in \operatorname{Coh}(X)$, we have $E^\bullet \in \operatorname{Coh}(X)$ which is flat over S; (2) is a consequence of the proof of (1).

(3) We take a resolution of F,

$$\begin{aligned} &\cdots \xrightarrow{d^{-3}} g^*(U_{-2}) \otimes \mathcal{A}(-n_2) \\ &\xrightarrow{d^{-2}} g^*(U_{-1}) \otimes \mathcal{A}(-n_1) \xrightarrow{d^{-1}} g^*(U_0) \otimes \mathcal{A}(-n_0) \to F \to 0, \end{aligned} \tag{1.96}$$

where U_i are locally free sheaves on S. Then we have a complex

$$\begin{aligned} &\cdots \xrightarrow{\tilde{d}^{-3}} f^*(U_{-2}) \otimes G(-n_2) \\ &\xrightarrow{\tilde{d}^{-2}} f^*(U_{-1}) \otimes G(-n_1) \xrightarrow{\tilde{d}^{-1}} f^*(U_0) \otimes G(-n_0). \end{aligned} \tag{1.97}$$

By the Morita equivalence (see Proposition 1.1.7), we have $\operatorname{im}\tilde{d}_s^{-i} = \ker \tilde{d}_s^{-i+1}$ in $\mathcal{C}_s$ for all $s \in S$. Let $\operatorname{coker}\tilde{d}^{-2}$ be the cokernel of $\tilde{d}^{-2}$ in $\operatorname{Coh}(X)$. Then by Lemma 1.3.8 below, $\operatorname{coker}\tilde{d}^{-2}$ is flat over S, $(\operatorname{coker}\tilde{d}^{-2})_s = \operatorname{coker}(\tilde{d}_s^{-2}) \in \mathcal{C}_s$, and

$$E := \operatorname{Cone}\big(\operatorname{coker}\tilde{d}^{-2} \to f^*(U_0) \otimes G(-n_0)\big) \tag{1.98}$$

is a family of objects in $\mathcal{C}_s$. By the construction, we have $E_s = \pi^{-1}(F_s) \otimes_{\pi^{-1}(\mathcal{A}_s)} G_s$. It is easy to see that the class of E in $\mathbf{D}(X)$ does not depend on the choice of the resolution (1.96) (cf. [BS, Lemma 14]). □

LEMMA 1.3.8

Let E^i, $0 \le i \le 3$, be coherent sheaves on X which are flat over S. Let

$$E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \tag{1.99}$$

be a complex in $\operatorname{Coh}(X)$.

(1) *If $\ker d_s^1 = \operatorname{im} d_s^0$ in $\operatorname{Coh}(X_s)$, then $(\operatorname{im} d^1)_s \to E_s^2$ is injective. In particular, if $\ker d_s^1 = \operatorname{im} d_s^0$ in $\operatorname{Coh}(X_s)$ for all $s \in S$, then $\operatorname{coker} d^1, \operatorname{im} d^1, \ker d^1$ in $\operatorname{Coh}(X)$ are flat over S and $\operatorname{im} d^0 = \ker d^1$.*

(2) *Assume that $E_s^i \in \mathcal{C}_s$ for all $s \in S$. We denote the kernel, cokernel, and the image of d_s^i in $\mathcal{C}_s$ by $\ker_{\mathcal{C}_s} d_s^i, \operatorname{coker}_{\mathcal{C}_s} d_s^i$, and $\operatorname{im}_{\mathcal{C}_s} d_s^i$, respectively. If $E_s^i \in \mathcal{C}_s$ and $\ker_{\mathcal{C}_s} d_s^i = \operatorname{im}_{\mathcal{C}_s} d_s^{i-1}$, $i = 1, 2$, in $\mathcal{C}_s$ for all s, then $\operatorname{im}_{\mathcal{C}_s} d_s^{i-1}$ coincide with the image of d_s^{i-1} in $\operatorname{Coh}(X_s)$ for $i = 1, 2$ and $\ker_{\mathcal{C}_s} d_s^1$ coincides with the kernel of d_s^1 in $\operatorname{Coh}(X_s)$. In particular, $\overline{E}^\bullet : E^2/d^1(E^1) \to E^3$ is a family of objects in $\mathcal{C}_s$, and we get an exact triangle:*

$$\ker d^0 \to E^\bullet \to \overline{E}^\bullet \to \ker d^0[1] \tag{1.100}$$

where $\ker d^0$ is the kernel of d^0 in $\operatorname{Coh}(X)$, which is flat over S.

Proof

(1) Let K be the kernel of $\xi : (\operatorname{im} d^1)_s \to E_s^2$. Then we have an exact sequence

$$(\ker d^1)_s \to \ker(d_s^1) \to K \to 0. \tag{1.101}$$

Since the image of $E_s^0 \to (\ker d^1)_s \to E_s^1$ is $d_s^0(E_s^0) = \ker(d_s^1)$, $K = 0$. The other claims easily follow from this.

(2) By our assumption, $\operatorname{im}_{\mathcal{C}_s} d_s^i = \operatorname{coker}_{\mathcal{C}_s} d_s^{i-1}$ for $i = 1, 2$. Since $\operatorname{im}_{\mathcal{C}_s} d_s^i$ is a subobject of E_s^{i+1} for $i = 0, 1, 2$, $\operatorname{im}_{\mathcal{C}_s} d_s^i \in \operatorname{Coh}(X_s)$ for $i = 0, 1, 2$ and $H^{-1}(\operatorname{coker}_{\mathcal{C}_s} d_s^{i-1}) = H^{-1}(\operatorname{im}_{\mathcal{C}_s} d_s^i) = 0$ for $i = 1, 2$. Then $H^0(\operatorname{im}_{\mathcal{C}_s} d_s^{i-1}) \to H^0(E_s^i)$ is injective for $i = 1, 2$, which implies that $\operatorname{im}_{\mathcal{C}_s} d_s^{i-1}$ is the image of d_s^{i-1} in $\operatorname{Coh}(X_s)$ for $i = 1, 2$. By the exact sequence

$$0 \to H^0(\ker_{\mathcal{C}_s} d_s^1) \to H^0(E_s^1) \to H^0(\operatorname{im}_{\mathcal{C}_s} d_s^1) \to 0 \tag{1.102}$$

and the injectivity of $H^0(\operatorname{im}_{\mathcal{C}_s} d_s^1) \to H^0(E_s^2)$, $\ker_{\mathcal{C}_s} d_s^1$ is the kernel of d_s^1 in $\operatorname{Coh}(X_s)$. Then the other claims follow from (1). □

1.3.1. Quot schemes

LEMMA 1.3.9

Let $\mathcal{A}$ be an $\mathcal{O}_Y$-algebra on Y which is flat over S. Let B be a coherent $\mathcal{A}$-module on Y which is flat over S. There is a closed subscheme $\mathrm{Quot}_{B/Y/S}^{\mathcal{A},P}$ of $Q := \mathrm{Quot}_{B/Y/S}^{P}$ parameterizing all quotient $\mathcal{A}_s$-modules F of B_s with $\chi(F(n)) = P(n)$.

Proof

Let $\mathcal{Q}$ and $\mathcal{K}$ be the universal quotient and the universal subsheaf of $B \otimes_{\mathcal{O}_S} \mathcal{O}_Q$:

$$0 \to \mathcal{K} \to B \otimes_{\mathcal{O}_S} \mathcal{O}_Q \to \mathcal{Q} \to 0. \tag{1.103}$$

Then we have a homomorphism

$$\mathcal{K} \otimes_{\mathcal{O}_S} \mathcal{A} \to B \otimes_{\mathcal{O}_S} \mathcal{O}_Q \otimes_{\mathcal{O}_S} \mathcal{A} \to B \otimes_{\mathcal{O}_S} \mathcal{O}_Q \to \mathcal{Q} \tag{1.104}$$

induced by the multiplication map $B \otimes_{\mathcal{O}_S} \mathcal{O}_Q \otimes_{\mathcal{O}_S} \mathcal{A} \to B \otimes_{\mathcal{O}_S} \mathcal{O}_Q$. Let $Z = \mathrm{Quot}_{B/Y/S}^{\mathcal{A},P}$ be the zero locus of this homomorphism. Then for an S-morphism $T \to Q$, $\mathcal{K} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ is an $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_T$-submodule of $B \otimes_{\mathcal{O}_S} \mathcal{O}_T$ if and only if $T \to Q$ factors through Z. $\square$

COROLLARY 1.3.10

Let G' be a family of objects in $\mathcal{C}_s$, $s \in S$. Then there is a quot scheme $\mathrm{Quot}_{G'/X/S}^{\mathcal{C},P}$ parameterizing all quotients $G'_s \to E$ in $\mathcal{C}_s$, where P is the G_s-twisted Hilbert polynomial of the quotient object $E, s \in S$.

Proof

We set $\mathcal{A} := \pi_*(G^\vee \otimes_{\mathcal{O}_X} G)$. Then $\mathcal{A}$ is a flat family of $\mathcal{O}_Y$-algebras on Y, and we have an equivalence between the category of $\mathcal{A}_T$-modules F flat over T and the category of families E of objects in $\mathcal{C}_t, t \in T$, by $F \mapsto \pi_T^{-1}(F) \otimes_{\pi^{-1}(\mathcal{A}_T)} G_T$. So the claim follows from Lemma 1.3.9 (cf. $B = \pi_*(G^\vee \otimes G')$). $\square$

1.4. Stability for perverse coherent sheaves

For a nonzero object $E \in \mathcal{C}_s$, $\chi(G_s, E(n)) = \chi(\mathbf{R}\pi_*(G_s^\vee \otimes E)(n)) > 0$ for $n \gg 0$ and there are integers $a_i(E)$ such that

$$\chi\big(G_s, E(n)\big) = \sum_i a_i(E) \binom{n+i}{i}. \tag{1.105}$$

DEFINITION 1.4.1 (SIMPSON)

Assume that $\mathcal{C}_s$ is a tilting of $\mathrm{Coh}(X_s)$ for all $s \in S$.

(1) An object $E \in \mathcal{C}_s$ is d-dimensional if $a_d(E) > 0$ and $a_i(E) = 0$, $i > d$.
(2) An object $E \in \mathcal{C}_s$ of dimension d is G_s-twisted semistable if

$$\chi\big(G_s, F(n)\big) \le \frac{a_d(F)}{a_d(E)} \chi\big(G_s, E(n)\big), \quad n \gg 0, \tag{1.106}$$

for all proper subobjects F of E.

(3) An object $E \in \mathcal{C}_s$ of dimension d is μ-semistable if E does not contain a subobject $F \neq 0$ with $a_d(F) = 0$ and

$$a_{d-1}(F) \leq \frac{a_d(F)}{a_d(E)} d_{d-1}(E), \tag{1.107}$$

for all proper subobjects F of E.

REMARK 1.4.2

(1) If $\dim E > \dim \pi(Z_s)$ and E is G_s-twisted semistable, then $H^{-1}(E) = 0$. Indeed $H^{-1}(E)[1]$ is a subobject of E with

$$\deg \chi\bigl(G_s, H^{-1}(E)(n)\bigr) \leq \dim \pi(Z_s) < \deg \chi\bigl(G_s, E(n)\bigr). \tag{1.108}$$

(2) Assume that $E \in \mathrm{Coh}(X_s) \cap \mathcal{C}_s$. For an exact sequence

$$0 \to F \to E \to F' \to 0 \tag{1.109}$$

in $\mathcal{C}_s$, we have an exact sequence in $\mathrm{Coh}(X_s)$,

$$H^{-1}(F') \xrightarrow{\varphi} H^0(F) \to H^0(E) \to H^0(F') \to 0. \tag{1.110}$$

Since $\chi(G_s, H^0(F)(n)) \leq \chi(G_s, (\operatorname{coker} \varphi)(n))$, in order to check the semistability of E, we may assume that $H^{-1}(F') = 0$.

PROPOSITION 1.4.3

There is a coarse moduli scheme $\overline{M}_{X/S}^{\mathcal{C},P} \to S$ of G_s-twisted semistable objects $E \in \mathcal{C}_s$ with the G_s-twisted Hilbert polynomial P. $\overline{M}_{X/S}^{\mathcal{C},P}$ is a projective scheme over S.

Proof

The claim is due to Simpson [S, Theorem 4.7]. We set $\mathcal{A} := \pi_*(G^\vee \otimes G)$. If we set $\Lambda_0 = \mathcal{O}_Y$ and $\Lambda_k = \mathcal{A}$ for $k \geq 1$, then a sheaf of $\mathcal{A}$-modules is an example of Λ-modules in [S]. Let Q^{ss} be an open subscheme of $\mathrm{Quot}_{\mathcal{A}(-n)\otimes V/Y/S}^{\mathcal{A},P}$ consisting of semistable $\mathcal{A}_s$-modules on Y_s, $s \in S$, where V is a vector space of dimension $P(n)$. Then we have the moduli space $\overline{M}_{Y/S}^{\mathcal{A},P} \to S$ of semistable $\mathcal{A}_s$-modules on Y_s as a geometric invariant theory (GIT) quotient $Q^{\mathrm{ss}} /\!/ \mathrm{GL}(V)$, where we use a natural polarization on the embedding of the quot scheme into the Grassmannian. By a standard argument due to Langton, we see that $\overline{M}_{Y/S}^{\mathcal{A},P}$ is projective over S. Since the semistable $\mathcal{A}_s$-modules correspond to G_s-twisted semistable objects via the Morita equivalence (see Proposition 1.3.7), we get the moduli space $\overline{M}_{X/S}^{\mathcal{C},P} \to S$, which is projective over S. $\square$

We consider a natural relative polarization on $\overline{M}_{X/S}^{\mathcal{C},P}$. Let Q^{ss} be the open subscheme of $\mathrm{Quot}_{G(-n)\otimes V/X/S}^{\mathcal{C},P} \cong \mathrm{Quot}_{\mathcal{A}(-n)\otimes V/Y/S}^{\mathcal{A},P}$ as in the above proof. Thus we have $\overline{M}_{X/S}^{\mathcal{C},P} = Q^{\mathrm{ss}} /\!/ \mathrm{GL}(V)$. Let $\mathcal{Q}$ be the universal quotient on $Q^{\mathrm{ss}} \times X$. Then $\mathcal{Q}_{|\{q\}\times X}$ is G-twisted semistable for all $q \in Q^{\mathrm{ss}}$. By the construction of the moduli space, we have a $\mathrm{GL}(V)$-equivariant isomorphism $V \to p_{Q^{\mathrm{ss}}*}(G^\vee \otimes \mathcal{Q}(n))$. We set

$$\begin{aligned}\mathcal{L}_{m,n} &:= \det p_{Q^{\mathrm{ss}}!}\big(G^\vee \otimes \mathcal{Q}(n+m)\big)^{\otimes P(n)} \\ &\qquad \otimes \det p_{Q^{\mathrm{ss}}!}\big(G^\vee \otimes \mathcal{Q}(n)\big)^{\otimes(-P(m+n))} \\ &= \det p_{Q^{\mathrm{ss}}!}\big(G^\vee \otimes \mathcal{Q}(n+m)\big)^{\otimes P(n)} \otimes \det V^{\otimes(-P(m+n))}.\end{aligned} \tag{1.111}$$

We note that $\mathbf{R}\pi_*(G^\vee \otimes \mathcal{Q})$ gives the universal quotient $\mathcal{A}$-module on $Y \times \mathrm{Quot}^{\mathcal{A},P}_{\mathcal{A}(-n)\otimes V/Y/S}$. By the construction of the moduli space, we get the following.

LEMMA 1.4.4
For $m \gg n \gg 0$, $\mathcal{L}_{m,n}$ is the pullback of a relatively ample line bundle on $\overline{M}^{\mathcal{C},P}_{X/S}$.

Assume that $S = \mathrm{Spec}(\mathbb{C})$ and $\dim X = 2$. We take $H \in |\mathcal{O}_X(1)|$.

DEFINITION 1.4.5
(1) For $\mathbf{e} \in K(X)_{\mathrm{top}}$, $\overline{M}^G_H(\mathbf{e})$ is the moduli space of G-twisted semistable objects E of $\mathcal{C}$ with $\tau(E) = \mathbf{e}$ and $M^G_H(\mathbf{e})$ the open subscheme consisting of G-twisted stable objects.

(2) Let $\mathcal{M}_H(\mathbf{e})^{\mu\text{-ss}}$ (resp., $\mathcal{M}^G_H(\mathbf{e})^{\mathrm{ss}}, \mathcal{M}^G_H(\mathbf{e})^s$) be the moduli stack of μ-semistable (resp., G-twisted semistable, G-twisted stable) objects E of $\mathcal{C}$ with $\tau(E) = \mathbf{e}$.

We set $r_0 := \mathrm{rk}\,\mathbf{e}$ and $\xi_0 := c_1(\mathbf{e})$. Then we see that

$$\begin{aligned}&\mathrm{ch}\big(P(n)G^\vee\big((n+m)H\big) - P(n+m)G^\vee(nH)\big) \\ &= m\Big[\frac{(\mathrm{rk}\,G)r_0}{2}(H^2)\big\{(m-2n)\,\mathrm{ch}\,G^\vee \\ &\qquad - n(n+m)\big((\mathrm{rk}\,G)H - \big(c_1(G),H\big)\varrho_X\big)\big\} \\ &\qquad + \Big(H, (\mathrm{rk}\,G)\xi_0 - r_0 c_1(G) - \frac{(\mathrm{rk}\,G)r_0}{2}K_X\Big) \\ &\qquad \times \Big(-\mathrm{ch}\,G^\vee + \frac{n(n+m)}{2}(H^2)(\mathrm{rk}\,G)\varrho_X\Big)\Big].\end{aligned} \tag{1.112}$$

LEMMA 1.4.6
We take $\zeta \in K(X)$ with $\mathrm{ch}(\zeta) = r_0 H + (\xi_0, H)\varrho_X$ and assume that $r_0 > 0$. Assume that $\tau(G) \in \mathbb{Z}\mathbf{e}$. If $\chi(\mathbf{e},\mathbf{e}) = 0$ and $E \cong E \otimes K_X$ for a G-twisted stable objects E with $\tau(E) = \mathbf{e}$ and $\mathcal{M}^G_H(\mathbf{e})^{\mathrm{ss}}$ is smooth at E, then $\det p_{Q^{\mathrm{ss}}!}(\mathcal{Q} \otimes \zeta^\vee) \cong \det p_{Q^{\mathrm{ss}}!}(\mathcal{Q}^\vee \otimes \zeta)^\vee$ is the pullback of an ample line bundle $\mathcal{L}(\zeta)$ on $\overline{M}^G_H(\mathbf{e})$.

Proof
We first show that $\det p_{Q^{\mathrm{ss}}!}(\mathcal{Q} \otimes E^\vee) \cong \mathcal{O}_{Q^{\mathrm{ss}}}$ as a $\mathrm{PGL}(V)$-equivariant line bundle for $E \in \mathcal{M}^G_H(\mathbf{e})^s$ with $E \cong E \otimes K_X$. We set $U := \{q \in Q^{\mathrm{ss}} \mid \mathcal{Q}_{|\{q\}\times X} \not\cong E\}$. Then $\mathrm{Hom}(\mathcal{Q}_{|\{q\}\times X}, E) = \mathrm{Hom}(E, \mathcal{Q}_{|\{q\}\times X}) = 0$ for $x \in U$. Since $\chi(\mathbf{e},\mathbf{e}) = 0$, we also

have $\mathrm{Ext}^1(E, \mathcal{Q}_{|\{q\}\times X}) = 0$. Hence $\det p_{Q^{ss}!}(\mathcal{Q} \otimes E^\vee)_{|U} \cong \mathcal{O}_U$. Since $\mathrm{codim}_{Q^{ss}}(Q^{ss} \setminus U) \geq 2$ and Q^{ss} is smooth in a neighborhood of $Q^{ss} \setminus U$, we have $\det p_{Q^{ss}!}(\mathcal{Q}\otimes E^\vee) \cong \mathcal{O}_{Q^{ss}}$ as a $\mathrm{PGL}(V)$-equivariant line bundle. We set $\tau(G) = \lambda \mathbf{e}$, $\lambda \in \mathbb{Z}_{>0}$. Since $r_0 K_X = 0$, we have $(H, (\mathrm{rk}\, G)\xi_0 - r_0 c_1(G) - \frac{(\mathrm{rk}\, G) r_0}{2} K_X) = 0$. Then we get $P(n)G^\vee((n+m)H) - P(n+m)G^\vee(nH) \equiv mn(n+m)\lambda\zeta^\vee \mod \mathbb{Z}\mathbf{e}^\vee$. By Lemma 1.4.4, we get our claim. □

REMARK 1.4.7

If $E \cong E \otimes K_X$ and $\mathcal{M}_H^G(\mathbf{e})^{ss}$ is smooth at E, then for an irreducible component $\mathcal{M}$ of $\mathcal{M}_H^G(\mathbf{e})^s$ containing E, we see that $\mathcal{M}$ is smooth and $E \cong E \otimes K_X$ for all $E \in \mathcal{M}$.

Indeed for $E' \in \mathcal{M}$, we have $\dim \mathrm{Ext}^1(E', E') = 1 + \dim \mathrm{Hom}(E', E' \otimes K_X)$. If $\mathrm{Hom}(E', E' \otimes K_X) = 0$ for $E' \in \mathcal{M}$, then $\dim \mathrm{Ext}^1(E', E') = 1$ and $\mathcal{M}$ is smooth of dimension zero at E'. Since $\mathcal{M}$ is irreducible and $\mathcal{M}$ is smooth of dimension 1 at E, we have $\mathrm{Hom}(E', E' \otimes K_X) \neq 0$ for all $E' \in \mathcal{M}$. Then we have $E' \cong E' \otimes K_X$ and $\dim \mathrm{Ext}^1(E', E') = 2$, which implies that $\mathcal{M}$ is smooth.

For a family $\mathcal{E}$ of G-twisted semistable objects on X parameterized by S, we have a morphism $f : S \to \overline{M}_H^G(\mathbf{e})$ such that $f(s)$ is the S-equivalence class of $\mathcal{E}_{|\{s\}\times X}$. Then we have $\det p_{S!}(\mathcal{E} \otimes \zeta^\vee) \cong f^*(\mathcal{L}(\zeta))$.

Indeed we have a morphism $S \to [Q^{ss}/\, \mathrm{GL}(V)] = \mathcal{M}_H^G(\mathbf{e})^{ss}$; that is, we have a principal $\mathrm{GL}(V)$-bundle $h : \mathcal{P} \to S$ and a $\mathrm{GL}(V)$-equivariant morphism $f : \mathcal{P} \to Q^{ss}$ which induces a $\mathrm{GL}(V)$-equivariant isomorphism $(h \times 1_X)^*(\mathcal{E}) \cong (f \times 1_X)^*(\mathcal{Q})$. Hence we have $\det p_{S!}(\mathcal{E} \otimes \zeta^\vee) \cong f^*(\mathcal{L}(\zeta))$.

More generally we assume that $\mathcal{E}$ is a family of G-twisted semistable objects as twisted objects, that is, $\mathcal{E}$ is a collection of families $\mathcal{E}_i$ on $S_i \times X$ such that

(i) $S = \bigcup_i S_i$ is an open covering of S;
(ii) there are isomorphisms $\varphi_{ij} : \mathcal{E}_{i|(S_i \cap S_j)\times X} \cong \mathcal{E}_{j|(S_i\cap S_j)\times X}$; and
(iii) $\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij}$ is a multiplication (see Section 1.7).

By these conditions, the collection of line bundles $\det p_{S_i!}(\mathcal{E}_i \otimes \zeta^\vee) \in \mathrm{Pic}(S_i)$ defines a line bundle on S. We denote this line bundle by $\det p_{S!}(\mathcal{E} \otimes \zeta^\vee)$.

LEMMA 1.4.8

For the morphism $f : S \to \overline{M}_H^G(\mathbf{e})$ such that $f(s)$ is the S-equivalence class of $\mathcal{E}_{|\{s\}\times X}$, we have an isomorphism

$$\det p_{S!}(\mathcal{E}^\vee \otimes \zeta)^\vee \cong \det p_{S!}(\mathcal{E} \otimes \zeta^\vee) \cong f^*\big(\mathcal{L}(\zeta)\big).$$

Proof

For the proof, we take an object $F \in \mathcal{C}$ such that $W_i := p_{S_i *}(\mathcal{E}_i \otimes F^\vee)$ are locally free. For the twisted sheaf $W := (\{W_i\}, \{\varphi_{ij}\})$ we consider the projective bundle $\phi : \mathbb{P}(W) \to S$. We set $P_i := \phi^{-1}(S_i)$. Since $\mathcal{O}_{P_i}(1)$ defines a twisted sheaf on $P = \mathbb{P}(W)$, we have a family of untwisted objects $\mathcal{E}'$ such that $(\phi \times 1)^*(\mathcal{E}_{|S_i \times X}) \cong$

$\mathcal{E}'_{|P_i \times X} \otimes \mathcal{O}_{P_i}(1)$. For the family $\mathcal{E}'$, we have a morphism $f' : P \to \overline{M}_H^G(\mathbf{e})$ such that $\det p_{P!}(\mathcal{E}' \otimes \zeta^\vee) \cong f'^*(\mathcal{L}(\zeta))$. We note that $\phi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(P)$ is injective. So we regard $\mathrm{Pic}(S)$ as a subset of $\mathrm{Pic}(P)$. It is easy to see that $\det p_{P!}(\mathcal{E}' \otimes \zeta^\vee) \cong \phi^*(\det p_{S!}(\mathcal{E} \otimes \zeta^\vee))$. Hence the claim holds. $\square$

DEFINITION 1.4.9

Assume that $\mathrm{rk}\,\mathbf{e} > 0$.

(1) $P(\mathbf{e})$ is the set of subobjects E' of $E \in \mathcal{M}_H(\mathbf{e})^{\mu-\mathrm{ss}}$ such that

$$\frac{(c_1(G^\vee \otimes E), H)}{\mathrm{rk}\,E} = \frac{(c_1(G^\vee \otimes E'), H)}{\mathrm{rk}\,E'}. \tag{1.113}$$

(2) For $E' \in P(\mathbf{e})$, we define a wall $W_{E'} \subset \mathrm{NS}(X) \otimes \mathbb{R}$ as the set of $\alpha \in \mathrm{NS}(X) \otimes \mathbb{R}$ satisfying

$$\left(\alpha, \frac{c_1(G^\vee \otimes E)}{\mathrm{rk}\,E} - \frac{c_1(G^\vee \otimes E')}{\mathrm{rk}\,E'}\right) + \left(\frac{\chi(G^\vee \otimes E)}{\mathrm{rk}\,E} - \frac{\chi(G^\vee \otimes E')}{\mathrm{rk}\,E'}\right) = 0. \tag{1.114}$$

Since $\{\tau(E') \mid E' \in P(\mathbf{e})\}$ is a finite set, $\bigcup_{E'} W_{E'}$ is finite. If $\alpha \in \mathrm{NS}(X) \otimes \mathbb{Q}$ does not lie on any $W_{E'}$, we say that α is general. If a local projective generator G' satisfies $\alpha := c_1(G')/\mathrm{rk}\,G' - c_1(G)/\mathrm{rk}\,G \notin \bigcup_{E'} W_{E'}$, then we also say that G' is general.

LEMMA 1.4.10

If G is general, that is, if $0 \notin \bigcup_{E'} W_{E'}$, then for $E' \in P(\mathbf{e})$,

$$\frac{\chi(G, \mathbf{e})}{\mathrm{rk}\,\mathbf{e}} = \frac{\chi(G, E')}{\mathrm{rk}\,E'} \iff \frac{\mathbf{e}}{\mathrm{rk}\,\mathbf{e}} = \frac{\tau(E')}{\mathrm{rk}\,E'} \in K(X)_{\mathrm{top}} \otimes \mathbb{Q}. \tag{1.115}$$

In particular, if $\mathbf{e}$ is primitive, then $\overline{M}_H^G(\mathbf{e}) = M_H^G(\mathbf{e})$ for a general G.

1.5. A generalization of stability for zero-dimensional objects

It is easy to see that every zero-dimensional object is G_s-twisted semistable. Our definition is not sufficient in order to get a good moduli space. So we introduce a refined version of twisted stability.

DEFINITION 1.5.1

Let G, G' be families of local projective generators of $\mathcal{C}_s$. A zero-dimensional object E is (G_s, G'_s)-twisted semistable if

$$\frac{\chi(G'_s, E_1)}{\chi(G_s, E_1)} \le \frac{\chi(G'_s, E)}{\chi(G_s, E)} \tag{1.116}$$

for all proper subobjects E_1 of E.

By a modification of Simpson's construction of moduli spaces, we can construct the coarse moduli scheme of (G_s, G'_s)-twisted semistable objects. From now on, we assume that $S = \mathrm{Spec}(\mathbb{C})$ for simplicity.

LEMMA 1.5.2

Let G be a locally free sheaf on X which is a local projective generator of $\mathcal{C}$.

(1) *Assume that there is an exact sequence in $\mathcal{C}$,*

$$0 \to E' \to V_0 \to V_1 \to \cdots \to V_r \to E \to 0 \tag{1.117}$$

such that V_i are local projective objects of $\mathcal{C}$. If $r \geq \dim X$, then E' is a local projective object of $\mathcal{C}$.

(2) *For $E \in K(X)$, there is a local projective generator G' of $\mathcal{C}$ such that $E = G' - NG(-n)$, where N and n are sufficiently large integers.*

Proof

(1) We first prove that $H^i(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E,F)) = 0$, $i > \dim X + 1$, for all $F \in \mathcal{C}$. Since $\mathcal{C}$ is a tilting of $\mathrm{Coh}(X)$ (see Proposition 1.1.13), $H^i(E) = H^i(F) = 0$ for $i \neq -1, 0$. By using a spectral sequence, we get

$$H^i\big(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(H^{-p}(E)[p], H^{-q}(F)[q]\big)\big) = 0 \tag{1.118}$$

for $i > \dim X + 1$. Hence we get $H^i(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E,F)) = 0$, $i > \dim X + 1$. Then we see that

$$H^i\big(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E',F)\big) \cong H^{i+r+1}\big(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E,F)\big) = 0 \tag{1.119}$$

for all integer with $i > \max\{\dim X - r, 0\} = 0$. Therefore E' is a local projective object.

(2) We first prove that there are local projective generators G_1, G_2 such that $E = G_1 - G_2$. We may assume that $E \in \mathcal{C}$. We take a resolution of E,

$$\begin{aligned} &0 \to E' \to G(-n_r)^{\oplus N_r} \xrightarrow{\phi} G(-n_{r-1})^{\oplus N_{r-1}} \to \cdots \\ &\quad \to G(-n_0)^{\oplus N_0} \to E \to 0. \end{aligned} \tag{1.120}$$

If $r \geq \dim X$, then (1) implies that E' is a local projective object. We set $r := 2j_0 + 1$. We set $G_1 := E' \oplus \bigoplus_{j=0}^{j_0} G(-n_{2j})^{\oplus N_{2j}}$ and $G_2 := \bigoplus_{j=0}^{j_0} G(-n_{2j+1})^{\oplus N_{2j+1}}$. Then G_1 and G_2 are local projective generators, and $E = G_1 - G_2$. We take a resolution

$$0 \to G_2' \to G(-n)^{\oplus N} \to G_2 \to 0 \tag{1.121}$$

such that $G_2' \in \mathcal{C}$. Then we see that $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(G_2', F) \in \mathrm{Coh}(Y)$ for any $F \in \mathcal{C}$. Since $E = (G_1 \oplus G_2') - G(-n)^{\oplus N}$ and $G_1 \oplus G_2'$ is a local projective generator, we get our claim. □

DEFINITION 1.5.3

Let A be an element of $K(X) \otimes \mathbb{Q}$, and let G be a local projective generator. A zero-dimensional object E is (G,A)-twisted semistable if

$$\frac{\chi(A,F)}{\chi(G,F)} \leq \frac{\chi(A,E)}{\chi(G,E)} \tag{1.122}$$

for all proper subobjects F of E.

By Lemma 1.5.2, we write $N'A = G' - NG(-n) \in K(X)$, where G' is a local projective generator and $n, N, N' > 0$. Then

$$\frac{\chi(G', E)}{\chi(G, E)} = N' \frac{\chi(A, E)}{\chi(G, E)} + N. \tag{1.123}$$

Hence E is (G, G')-twisted semistable if and only if E is (G, A)-twisted semistable. Thus we get the following proposition.

PROPOSITION 1.5.4

Assume that $\dim X = 2$. *Let A be an element of* $K(X) \otimes \mathbb{Q}$, *and let G be a local projective generator. Let v be a Mukai vector of a zero-dimensional object.*

(1) *There is a coarse moduli scheme* $\overline{M}_{\mathcal{O}_X(1)}^{G,A}(v)$ *of (G, A)-twisted semistable objects of $\mathcal{C}$.*

(2) *If v is primitive and A is general in* $K(X) \otimes \mathbb{Q}$, *then* $\overline{M}_{\mathcal{O}_X(1)}^{G,A}(v)$ *consists of (G, A)-twisted stable objects. Moreover,* $\overline{M}_{\mathcal{O}_X(1)}^{G,A}(v)$ *is a fine moduli space.*

REMARK 1.5.5

As is well known, if there is $E \in K(X)$ with $\chi(E, v) = 1$, then there is a universal family. If particular, for $v = \varrho_X$, we have a universal family. If $v \neq \varrho_X$, then the moduli space is a point. So obviously we have a universal family.

REMARK 1.5.6

If $v(E) = \varrho_X$ and $\operatorname{rk} A = 0$, then E is (G, A)-twisted semistable if and only if $\chi(A, E') \leq 0$ for all subobjects E' of E in $\mathcal{C}$. Thus the semistability does not depend on the choice of G.

REMARK 1.5.7

In Section 1.7, we deal with the twisted sheaves. In this case, we still have the moduli spaces of zero-dimensional stable objects, but $\overline{M}_{\mathcal{O}_X(1)}^{G,A}(\varrho_X)$ does not have a universal family.

DEFINITION 1.5.8

$\mathcal{M}_{\mathcal{O}_X(1)}^{G,A}(v)^{\mathrm{ss}}$ denotes the moduli stack of (G, A)-twisted semistable objects E with $v(E) = v$.

1.6. Construction of the moduli spaces of $\mathcal{A}$-modules of dimension zero

By Proposition 1.1.7, we have an equivalence $\mathcal{C} \to \operatorname{Coh}_{\mathcal{A}}(Y)$. We set $\mathcal{B} := \pi_*(G^\vee \otimes G')$. Then $\mathcal{B}$ is a local projective generator of $\operatorname{Coh}_{\mathcal{A}}(Y)$. For all $F \in \operatorname{Coh}_{\mathcal{A}}(Y)$, $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{B}, F) = \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, F)$ and $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{B}, F) = 0$ if and only if $F = 0$. In particular, we have a surjective morphism

$$\phi : \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, F) \otimes_{\mathcal{A}} \mathcal{B} \to F. \tag{1.124}$$

For $F \in \mathrm{Coh}_{\mathcal{A}}(Y)$, we set

$$\chi_{\mathcal{A}}(\mathcal{B}, F) := \chi\big(\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{B}, F)\big). \tag{1.125}$$

For $F \in \mathrm{Coh}_{\mathcal{A}}(Y)$, $\pi^{-1}(F) \otimes_{\pi^{-1}(\mathcal{A})} G$ is (G, G')-twisted semistable if

$$\frac{\chi_{\mathcal{A}}(\mathcal{B}, F_1)}{\chi(F_1)} \leq \frac{\chi_{\mathcal{A}}(\mathcal{B}, F)}{\chi(F)} \tag{1.126}$$

for all proper sub-$\mathcal{A}$-modules F_1 of F. We define the $(\mathcal{A}, \mathcal{B})$-twisted semistability by this inequality.

PROPOSITION 1.6.1

There is a coarse moduli scheme of $(\mathcal{A}, \mathcal{B})$-twisted semistable $\mathcal{A}$-modules of dimension zero.

Proof of Proposition 1.6.1

Let F be an $\mathcal{A}$-module of dimension zero. Then $\mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, F) \otimes \mathcal{B} \to F$ is surjective. Hence all zero-dimensional objects F are parameterized by a quot scheme $Q := \mathrm{Quot}^{\mathcal{A},m}_{V \otimes \mathcal{B}/Y/\mathbb{C}}$, where $m = \chi(F)$ and $\dim V = \chi_{\mathcal{A}}(\mathcal{B}, F)$. Let $V \otimes \mathcal{O}_Q \otimes \mathcal{B} \to \mathcal{F}$ be the universal quotient. For simplicity, we set $\mathcal{F}_q := \mathcal{F}_{|\{q\} \times Y}$, $q \in Q$. For a sufficiently large integer n, we have a quotient $V \otimes H^0(Y, \mathcal{B}(n)) \to H^0(Y, F(n))$. We set $W := H^0(Y, \mathcal{B}(n))$. Then we have an embedding

$$\mathrm{Quot}^{\mathcal{A},m}_{V \otimes \mathcal{B}/Y/\mathbb{C}} \hookrightarrow \mathrm{Gr}(V \otimes W, m). \tag{1.127}$$

This embedding is equivariant with respect to the natural action of $\mathrm{PGL}(V)$. The following is well known.

LEMMA 1.6.2

Let $\alpha : V \otimes W \to U$ be a point of $\mathfrak{G} := \mathrm{Gr}(V \otimes W, m)$. Then α belongs to the set $\mathfrak{G}^{\mathrm{ss}}$ of semistable points if and only if

$$\frac{\dim U}{\dim V} \leq \frac{\dim \alpha(V_1 \otimes W)}{\dim V_1} \tag{1.128}$$

for all proper subspaces $V_1 \neq 0$ of V. If the inequality is strict for all V_1, then α is stable.

We set

$$Q^{\mathrm{ss}} := \big\{ q \in Q \,\big|\, \mathcal{F}_q \text{ is } (\mathcal{A}, \mathcal{B})\text{-twisted semistable} \big\}. \tag{1.129}$$

For $q \in Q^{\mathrm{ss}}$, $V \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, F)$ is an isomorphism. We only prove that $Q^{\mathrm{ss}} = \mathfrak{G}^{\mathrm{ss}} \cap Q$. Then Proposition 1.6.1 easily follows.

For an $\mathcal{A}$-submodule F_1 of F, we set $V_1 := \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, F_1)$. Then we have a surjective homomorphism $V_1 \otimes \mathcal{B} \to F_1$. Conversely, for a subspace V_1 of V, we set $F_1 := \mathrm{im}(V_1 \otimes \mathcal{B} \to F)$. Then $V_1 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, F_1)$ is injective.

We set

$$\mathfrak{F} := \big\{ \mathrm{im}(V_1 \otimes \mathcal{B} \to \mathcal{F}_q) \,\big|\, q \in Q, V_1 \subset V \big\}. \tag{1.130}$$

Since $\mathfrak{F}$ is bounded, we can take an integer n in the definition of W such that $V_1 \otimes W \to H^0(Y, F_1)$ is surjective for all $F_1 \in \mathfrak{F}$. Assume that $\mathcal{F}_q$ is $(\mathcal{A}, \mathcal{B})$-twisted semistable. For any $V_1 \subset V$, we set $F_1 := \operatorname{im}(V_1 \otimes \mathcal{B} \to \mathcal{F}_q)$. Then $\alpha(V_1 \otimes W) = H^0(Y, F_1)$. Hence

$$
\begin{aligned}
\frac{\dim \alpha(V_1 \otimes W)}{\dim V_1} &\geq \frac{\chi(F_1)}{\dim \operatorname{Hom}_{\mathcal{A}}(\mathcal{B}, F_1)} \\
&= \frac{\chi(F_1)}{\chi_{\mathcal{A}}(\mathcal{B}, F_1)} \geq \frac{\chi(\mathcal{F}_q)}{\chi_{\mathcal{A}}(\mathcal{B}, \mathcal{F}_q)} = \frac{\dim \alpha(V \otimes W)}{\dim V}.
\end{aligned}
\tag{1.131}
$$

Thus $q \in \mathfrak{G}^{ss}$.

We take a point $q \in \mathfrak{G}^{ss} \cap Q$. We first prove that $\psi : V \to \operatorname{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{F}_q)$ is an isomorphism. We set $V_1 := \ker \psi$. Since $V_1 \otimes \mathcal{B} \to \mathcal{F}_q$ is zero, we get $\alpha(V_1 \otimes W) = 0$. Then

$$
\frac{\dim U}{\dim V} \leq \frac{\dim \alpha(V_1 \otimes W)}{\dim V_1} = 0,
\tag{1.132}
$$

which is a contradiction. Therefore ψ is injective. Since $\dim V = \dim \operatorname{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{F}_q)$, ψ is an isomorphism. Let $F_1 \neq 0$ be a proper $\mathcal{A}$-submodule of $\mathcal{F}_q$. We set $V_1 := \operatorname{Hom}_{\mathcal{A}}(\mathcal{B}, F_1)$. Then

$$
\frac{\chi(F_1)}{\dim \operatorname{Hom}_{\mathcal{A}}(\mathcal{B}, F_1)} \geq \frac{\dim \alpha(V_1 \otimes W)}{\dim V_1} \geq \frac{\dim \alpha(V \otimes W)}{\dim V} = \frac{\chi(\mathcal{F}_q)}{\chi_{\mathcal{A}}(\mathcal{B}, \mathcal{F}_q)}.
\tag{1.133}
$$

Hence $\mathcal{F}_q$ is $(\mathcal{A}, \mathcal{B})$-twisted semistable. If q is a stable point, then we also see that $\mathcal{F}_q$ is $(\mathcal{A}, \mathcal{B})$-twisted stable.

1.7. Twisted case

1.7.1. Definition

Let $X = \bigcup_i X_i$ be an analytic open covering of X, and let $\beta = \{\beta_{ijk} \in H^0(X_i \cap X_j \cap X_k, \mathcal{O}_X^\times)\}$, a Cech 2-cocycle of $\mathcal{O}_X^\times$. We assume that β defines a torsion element $[\beta]$ of $H^2(X, \mathcal{O}_X^\times)$.

DEFINITION 1.7.1

A coherent β-twisted sheaf E consists of $(\{E_i\}, \{\varphi_{ij}\})$ such that

(i) E_i is a coherent sheaf on X_i;
(ii) $\varphi_{ij} : E_{i|X_i \cap X_j} \to E_{j|X_i \cap X_j}$ is an isomorphism;
(iii) $\varphi_{ji} = \varphi_{ij}^{-1}$;
(iv) $\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} = \beta_{ijk} \operatorname{id}_{X_i \cap X_j \cap X_k}$.

Let G be a locally free β-twisted sheaf of rank r, and let $P := \mathbb{P}(G^\vee)$ be the associated projective bundle over X (cf. [Y4, Section 1.1]). Let $w(P) \in H^2(X, \mathbb{Z}/r\mathbb{Z})$ be the characteristic class of P (see [Y4, Definition 1.2]). Then $[\beta]$ is trivial if and only if $w(P) \in \operatorname{im}(\operatorname{NS}(X) \to H^2(X, \mathbb{Z}/r\mathbb{Z}))$ (see [Y4, Lemma 1.4]).

Let $\operatorname{Coh}^\beta(X)$ be the category of coherent β-twisted sheaves on X, and let $\mathbf{D}^\beta(X)$ be the bounded derived category of $\operatorname{Coh}^\beta(X)$. Let $K^\beta(X)$ be the Grothendieck group of $\operatorname{Coh}^\beta(X)$. Then similar statements in Lemma 1.1.11 hold for $\mathrm{Coh}^\beta(X)$. Then all results in Sections 1.3 and 1.4 hold. In particular, if a locally free β-twisted sheaf G defines a torsion pair, then we have the moduli of G-twisted semistable objects. Replacing $\zeta \in K(X)$ by $\zeta \in K^\beta(X)$ with $c_1(\zeta) = r_0 H$ and $\chi(G \otimes \zeta^\vee) = 0$, Lemma 1.4.6 also holds.

1.7.2. Chern character
We have a homomorphism

$$\begin{aligned} \mathrm{ch}_G : \mathbf{D}^\beta(X) &\to H^{ev}(X,\mathbb{Q}), \\ E &\mapsto \frac{\mathrm{ch}(G^\vee \otimes E)}{\sqrt{\mathrm{ch}(G^\vee \otimes G)}}. \end{aligned} \tag{1.134}$$

Obviously $\mathrm{ch}_G(E)$ depends only on the class in $K^\beta(X)$. Since

$$\mathrm{ch}_G(E)^\vee \mathrm{ch}_G(F) = \frac{\mathrm{ch}((G^\vee \otimes E)^\vee \otimes (G^\vee \otimes F))}{\mathrm{ch}(G^\vee \otimes G)} = \mathrm{ch}(E^\vee \otimes F), \tag{1.135}$$

we have the following Riemann–Roch formula:

$$\chi(E,F) = \int_X \mathrm{ch}_G(E)^\vee \mathrm{ch}_G(F)\mathrm{td}_X. \tag{1.136}$$

Assume that X is a surface. For a torsion G-twisted sheaf E, we can attach the codimension 1 part of the scheme-theoretic support $\mathrm{Div}(E)$ as in the usual sheaves. Then we see that

$$\mathrm{ch}_G(E) = \big(0, \big[\mathrm{Div}(E)\big], a\big), \quad a \in \mathbb{Q}, \tag{1.137}$$

where $[\mathrm{Div}(E)]$ denotes the homology class of the divisor $\mathrm{Div}(E)$, and we regard it as an element of $H^2(X,\mathbb{Z})$ by the Poincaré duality. More generally, if $E \in \mathbf{D}^\beta(X)$ satisfies $\mathrm{rk}\, H^i(E) = 0$ for all i, then

$$\mathrm{ch}_G(E) = \Big(0, \sum_i (-1)^i \big[\mathrm{Div}\big(H^i(E)\big)\big], a\Big), \quad a \in \mathbb{Q}. \tag{1.138}$$

We set $c_1(E) := \sum_i (-1)^i [\mathrm{Div}(H^i(E))]$.

REMARK 1.7.2
If $H^3(X,\mathbb{Z})$ is torsion free, then we have an automorphism η of $H^*(X,\mathbb{Q})$ such that the image of $\eta \circ \mathrm{ch}_G$ is contained in $\mathrm{ch}(K(X)) \subset \mathbb{Z} \oplus H^2(X,\mathbb{Z}) \oplus H^4(X, (1/2)\mathbb{Z})$ and (1.136) holds if we replace ch_G by $\eta \circ \mathrm{ch}_G$ (cf. [Y4]). We first note that

$$\mathrm{ch}\big(K(X)\big) = \big\{(r,D,a) \,\big|\, r \in \mathbb{Z}, D \in H^2(X,\mathbb{Z}), a - (D,K_X)/2 \in \mathbb{Z}\big\}. \tag{1.139}$$

Replacing the statement of [Y4, Lemma 3.1] by

$$\begin{aligned} &c_2(E^\vee \otimes E) + r(r-1)\big(w(E),K_X\big) \\ &\quad \equiv -(r-1)\big(\big(w(E)^2\big) - r\big(w(E),K_X\big)\big) \mod 2r, \end{aligned} \tag{1.140}$$

we can prove a claim similar to [Y4, Lemma 3.3].

LEMMA 1.7.3

Let E be a β-twisted sheaf of $\operatorname{rk} E = 0$. *Then*

$$\big[\chi(G,E) \mod r\mathbb{Z}\big] \equiv -w(P) \cap \big[\operatorname{Div}(E)\big], \tag{1.141}$$

where we identified $H_0(X, \mathbb{Z}/r\mathbb{Z})$ with $\mathbb{Z}/r\mathbb{Z}$.

Proof

Since $\chi(G,E)$ and $[\operatorname{Div}(E)]$ are additive, it is sufficient to prove the claim for pure sheaves. If $\dim E = 0$ as an object of $\operatorname{Coh}^\beta(X)$, then $r \mid \chi(G,E)$ and $\operatorname{Div}(E) = 0$. Hence the claim holds. We assume that E is purely 1-dimensional. Then E is a twisted sheaf on $C := \operatorname{Div}(E)$. Since C is a curve, there is a β-twisted line bundle L on C, and we have an equivalence

$$\begin{aligned} \varphi : \operatorname{Coh}^\beta(C) &\to \operatorname{Coh}(C), \\ E &\mapsto E \otimes L^\vee. \end{aligned} \tag{1.142}$$

Then we can take a filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$ of E such that $\operatorname{Div}(F_i/F_{i-1})$ are reduced and irreducible curves and F_i/F_{i-1} are torsion-free β-twisted sheaves of rank 1 on $\operatorname{Div}(F_i/F_{i-1})$. Replacing E by F_i/F_{i-1}, we may assume that E is a twisted sheaf of rank 1 on an irreducible and reduced curve $C = \operatorname{Div}(E)$. Then $\chi(G,E) = \chi(\varphi(G_{|C})^\vee \otimes \varphi(E)) = \int_C c_1(\varphi(G_{|C})^\vee) + r\chi(\varphi(E))$. Since $w(P)_{|C} = w(P_{|C}) = c_1(\varphi(G_{|C})) \mod r\mathbb{Z}$, $[\chi(G,E) \mod r\mathbb{Z}] \equiv -w(P) \cap [C]$. $\square$

COROLLARY 1.7.4

For an object E of $\mathbf{D}^\beta(X)$, assume that $\operatorname{rk} H^i(E) = 0$ *for all i. Then*

$$\big[\chi(G,E) \mod r\mathbb{Z}\big] \equiv -w(P) \cap \big[\operatorname{Div}(E)\big]. \tag{1.143}$$

Moreover if $c_1(E) = 0$, then $\operatorname{ch}_G(E) \in \mathbb{Z}\varrho_X$.

Proof

The second claim follows from $\int_X \operatorname{ch}_G(E) = \chi(G,E)/r = (\chi(G,E)/r)\int_X \varrho_X$. $\square$

2. Perverse coherent sheaves for the resolution of rational double points

2.1. Perverse coherent sheaves on the resolution of rational singularities

Let Y be a projective normal surface with at worst rational singularities, and let $\pi : X \to Y$ be the minimal resolution. Let p_i, $i = 1, 2, \ldots, n$ be the singular points of Y, and let $Z_i := \pi^{-1}(p_i) = \sum_{j=1}^{t_i} a_{ij} C_{ij}$ be their fundamental cycles. By the assumption, we have $R^1\pi_*(\mathcal{O}_X) = 0$, and C_{ij} are smooth rational curves on X.

Let β be a 2-cocycle of $\mathcal{O}_X^\times$ whose image in $H^2(X, \mathcal{O}_X^\times)$ is a torsion element. For β-twisted line bundles L_{ij} on C_{ij}, we shall define abelian categories $\operatorname{Per}(X/Y, \{L_{ij}\})$ and $\operatorname{Per}(X/Y, \{L_{ij}\})^*$. Let A_{p_i} be the unique line bundle on Z_i such that $A_{p_i|C_{ij}} = L_{ij}(1)$ (see Lemma 1.2.16).

PROPOSITION 2.1.1

(1) *There is a locally free sheaf G such that $\mathbf{R}\pi_*(G^\vee \otimes L_{ij}) = 0$ for all i, j.*

(2) $\mathcal{C}(G)$ *is the tilting of* $\operatorname{Coh}^\beta(X)$ *with respect to the torsion pair* (T,S) *such that*

$$\begin{aligned} S &:= \{E \in \operatorname{Coh}^\beta(X) \mid E \text{ is generated by subsheaves of } L_{ij}\}, \\ T &:= \{E \in \operatorname{Coh}^\beta(X) \mid \operatorname{Hom}(E, L_{ij}) = 0\}. \end{aligned} \tag{2.1}$$

(3) $\mathcal{C}(G)^*$ *is the tilting of* $\operatorname{Coh}^\beta(X)$ *with respect to the torsion pair* (T^*,S^*) *such that*

$$\begin{aligned} S^* &:= \{E \in \operatorname{Coh}^\beta(X) \mid E \text{ is generated by subsheaves of } A_{p_i} \otimes \omega_{Z_i}\}, \\ T^* &:= \{E \in \operatorname{Coh}^\beta(X) \mid \operatorname{Hom}(E, A_{p_i} \otimes \omega_{Z_i}) = 0\}. \end{aligned} \tag{2.2}$$

For the proof of (1), we shall use the deformation theory of a coherent twisted sheaf.

DEFINITION 2.1.2

For a coherent β-twisted sheaf E on a scheme W, $\operatorname{Def}(W,E)$ denotes the local deformation space of E fixing $\det E$.

For a complex $E \in \mathbf{D}^\beta(X)$, let

$$\operatorname{Ext}^i(E,E)_0 := \ker\big(\operatorname{Ext}^i(E,E) \xrightarrow{\operatorname{tr}} H^i(X,\mathcal{O}_X)\big) \tag{2.3}$$

be the kernel of the trace map. If $\operatorname{Ext}^2(E,E)_0 = 0$, then $\operatorname{Def}(W,E)$ is smooth and the Zariski tangent space at E is $\operatorname{Ext}^1(E,E)_0$. The following is well known.

LEMMA 2.1.3

Let D be a divisor on X. For $E \in \operatorname{Coh}^\beta(X)$ with $\operatorname{rk} E > 0$, we have a torsion-free β-twisted sheaf E' such that $\tau(E') = \tau(E) - n\tau(\mathbb{C}_x)$ and $\operatorname{Ext}^2(E', E'(D))_0 = 0$.

Proof

For a locally free β-twisted sheaf E, we consider a general surjective homomorphism $\phi : E \to \bigoplus_{i=1}^n \mathbb{C}_{x_i}$, $x_i \in X$. If n is sufficiently large, then $E' := \ker\phi$ satisfies the claim. □

LEMMA 2.1.4

Let C be an effective divisor on X. For $(r,\mathcal{L}) \in \mathbb{Z}_{>0} \times \operatorname{Pic}(C)$, the moduli stack of locally free sheaves E on C such that $(\operatorname{rk} E, \det E) = (r, \mathcal{L})$ is irreducible.

Proof

For a locally free sheaf E on C we consider $\phi : H^0(X, E(k)) \otimes \mathcal{O}_C(-k) \to E$. Assume that ϕ is surjective. Then there is a subvector space $V \subset H^0(X, E(k))$ of $\dim V = r-1$ such that $\psi : V \otimes \mathcal{O}_C(-k) \to E$ is injective for any point of C. Then $\operatorname{coker}\psi$ is a line bundle which is isomorphic to $\det(E) \otimes \mathcal{O}_C((r-1)k)$. Hence E is parameterized an affine space $\operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{L} \otimes \mathcal{O}_C((r-1)k), \mathcal{O}_C(-k) \otimes V) =$

$H^1(C, \mathcal{L}^\vee(-rk) \otimes V)$. Since the surjectivity of ϕ is an open condition and ϕ is surjective for $k \gg 0$, we get our claim. $\square$

Proof of Proposition 2.1.1

(1) For a locally free β-twisted sheaf G_1 on X, we set $g_{ij} := \chi(G_1, L_{ij})$. Let $\alpha \in \bigoplus_{i=1}^n \bigoplus_{j=1}^{t_i} \mathbb{Q}[C_{ij}]$ be a $\mathbb{Q}$-divisor such that $\operatorname{rk} G_1(\alpha, C_{ij}) = g_{ij}$. We take a locally free sheaf $A \in \operatorname{Coh}(X)$ such that $c_1(A)/\operatorname{rk} A = \alpha$. Then $\chi(G_1 \otimes A, L_{ij}) = \operatorname{rk} A(g_{ij} - \operatorname{rk} G_1(\alpha, C_{ij})) = 0$ for all i, j. By Lemma 2.1.3, there is a torsion-free β-twisted sheaf G on X such that $\tau(G) = \tau(G_1 \otimes A) - k\tau(\mathbb{C}_x)$ and $\operatorname{Hom}(G, G(K_X + C_{ij}))_0 = 0$ for all i, j. We consider the restriction morphism

$$\phi_{ij} : \operatorname{Def}(X, G) \to \operatorname{Def}(C_{ij}, G_{|C_{ij}}). \tag{2.4}$$

Since $\operatorname{Ext}^2(G, G(-C_{ij}))_0 = 0$, we get $\operatorname{Ext}^2(G, G)_0 = 0$. Thus $\operatorname{Def}(X, G)$ is smooth. We also have the smoothness of $\operatorname{Def}(C_{ij}, G_{|C_{ij}})$, by the local freeness of $G_{|C_{ij}}$. We consider the homomorphism of the tangent spaces

$$\operatorname{Ext}^1_{\mathcal{O}_X}(G, G)_0 \to \operatorname{Ext}^1_{\mathcal{O}_{C_{ij}}}(G_{|C_{ij}}, G_{|C_{ij}})_0. \tag{2.5}$$

Then it is surjective by $\operatorname{Ext}^2(G, G(-C_{ij}))_0 = 0$. Therefore ϕ is submersive. By the equivalence $\varphi : \operatorname{Coh}^\beta(C_{ij}) \to \operatorname{Coh}(C_{ij})$ in (1.142), we have an isomorphism $\operatorname{Def}(C_{ij}, G_{|C_{ij}}) \to \operatorname{Def}(C_{ij}, \varphi(G_{|C_{ij}}))$. Since $\chi(G, L_{ij}) = 0$, $\det(G_{|C_{ij}} \otimes L_{ij}^\vee) = \mathcal{O}_{C_{ij}}(\operatorname{rk} G)$. Then Lemma 2.1.4 implies that G deforms to a β-twisted sheaf such that $G_{|C_{ij}} \cong L_{ij}(1)^{\oplus \operatorname{rk} G}$. Since these conditions are open, there is a locally free β-twisted sheaf G such that $G_{|C_{ij}} \cong L_{ij}(1)^{\oplus \operatorname{rk} G}$ for all i, j. By taking the double dual of G, we get (1).

(2) Note that $L_{ij} = A_{p_i} \otimes \mathcal{O}_{C_{ij}}(-1)$. By Propositions 1.2.23 and 1.1.26, we get the claim. For (3), we use Propositions 1.2.26 and 1.1.26. $\square$

DEFINITION 2.1.5

(1) We set $\operatorname{Per}(X/Y, \{L_{ij}\}) := \mathcal{C}(G)$ and $\operatorname{Per}(X/Y, \{L_{ij}\})^* := \mathcal{C}(G)^*$.

(2) If β is trivial, then we can write $L_{ij} = \mathcal{O}_{C_{ij}}(b_{ij})$. In this case, we set $\operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n) := \operatorname{Per}(X/Y, \{L_{ij}\})$ and $\operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)^* := \operatorname{Per}(X/Y, \{L_{ij}\})^*$, where $\mathbf{b}_i := (b_{i1}, b_{i2}, \dots, b_{it_i})$.

REMARK 2.1.6

If $\mathbf{b}_i = (-1, -1, \dots, -1)$ for all i, then $\operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n) = {}^{-1}\operatorname{Per}(X/Y)$.

DEFINITION 2.1.7

We set

$$\begin{aligned} A_0(\mathbf{b}_i) &:= A_{p_i}, \\ A_0(\mathbf{b}_i)^* &:= A_{p_i} \otimes \omega_{Z_i}. \end{aligned} \tag{2.6}$$

We collect easy facts on $A_0(\mathbf{b}_i)$ and $A_0(\mathbf{b}_i)^*$ which follow from Lemmas 1.2.22 and 1.2.27.

LEMMA 2.1.8

(1) (a) *For* $E = A_0(\mathbf{b}_i)$*, we have*

$$\operatorname{Hom}\bigl(E, \mathcal{O}_{C_{ij}}(b_{ij})\bigr) = \operatorname{Ext}^1\bigl(E, \mathcal{O}_{C_{ij}}(b_{ij})\bigr) = 0, \quad 1 \le j \le t_i, \tag{2.7}$$

and there is an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathbb{C}_x \longrightarrow 0 \tag{2.8}$$

such that F *is a successive extension of* $\mathcal{O}_{C_{ij}}(b_{ij})$ *and* $x \in Z_i$*.*

(b) *Conversely, if* E *satisfies these conditions, then* $E \cong A_0(\mathbf{b}_i)$*.*

(2) (a) *For* $E = A_0(\mathbf{b}_i)^*$*, we have*

$$\operatorname{Hom}\bigl(\mathcal{O}_{C_{ij}}(b_{ij}), E\bigr) = \operatorname{Ext}^1\bigl(\mathcal{O}_{C_{ij}}(b_{ij}), E\bigr) = 0, \quad 1 \le j \le t_i, \tag{2.9}$$

and there is an exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathbb{C}_x \longrightarrow 0 \tag{2.10}$$

such that F *is a successive extension of* $\mathcal{O}_{C_{ij}}(b_{ij})$ *and* $x \in Z_i$*.*

(b) *Conversely, if* E *satisfies these conditions, then* $E \cong A_0(\mathbf{b}_i)^*$*.*

2.2. Moduli spaces of zero-dimensional objects

Let $\pi : X \to Y$ be the minimal resolution of a normal projective surface Y, and let $p_1, p_2, \ldots, p_n$ be the rational double points of Y as in Section 2.1. We set $Z := \bigcup_i Z_i$. Let G be a locally free sheaf on X which is a tilting generator of the category $\mathcal{C} := \mathcal{C}(G)$ in Lemma 1.1.11. For $\alpha \in \operatorname{NS}(X) \otimes \mathbb{Q}$, we define α-twisted semistability as (G, A)-twisted stability in Definition 1.5.3 with $\gamma(A) = (0, \alpha, 0)$, where γ is the homomorphism (0.8). Since $\operatorname{rk} A = 0$, $\gamma(A)$ is nothing but the Mukai vector $v(A)$ of A. In this subsection, we shall study the moduli of α-twisted semistable objects. For brevity, we say that α-twisted semistability is α-semistability.

DEFINITION 2.2.1

For simplicity, we set $X^\alpha := \overline{M}_{\mathcal{O}_X(1)}^{G,\alpha}(\varrho_X)$. We also set $\mathcal{X}^\alpha := \mathcal{M}_{\mathcal{O}_X(1)}^{G,\alpha}(\varrho_X)^{\mathrm{ss}}$.

Zero-semistability means that the inequality (1.122) holds for $A = 0$. Hence every zero-dimensional object is zero-semistable, and we have a natural morphism $\pi_\alpha : X^\alpha \to X^0$. We also see that zero-stable objects correspond to irreducible objects of $\mathcal{C}$.

LEMMA 2.2.2

For a zero-dimensional object E *of* $\mathcal{C}$*, there is a proper subspace* $T(E)$ *of* $\operatorname{Ext}^2(E, E)$ *such that all obstructions for infinitesimal deformations of* E *belong to* $T(E)$*.*

Proof

Let E be a zero-dimensional object of $\mathcal{C}$. We first assume that there is a curve $C \in |K_X|$ such that $C \cap \operatorname{Supp}(E) = \emptyset$. Then $H^0(X, K_X) \to \operatorname{Hom}(E, E(K_X))$ is nontrivial, which implies that the trace map

$$\mathrm{tr} : \mathrm{Ext}^2(E,E) \to H^2(X,\mathcal{O}_X) \tag{2.11}$$

is nontrivial. Since the obstruction for infinitesimal deformations of E lives in $\ker \mathrm{tr}$, $T(E) = \ker \mathrm{tr}$ is a proper subspace of $\mathrm{Ext}^2(E,E)$. For a general case, we use the covering trick. Let D be a very ample divisor on Y such that there is a smooth curve $B \in |2D|$ with $B \cap \pi(\mathrm{Supp}(E) \cup Z) = \emptyset$ and $|K_Y + D|$ contains a curve C with $C \cap \pi(\mathrm{Supp}(E) \cup Z) = \emptyset$. Since π is isomorphic over $Y \setminus \pi(Z)$, we may regard B and C as divisors on X. Let $\phi : \widetilde{Y} \to Y$ be the double covering branched along B, and set $\widetilde{X} = X \times_Y \widetilde{Y}$. We also denote $\widetilde{X} \to X$ by ϕ. Then $|K_{\widetilde{X}}| = |\phi^*(K_X + D)|$ contains $\phi^*(C)$. Since ϕ is étale over $Y \setminus B$, we have a decomposition $\pi^*(E) = E_1 \oplus E_2$, and $\mathrm{Ext}^2(E,E) \to \mathrm{Ext}^2(E_i,E_i)$ are isomorphisms for $i = 1,2$. Under these isomorphisms, $T(E)$ is mapped into $T(E_i)$. Since $\mathrm{tr}_i : \mathrm{Ext}^2(E_i,E_i) \to H^2(\widetilde{X},\mathcal{O}_{\widetilde{X}})$ are nontrivial, $\ker \mathrm{tr}_i$ are proper subspaces of $\mathrm{Ext}^2(E_i,E_i)$. Hence $T(E)$ is a proper subspace of $\mathrm{Ext}^2(E,E)$. □

PROPOSITION 2.2.3

(1) *For a zero-dimensional object E of $\mathcal{C}$, $E \otimes K_X \cong E$. In particular,* $\mathrm{Ext}^2(E, E) \cong \mathrm{Hom}(E,E)^\vee$.

(2) *For a zero-dimensional Mukai vector v, $M^{G,\alpha}_{\mathcal{O}_X(1)}(v)$ is smooth of dimension $\langle v^2 \rangle + 2$.*

Proof

(1) Since $K_X = \pi^*(K_Y)$ and $\dim \pi(\mathrm{Supp}(E)) = 0$, we get $E \otimes K_X \cong E$. (2) For $E \in M^{G,\alpha}_{\mathcal{O}_X(1)}(v)$, we have $\mathrm{Hom}(E,E) = \mathbb{C}$. Then Lemma 2.2.2 implies that $T(E) = 0$. Since $\dim \mathrm{Ext}^1(E,E) = \langle v^2 \rangle + 2$, $M^{G,\alpha}_{\mathcal{O}_X(1)}(v)$ is smooth of dimension $\langle v^2 \rangle + 2$. □

REMARK 2.2.4

There is another argument to prove the smoothness due to Bridgeland [Br1]. We shall use the argument later. So for stable objects, we do not need Lemma 2.2.2, but it is necessary for the study of properly semistable objects (see Proposition 2.2.8).

LEMMA 2.2.5

Assume that $\alpha \in \mathrm{NS}(X) \otimes \mathbb{Q}$ satisfies

$$(\alpha, D) \neq 0 \quad \textit{for all } D \in \mathrm{NS}(X) \textit{ with } (D^2) = -2 \qquad \textit{and} \\ \big(c_1(\mathcal{O}_X(1)), D\big) = 0. \tag{2.12}$$

Then $X^\alpha = M^{G,\alpha}_{\mathcal{O}_X(1)}(\varrho_X)$; that is, X^α consists of α-stable objects.

Proof

Assume that $E \in X^\alpha$ is S-equivalent to $\bigoplus_{i=1}^t E_i$, where E_i are α-stable objects. Then $(\alpha, c_1(E_i)) = 0$, $(c_1(\mathcal{O}_X(1)), c_1(E_i)) = 0$, and $(c_1(E_i)^2) = \langle v(E_i)^2 \rangle \geq -2$ for all i. Since $\langle v(E_i), v(E_j)\rangle \geq 0$ for $E_i \not\cong E_j$ and $\sum_{i,j}\langle v(E_i), v(E_j)\rangle = \langle v(E)^2\rangle = 0$,

(i) $\langle v(E_i)^2\rangle = -2$ for an i, or
(ii) $\langle v(E_i)^2\rangle = 0$ for all i.

By our choice of α, case (i) does not occur. In the second case, we see that $v(E_i) = a_i \varrho_X$, $a_i \in \mathbb{Z}$. Since $\chi(G, E_i) = \operatorname{rk} G a_i > 0$, we have $a_i > 0$. Then $\varrho_X = (\sum_i a_i)\varrho_X$ implies $t = 1$. Therefore E is α-stable. □

LEMMA 2.2.6

Let $\mathcal{E}$ be an object of $\mathbf{D}(X \times X')$ such that $\Phi^{\mathcal{E}^\vee}_{X \to X'} : \mathbf{D}(X) \to \mathbf{D}(X')$ is an equivalence, $\mathcal{E}_{|X \times \{x'\}} \in \mathcal{C}$ for all $x' \in X'$, and $v(\mathcal{E}_{|X \times \{x'\}}) = \varrho_X$. Then every irreducible object of $\mathcal{C}$ appears as a direct summand of the S-equivalence class of $\mathcal{E}_{|X \times \{x'\}}$.

Proof

Let E be an irreducible object of $\mathcal{C}$. If $\operatorname{Supp}(E) \not\subset Z$, then we have a nontrivial morphism $E \to \mathbb{C}_x$, $x \notin Z$. Since $(\mathcal{C})_{|X \setminus Z} = \operatorname{Coh}(X \setminus Z)$, $\mathbb{C}_x$ is an irreducible object. Hence $E \cong \mathbb{C}_x$. Since $\chi(\mathcal{E}_{|X \times \{x'\}}, \mathbb{C}_x) = 0$ and $\Phi^{\mathcal{E}^\vee}_{X \to X'}$ is an equivalence, there is a point $x' \in X'$ such that $\operatorname{Hom}(\mathcal{E}_{|X \times \{x'\}}, \mathbb{C}_x) \neq 0$ or $\operatorname{Hom}(\mathbb{C}_x, \mathcal{E}_{|X \times \{x'\}}) \neq 0$. Since $v(\mathbb{C}_x) = v(\mathcal{E}_{|X \times \{x'\}}) = \varrho_X$, we get $\mathbb{C}_x \cong \mathcal{E}_{|X \times \{x'\}}$. If $\operatorname{Supp}(E) \subset \bigcup_i Z_i$, then we still have $\chi(\mathcal{E}_{|X \times \{x'\}}, E) = 0$, since $\mathcal{E}_{|X \times \{x'\}} = \mathbb{C}_x$, $x \notin Z$, for a point $x' \in X'$. Then we have $\operatorname{Hom}(\mathcal{E}_{|X \times \{x'\}}, E) \neq 0$ or $\operatorname{Hom}(E, \mathcal{E}_{|X \times \{x'\}}) \neq 0$. Therefore our claim holds. □

LEMMA 2.2.7

If α is general, then X^α is irreducible.

Proof

Let X' be a connected component of X^α, and let $\mathcal{E}$ be a universal family on $X \times X'$. By Proposition 2.2.3, $\mathcal{E}_{|X \times \{x'\}} \otimes K_X \cong \mathcal{E}_{|X \times \{x'\}}$ for all $x' \in X'$. Then we have an equivalence $\Phi^{\mathcal{E}^\vee}_{X \to X'} : \mathbf{D}(X) \to \mathbf{D}(X')$. By the same argument as in the proof of Lemma 2.2.6, we see that every $E \in X^\alpha$ belongs to X'. □

PROPOSITION 2.2.8

We have that $\mathcal{X}^0$ is a locally complete intersection stack of dimension 1 *and irreducible. In particular, $\mathcal{X}^0$ is a reduced stack.*

Proof

Let Q be an open subscheme of a perverse quot scheme such that X^0 is a GIT quotient of a suitable $\operatorname{GL}(N)$-action. Then $\mathcal{X}^0$ is the quotient stack $[Q/\operatorname{GL}(N)]$. Let $\mathcal{E}$ be the family of zero-dimensional objects of $\mathcal{C}$ on $Q \times X$. For any point $q \in Q$, we set $n_1 := \dim \operatorname{Hom}(\mathcal{K}_{|\{q\} \times X}, \mathcal{E}_{|\{q\} \times X})$ and $n_2 := \dim T(\mathcal{E}_{|\{q\} \times X})$, where $\mathcal{K}$ is the universal subobject on $Q \times X$. Then an analytic neighborhood of Q is an intersection of n_2 hypersurfaces in $\mathbb{C}^{n_1}$. Hence $\dim Q \geq n_1 - n_2$ and $\dim[Q/\operatorname{GL}(N)] \geq -\chi(\mathcal{E}_{|\{q\} \times X}, \mathcal{E}_{|\{q\} \times X}) + 1 = 1$. We take a general α and set

$Q^u := \{q \in Q | \mathcal{E}_{|\{q\}\times X}$ is not α-semistable$\}$. By the proof of [OY, Proposition 2.16], we see that $\dim[Q^u/\operatorname{GL}(N)] = 0$. Since $[(Q \setminus Q^u)/\operatorname{GL}(N)]$ is the moduli stack of α-stable objects, it is a smooth and irreducible stack of dimension 1. Hence $[Q/\operatorname{GL}(N)]$ is a locally complete intersection stack of dimension 1 and irreducible. In particular $[Q/\operatorname{GL}(N)]$ is a reduced stack. □

LEMMA 2.2.9

Let E be a zero-semistable object with $v(E) = \varrho_X$. Then $\operatorname{Supp}(\pi_*(G^\vee \otimes E))$ is a point of Y.

Proof

For E, we have a decomposition $E = \bigoplus_{i=1}^t E_i$ such that $\operatorname{Supp}(\pi_*(G^\vee \otimes E_i))$, $i = 1, \dots, t$ are distinct t-points of Y. We set $v(E_i) = (0, D_i, a_i)$. Since D_i are contained in the exceptional loci, $0 = \langle v(E)^2 \rangle = \sum_i (D_i^2)$ implies that $(D_i^2) = 0$ for all i. Thus we have $v(E_i) = a_i \varrho_X$ for all i, which implies that $\varrho_X = (\sum_i a_i)\varrho_X$. Since $\chi(G, E_i) > 0$, we have $a_i > 0$. Therefore $t = 1$. □

By Lemma 1.1.21, we get the following.

LEMMA 2.2.10

(1) *We have $\mathbb{C}_x \in \mathcal{C}$ for all $x \in X$. In particular, we have a morphism $\varphi : X \to X^0$ by sending $x \in X$ to the S-equivalence class of $\mathbb{C}_x$;*

(2) *$\varphi(Z_i)$ is a point.*

PROPOSITION 2.2.11

There is an isomorphism $\psi : X^0 \to Y$ such that $\psi \circ \varphi : X \to Y$ coincides with π. In particular, X^0 is a normal projective surface.

Proof

We keep the notation in the proof of Proposition 2.2.8. By Lemma 2.2.9, $\mathcal{F} := \pi_*(G^\vee \otimes \mathcal{E})$ is a flat family of coherent sheaves on Y such that $\operatorname{Supp}(\mathcal{F}_q)$ is a point for every $q \in Q$. Since the characteristic of the base field is zero, we have a morphism $Q \to S^r Y$, where $r = \operatorname{rk} G$ (cf. [F1], [F2]). Since the image is contained in the diagonal Y, we have a morphism $Q \to Y$. Hence we have a morphsim $\psi : X^0 \to Y$. By the construction of φ and ψ, $\pi = \psi \circ \varphi$. Since φ and ψ are projective birational morphisms between irreducible surfaces, φ and ψ are contractions. By using Lemma 2.2.10, we see that ψ is injective. Hence ψ is a finite morphism. Since Y is normal, ψ is an isomorphism. □

LEMMA 2.2.12

(1) *Assume that $p_i \in Y$ corresponds to $\bigoplus_{j=0}^{s_i} E_{ij}^{\oplus a_{ij}}$ via ψ, where E_{ij} are zero-stable objects. Then $\mathbb{C}_x, x \in Z_i$, are S-equivalent to $\bigoplus_{j=0}^{s_i} E_{ij}^{\oplus a_{ij}}$. We also have $E_{ij} \in \mathcal{C}_{p_i}$.*

(2) *Let $E \in \mathcal{C}$ be a zero-twisted stable object. Then E is one of the following:*

$$\mathbb{C}_x \quad (x \in X \setminus Z), \qquad E_{ij} \quad (1 \le i \le n, 0 \le j \le s_i). \tag{2.13}$$

(3) *Every zero-dimensional object is generated by* (2.13).

Proof

By Proposition 2.2.11 and Lemma 2.2.9, (1) holds. We shall apply Lemma 2.2.6 to $\mathcal{E} = \mathcal{O}_\Delta \in \mathbf{D}(X \times X)$. Then (2) is a consequence of (1). It also follows from Lemma 1.1.21(3); (3) follows from (2). □

REMARK 2.2.13

If $\mathcal{C} = {}^{-1}\operatorname{Per}(X/Y)$, then $\pi_*(\mathcal{E})$ is a flat family of coherent sheaves on Y such that $\pi^*(\mathcal{E})_{|\{q\}\times Y}$ is a point sheaf. Then we have a morphism $Q \to Y$. Thus we do not need the reducedness of Q in this case.

Thanks to Lemma 2.2.12, we introduce the following definition.

DEFINITION 2.2.14

(1) E_{ij} (i.e., $\mathbf{I}_{p_i j}$ in Definition 1.1.29), $0 \le j \le s_i$, denotes the zero-stable objects of $\mathcal{C}_{p_i}$;

(2) a_{ij}, $0 \le j \le s_i$, are positive integers such that $\bigoplus_{j=0}^{s_i} E_{ij}^{\oplus a_{ij}}$ and $\mathbb{C}_x$ $(x \in Z_i)$ are S-equivalent.

LEMMA 2.2.15

Assume that $\alpha \in \operatorname{NS}(X) \otimes \mathbb{Q}$ satisfies (2.12). *Then K_{X^α} is the pullback of a line bundle on X^0.*

Proof

Let $\mathcal{E}$ be the universal family on $X^\alpha \times X$. Let $p_S : S \times X \to S$ be the projection. Since X^α is smooth, the base-change theorem implies that $\operatorname{Ext}^i_{p_{X^\alpha}}(\mathcal{E},\mathcal{E})$, $i = 0,1,2$, are locally free sheaves on X^α and compatible with base changes. Since $\operatorname{Ext}^1_{p_{X^\alpha}}(\mathcal{E},\mathcal{E})$ is the tangent bundle of X^α, we show that there is a symplectic form on $\operatorname{Ext}^1_{p_{X^\alpha}}(\mathcal{E},\mathcal{E})$. For any point $y \in Y$, we take a very ample divisor D_2 on Y such that $y \notin D_2$, $|K_Y + D_2|$ contains a divisor D_1 with $y \notin D_1$. We set $U := Y \setminus (D_1 \cup D_2)$. Then U is an open neighborhood of y such that K_Y is trivial over U. Let $\widetilde{D}_i$ be the pullback of D_i to X. Then we have $K_X = \mathcal{O}_X(\widetilde{D}_1 - \widetilde{D}_2)$. We set $V := \pi_\alpha^{-1}(\psi^{-1}(U))$. We shall prove that (i) the alternating pairing

$$\operatorname{Ext}^1_{p_V}(\mathcal{E},\mathcal{E}) \times \operatorname{Ext}^1_{p_V}(\mathcal{E},\mathcal{E}) \to \operatorname{Ext}^2_{p_V}(\mathcal{E},\mathcal{E}) \tag{2.14}$$

is nondegenerate and (ii) $\operatorname{Ext}^2_{p_V}(\mathcal{E},\mathcal{E}) \cong \mathcal{O}_V$. Since $\operatorname{Ext}^1_{p_{X^\alpha}}(\mathcal{E},\mathcal{E})$ is the tangent bundle, this means that $K_V \cong \mathcal{O}_V$. Thus the claim holds.

We first note that there are isomorphisms

$$\operatorname{Ext}^i_{p_V}(\mathcal{E},\mathcal{E}) \cong \operatorname{Ext}^i_{p_V}\big(\mathcal{E},\mathcal{E}(\widetilde{D}_1)\big) \cong \operatorname{Ext}^i_{p_V}\big(\mathcal{E},\mathcal{E}(\widetilde{D}_1 - \widetilde{D}_2)\big), \quad i = 0,1,2, \tag{2.15}$$

which is compatible with the base change. By the Serre duality, the trace map $\mathrm{tr} : \mathrm{Ext}^2(\mathcal{E}_y, \mathcal{E}_y(K_X)) \to H^2(X, K_X)$ is an isomorphism for $y \in V$. Hence (ii) holds, where $\mathcal{E}_y := \mathcal{E}_{|\{y\}\times X}$. By the Serre duality, the pairing $\mathrm{Ext}^1(\mathcal{E}_y, \mathcal{E}_y) \times \mathrm{Ext}^1(\mathcal{E}_y, \mathcal{E}_y(K_X)) \to \mathrm{Ext}^2(\mathcal{E}_y, \mathcal{E}_y(K_X)) \cong H^2(X, K_X)$ is nondegenerate. Combining this with (2.15), we get (i). □

DEFINITION 2.2.16

We set $Z_i^\alpha := \pi_\alpha^{-1}(\bigoplus_j E_{ij}^{\oplus a_{ij}}) = \pi_\alpha^{-1} \circ \psi^{-1}(p_i)$ and $Z^\alpha := \bigcup_i Z_i^\alpha$.

LEMMA 2.2.17 (CF. [OY, LEMMA 2.4])

Assume that $-(\alpha, c_1(E_{ij})) > 0$ *for all* $j > 0$. *Let* F *be a zero-semistable object such that* $v(F) = v(E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j})$, $0 \le b_j \le a_{ij}$.

(1) *If* $v(F) \ne \varrho_X$, *then* F *is* S*-equivalent to* $E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j}$ *with respect to zero-stability.*
(2) *Assume that* F *is* S*-equivalent to* $E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j}$. *Then the following conditions are equivalent:*
   (a) F *is* α*-stable;*
   (b) F *is* α*-semistable;*
   (c) $\mathrm{Hom}(E_{ij}, F) = 0$ *for all* $j > 0$.
(3) *Assume that* F *is* α*-stable. For a nonzero homomorphism* $\phi : F \to E_{ij}$, $j > 0$, ϕ *is surjective and* $F' := \ker \phi$ *is an* α*-stable object.*
(4) *If there is a nontrivial extension*

$$0 \to F \to F'' \to E_{ij} \to 0 \tag{2.16}$$

and $b_k + \delta_{jk} \le a_{ik}$, *then* F'' *is an* α*-stable object, where* $\delta_{jk} = 0, 1$ *according as* $j \ne k$, $j = k$.

Proof

(1) Since $E := F \oplus \bigoplus_{j>0} E_{ij}^{\oplus(a_{ij}-b_j)}$ is a zero-semistable object with $v(E) = \varrho_X$ and $\mathrm{Supp}(\pi_*(G^\vee \otimes E)) = \mathrm{Supp}(\pi_*(G^\vee \otimes F)) \cup \{p_i\}$, Lemma 2.2.9 and Proposition 2.2.11 imply that the S-equivalence class of E corresponds to $p_i \in Y$. Hence E is S-equivalent to $\bigoplus_{j\ge 0} E_{ij}^{\oplus a_{ij}}$, which implies that F is S-equivalent to $E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j}$.

(2) It is sufficient to prove that (c) implies (a). Let $\psi : F \to I$ be a quotient of F. Since I and $\ker \psi$ are zero-dimensional objects, they are zero-semistable. Since $\mathrm{Hom}(E_{ij}, \ker \psi) = 0$ for $j > 0$, (1) implies that E_{i0} is a subobject of $\ker \psi$. Hence $v(I) = \sum_{j>0} b'_j v_{ij}$, which implies that F is α-stable.

(3) Since E_{ij} is irreducible, ϕ is surjective. By (1), $\ker \phi$ also satisfies the assumption of (2). Let $\psi : \ker \phi \to I$ be a quotient object. Since $\mathrm{Hom}(E_{ik}, F) = 0$ for $k > 0$, (2) implies that $\ker \phi$ is α-stable.

(4) Since $v(F) \ne \varrho_X$, (1) implies that F'' satisfies the assumption of (2). If $\mathrm{Hom}(E_{ik}, F'') \ne 0$ for $k > 0$, then $\mathrm{Hom}(E_{ik}, F) = 0$ implies that $k = j$ and we

have a splitting of the exact sequence. Hence $\mathrm{Hom}(E_{ik}, F'') = 0$ for $k > 0$. Then (2) implies the claim. □

COROLLARY 2.2.18

Assume that $-(\alpha, c_1(E_{ij})) > 0$ *for all* $j > 0$. *We set* $v := v(E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j})$, $0 \le b_j \le a_{ij}$, *with* $\langle v^2 \rangle = -2$.

(1) *We have* $\dim \mathrm{Hom}(E, E_{ij}) = \max\{-\langle v, v(E_{ij})\rangle, 0\}$.

(2) *If* $-\langle v, v(E_{ij})\rangle > 0$, *then* $M^{G,\alpha}_{\mathcal{O}_X(1)}(v) \cong M^{G,\alpha}_{\mathcal{O}_X(1)}(w)$, *where* $w = v + \langle v, v(E_{ij})\rangle v(E_{ij})$.

Proof

(1) For $E \in M^{G,\alpha}_{\mathcal{O}_X(1)}(v)$, we set $n := \dim \mathrm{Hom}(E, E_{ij})$. Then we have a surjective morphism $\phi : E \to E_{ij}^{\oplus n}$. Then $F := \ker \phi$ is α-stable. Since $-2 \le \langle v(F)^2 \rangle = \langle v(E)^2 \rangle - 2n(n + \langle v, v(E_{ij})\rangle)$, $n = -\langle v, v(E_{ij})\rangle$ or $n = 0$.

(2) If $-\langle v, v(E_{ij})\rangle > 0$, then $\dim \mathrm{Hom}(E, E_{ij}) = -\langle v, v(E_{ij})\rangle$, $\mathrm{Ext}^p(E, E_{ij}) = 0$, $p > 0$, and we have a morphism $\sigma : M^{G,\alpha}_{\mathcal{O}_X(1)}(v) \to M^{G,\alpha}_{\mathcal{O}_X(1)}(w)$ by sending $E \in M^{G,\alpha}_{\mathcal{O}_X(1)}(v)$ to $F := \ker(E \to E \otimes \mathrm{Hom}(E, E_{ij})^\vee)$. Conversely, for $F \in M^{G,\alpha}_{\mathcal{O}_X(1)}(w)$, $\langle v(F), v(E_{ij})\rangle = -\langle v, v(E_{ij})\rangle > 0$. Hence $\mathrm{Hom}(F, E_{ij}) = 0$, which implies that $\dim \mathrm{Ext}^1(E_{ij}, F) = \langle v(F), v(E_{ij})\rangle$ and the universal extension gives an α-stable object E with $v(E) = v$. Therefore we also have the inverse of σ. □

We come to the main result of this subsection.

THEOREM 2.2.19 (CF. [OY, THEOREM 0.1])

(1) *We have* $X^0 \cong Y$, *and the singular points* $p_1, p_2, \dots, p_n$ *of* X^0 *correspond to the* S*-equivalence classes of properly zero-twisted semistable objects.*

(2) *Assume that* α *satisfies that* $(\alpha, D) \ne 0$ *for all* $D \in \mathrm{NS}(X)$ *with* $(D^2) = -2$ *and* $(c_1(\mathcal{O}_X(1)), D) = 0$. *Then* $X^\alpha = M^{G,\alpha}_{\mathcal{O}_X(1)}(\varrho_X)$; *that is, the* α*-semistability and the* α*-stability are equivalent. In particular,* $\pi_\alpha : X^\alpha \to X^0$ *is the minimal resolution of the singularities.*

(3) *Let* $\bigoplus_{j=0}^{s_i} E_{ij}^{\oplus a_{ij}}$ *be the* S*-equivalence class corresponding to* p_i *as in Definition 2.2.14. Then the matrix* $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\ge 0}$ *is of affine type* $\tilde{A}, \tilde{D}, \tilde{E}$. *Assume that* $a_{i0} = 1$ *(cf. Lemma A.1.1(1)). Then the singularity of* X^0 *at* p_i *is a rational double point of type* A, D, E *according to the type of the matrix* $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\ge 1}$.

(4) *We have* $s_i = t_i$, *that is*

$$\#\{\textit{irreducible objects of } \mathcal{C}_{p_i}\} = \#\{\textit{irreducible components of } Z_i\} + 1.$$

Proof

(1) By Proposition 2.2.11, $X^0 \cong Y$. Since $\varphi : X \to X^0$ is surjective, $y \in Y$ corresponds to the S-equivalence class of $\mathbb{C}_x$, $x \in \pi^{-1}(y)$. By Lemma 2.2.10, $\mathbb{C}_x$, $x \in \pi^{-1}(p_i)$, is not irreducible. Hence p_i corresponds to a properly zero-semistable object. For a smooth point $y \in Y$, $\mathbb{C}_x$, $x \in \pi^{-1}(y)$, is irreducible. Therefore the

second claim also holds. The proof of (2) is a consequence of Proposition 2.2.3 and Lemma 2.2.15.

(3) We note that

$$
\begin{aligned}
\chi(G, E_{ij}) &> 0, \\
\big\langle \varrho_X, v(E_{ij}) \big\rangle &= 0, \\
\big\langle v(E_{ij}), v(E_{ij}) \big\rangle &= -2, \\
\big\langle v(E_{ij}), v(E_{kl}) \big\rangle &\geq 0 \quad (E_{ij} \neq E_{kl}).
\end{aligned}
\tag{2.17}
$$

As we see in Example A.1.3 in the appendix, we can apply Lemma A.1.1(1) to our situation. Hence the matrix $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\geq 0}$ is of affine type $\tilde{A}, \tilde{D}, \tilde{E}$. Then we may assume that $a_{i0} = 1$ for all i. By Lemma A.1.1(2), we can choose an α with $-\langle v(E_{ij}), \alpha\rangle > 0$ for all $j > 0$. Let $\mathcal{E}^\alpha$ be the universal family on $X \times X^\alpha$. The claim (3) is a consequence of the following lemma. The claim (4) is a consequence of (3) and the uniqueness of the minimal resolution. Since the first part of (3) implies that the rank of $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\geq 0}$ is s_i, (4) also follows from $\sum_{j=0}^{s_i} \mathbb{Z}E_{ij} = (\sum_{j=1}^{t_i} \mathbb{Z}\mathcal{O}_{C_{ij}}) + \mathbb{Z}\mathbb{C}_x$, where we identify $\mathrm{Coh}^\beta(Z_i)$ with $\mathrm{Coh}(Z_i)$ via (1.142). □

REMARK 2.2.20

For α satisfying $-\langle v(E_{ij}), \alpha\rangle > 0$ for all $j > 0$, Lemma 2.2.22 also shows that $\pi^\alpha : X^\alpha \to X^0$ is the minimal resolution.

DEFINITION 2.2.21

From now on, we assume that $a_{i0} = 1$ for all i.

LEMMA 2.2.22

Let $\mathcal{E}$ be a universal family on $X \times X^\alpha$. Assume that α satisfies $-\langle v(E_{ij}), \alpha\rangle > 0$ for all $j > 0$.

(1) *We set*

$$
C_{ij}^\alpha := \big\{ x^\alpha \in X^\alpha \bigm| \mathrm{Hom}(\mathcal{E}_{|X\times\{x^\alpha\}}, E_{ij}) \neq 0 \big\}, \quad j > 0.
\tag{2.18}
$$

Then C_{ij}^α is a smooth rational curve.

(2) *We have*

$$
Z_i^\alpha = \big\{ x^\alpha \in X^\alpha \bigm| \mathrm{Hom}(E_{i0}, \mathcal{E}_{|X\times\{x^\alpha\}}) \neq 0 \big\} = \bigcup_j C_{ij}^\alpha .
\tag{2.19}
$$

(3) *We have that $\bigcup_j C_{ij}^\alpha$ is simple normal crossing and $(C_{ij}^\alpha, C_{ik}^\alpha) = \langle v(E_{ij}), v(E_{ik})\rangle$.*

Proof

(1) By our choice of α, $\mathrm{Hom}(E_{ij}, \mathcal{E}_{|X\times\{x^\alpha\}}) = 0$ for all $x^\alpha \in X^\alpha$. If $C_{ij}^\alpha = \emptyset$, then $\chi(E_{ij}, \mathcal{E}_{|X\times\{x^\alpha\}}) = 0$ implies that $\mathrm{Hom}(\mathcal{E}_{|X\times\{x^\alpha\}}, E_{ij}) = \mathrm{Ext}^1(\mathcal{E}_{|X\times\{x^\alpha\}}, E_{ij}) = 0$. Then $\Phi_{X\to X^\alpha}^{\mathcal{E}^\vee}(E_{ij}) = 0$, which is a contradiction. Therefore $C_{ij}^\alpha \neq \emptyset$. In order to

prove the smoothness, we consider the moduli space of coherent systems

$$N\big(\varrho_X, v(E_{ij})\big) := \big\{(E,V) \bigm| E \in X^\alpha, V \subset \mathrm{Hom}(E, E_{ij}), \dim_{\mathbb{C}} V = 1\big\}. \tag{2.20}$$

We have a natural projection $\iota : N(\varrho_X, v(E_{ij})) \to X^\alpha$ whose image is C_{ij}^α. For $(E,V) \in N(\varrho_X, v(E_{ij}))$, we have a homomorphism $\xi : E \to E_{ij} \otimes V^\vee$. The Zariski tangent space at (E,V) is $\mathrm{Hom}(E, E \to E_{ij} \otimes V^\vee)$. By Lemma 2.2.17(3), ξ is surjective and $\ker \xi \in M_{\mathcal{O}_X(1)}^{G,\alpha}(\varrho_X - v(E_{ij}))$. In particular, $\mathrm{Hom}(E, E \to E_{ij} \otimes V^\vee) \cong \mathrm{Ext}^1(E, \ker \xi)$. Conversely, for $F \in M_{\mathcal{O}_X(1)}^{G,\alpha}(\varrho_X - v(E_{ij}))$ and a nontrivial extension

$$0 \to F \to E \to E_{ij} \to 0, \tag{2.21}$$

Lemma 2.2.17(4) implies that $E \in X^\alpha$ and $E \to E_{ij}$ defines an element of $N(\varrho_X, v(E_{ij}))$. By Corollary 2.2.18(1) and our choice of α, $\mathrm{Hom}(F, E_{ij}) = \mathrm{Hom}(E_{ij}, F) = 0$. Hence $\dim \mathrm{Ext}^1(E_{ij}, F) = 2$. Since $M_{\mathcal{O}_X(1)}^{G,\alpha}(\varrho_X - v(E_{ij}))$ is a reduced one point, we see that $N(\varrho_X, v(E_{ij}))$ is isomorphic to $\mathbb{P}^1$. We show that $\iota : N(\varrho_X, v(E_{ij})) \to X^\alpha$ is a closed immersion. For $(E,V) \in N(\varrho_X, v(E_{ij}))$, $\dim \mathrm{Hom}(E, E_{ij}) = \dim \mathrm{Hom}(\ker \xi, E_{ij}) + 1 = 1$. Hence ι is injective. We also see that $\iota_* : \mathrm{Ext}^1(E, \ker \xi) \to \mathrm{Ext}^1(E, E)$ is injective. Therefore ι is a closed immersion.

(2) By our choice of α, $\mathrm{Hom}(E_{i0}, \mathcal{E}_{|X \times \{x^\alpha\}}) \neq 0$ for $x^\alpha \in Z_i^\alpha$. Conversely, if $\mathrm{Hom}(E_{i0}, \mathcal{E}_{|X \times \{x^\alpha\}}) \neq 0$, then Lemma 2.2.9 implies that $\mathrm{Supp}(\pi_*(G^\vee \otimes \mathcal{E}_{|X \times \{x^\alpha\}})) = \{p_i\}$. Since $\mathrm{Supp}(\pi_*(G^\vee \otimes \mathcal{E}_{|X \times \{x^\alpha\}}))$ depends only on the S-equivalence class of $\mathcal{E}_{|X \times \{x^\alpha\}}$, we have $\psi(\pi_\alpha(x^\alpha)) = p_i$. Thus $x^\alpha \in Z_i^\alpha$. Therefore we have the first equality. By the choice of α, we also get $Z_i^\alpha \subset \bigcup_j C_{ij}^\alpha$. If $\mathrm{Hom}(\mathcal{E}_{|X \times \{x^\alpha\}}, E_{ij}) \neq 0$, $j > 0$, then we see that $\mathrm{Supp}(\pi_*(G^\vee \otimes \mathcal{E}_{|X \times \{x^\alpha\}})) = \{p_i\}$, which implies that $x^\alpha \in Z_i^\alpha$. Thus the second claim also holds.

(3) Since $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k \geq 1}$ is of ADE-type, by using Corollary 2.2.18, we can show that $M_{\mathcal{O}_X(1)}^{G,\alpha}(v) \cong M_{\mathcal{O}_X(1)}^{G,\alpha}(v(E_{i0}))$ for $v = v(E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j})$, $0 \leq b_j \leq a_{ij}$, with $\langle v^2 \rangle = -2$. In particular, they are nonempty. Then by similar arguments in [OY, Proposition 2.9], we can also show that $\bigcup_j C_{ij}^\alpha$ is simple normal crossing and $(C_{ij}^\alpha, C_{ik}^\alpha) = \langle v(E_{ij}), v(E_{ik})\rangle$. (For another proof, see Corollary 2.3.12.) $\square$

2.3. Fourier–Mukai transforms on X

We keep the notation in Section 2.2. Assume that X^α consists of α-stable objects. Let $\mathcal{E}^\alpha$ be a universal family on $X \times X^\alpha$. We have an equivalence $\Phi_{X \to X^\alpha}^{(\mathcal{E}^\alpha)^\vee} : \mathbf{D}(X) \to \mathbf{D}(X^\alpha)$. If $\mathcal{F}^\alpha$ is another universal family, then we see that

$$\Phi_{X \to X^\alpha}^{(\mathcal{E}^\alpha)^\vee} \circ \Phi_{X^\alpha \to X}^{\mathcal{F}^\alpha} = \Phi_{X^\alpha \to X^\alpha}^{\mathcal{O}_\Delta(L)}[-2], \quad L \in \mathrm{Pic}(X^\alpha). \tag{2.22}$$

Let Γ^α be the closure of the graph of the rational map $\pi_\alpha^{-1} \circ \pi$:

$$\begin{array}{ccc} \Gamma^\alpha & \longrightarrow & X^\alpha \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_\alpha} \\ X & \xrightarrow[\pi]{} & Y \end{array} \tag{2.23}$$

LEMMA 2.3.1

(1) *We may assume that* $\mathcal{E}^\alpha_{|X\times(X^\alpha\setminus Z^\alpha)} \cong \mathcal{O}_{\Gamma^\alpha|X\times(X^\alpha\setminus Z^\alpha)}$.

(2) $\mathcal{E}^\alpha$ *is characterized by* $\mathcal{E}^\alpha_{|X\times(X^\alpha\setminus Z^\alpha)}$ *and* $\det\Phi^{(\mathcal{E}^\alpha)^\vee}_{X\to X^\alpha}(G)$.

Proof

(1) We note that $\mathcal{E}^\alpha_{|X\times(X^\alpha\setminus Z^\alpha)} \cong (\mathcal{O}_{\Gamma^\alpha}\otimes p^*_{X^\alpha}(L))_{|X\times(X^\alpha\setminus Z^\alpha)}$, where $L\in \operatorname{Pic}(X^\alpha\setminus Z^\alpha)$. We also denote an extension of L to X^α by L. Then $\mathcal{E}^\alpha\otimes p^*_{X^\alpha}(L^\vee)$ is a desired universal family.

(2) Assume that $\mathcal{E}^\alpha_{|X\times(X^\alpha\setminus Z^\alpha)} \cong (\mathcal{E}^\alpha\otimes p^*_{X^\alpha}(L))_{|X\times(X^\alpha\setminus Z^\alpha)}$ and $\det\Phi^{(\mathcal{E}^\alpha)^\vee}_{X\to X^\alpha}(G)\cong\det\Phi^{(\mathcal{E}^\alpha\otimes p^*_{X^\alpha}(L))^\vee}_{X\to X^\alpha}(G)$. Then $L_{|X^\alpha\setminus Z^\alpha}\cong\mathcal{O}_{X^\alpha\setminus Z^\alpha}$ and $L^{\otimes \operatorname{rk} G}\cong \mathcal{O}_{X^\alpha}$. In order to prove $L\cong\mathcal{O}_{X^\alpha}$, it is sufficient to prove the injectivity of the restriction map

$$r:\operatorname{Pic}(X^\alpha)\to\operatorname{Pic}(X^\alpha\setminus Z^\alpha)\times\prod_{i,j}\operatorname{Pic}(C^\alpha_{ij}). \tag{2.24}$$

If $L_{|X^\alpha\setminus Z^\alpha}\cong\mathcal{O}_{X^\alpha\setminus Z^\alpha}$, then we can write $L=\mathcal{O}_X(\sum_{i,j} r_{ij}C^\alpha_{ij})$. Since the intersection matrix $((C^\alpha_{ij},C^\alpha_{ik}))_{j,k}$ is negative definite, $\deg(L_{|C^\alpha_{ij}})=\sum_k r_{ik}(C^\alpha_{ik},C^\alpha_{ij})=0$ for all i,j implies that $r_{ij}=0$ for all i,j. Thus r is injective. □

DEFINITION 2.3.2

We set $\Lambda^\alpha:=\Phi^{(\mathcal{E}^\alpha)^\vee}_{X\to X^\alpha}[2]$.

LEMMA 2.3.3

Let $\mathcal{O}_X(C)$ *and* $\mathcal{O}_{X^\alpha}(C)$ *be the pullbacks of a line bundle* $\mathcal{O}_Y(C)$ *on* Y. *Then*

$$\Lambda^\alpha\circ\big(\mathcal{O}_X(C)\otimes\bullet\big)=\big(\mathcal{O}_{X^\alpha}(C)\otimes\bullet\big)\circ\Lambda^\alpha.$$

Proof

Let D be an effective divisor on X such that $D\cap Z=\emptyset$. It is sufficient to prove that

$$\mathcal{E}^\alpha\otimes\big(\mathcal{O}_X(-D)\boxtimes\mathcal{O}_{X^\alpha}(D)\big)\cong\mathcal{E}^\alpha. \tag{2.25}$$

We note that $\mathcal{E}^\alpha\cong\mathcal{O}_{\Gamma^\alpha}$ over $X^\alpha\setminus Z^\alpha$. Obviously the claim holds over $X^\alpha\setminus Z^\alpha$. By Lemma 2.3.1, we shall show that $\det\Lambda^\alpha(G(D))\cong\det(\Lambda^\alpha(G)(D))$. We have an exact triangle

$$(\mathcal{E}^\alpha)^\vee\to(\mathcal{E}^\alpha)^\vee(D)\to(\mathcal{E}^\alpha)^\vee_{|D}(D)\to(\mathcal{E}^\alpha)^\vee[1]. \tag{2.26}$$

Since $(\mathcal{E}^\alpha)^\vee_{|D}(D)\cong\mathcal{O}_{\Delta|D}(D)[-2]$, we have an exact triangle

$$\Lambda^\alpha(G)\to\Lambda^\alpha\big(G(D)\big)\to G_{|D}(D)\to\Lambda^\alpha(G)[1]. \tag{2.27}$$

Hence we get $\det\Lambda^\alpha(G(D))\cong(\det\Lambda^\alpha(G))((\operatorname{rk}G)D)\cong\det(\Lambda^\alpha(G)(D))$. □

PROPOSITION 2.3.4

(1) *We set* $G^\alpha:=\Lambda^\alpha(G)$. *Then* G^α *is a locally free sheaf, and* $\mathbf{R}\pi_{\alpha*}(G^{\alpha\vee}\otimes G^\alpha)=\pi_{\alpha*}(G^{\alpha\vee}\otimes G^\alpha)$.

(2) *For the zero-stable objects* E_{ij}, $\Lambda^\alpha(E_{ij})[k]$ *is a sheaf, where* $k = -1$ *or* 0 *according as* $(\alpha, c_1(E_{ij})) < 0$ *or* $(\alpha, c_1(E_{ij})) > 0$.

(3) *We set* $\mathcal{A}^\alpha := \pi_{\alpha *}(G^{\alpha\vee} \otimes G^\alpha)$. *Then* $\mathcal{A}^\alpha$ *is a reflexive sheaf on* Y. *Under the identification* $X^\alpha \setminus Z^\alpha \cong X \setminus Z$, $G^\alpha_{|X^\alpha \setminus Z^\alpha}$ *corresponds to* $G_{|X \setminus Z}$. *Hence we have an isomorphism* $\mathcal{A} \cong \mathcal{A}^\alpha$.

(4) *We identify* $\mathrm{Coh}_{\mathcal{A}}(Y)$ *with* $\mathrm{Coh}_{\mathcal{A}^\alpha}(Y)$ *via* $\mathcal{A} \cong \mathcal{A}^\alpha$. *Then we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Lambda^\alpha} & \Lambda^\alpha(\mathcal{C}) \\ {\scriptstyle \mathbf{R}\pi_* \mathcal{H}om_{\mathcal{O}_X}(G,\ )} \downarrow & & \downarrow {\scriptstyle \mathbf{R}\pi_{\alpha *} \mathcal{H}om_{\mathcal{O}_{X^\alpha}}(G^\alpha,\ )} \\ \mathrm{Coh}_{\mathcal{A}}(Y) & = & \mathrm{Coh}_{\mathcal{A}^\alpha}(Y) \end{array} \tag{2.28}$$

In particular, G^α *gives a local projective generator of* $\Lambda^\alpha(\mathcal{C})$.

(5) *We set*

$$\begin{aligned} S^\alpha &:= \{\Lambda^\alpha(E_{ij})[-1] \mid i, j\} \cap \mathrm{Coh}(X^\alpha), \\ \mathcal{T}^\alpha &:= \{E \in \mathrm{Coh}(X^\alpha) \mid \mathrm{Hom}(E, c) = 0, c \in S^\alpha\}, \\ \mathcal{S}^\alpha &:= \{E \in \mathrm{Coh}(X^\alpha) \mid E \textit{ is a successive extension} \\ &\qquad \textit{of subsheaves of } c \in S^\alpha\}. \end{aligned} \tag{2.29}$$

Then $(\mathcal{T}^\alpha, \mathcal{S}^\alpha)$ *is a torsion pair of* $\mathrm{Coh}(X^\alpha)$, *and* $\Lambda^\alpha(\mathcal{C})$ *is the tilting of* $\mathrm{Coh}(X^\alpha)$ *with respect to* $(\mathcal{T}^\alpha, \mathcal{S}^\alpha)$.

(6) *Let* G' *be a local projective generator of* $\mathcal{C}$. *For* $\mathbf{e} \in K(X)_{\mathrm{top}}$, Λ^α *induces an isomorphism* $\mathcal{M}^{G'}_{\mathcal{O}_X(1)}(\mathbf{e})^{ss} \to \mathcal{M}^{\Lambda^\alpha(G')}_{\mathcal{O}_{X^\alpha}(1)}(\Lambda^\alpha(\mathbf{e}))^{ss}$.

Proof

(1) We note that $\mathrm{Hom}(\mathcal{E}^\alpha_{|X \times \{x^\alpha\}}, G[i]) \cong \mathrm{Hom}(G, \mathcal{E}^\alpha_{|X \times \{x^\alpha\}}[2-i])^\vee = 0$ for $i \neq 2$ and $x^\alpha \in X^\alpha$. By the base-change theorem, G^α is a locally free sheaf. By using Lemma 2.3.3 and the ampleness of $\mathcal{O}_Y(1)$, we have

$$\begin{aligned} &H^0(Y, R^i\pi_{\alpha *}(G^{\alpha\vee} \otimes G^\alpha)(n)) \\ &\quad = \mathrm{Hom}(\Lambda^\alpha(G), \Lambda^\alpha(G)(n)[i]) \\ &\quad = \mathrm{Hom}(\Lambda^\alpha(G), \Lambda^\alpha(G(n))[i]) \\ &\quad = \mathrm{Hom}(G, G(n)[i]) = H^0(Y, R^i\pi_*(G^\vee \otimes G)(n)) = 0 \end{aligned} \tag{2.30}$$

for $n \gg 0$ and $i \neq 0$. Therefore $R^i\pi_*(G^{\alpha\vee} \otimes G^\alpha) = 0$, $i \neq 0$, and the claim holds.

(2) If $(\alpha, c_1(E_{ij})) < 0$, then $\mathrm{Hom}(\mathcal{E}^\alpha_{|X \times \{x^\alpha\}}, E_{ij}[2]) \cong \mathrm{Hom}(E_{ij}, \mathcal{E}^\alpha_{|X \times \{x^\alpha\}})^\vee = 0$ for $x^\alpha \in X^\alpha$. Since $\mathrm{Hom}(\mathcal{E}^\alpha_{X \times |\{x^\alpha\}}, E_{ij}) = 0$ if $x^\alpha \notin Z^\alpha_i$, we see that $\Lambda^\alpha(E_{ij})[-1]$ is a torsion sheaf whose support is contained in Z^α_i.

If $(\alpha, c_1(E_{ij})) > 0$, then $\mathrm{Hom}(\mathcal{E}^\alpha_{|X \times \{x^\alpha\}}, E_{ij}) = 0$ for $x^\alpha \in X^\alpha$. Since $\mathrm{Hom}(\mathcal{E}^\alpha_{|X \times \{x^\alpha\}}, E_{ij}[2]) = 0$ if $x^\alpha \notin Z^\alpha_i$, we see that $\Lambda^\alpha(E_{ij})$ is a torsion sheaf whose support is contained in Z^α_i.

(3) By claim (1) and [E, Lemma 2.1], $\mathcal{A}^\alpha$ is a reflexive sheaf. Since $\mathcal{E}^\alpha$ is isomorphic to $\mathcal{O}_{\Gamma^\alpha}$ over $X^\alpha \setminus Z^\alpha$, we get $\Lambda^\alpha(G)_{|X^\alpha \setminus Z^\alpha} \cong \pi_\alpha^{-1} \circ \pi(G_{|X \setminus Z})$. Hence the second claim also follows.

(4) For $E \in \mathcal{C}$, we first prove that $\mathbf{R}\pi_{\alpha *}(G^{\alpha \vee} \otimes \Lambda^\alpha(E)) \in \mathrm{Coh}_{\mathcal{A}^\alpha}(Y)$. As in the proof of (1), we have

$$\begin{aligned} H^i\big(Y, \mathbf{R}\pi_{\alpha *}\big(G^{\alpha\vee} \otimes \Lambda^\alpha(E)\big)(n)\big) &= \mathrm{Hom}\big(G^\alpha, \Lambda^\alpha(E)(n)[i]\big) \\ &= \mathrm{Hom}\big(G, E(n)[i]\big) = 0 \end{aligned} \tag{2.31}$$

for $i \neq 0$, $n \gg 0$. Therefore $H^i(\mathbf{R}\pi_{\alpha *}(G^{\alpha\vee} \otimes \Lambda^\alpha(E))) = 0$ for $i \neq 0$. For $E \in \mathcal{C}$, we take an exact sequence

$$G(-m)^{\oplus M} \to G(-n)^{\oplus N} \to E \to 0. \tag{2.32}$$

Then we have a diagram

$$\begin{array}{ccccccc} \mathcal{A}(-m)^{\oplus M} & \longrightarrow & \mathcal{A}(-n)^{\oplus N} & \longrightarrow & \pi_*(G^\vee \otimes E) & \longrightarrow & 0 \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} & & & & \\ \mathcal{A}^\alpha(-m)^{\oplus M} & \longrightarrow & \mathcal{A}^\alpha(-n)^{\oplus N} & \longrightarrow & \pi_{\alpha *}\big(G^{\alpha\vee} \otimes \Lambda^\alpha(E)\big) & \longrightarrow & 0 \end{array} \tag{2.33}$$

which is commutative over $Y^* := Y \setminus \{p_1, p_2, \dots, p_n\}$, where ϕ and ψ are the isomorphisms induced by $\mathcal{A} \cong \mathcal{A}^\alpha$. Let $j : Y^* \hookrightarrow Y$ be the inclusion. Since $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}, \mathcal{A}^\alpha) \to j_* j^* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}, \mathcal{A}^\alpha)$ is an isomorphism, (2.33) is commutative, which induces an isomorphism $\xi : \pi_*(G^\vee \otimes E) \to \pi_{\alpha *}(G^{\alpha\vee} \otimes \Lambda^\alpha(E))$. It is easy to see that the construction of ξ is functorial and defines an isomorphism $\mathbf{R}\pi_* \mathcal{H}om_{\mathcal{O}_X}(G, \ ) \cong \mathbf{R}\pi_{\alpha *} \mathcal{H}om_{\mathcal{O}_{X^\alpha}}(G^\alpha, \ ) \circ \Lambda^\alpha$.

(5) Since Λ^α is an equivalence, $\Lambda^\alpha(E_{ij})$ are irreducible objects of $\Lambda^\alpha(\mathcal{C})$. By Propositions 1.1.13 and 1.1.26, we get the claim.

(6) We note that the proof of (1) implies that $\Lambda^\alpha(G')$ is a local projective generator of $\Lambda^\alpha(\mathcal{C})$. By Lemma 2.3.3, $\chi(G', E(n)) = \chi(\Lambda^\alpha(G'), \Lambda^\alpha(E)(n))$. Hence the claim holds. □

REMARK 2.3.5

If $\mathcal{C} = {}^{-1}\mathrm{Per}(X/Y)$, then $\mathcal{O}_X \in {}^{-1}\mathrm{Per}(X/Y)$ and $\Lambda^\alpha(\mathcal{O}_X)$ is a line bundle on X^α. Hence we may assume that $\Lambda^\alpha(\mathcal{O}_X) \cong \mathcal{O}_{X^\alpha}$. Then $\mathrm{Hom}(\mathcal{O}_{X^\alpha}, \Lambda^\alpha(\mathcal{O}_{C_{ij}}(-1))[n]) = 0$ for all n. Thus $\Lambda^\alpha(\mathcal{O}_{C_{ij}}(-1))[n]$ is a successive extension of $\mathcal{O}_{C_{ik}}(-1)$. We also get $\mathrm{Hom}(\mathcal{O}_{X^\alpha}, \Lambda^\alpha(\mathcal{O}_{Z_i})) = \mathbb{C}$ and $\mathrm{Hom}(\mathcal{O}_{X^\alpha}, \Lambda^\alpha(\mathcal{O}_{Z_i})[n]) = 0$ for $n \neq 0$.

Since Λ^α is an equivalence with $\Lambda^\alpha(\varrho_X) = \varrho_{X^\alpha}$, we have the following corollary.

COROLLARY 2.3.6

For a general α, the equivalence

$$\Lambda^\alpha : \mathcal{C} \to \Lambda^\alpha(\mathcal{C})$$

induces an isomorphism

$$\Lambda^\alpha : \mathcal{M}^{G,\beta}_{\mathcal{O}_X(1)}(\varrho_X)^{\mathrm{ss}} \to \mathcal{M}^{G^\alpha,\Lambda^\alpha(\beta)}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha})^{\mathrm{ss}},$$

where $\beta \in \varrho_X^\perp$.

2.3.1. Wall and chambers

For the zero-stable objects E_{ij} in Theorem 2.2.19, we set $v_{ij} := v(E_{ij})$. By Lemma 2.2.6, $\{E_{ij}\}$ is the set of irreducible objects E with $\mathrm{Supp}(E) \subset \bigcup_i Z_i$. Let $\mathfrak{g}_i$ be the finite-dimensional Lie algebra whose Cartan matrix is $(-\langle v_{ij}, v_{ik}\rangle_{j,k\geq 1})$ and

$$R_i := \Big\{u = \sum_{j>0} n'_{ij} v_{ij} \;\Big|\; \langle u^2\rangle = -2, n'_{ij} \geq 0\Big\}. \tag{2.34}$$

Then R_i is identified with the set of positive roots of $\mathfrak{g}_i$. In particular, R_i is a finite set.

DEFINITION 2.3.7

Let v be the Mukai vector of a zero-dimensional object E, which is primitive. For $u \in \bigcup_i R_i$, we define the wall as

$$W_u := \Big\{\alpha \in \mathrm{NS}(X) \otimes \mathbb{R} \;\Big|\; \frac{\langle u, \alpha\rangle}{\langle u, v(G)\rangle} = \frac{\langle v, \alpha\rangle}{\langle v, v(G)\rangle}\Big\}. \tag{2.35}$$

A connected component of $\mathrm{NS}(X) \otimes \mathbb{R} \setminus \bigcup_u W_u$ is called a chamber.

REMARK 2.3.8

If $v = \varrho_X$, then $W_u = u^\perp$.

LEMMA 2.3.9

Let v be the Mukai vector of a zero-dimensional object E, which is primitive.

(1) $\overline{M}^{G,\alpha}_{\mathcal{O}_X(1)}(v)$ *consists of α-twisted stable objects if and only if* $\alpha \notin \bigcup_u W_u$. *We say that α is general with respect to v.*

(2) *If α is general with respect to v, then the virtual Hodge number of* $M^{G,\alpha}_{\mathcal{O}_X(1)}(v)$ *does not depend on the choice of α. In particular, the nonemptyness of* $M^{G,\alpha}_{\mathcal{O}_X(1)}(v)$ *does not depend on the choice of α.*

Proof

(1) For $E \in \overline{M}^{G,\alpha}_{\mathcal{O}_X(1)}(v)$, we assume that E is S-equivalent to $\bigoplus_{i=1}^k E_i$, where E_i are α-stable. If $\langle v(E_i)^2\rangle = 0$ for all i, then $v(E_i) \in \mathbb{Z}_{>0}\varrho_X$. Hence $v = \sum_{i=1}^k v(E_i)$ is not primitive. Therefore we may assume that $\langle v(E_1)^2\rangle = -2$. By the α-stability of E_1, $\mathrm{Supp}(E_1) \subset Z_i$ for an i. Since E_1 is generated by $\{E_{ij} \mid 0 \leq j \leq s_i\}$, $v(E_1) \in \bigoplus_{j=0}^{s_i} \mathbb{Z}_{\geq 0} v_{ij}$. Then we see that $v(E_1) \in \pm R_i + \mathbb{Z}\varrho_X$. Therefore the claim holds.

(2) The proof is similar to that of [Y3, Proposition 2.6]. $\square$

LEMMA 2.3.10

(1) *Let* $w_1 := v_{i0} + \sum_{j=1}^{s_i} n_{ij} v_{ij}$, $n_{ij} \geq 0$, *be a Mukai vector with* $\langle w_1^2 \rangle \geq -2$. *Then there is an* α*-twisted stable object* E *with* $v(E) = w_1$ *for a general* α.

(2) *Let* $w_2 \in R_i$ *be a nonzero Mukai vector. Then there is an* α*-twisted stable object* E *with* $v(E) = w_2$ *for a general* α.

Proof

(1) By Proposition 2.3.16 below and Corollary 2.3.6, we may assume that $\mathcal{C} = \mathrm{Per}(X'/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$. The claim follows from Lemma 2.3.19 below and Lemma 2.3.9(2). Instead of using Lemma 2.3.19, we can also use Corollary 2.2.18 to show the claim for a special α.

(2) We set $w_1 := \sum_{j=0}^{s_i} a_{ij} v_{ij} - w_2$. Then w_1 is the Mukai vector in (1). We can take a general element $\alpha \in \mathrm{NS}(X) \otimes \mathbb{Q}$ such that $\langle \alpha, w_1 \rangle = 0$. Then α is general with respect to w_1 and we have a α-twisted stable object E with $v(E) = w_1$. We consider $X^{\alpha'}$ such that α' is sufficiently close to α and $\langle \alpha', v(E) \rangle > 0$. Since $\Lambda^{\alpha'}$ is an equivalence, there is a morphism $\phi : E \to \mathcal{E}^{\alpha'}_{|\{y\} \times X}$, where $y \in X^{\alpha'}$. By our choice of α, $\operatorname{coker} \phi$ is an α-twisted stable object with $v(\operatorname{coker} \phi) = w_2$. Then the claim follows from Lemma 2.3.9(2). □

2.3.2. A special chamber

We take $\alpha \in \varrho_X^{\perp}$ with $-\langle v(E_{ij}), \alpha \rangle > 0$, $j > 0$.

LEMMA 2.3.11

For $j > 0$, $\Lambda^\alpha(E_{ij})[-1]$, $j > 0$, *is a line bundle on* C_{ij}^α. *We set* $\Lambda^\alpha(E_{ij}) := \mathcal{O}_{C_{ij}^\alpha}(b_{ij}^\alpha)[1]$.

Proof

We note that $\Lambda^\alpha(E_{ij}) \overset{\mathbf{L}}{\otimes} \mathbb{C}_{x^\alpha} = \mathbf{R}\mathrm{Hom}(\mathcal{E}^\alpha_{|X \times \{x^\alpha\}}, E_{ij}[2])$. Then $H^k(\Lambda(E_{ij}) \overset{\mathbf{L}}{\otimes} \mathbb{C}_{x^\alpha}) = 0$ for $k \neq -1, -2$. Hence $H^k(\Lambda^\alpha(E_{ij})) = 0$ for $k \neq -1, -2$ and $H^{-2}(\Lambda^\alpha(E_{ij}))$ is a locally free sheaf. By the proof of Theorem 2.2.19(3), $\mathrm{Supp}(H^k(\Lambda^\alpha(E_{ij}))) \subset C_{ij}^\alpha$ for all k. Hence $H^{-2}(\Lambda^\alpha(E_{ij})) = 0$, which implies that $\Lambda^\alpha(E_{ij})[-1] \in \mathrm{Coh}(X^\alpha)$. Since $\mathrm{Hom}(\mathbb{C}_{x^\alpha}, \Lambda^\alpha(E_{ij})[-1]) = \mathrm{Hom}(\mathcal{E}^\alpha_{|X \times \{x^\alpha\}}, E_{ij}[-1]) = 0$, $\Lambda^\alpha(E_{ij})[-1]$ is purely 1-dimensional. We set $C := \mathrm{Div}(\Lambda^\alpha(E_{ij})[-1])$. Then $(C^2) = \langle v(\Lambda^\alpha(E_{ij})[-1])^2 \rangle = \langle v(E_{ij})^2 \rangle = -2$, which implies that $C = C_{ij}^\alpha$. Therefore $\Lambda^\alpha(E_{ij})[-1]$ is a line bundle on C_{ij}^α. □

COROLLARY 2.3.12

(1) *We have* $(C_{ij}^\alpha, C_{i'j'}^\alpha) = \langle v(E_{ij}), v(E_{i'j'}) \rangle$.

(2) $\{C_{ij}^\alpha\}$ *is a simple normal crossing divisor.*

Proof

(1) By Lemma 2.3.11, $(C_{ij}^\alpha, C_{i'j'}^\alpha) = \langle v(\Lambda^\alpha(E_{ij})), v(\Lambda^\alpha(E_{i'j'})) \rangle = \langle v(E_{ij}), v(E_{i'j'}) \rangle$. Then (2) also follows. □

We have that E_{i0} is a subobject of $\mathcal{E}_{|X\times\{x^\alpha\}}$ for $x^\alpha \in Z_i^\alpha$, and we have an exact sequence

$$0 \to E_{i0} \to \mathcal{E}_{|X\times\{x^\alpha\}} \to F \to 0, \quad x^\alpha \in Z_i^\alpha, \tag{2.36}$$

where F is a zero-semistable object with $\mathrm{gr}(F) = \bigoplus_{j=1}^{s_i} E_{ij}^{\oplus a_{ij}}$. Then we get an exact sequence

$$0 \to \Lambda^\alpha(F)[-1] \to \Lambda^\alpha(E_{i0}) \to \mathbb{C}_{x^\alpha} \to 0 \tag{2.37}$$

in $\mathrm{Coh}(X^\alpha)$. Thus $\Lambda^\alpha(E_{i0}) \in \mathrm{Coh}(X^\alpha)$.

DEFINITION 2.3.13
We set $A_{i0}^\alpha := \Lambda^\alpha(E_{i0})$ and $A_{ij}^\alpha := \Lambda^\alpha(E_{ij}) = \mathcal{O}_{C_{ij}^\alpha}(b_{ij}^\alpha)[1]$ for $j>0$.

LEMMA 2.3.14
(1) *We have* $\mathrm{Hom}(A_{i0}^\alpha, A_{ij}^\alpha[-1]) = \mathrm{Ext}^1(A_{i0}^\alpha, A_{ij}^\alpha[-1]) = 0$.
(2) *We set* $\mathbf{b}_i^\alpha := (b_{i1}^\alpha, b_{i2}^\alpha, \dots, b_{is_i}^\alpha)$. *Then* $A_{i0}^\alpha \cong A_0(\mathbf{b}_i^\alpha)$. *In particular,* $\mathrm{Hom}(A_{i0}^\alpha, \mathbb{C}_{x^\alpha}) = \mathbb{C}$ *for* $x^\alpha \in Z_i^\alpha$.

Proof
(1) We have

$$\begin{aligned}\mathrm{Hom}\big(A_{i0}^\alpha, A_{ij}^\alpha[k]\big) &= \mathrm{Hom}\big(\Lambda^\alpha(E_{i0}), \Lambda^\alpha(E_{ij})[k]\big)\\ &= \mathrm{Hom}\big(E_{i0}, E_{ij}[k]\big) = 0\end{aligned} \tag{2.38}$$

for $k=-1,0$.
(2) By (2.37) and (1), we can apply Lemma 1.2.22 and get $A_{i0}^\alpha = A_0(\mathbf{b}_i^\alpha) = A_{p_i}$. ∎

REMARK 2.3.15
Assume that $\alpha \in v_0^\perp$ satisfies $-\langle v(E_{ij}), \alpha\rangle < 0$, $j>0$. Then $\Phi(E_{ij})[2] = \mathcal{O}_{C_{ij}^\alpha}(b_{ij}'')$, $j>0$, and $\Phi(E_{i0})[2] = A_0(\mathbf{b}_i'')[1]$ belong to $\mathrm{Per}(X^\alpha/Y, \mathbf{b}_1'', \dots, \mathbf{b}_n'')^*$, where $\mathbf{b}_i'' := (b_{i1}'', \dots, b_{is_i}'')$.

By Proposition 2.3.4, we have the following result.

PROPOSITION 2.3.16
If $-\langle \alpha, v(E_{ij})\rangle > 0$ *for all* $j>0$, *then* Λ^α *induces an equivalence*

$$\mathcal{C} \to \mathrm{Per}(X^\alpha/Y, \mathbf{b}_1^\alpha, \dots, \mathbf{b}_n^\alpha),$$

where $\mathbf{b}_i^\alpha = (b_{i1}^\alpha, \dots, b_{is_i}^\alpha)$.

PROPOSITION 2.3.17
Assume that there is a $\beta \in \varrho_X^\perp$ *such that* $\mathbb{C}_x$ *are* β*-stable for all* $x \in X$.

(1) *We set* $\mathcal{F} := \mathcal{E}^{\alpha\vee}[2]$. *Then we have an isomorphism*

$$X \to M^{G^\alpha,\Lambda^\alpha(\beta)}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha}) = (X^\alpha)^{\Lambda^\alpha(\beta)},$$
$$x \mapsto \mathcal{F} \overset{\mathbf{L}}{\otimes} \mathbb{C}_x. \tag{2.39}$$

Since $\Phi^{\mathcal{F}^\vee[2]}_{X^\alpha\to X} = \Phi^{\mathcal{E}^\alpha}_{X^\alpha\to X}$, *we have* $\mathcal{C} = \Phi^{\mathcal{F}^\vee[2]}_{X^\alpha\to X}(\mathrm{Per}(X^\alpha/Y, \mathbf{b}^\alpha_1,\dots,\mathbf{b}^\alpha_n))$.

(2) *We also have an isomorphism*

$$X \to M^{(G^\alpha)^\vee,-D_{X^\alpha}\circ\Lambda^\alpha(\beta)}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha}),$$
$$x \mapsto \mathcal{E}^\alpha \overset{\mathbf{L}}{\otimes} \mathbb{C}_x, \tag{2.40}$$

where $M^{(G^\alpha)^\vee,-D_{X^\alpha}\circ\Lambda^\alpha(\beta)}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha})$ *is the moduli of stable objects of* $\Lambda^\alpha(\mathcal{C})^D$.

Thus X *and* X^α *are Fourier–Mukai dual.*

Proof

(1) This is a consequence of Corollary 2.3.6.

(2) This is a consequence of (1) and the isomorphism $\mathcal{M}^{G^\alpha,\gamma}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha})^{\mathrm{ss}} \to \mathcal{M}^{(G^\alpha)^\vee,-D_{X^\alpha}(\gamma)}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha})^{\mathrm{ss}}$ defined by $E \mapsto D_{X^\alpha}(E)[2]$. □

The following proposition explains the condition of the stability of $\mathbb{C}_x$.

PROPOSITION 2.3.18

There exist X' *and* γ *such that* $\mathcal{C} = \Lambda^\gamma(\mathrm{Per}(X'/Y, \mathbf{b}_1,\dots,\mathbf{b}_n))$ *with* $X = (X')^\gamma$ *if and only if there is a* $\beta \in \varrho_X^\perp$ *such that* $\mathbb{C}_x$ *are* β*-stable for all* $x \in X$.

Proof

For $X = (X')^\gamma$, γ-stability of $\mathcal{E}^\gamma_{|X'\times\{x\}}$ and Corollary 2.3.6 imply the β-stability of $\mathbb{C}_x$, where $\beta := \Lambda^\gamma(\gamma)$. Conversely, if $\mathbb{C}_x$ are β-stable for all $x \in X$, then Proposition 2.3.17(1) implies the claim, where $X' := X^\alpha$ in Proposition 2.3.16 and $\gamma := \Lambda^\alpha(\beta)$. □

We give two examples of $\mathcal{C}$ satisfying the stability condition of $\mathbb{C}_x$.

LEMMA 2.3.19

(1) *Assume that* $\mathcal{C} = \mathrm{Per}(X/Y, \mathbf{b}_1,\dots,\mathbf{b}_n)$. *If* $-\langle\alpha, v(\mathcal{O}_{C_{ij}}(b_{ij})[1])\rangle > 0$ *for all* $j > 0$, *then* $X \cong X^\alpha$ *by sending* $x \in X$ *to* $\mathbb{C}_x \in X^\alpha$. *Moreover* $A_{p_i} \otimes \mathcal{O}_C$ *such that* $\mathcal{O}_C$ *is a purely* 1*-dimensional* $\mathcal{O}_{Z_i}$*-module with* $\chi(\mathcal{O}_C) = 1$ *are* α*-stable.*

(2) *Assume that* $\mathcal{C} = \mathrm{Per}(X/Y, \mathbf{b}_1,\dots,\mathbf{b}_n)^*$. *If* $-\langle\alpha, v(\mathcal{O}_{C_{ij}}(b_{ij}))\rangle < 0$ *for all* $j > 0$, *then* $X \cong X^\alpha$ *by sending* $x \in X$ *to* $\mathbb{C}_x \in X^\alpha$.

Proof

We only prove (1). Since $\mathbb{C}_x$, $x \in X \setminus \bigcup_{i=1}^n Z_i$ is irreducible, it is α-stable for any α. For $x \in Z_i$, assume that there is an exact sequence

$$0 \to E_1 \to \mathbb{C}_x \to E_2 \to 0 \tag{2.41}$$

such that $E_1 \neq 0$, $E_2 \neq 0$ and $-\langle \alpha, v(E_1)\rangle = \chi(v^{-1}(\alpha), E_1) \geq 0$. We note that $-\langle \alpha, v(E_{ij})\rangle > 0$ for all $j > 0$. Since $\langle \alpha, \varrho_X\rangle = 0$, $\langle \alpha, v(A_0(\mathbf{b}_i))\rangle = -\sum_{j>0} a_{ij}\langle \alpha, v(E_{ij})\rangle$. As a zero-semistable object, E_1 is S-equivalent to $\bigoplus_{j>0} \mathcal{O}_{C_{ij}}(b_{ij})[1]^{\oplus a'_{ij}}$, $a'_{ij} \leq a_{ij}$. Since $\operatorname{Hom}(\mathcal{O}_{C_{ij}}(b_{ij})[1], \mathbb{C}_x) = 0$, this is impossible. Therefore $\mathbb{C}_x$ is α-twisted stable. Then we have an injective morphism $\phi : X \to X^\alpha$ by sending $x \in X$ to $\mathbb{C}_x$. By using the Fourier–Mukai transform $\Phi_{X\to X}^{\mathcal{O}_\Delta^\vee} : \mathbf{D}(X) \to \mathbf{D}(X)$, we see that ϕ is surjective. Since both spaces are smooth, ϕ is an isomorphism. The last claim also follows by a similar argument. □

2.3.3. Relation with the twist functor (see [ST])

Let F be a spherical object of $\mathbf{D}(X)$, and set

$$\mathcal{E} := \operatorname{Cone}(F^\vee \boxtimes F \to \mathcal{O}_\Delta)[1]. \tag{2.42}$$

Then $T_F := \Phi_{X\to X}^{\mathcal{E}}$ is an autoequivalence of $\mathbf{D}(X)$.

LEMMA 2.3.20

Let $\Pi : \mathbf{D}(X) \to \mathbf{D}(Y)$ be a Fourier–Mukai transform. Then

$$\Pi \circ T_F \cong T_{\Pi(F)} \circ \Pi. \tag{2.43}$$

Proof

Let $\mathbf{E} \in \mathbf{D}(X \times Y)$ be an object such that $\Pi = \Phi_{X\to Y}^{\mathbf{E}}$. It is sufficient to prove $\Pi(\mathcal{E}) \cong T_{\Pi(F)}(\mathbf{E})$. We set $X_i := X$, $i = 1, 2$. We note that $F^\vee \cong \operatorname{Hom}_p(\mathcal{O}_{X_1} \boxtimes F, \mathcal{O}_\Delta)$, where $p : X_1 \times X_2 \to X_1$ is the projection and $\Delta \subset X_1 \times X_2$ the diagonal. Then

$$\mathcal{E} \cong \operatorname{Cone}\big(\operatorname{Hom}_p(\mathcal{O}_{X_1} \boxtimes F, \mathcal{O}_\Delta) \boxtimes F \to \mathcal{O}_\Delta\big)[1]. \tag{2.44}$$

Let $p_{X_2} : Y \times X_2 \to X_2$, $p_Y : Y \times X_2 \to Y$, and $q : X_1 \times Y \to X_1$ be the projections. We have a morphism

$$\begin{aligned}\operatorname{Hom}_p(\mathcal{O}_{X_1} \boxtimes F, \mathcal{O}_\Delta) &\to \operatorname{Hom}_{q'}\big(\mathcal{O}_{X_1} \boxtimes \big(\mathbf{E} \otimes p_{X_2}^*(F)\big), (\mathcal{O}_{X_1} \boxtimes \mathbf{E})_{|\Delta'}\big) \\ &\to \operatorname{Hom}_q\big(\mathcal{O}_{X_1} \boxtimes \mathbf{R}p_{Y*}\big(\mathbf{E} \otimes p_{X_2}^*(F)\big), \mathbf{E}\big),\end{aligned} \tag{2.45}$$

where $\Delta' = \Delta \times Y$ and $q' : X_1 \times Y \times X_2 \to X_1$ is the projection. We also have a commutative diagram in $\mathbf{D}(Y \times X_1)$:

$$\begin{tikzcd} \operatorname{Hom}_p(\mathcal{O}_{X_1} \boxtimes F, \mathcal{O}_\Delta) \boxtimes \Pi(F) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \mathbf{E} \arrow[d, equal] \\ \operatorname{Hom}_q(\mathcal{O}_{X_1} \boxtimes \Phi_{X\to Y}^{\mathbf{E}}(F), \mathbf{E}) \boxtimes \Pi(F) \arrow[r, "\beta"] & \mathbf{E} \end{tikzcd} \tag{2.46}$$

Since Π is an equivalence, γ is an isomorphism. Since $\Pi(\mathcal{E}) \cong \operatorname{Cone}(\alpha)[1]$ and $T_{\Pi(F)}(\mathbf{E}) \cong \operatorname{Cone}(\beta)[1]$, we get $\Pi(\mathcal{E}) \cong T_{\Pi(F)}(\mathbf{E})$. □

COROLLARY 2.3.21

Assume that $\operatorname{Supp}(H^i(F)) \subset Z$ for all i. Let D be the pullback of a divisor on Y. Then $T_F(E(D)) \cong T_F(E)(D)$.

Proof
We apply Lemma 2.3.20 to $\Pi = \Phi_{X \to X}^{\mathcal{O}_\Delta(D)}$. Since $\Pi(F) \cong F$, we get our claim. □

PROPOSITION 2.3.22
Assume that $G^\vee \otimes G$ satisfies $R^1\pi_*(G^\vee \otimes G) = 0$. Assume that $G' := T_F(G)$ is a locally free sheaf up to shift.

(1) *We have $\mathbf{R}^1\pi_*(G'^\vee \otimes G') = 0$ and $\pi_*(G'^\vee \otimes G') \cong \pi_*(G^\vee \otimes G)$.*
(2) *We set $\mathcal{A}' := \pi_*(G'^\vee \otimes G')$. We identify $\mathrm{Coh}_{\mathcal{A}}(Y)$ with $\mathrm{Coh}_{\mathcal{A}'}(Y)$ via $\mathcal{A} \cong \mathcal{A}'$. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n) & \xrightarrow{T_F} & T_F\big(\mathrm{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)\big) \\ {\scriptstyle \mathbf{R}\pi_*\mathcal{H}om_{\mathcal{O}_X}(G,\ )}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{R}\pi_*\mathcal{H}om_{\mathcal{O}_X}(G',\ )} \\ \mathrm{Coh}_{\mathcal{A}}(Y) & = & \mathrm{Coh}_{\mathcal{A}'}(Y) \end{array} \tag{2.47}$$

Proof
The proof is almost the same as that of Proposition 2.3.4. □

For an $\alpha \in H^\perp \otimes \mathbb{Q}$, let F be an α-stable object such that

(i) $\langle v(F)^2 \rangle = -2$ and
(ii) $\langle \alpha, v(F) \rangle = 0$.

By (i), F is a spherical object. By the same proof of [OY, Proposition 1.12], we have the following result.

PROPOSITION 2.3.23
We set $\alpha^\pm := \pm\epsilon v(F) + \alpha$, where $0 < \epsilon \ll 1$. Then T_F induces an isomorphism

$$\begin{aligned} \mathcal{X}^{\alpha^-} &\to \mathcal{X}^{\alpha^+}, \\ E &\mapsto T_F(E) \end{aligned} \tag{2.48}$$

which preserves the S-equivalence classes. Hence we have an isomorphism

$$X^{\alpha^-} \to X^{\alpha^+}. \tag{2.49}$$

Combining Proposition 2.3.23 with Lemma 2.3.20, we get the following corollary.

COROLLARY 2.3.24
Assume that α belongs to exactly one wall defined by F. Then T_F induces an isomorphism $X^{\alpha^-} \to X^{\alpha^+}$. Under this isomorphism, we have

$$\Phi_{X^{\alpha^-} \to X}^{\mathcal{E}^{\alpha^+}} \cong T_F \circ \Phi_{X^{\alpha^-} \to X}^{\mathcal{E}^{\alpha^-}} \cong \Phi_{X^{\alpha^-} \to X}^{\mathcal{E}^{\alpha^-}} \circ T_A, \tag{2.50}$$

where $A := \Phi_{X \to X^{\alpha^-}}^{(\mathcal{E}^{\alpha^-})^\vee[2]}(F)$.

2.3.4. More results on the structure of $\mathcal{C}$

Let $\mathcal{C}$ be the category of perverse coherent sheaves in Lemma 1.1.11. Assume that there is $\beta \in \mathrm{NS}(X) \otimes \mathbb{Q}$ such that $\mathbb{C}_x$ is β-stable for all $x \in X$. By Proposition 2.3.18, $\mathcal{C} = \Lambda^\alpha(\mathrm{Per}(X'/Y, \mathbf{b}_1, \dots, \mathbf{b}_n))$. So we first assume that $\mathcal{C} = \mathrm{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$ and set

$$E_{ij} := \begin{cases} \mathcal{O}_{C_{ij}}(b_{ij})[1], & j > 0, \\ A_0(\mathbf{b}), & j = 0. \end{cases} \tag{2.51}$$

We set $v_{ij} := v(E_{ij})$. Let u_0 be an isotropic Mukai vector such that $r_0 := \mathrm{rk}\, u_0 > 0$, $\langle u_0, v_{ij} \rangle = 0$ for all i, j. We set

$$L := \mathbb{Z}u_0 + \sum_{i=1}^{n} \sum_{j=0}^{s_i} \mathbb{Z}v_{ij}. \tag{2.52}$$

Then L is a sublattice of $H^*(X, \mathbb{Z})$, and we have a decomposition

$$L = (\mathbb{Z}u_0 \oplus \mathbb{Z}\varrho_X) \perp \Big(\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{s_i} \mathbb{Z}v_{ij} \Big). \tag{2.53}$$

We set

$$\begin{aligned} T_i &:= \bigoplus_{j=1}^{s_i} \mathbb{Z}C_{ij}, \\ T &:= \bigoplus_{i=1}^{n} T_i. \end{aligned} \tag{2.54}$$

Then we have an isometry

$$\begin{aligned} \psi : \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{s_i} \mathbb{Z}v_{ij} &\to T \\ v &\mapsto c_1(v). \end{aligned} \tag{2.55}$$

Combining the isometry $\mathbb{Z}u_0 \oplus \mathbb{Z}\varrho_X \to \mathbb{Z}r_0 \oplus \mathbb{Z}\varrho_X$ $(xu_0 + z\varrho_X \mapsto xr_0 + z\varrho_X)$, we also have an isometry

$$\widetilde{\psi} : (\mathbb{Z}u_0 \oplus \mathbb{Z}\varrho_X) \perp \Big(\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{s_i} \mathbb{Z}v_{ij} \Big) \to (\mathbb{Z}r_0 \oplus \mathbb{Z}\varrho_X) \perp T. \tag{2.56}$$

Let $\mathfrak{g}_i$ (resp., $\widehat{\mathfrak{g}}_i$) be the finite-dimensional Lie algebra (resp., affine Lie algebra) associated to the lattice $\bigoplus_{j=1}^{s_i} \mathbb{Z}v_{ij}$ (resp., $\bigoplus_{j=0}^{s_i} \mathbb{Z}v_{ij}$). Let $\mathfrak{g}$ (resp., $\widehat{\mathfrak{g}}$) be the Lie algebra associated to $\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{s_i} \mathbb{Z}v_{ij}$ (resp., $\bigoplus_{i=1}^{n} \bigoplus_{j=0}^{s_i} \mathbb{Z}v_{ij}$). Since the centers of $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}_i$ are 1-dimensional, $\widehat{\mathfrak{g}}$ is smaller than $\bigoplus_i \widehat{\mathfrak{g}}_i$.

Let $W(\mathfrak{g}_i)$ (resp., $W(\mathfrak{g})$) be the Weyl group of $\mathfrak{g}_i$ (resp., $\mathfrak{g}$), and let $\mathcal{W}_i$ (resp., $\mathcal{W}$) be the set of Weyl chambers of $W(\mathfrak{g}_i)$ (resp., $W(\mathfrak{g})$). Since $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$, $W(\mathfrak{g}) = \prod_{i=1}^{n} W(\mathfrak{g}_i)$, and $\mathcal{W} = \prod_{i=1}^{n} \mathcal{W}_i$. By the action of $W(\mathfrak{g})$, $\mathbb{Q}u_0 + \mathbb{Q}\varrho_X$ is fixed. Let $W(\widehat{\mathfrak{g}}_i)$ (resp., $W(\widehat{\mathfrak{g}})$) be the Weyl group of $\widehat{\mathfrak{g}}_i$ (resp., $\widehat{\mathfrak{g}}$). We have the following decompositions:

$$W(\widehat{\mathfrak{g}}_i) = T_i \rtimes W(\mathfrak{g}_i),$$
$$W(\widehat{\mathfrak{g}}) = T \rtimes W(\mathfrak{g}), \tag{2.57}$$

and the action of $D \in T$ on L is the multiplication by e^D. Indeed

$$T_{\mathcal{O}_{C_{ij}}(b_{ij}+1)} \circ T_{\mathcal{O}_{C_{ij}}(b_{ij})[1]} = e^{-C_{ij}}$$

as an isometry of L.

We shall study the category $\Lambda^\alpha(\mathcal{C})$. We may assume that $\alpha \in \mathrm{NS}(X) \otimes \mathbb{Q}$ is $\alpha = \sum_i \alpha_i$ with $\alpha_i \in T_i \otimes \mathbb{Q}$. Via the identification ψ, we have an action of W on $T \otimes \mathbb{Q}$. We set

$$C_i^{\mathrm{fund}} := \{\alpha \in T_i \otimes \mathbb{R} \mid (\alpha, C_{ij}) > 0, 1 \le j \le s_i\},$$
$$C^{\mathrm{fund}} := \prod_{i=1}^n C_i^{\mathrm{fund}}. \tag{2.58}$$

C^{fund} is the fundamental Weyl chamber. If $\alpha \in C^{\mathrm{fund}}$, then Lemma 2.3.19 implies that $\mathbb{C}_x$ is α-stable for all $x \in X$. By the action of $W(\mathfrak{g}_i)$, we have $\mathcal{W}_i = W(\mathfrak{g}_i)C_i^{\mathrm{fund}}$. We also set

$$C_{\mathrm{alcove}}^{\mathrm{fund}} := \{\alpha \in T \otimes \mathbb{R} \mid (\alpha, C_{ij}) > 0, 1 \le j \le s_i, (\alpha, Z_i) < 1\}. \tag{2.59}$$

By the isometry $\widetilde{\psi}^{-1}$, we have

$$\begin{aligned}(\alpha, C_{ij}) &= -\langle \psi^{-1}(\alpha), v_{ij}\rangle \\ &= -\Big\langle \Big(\frac{u_0}{\operatorname{rk} u_0} + \psi^{-1}(\alpha) + \frac{(\alpha^2)}{2}\varrho_X\Big), v_{ij}\Big\rangle \\ &= -\langle e^{(c_1(u_0)/\operatorname{rk} u_0)+\alpha}, v_{ij}\rangle\end{aligned} \tag{2.60}$$

for $j > 0$ and $1 - (\alpha, Z_i) = 1 + \sum_{j=1}^{s_i} a_{ij}\langle e^{(c_1(u_0)/\operatorname{rk} u_0)+\alpha}, v_{ij}\rangle = -\langle e^{(c_1(u_0)/\operatorname{rk} u_0)+\alpha}, v_{i0}\rangle$. Hence we have

$$C_{\mathrm{alcove}}^{\mathrm{fund}} = \{\alpha \in T \otimes \mathbb{R} \mid -\langle e^{(c_1(u_0)/\operatorname{rk} u_0)+\alpha}, v_{ij}\rangle > 0\}. \tag{2.61}$$

Applying Corollary 2.3.24 successively, we get the following result.

PROPOSITION 2.3.25

If $\alpha \in T \otimes \mathbb{Q}$ belongs to a chamber $C = \prod_{i=1}^n C_i$, $C_i \subset T_i \otimes \mathbb{Q}$, then there are rigid objects $F_1, \ldots, F_n \in \mathcal{C}$ such that $X^\alpha \cong X$ and $\Phi_{X \to X}^{\mathcal{E}^\alpha} = T_{F_n} \circ T_{F_{n-1}} \circ \cdots \circ T_{F_1}$. Thus $\Lambda^\alpha = (\Phi_{X \to X}^{\mathcal{E}^\alpha})^{-1}$ induces an isometry $w(\alpha)$ of L.

Then we have a map

$$\phi : \mathcal{W} \to W(\widehat{\mathfrak{g}})/T,$$
$$C(\alpha) \mapsto [w(\alpha) \mod T], \tag{2.62}$$

where $C(\alpha)$ is the chamber containing α.

LEMMA 2.3.26

The map $\phi : \mathcal{W} \to W(\widehat{\mathfrak{g}})/T \cong W(\mathfrak{g})$ *is bijective.*

Proof

There is an element α_0 in the fundamental Weyl chamber such that $\alpha = \Phi_{X\to X}^{\mathcal{E}^\alpha}(\alpha_0)$. Hence $w(\alpha)(C(\alpha)) = C(\alpha_0)$. Thus ϕ is injective. Since $\#\mathcal{W}_i = \#W(\mathfrak{g}_i)$, ϕ is bijective. □

We set

$$T^* := \{D \in T \otimes \mathbb{Q} \mid (D, C_{ij}) \in \mathbb{Z}\}. \tag{2.63}$$

Then $\widetilde{W} := T^* \rtimes W(\mathfrak{g})$ is the extended Weyl group. By the action of $\widetilde{W}$, we can change $(\mathbf{b}_1, \dots, \mathbf{b}_n)$ to any sequence $(\mathbf{b}'_1, \dots, \mathbf{b}'_n)$.

PROPOSITION 2.3.27

Let $\mathcal{C}$ *be the category in Lemma 1.1.11, and assume that there is* $\beta \in \mathrm{NS}(X) \otimes \mathbb{Q}$ *such that* $\mathbb{C}_x$ *is* β*-stable for all* $x \in X$*. Then* $\mathcal{C}$ *is equivalent to* ${}^{-1}\operatorname{Per}(X/Y)$*. In particular,* $\operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n) \cong {}^{-1}\operatorname{Per}(X/Y)$.

Proof

We may assume that $\mathcal{C} = \operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$. We set

$$u_{ij} := \begin{cases} v(\mathcal{O}_{C_{ij}}(-1)[1]), & j > 0, \\ v(\mathcal{O}_{Z_i}), & j = 0. \end{cases} \tag{2.64}$$

By the theory of affine Lie algebras, there is an element $w \in W(\widehat{\mathfrak{g}})$ such that

$$\begin{aligned} &w\big(\{\beta \in T \otimes \mathbb{R} \mid -\langle e^\beta, v_{ij}\rangle > 0, i, j \geq 0\}\big) \\ &\quad = \{\beta \in T \otimes \mathbb{R} \mid -\langle e^\beta, u_{ij}\rangle > 0, i, j \geq 0\}. \end{aligned} \tag{2.65}$$

Then we have

$$\{w(v_{ij}) \mid 0 \leq j \leq s_i\} = \{u_{ij} \mid 0 \leq j \leq s_i\}$$

for all i.

For each i, there is an integer j_i such that (1) $c_1(w(v_{ij_i}))$ is effective and (2) $-c_1(w(v_{ij}))$, $j \neq j_i$ are effective. By Lemma 2.3.26, we have $w = e^D \phi(\alpha)$, $D, \alpha \in T$. Since $v(\Lambda^\alpha(E_{ij}) \otimes \mathcal{O}_X(D)) = e^D v(\Lambda^\alpha(E_{ij})) = e^D \phi(\alpha)(v_{ij})$, Proposition 2.3.4(2) implies that $-(\alpha, c_1(E_{ij})) > 0$ unless $j = j_i$. By Lemmas 2.2.22 and 2.3.11, $\Lambda^\alpha(E_{ij})[-1]$, $j \neq j_i$, is a line bundle on a smooth rational curve and $\Lambda^\alpha(E_{ij_i})$ is a line bundle on Z_i. Thus

$$\begin{aligned} \{\Lambda^\alpha(E_{ij}) \otimes \mathcal{O}_X(D) \mid j \neq j_i\} &= \{\mathcal{O}_{C_{ij}}(-1)[1] \mid 0 < j \leq s_i\}, \\ \Lambda^\alpha(E_{ij_i}) \otimes \mathcal{O}_X(D) &= \mathcal{O}_{Z_i}. \end{aligned} \tag{2.66}$$

By Proposition 2.3.4(5), we get $\Lambda^\alpha(\mathcal{C}) \otimes \mathcal{O}_X(D) \cong {}^{-1}\operatorname{Per}(X/Y)$. □

REMARK 2.3.28

For the derived category of coherent twisted sheaves, we also see that the equivalence classes of $\operatorname{Per}(X/Y, \{L_{ij}\})$ do not depend on the choice of $\{L_{ij}\}$.

2.4. Construction of a local projective generator

We return to the general situation in Section 2.1. We shall construct local projective generators for $\operatorname{Per}(X/Y, \{L_{ij}\})$.

PROPOSITION 2.4.1

Let β be a 2-cocycle of $\mathcal{O}_X^\times$ defining a torsion element of $H^2(X, \mathcal{O}_X^\times)$. Assume that $E \in K^\beta(X)$ satisfies

$$\begin{aligned} &0 \le -\chi(E, L_{ij}), \quad 1 \le j \le s_i, \\ &-\sum_j a_{ij}\chi(E, L_{ij}) \le r \end{aligned} \tag{2.67}$$

for all i.

(1) *There is a locally free β-twisted sheaf G on X such that $R^1\pi_*(G^\vee \otimes G) = 0$, $\mathbf{R}\pi_*(G^\vee \otimes F) \in \operatorname{Coh}(Y)$ for $F \in \operatorname{Per}(X/Y, \{L_{ij}\})$, G is μ-stable, and $\tau(G) = \tau(E) - k\tau(\mathbb{C}_x)$, $k \gg 0$.*

(2) *There is a locally free β-twisted sheaf G on X such that $R^1\pi_*(G^\vee \otimes G) = 0$, $\mathbf{R}\pi_*(G^\vee \otimes F) \in \operatorname{Coh}(Y)$ for $F \in \operatorname{Per}(X/Y, \{L_{ij}\})$, and $\tau(G) = 2\tau(E)$.*

(3) *Moreover, if the inequalities in* (2.67) *are strict, then G in (1) and (2) are local projective generators of* $\operatorname{Per}(X/Y, \{L_{ij}\})$.

COROLLARY 2.4.2

Assume that $(r, \xi) \in \mathbb{Z}_{>0} \oplus \operatorname{NS}(X)$ satisfies

$$\begin{aligned} &0 < (\xi, C_{ij}) - r(b_{ij} + 1), \quad 1 \le j \le s_i, \\ &\sum_j a_{ij}(\xi, C_{ij}) - r\sum_j a_{ij}(b_{ij} + 1) < r, \end{aligned} \tag{2.68}$$

for all i.

(1) *For any sufficiently large $c_2 \in \mathbb{Z}$, there is a local projective generator G of* $\operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$ *such that G is a μ-stable sheaf with respect to H and* $(\operatorname{rk} G, c_1(G), c_2(G)) = (r, \xi, c_2)$.

(2) *For any $\mathbf{e} \in K(X)_{\mathrm{top}}$ with $(\operatorname{rk}\mathbf{e}, c_1(\mathbf{e})) = (r, \xi)$, there is a local projective generator G such that $\tau(G) = 2\mathbf{e}$.*

Proof of Proposition 2.4.1

(1) We assume that H is represented by a smooth connected curve with $Z \cap H = \emptyset$, where $Z = \sum_{i=1}^n Z_i$. We take a torsion-free sheaf E such that $\operatorname{Ext}^2(E, E(-Z-H))_0 = 0$ by using Lemma 2.1.3. By the construction of E, we may assume that E is locally free on $Z \cup H$. We consider the restriction morphism of

the local deformation spaces

$$\phi : \mathrm{Def}(X,E) \to \mathrm{Def}(Z,E_{|Z}) \times \mathrm{Def}(H,E_{|H}). \tag{2.69}$$

Then $\mathrm{Def}(X,E)$ and $\mathrm{Def}(Z,E_{|Z}) \times \mathrm{Def}(H,E_{|H})$ are smooth, and ϕ is submersive. In particular, by using Lemma 2.4.3 below, we see that E deforms to a locally free β-twisted sheaf G such that G is μ-stable with respect to H and $\mathrm{Hom}(G,L_{ij}) = \mathrm{Ext}^1(G,A_{p_i}) = 0$ for all i,j. By Remark 1.1.35, Proposition 2.4.1(1) holds.

(2) By (1), we have locally free sheaves E_i, $i=1,2$, such that $R^1\pi_*(E_i^\vee \otimes E_i) = 0$, $\mathbf{R}\pi_*(G_i^\vee \otimes F) \in \mathrm{Coh}(Y)$ for $F \in \mathrm{Per}(X/Y,\{L_{ij}\})$, $\tau(E_i) = \tau(E) - k_i\tau(\mathbb{C}_x)$, and $k_1 + k_2 = k^2(H^2)\,\mathrm{rk}\,E$. Then $G = E_1(kH) \oplus E_2(-kH)$ satisfies the claim.

(3) The claim follows from Proposition 1.1.33. □

LEMMA 2.4.3

(1) $E_{|Z}$ *deforms to a locally free* β*-twisted sheaf such that*

$$H^0(C_{ij}, E^\vee \otimes L_{ij}) = H^1(Z_i, E^\vee \otimes A_{p_i}) = 0 \tag{2.70}$$

for all i,j.

(2) $E_{|H}$ *deforms to a* μ*-stable locally free* β*-twisted sheaf on* H.

Proof

(1) Since $E_{|Z} = \bigoplus_{i=1}^n E_{|Z_i}$, we shall prove the claims for each $E_{|Z_i}$. Since $H^2(Z,\mathcal{O}_Z^\times) = \{1\}$, there is a β-twisted line bundle $\mathcal{L}$ on Z_i which induces an equivalence $\varphi : \mathrm{Coh}^\beta(Z) \cong \mathrm{Coh}(Z)$ in (1.142). Since $\mathrm{Pic}(Z_i) \to \mathbb{Z}^{s_i}$ $(L \mapsto \prod_{j=1}^{s_i} \deg(L_{|C_{ij}}))$ is an isomorphism, we may assume that $\varphi(L_{ij}) = \mathcal{O}_{C_{ij}}(-1)$. Thus we may assume that β is trivial and $L_{ij} = \mathcal{O}_{C_{ij}}(-1)$. In this case, we have $A_{p_i} = \mathcal{O}_{Z_i}$. Then we have $\deg(E_{|C_{ij}}) \geq 0$ for all $j > 0$ and $\deg(E_{|Z_i}) \leq r$. Let D be an effective Cartier divisor on Z_i such that $(D, C_{ij}) = \deg(E_{|C_{ij}})$. Then $\mathcal{O}_{Z_i}(D) \cong \det E_{|Z_i}$, and

$$K := \ker\bigl(H^0(\mathcal{O}_{Z_i \cap D}) \otimes \mathcal{O}_{Z_i} \to \mathcal{O}_{Z_i \cap D}\bigr) \tag{2.71}$$

is a locally free sheaf on Z_i such that $H^1(Z_i, K) = 0$ and $H^0(C_{ij}, K_{|C_{ij}}(-1)) = 0$. Since $\mathrm{rk}\,K = \dim H^0(\mathcal{O}_{Z_i \cap D}) = \deg(D) = \deg(E_{|Z_i}) \leq r$, we set $F := K \oplus \mathcal{O}_{Z_i}^{\oplus(\mathrm{rk}\,E - \mathrm{rk}\,K)}$. Since F is a locally free sheaf with $(\mathrm{rk}\,F^\vee, \det(F^\vee)) = (\mathrm{rk}\,E_{|Z_i}, \det(E_{|Z_i}))$, we get the claim by Lemma 2.1.4 and the openness of the condition (2.70).

(2) This is well known. □

COROLLARY 2.4.4

Let $\mathcal{C}$ *be the category of perverse coherent sheaves on* X*, and let* E_{ij}, $1 \leq i \leq n$, $0 \leq j \leq s_i$, *be the zero-stable objects in Definition 2.2.14. For an element* $E \in K(X)$ *satisfying* $\chi(E, E_{ij}) > 0$ *for all* i,j*, there is a local projective generator* G *of* $\mathcal{C}$ *such that* $\tau(G) = 2\tau(E)$.

Proof

We consider the equivalence Λ^α in Proposition 2.3.16. Then since $\chi(\Lambda^\alpha(E), \Lambda^\alpha(E_{ij})) > 0$ for all i, j, Proposition 2.4.1 implies that there is a local projective generator G^α of $\Lambda^\alpha(\mathcal{C})$ such that $\tau(G^\alpha) = 2\tau(\Lambda^\alpha(E))$. We then also set $G := (\Lambda^\alpha)^{-1}(G^\alpha) \in \mathcal{C}$. Then

$$
\begin{aligned}
H^0\big(X, H^k(G \overset{\mathbf{L}}{\otimes} \mathbb{C}_x)\big) &= H^k(X, G \overset{\mathbf{L}}{\otimes} \mathbb{C}_x) \\
&= \operatorname{Hom}\big(\mathbb{C}_x, G[k+2]\big) \\
&= \operatorname{Hom}\big(\Lambda^\alpha(\mathbb{C}_x), G^\alpha[k+2]\big) \\
&= \operatorname{Hom}\big(G^\alpha, \Lambda^\alpha(\mathbb{C}_x)[-k]\big)^\vee = 0
\end{aligned} \tag{2.72}
$$

for all $x \in X$ and $k \neq 0$, where we used the fact that Λ^α is an equivalence and [Br2, Theorem 1.1] to show that $\Lambda^\alpha(\mathbb{C}_x)(K_{X^\alpha}) \cong \Lambda^\alpha(\mathbb{C}_x)$. The same claim also follows from Lemma 2.2.15 and the proof of Lemma 2.3.3.

Therefore G is a locally free sheaf on X. Since G^α is a local projective generator of $\Lambda^\alpha(\mathcal{C})$ and Λ^α is an equivalence, G is a local projective generator of $\mathcal{C}$. □

PROPOSITION 2.4.5

We set $v = (r, \xi, a) \in H^{ev}(X, \mathbb{Z})_{\mathrm{alg}}$, $r > 0$. Assume that $(\xi, D) \notin r\mathbb{Z}$ for all $D \in \bigoplus_{i,j} \mathbb{Z}[C_{ij}]$ with $(D^2) = -2$. Then there is a category of perverse coherent sheaves $\mathcal{C}(v)$ and a locally free sheaf G on X such that G is a local projective generator of $\mathcal{C}(v)$ with $v(G) = 2v$. We also have a local projective generator G' of $\mathcal{C}(v)$ such that G' is μ-stable with respect to H and $v(G') = v - b\varrho_X$, $b \gg 0$. Moreover, there is $\beta \in \varrho_X^\perp$ such that $\mathbb{C}_x \in \mathcal{C}(v)$ is β-stable for all $x \in X$.

Proof

We set $\mathcal{C} = \operatorname{Per}(X/Y, \mathbf{b}_1, \ldots, \mathbf{b}_n)$ and keep the notation as above. By our assumption, $\langle v, u\rangle \notin r\mathbb{Z}$ for all (-2)-vectors $u \in L$. Then there is $w \in W(\widehat{\mathfrak{g}})$ such that $v = w(v_f)$ and v_f/r belongs to the fundamental alcove, that is, $-\langle v_f/r, v_{ij}\rangle > 0$ for all i, j. By Lemma 2.3.26, we have an element α such that $w = e^D\phi(\alpha)$, $D \in T$. By Proposition 2.4.1, there is a local projective generator G_f of $\mathcal{C}$ such that $v(G_f) = 2v_f$. We set $\mathcal{C}(v) := \Lambda^\alpha(\mathcal{C}) \otimes \mathcal{O}_X(D)$. Then $G^\alpha := \Lambda^\alpha(G_f)$ is a local projective generator of $\mathcal{C}(v) \otimes \mathcal{O}_X(-D)$. Hence $G := G^\alpha(D)$ is a local projective generator of $\mathcal{C}(v)$ such that $v(G) = 2v$. The last claim follows from Proposition 2.3.18. □

REMARK 2.4.6

If v is a Mukai vector of a twisted sheaf, then replacing $\operatorname{Per}(X/Y, \mathbf{b}_1, \ldots, \mathbf{b}_n)$ by $\operatorname{Per}(X/Y, \{L_{ij}\})$, the same claim holds.

2.5. Deformation of a local projective generator

Let $f : (\mathcal{X}, \mathcal{L}) \to S$ be a flat family of polarized surfaces over S. For a point $s_0 \in S$, we set $X := \mathcal{X}_{s_0}$. Let $\mathcal{H}$ be a relative Cartier divisor on X such that $H := \mathcal{H}_{s_0}$

gives a contraction $f: X \to Y$ to a normal surface Y with $\mathbf{R}\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$. We construct a family of contractions $f: \mathcal{X} \to \mathcal{Y}$ over a neighborhood of s_0.

Replacing H by mH, we may assume that $H^i(X, \mathcal{O}_X(mH)) = H^i(Y, \mathcal{O}_Y(mH)) = 0$ for $m > 0$. We shall find an open neighborhood S_0 of s_0 such that $R^i f_*(\mathcal{O}_{\mathcal{X}_{S_0}}(m\mathcal{H})) = 0$, $i > 0, m > 0$, and $f_*(\mathcal{O}_{\mathcal{X}_{S_0}}(m\mathcal{H}))$ is locally free. We consider the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(m\mathcal{H}) \to \mathcal{O}_{\mathcal{X}}\big((m+1)\mathcal{H}\big) \to \mathcal{O}_{\mathcal{H}}\big((m+1)\mathcal{H}\big) \to 0. \tag{2.73}$$

Since $\mathcal{H} \to S$ is a flat morphism, the base-change theorem implies that $R^i f_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{H})) \to R^i f_*(\mathcal{O}_{\mathcal{X}}((m+1)\mathcal{H}))$ $(i > 0)$ is surjective if $(m+1)(H^2) > (H^2) + (H, K_X)$. We take an open neighborhood S_0 of s_0 such that $R^i f_*(\mathcal{O}_{\mathcal{X}_{S_0}}(m\mathcal{H})) = 0$, $i > 0, (H, K_X)/(H^2) \geq m > 0$. Then the claim holds. We replace S by S_0 and set $\mathcal{Y} := \operatorname{Proj}(\bigoplus_m f_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{H})))$. Then $\mathcal{Y}$ is flat over S and $\mathcal{Y}_{s_0} \cong Y$. By the construction, $\mathcal{Y} \to S$ is a flat family of normal surfaces.

Let $\mathcal{Z} := \{x \in \mathcal{X} \mid \dim \pi^{-1}(\pi(x)) \geq 1\}$ be the exceptional locus. Then $\{(\mathcal{Z}_s, \mathcal{L}_s) \mid s \in S\}$ is a bounded set. Hence $\mathcal{D} := \{D \in \operatorname{NS}(\mathcal{X}_s) \mid s \in S, (D, \mathcal{H}_s) = 0\}$ is a finite set. Replacing S by an open neighborhood of s_0, we may assume that $D \in \mathcal{D}$ is a deformation of $D_0 \in \operatorname{NS}(X)$ (i.e., D belongs to $\operatorname{NS}(X)$ via the identification $H^2(\mathcal{X}_s, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$).

LEMMA 2.5.1

Assume that there is a locally free sheaf G on $\mathcal{X}$ such that $R^1\pi_*(G^\vee \otimes G) = 0$ and $\operatorname{rk} G \nmid (c_1(G)_{s_0}, D)$ for all (-2)-curves with $(D, \mathcal{H}_{s_0}) = 0$. Then replacing S by an open neighborhood of s_0, we may assume that $\operatorname{rk} G \nmid (c_1(G)_s, D)$ for all (-2)-curves with $(D, \mathcal{H}_s) = 0$. Thus G is a family of tilting generators.

As an example, we consider a family of $K3$ surfaces. Let X be a $K3$ surface, and let $\pi: X \to Y$ be a contraction. Let p_i, $i = 1, 2, \ldots, n$ be the singular points, and let $Z_i := \sum_j a_{ij} C_{ij}$ be their fundamental cycles. Let H be the pullback of an ample divisor on Y. Assume that $(r, \xi) \in \mathbb{Z}_{>0} \times \operatorname{NS}(X)$ satisfies $r \nmid (\xi, D)$ for all (-2)-curves D with $(D, H) = 0$. By Proposition 2.4.5, there is a category of perverse coherent sheaves $\mathcal{C}$ and a local projective generator G of $\mathcal{C}$ such that G is μ-stable with respect to H and $(\operatorname{rk} G, c_1(G)) = (r, \xi)$. Replacing G by $G \otimes L^{\otimes m}$ and $L \in \operatorname{Pic}(X)$ and $\mathcal{C}$ by $\mathcal{C} \otimes L^{\otimes m}$, we assume that ξ is ample. If $(\mathbb{Q}\xi + \mathbb{Q}H) \cap H^\perp$ does not contain a (-2)-curve, then we have a deformation $(\mathcal{X}, \mathcal{L}) \to S$ of (X, ξ) such that $\mathcal{H}_s$ is ample for a general $s \in S$. Since G is simple, replacing S by a smooth covering $S' \to S$, we also have a deformation $\mathcal{G}$ of G over S. By shrinking S, we may assume that $\mathcal{G}$ is a family of tilting generators. Then we can construct a family of moduli spaces $f: \overline{M}^{\mathcal{G}}_{(\mathcal{X},\mathcal{H})/S}(v) \to S$ of $\mathcal{G}_s$-twisted semistable objects on $\mathcal{X}_s$, $s \in S$ (for the twisted cases, see steps (3), (4) of the proof of [Y4, Theorem 3.16]). By our assumption, a general fiber of f is the moduli space of $\mathcal{G}_s$-twisted semistable sheaves, which is nonempty by Lemma A.2.4. Hence we get the following lemma.

LEMMA 2.5.2

Assume that v is primitive and $\langle v^2 \rangle \geq -2$. Then f is surjective. In particular, $\overline{M}^{\mathcal{G}}_{(\mathcal{X},\mathcal{H})/S}(v)_{s_0} \neq \emptyset$.

REMARK 2.5.3

We note that $R := \{C \in \mathrm{NS}(X) \mid (C,H) = 0, (C^2) = -2\}$ is a finite set. If $\rho(X) \geq 3$, then $\bigcup_{C \in R}(\mathbb{Q}H + \mathbb{Q}C)$ is a proper subset of $\mathrm{NS}(X) \otimes \mathbb{Q}$. Hence $(\mathbb{Q}\xi + \mathbb{Q}H) \cap R = \emptyset$ for a general ξ. In general, we have a deformation $(\mathcal{X},\mathcal{L}) \to S$ of (X,ξ) such that $\mathcal{G}$ is a family of tilting generators and $\rho(\mathcal{X}_s) \geq 3$ for infinitely many points $s \in S$.

REMARK 2.5.4

By the usual deformation theory of objects, we note that $M^{\mathcal{G}}_{(\mathcal{X},\mathcal{H})/S}(v) \to S$ is a smooth morphism. If $\overline{M}^{\mathcal{G}}_{(\mathcal{X},\mathcal{H})/S}(v)_{s_0} = M^{\mathcal{G}}_{(\mathcal{X},\mathcal{H})/S}(v)_{s_0}$, then we have a smooth deformation $\overline{M}^{\mathcal{G}}_{(\mathcal{X},\mathcal{H})/S}(v) \to S$ of $\overline{M}^{\mathcal{G}}_{(\mathcal{X},\mathcal{H})/S}(v)_{s_0}$. In particular, $\overline{M}^{\mathcal{G}}_{(\mathcal{X},\mathcal{H})/S}(v)_{s_0}$ deforms to a usual moduli of semistable sheaves.

COROLLARY 2.5.5

Let $v_0 = (r,\xi,a)$ be a primitive isotropic Mukai vector such that $r \not| (\xi,D)$ for all (-2)-curves D with $(D,H) = 0$. Let $\mathcal{C}$ be the category in Proposition 2.4.5. Then $M^{v_0}_H(v_0) \neq \emptyset$.

Proof

By Lemma 2.5.2 and Remark 2.5.3, we see that $\overline{M}^{v_0}_H(v_0) \neq \emptyset$. By the same proof as that of [OY, Lemma 2.17], we see that $\overline{M}^{v_0+\alpha}_H(v_0) \neq \emptyset$ for a general α. Then $\overline{M}^{v_0+\alpha}_H(v_0)$ is a $K3$ surface. In the same way as in the proof of [OY, Proposition 2.11], we see that $M^{v_0}_H(v_0) \neq \emptyset$. □

Appendix

A.1 Elementary facts on lattices

LEMMA A.1.1

Assume that $L \cong \mathbb{Z}^n$ has an integral bilinear form $(\ ,\ )$ and a linear map $f : L \otimes \mathbb{Q} \to \mathbb{Q}$. Let v be a primitive element of L such that $(v,v) = 0$, $(v,w) = (w,v)$ for any w. We set $v^\perp := \{x \in L \mid (v,x) = 0\}$. Assume that $(\ ,\ )_{|v^\perp}$ is symmetric, that there is an element $u \in L \otimes \mathbb{Q}$ such that $(u,v) = 0$ and that $(v^\perp \cap u^\perp)/\mathbb{Z}v$ is negative definite.

(1) *If $v = \sum_{i=0}^{s} a_i v_i$, $a_i \in \mathbb{Z}_{>0}$, such that*

   (i) $v_i \in v^\perp \cap u^\perp$, $i = 0,1,\ldots,s$,
   (ii) $(v_i^2) = -2$,
   (iii) $(v_i,v_j) \geq 0$ *for* $i \neq j$, *and*
   (iv) $f(v_i) > 0$ *for* $0 \leq i \leq s$.

Then the matrix $(-(v_i,v_j)_{i,j})$ *is of affine type* $\widetilde{A},\widetilde{D},\widetilde{E}$, *and* $1 \in \{a_0,a_1,\dots,a_s\}$.

(2) *Assume that* v *has two expressions*

$$v=\sum_{i=0}^{s} a_i v_i=\sum_{i=0}^{t} a_i' v_i', \quad a_i,a_i' \in \mathbb{Z}_{>0}, \tag{A.1}$$

such that $v_i,v_i' \in v^\perp \cap u^\perp$, $(v_i^2)=((v_i')^2)=-2$, $f(v_i),f(v_i')>0$, *and* $(w_1,w_2)\geq 0$ *for different* $w_1,w_2 \in V_1 \cup V_2$, *where* $V_1:=\{v_0,v_1,\dots,v_s\}$ *and* $V_2:=\{v_0',v_1',\dots,v_t'\}$, $V_1=V_2$, *or* $\bigoplus_i \mathbb{Z}v_i \perp \bigoplus_i \mathbb{Z}v_i'$.

Proof

(1) We shall show that the dual graph of $\{v_0,v_1,\dots,v_s\}$ is connected. If we have a decomposition $v=(\sum_{i\in I_1} a_iv_i)+(\sum_{i\in I_2} a_iv_i)$ such that $(v_i,v_j)=0$ for $i\in I_1$, $j\in I_2$, then $0=(v^2)=(\sum_{i\in I_1} a_iv_i)^2+(\sum_{i\in I_2} a_iv_i)^2$. Hence $\sum_{i\in I_1} a_iv_i, \sum_{i\in I_2} a_iv_i \in \mathbb{Z}v$, which implies that the graph is connected by (iv). Then the standard arguments show the claims.

(2) We set $I:=\{i \mid v_i' \in V_1\}$ and $J:=\{i \mid v_i' \notin V_1\}$. Then $v=(\sum_{i\in I} a_i'v_i')+(\sum_{i\in J} a_i'v_i')$. If $i\in J$, then $0=(v_i,v)=\sum_j a_j(v_i',v_j)\geq 0$. Hence $v_i' \in (\bigoplus_i \mathbb{Z}v_i)^\perp$. Then $0=(v^2)=((\sum_{i\in I} a_i'v_i')^2)+((\sum_{i\in J} a_i'v_i')^2)$. Hence $\sum_{i\in I} a_i'v_i', \sum_{i\in J} a_i'v_i' \in \mathbb{Z}v$, which implies that $I=\emptyset$ or $J=\emptyset$. If $J=\emptyset$, then $V_2\subset V_1$, and we see that $V_1=V_2$. If $I=\emptyset$, then all v_i' belong to $\bigoplus_i \mathbb{Z}v_i$. Thus $\bigoplus_i \mathbb{Z}v_i \perp \bigoplus_i \mathbb{Z}v_i'$. □

REMARK A.1.2

If the dual graph of $\{v_0,v_1,\dots,v_s\}$ is connected, then we do not need the existence of u and f to show (1). Thus if $v=\sum_{i=0}^s a_iv_i$, $a_i\in\mathbb{Z}_{>0}$, such that

(i) $v_i \in v^\perp$, $i=0,1,\dots,s$,
(ii) $(v_i^2)=-2$,
(iii) $(v_i,v_j)\geq 0$ for $i\neq j$,

then the matrix $(-(v_i,v_j)_{i,j})$ is of affine type $\widetilde{A},\widetilde{D},\widetilde{E}$ and $1\in\{a_0,a_1,\dots,a_s\}$ (cf. [Ko, proof of Theorem 6.2]).

If the dual graphs of $\{v_0,v_1,\dots,v_s\}$ and $\{v_0',v_1',\dots,v_t'\}$ are connected, then (2) also holds under the assumption $v_i,v_i'\in v^\perp$, $(v_i^2)=((v_i')^2)=-2$, and $(w_1,w_2)\geq 0$ for different $w_1,w_2\in V_1\cup V_2$.

EXAMPLE A.1.3

Let X be a smooth projective surface, and let H be a divisor on X with $(H^2)>0$. We set $L:=\operatorname{ch}(K(X))$ and $(x,y):=-\int_X x^\vee y \operatorname{td}_X$, $x,y\in L$. Then $\varrho_X=\operatorname{ch}(\mathbb{C}_x)$ is primitive in L. Since $\mathbb{C}_x\otimes K_X\cong\mathbb{C}_x$, $(\varrho_X,x)=(x,\varrho_X)$. Moreover, $(\ ,\ )_{|\varrho_X^\perp}$ is symmetric. Since $(\varrho_X^\perp\cap\operatorname{ch}(\mathcal{O}_H)^\perp)/\mathbb{Z}\varrho_X\cong\{D\in\operatorname{NS}(X)_f \mid (H,D)=0\}$, it is negative definite, where $\operatorname{NS}(X)_f$ is the torsion-free quotient of $\operatorname{NS}(X)$.

A.2 Existence of twisted semistable sheaves

Let X be a smooth projective surface, and let H be an ample divisor on X. Let $\mathbf{e} \in K(X)_{\text{top}}$ be a toplogical invariant of a coherent sheaf on X.

DEFINITION A.2.1

A polarization H on X is general with respect to $\mathbf{e}$ if for every μ-semistable sheaf E with $\tau(E) = \mathbf{e}$ and a subsheaf $F \neq 0$ of E,

$$\text{(A.2)} \qquad \frac{(c_1(F), H)}{\operatorname{rk} F} = \frac{(c_1(E), H)}{\operatorname{rk} E} \quad \text{if and only if} \quad \frac{c_1(F)}{\operatorname{rk} F} = \frac{c_1(E)}{\operatorname{rk} E}.$$

If H is general with respect to $\mathbf{e}$, then the G-twisted semistability does not depend on the choice of G.

DEFINITION A.2.2

We let $\mathcal{M}_H^G(\mathbf{e})^{\text{ss}}$ (resp., $\mathcal{M}_H^G(\mathbf{e})^s$) denote the moduli stack of G-twisted semistable sheaves (resp., G-twisted stable sheaves). $\mathcal{M}_H(\mathbf{e})^{\mu\text{-ss}}$ (resp., $\mathcal{M}_H(\mathbf{e})^{\mu\text{-}s}$) denotes the moduli stack of μ-semistable sheaves (resp., μ-stable sheaves).

The following is [MW, Lemma 3.6]. For the sake of convenience, we give a proof.

LEMMA A.2.3

Assume that H is not general with respect to $\mathbf{e}$, and let ϵ be a sufficiently small $\mathbb{Q}$-divisor such that $H + \epsilon$ is general with respect to $\mathbf{e}$. Then there is a locally free sheaf G such that $\mathcal{M}_H^G(\mathbf{e})^{\text{ss}} = \mathcal{M}_{H+\epsilon}(\mathbf{e})^{\text{ss}}$.

Proof

We set

$$\text{(A.3)} \qquad \mathcal{F}(\mathbf{e}) := \left\{ (F, E) \;\middle|\; \begin{array}{l} E \in \mathcal{M}_H(\mathbf{e})^{\mu-\text{ss}}, F \subset E,\ E/F \text{ is torsion-free}, \\ (c_1(F), H)/\operatorname{rk} F = (c_1(E), H)/\operatorname{rk} E \end{array} \right\}.$$

Since $\mathcal{F}(\mathbf{e})$ is a bounded set, we have

$$\text{(A.4)} \qquad B := \max\left\{ \left| \frac{\chi(E)}{\operatorname{rk} E} - \frac{\chi(F)}{\operatorname{rk} F} \right| \;\middle|\; (F, E) \in \mathcal{F}(\mathbf{e}) \right\} < \infty.$$

Assume that $N\epsilon \in \operatorname{NS}(X)$. Take $m \geq (\operatorname{rk} \mathbf{e})^2 NB$, and take a locally free sheaf G with $c_1(G)/\operatorname{rk} G = -m\epsilon$. Then for $(F, E) \in \mathcal{F}(\mathbf{e})$,

$$\text{(A.5)} \qquad \begin{aligned} &\frac{\chi(G, E(nH))}{\operatorname{rk} E} - \frac{\chi(G, F(nH))}{\operatorname{rk} F} \\ &\quad = m\Big(\frac{c_1(E)}{\operatorname{rk} E} - \frac{c_1(F)}{\operatorname{rk} F}, \epsilon\Big) + \frac{\chi(E)}{\operatorname{rk} E} - \frac{\chi(F)}{\operatorname{rk} F} \geq 0 \end{aligned}$$

if and only if

(1) we have

$$\text{(A.6)} \qquad \Big(\frac{c_1(E)}{\operatorname{rk} E} - \frac{c_1(F)}{\operatorname{rk} F}, \epsilon\Big) \geq 0,$$

or

(2) we have

$$\text{(A.7)} \qquad \frac{c_1(E)}{\operatorname{rk} E} - \frac{c_1(F)}{\operatorname{rk} F} = 0, \qquad \frac{\chi(E)}{\operatorname{rk} E} - \frac{\chi(F)}{\operatorname{rk} F} \geq 0,$$

which is the semistability of E with respect to $H + \epsilon$. Therefore the claim holds. □

LEMMA A.2.4

Let (X, H) be a polarized $K3$ surface, and let $v = r + \xi + a\varrho_X$, $\xi \in \mathrm{NS}(X)$, be a primitive Mukai vector with $\langle v^2 \rangle \geq -2$. Then there is a G-twisted semistable sheaf E with $v(E) = v$ for any G.

Proof

We first assume that $r > 0$. If H is general with respect to v, then there is a stable sheaf E with $v(E) = v$ by [Y1, Theorem 8.1]. Obviously E is G-twisted stable for any G. If H is not general with respect to v, then Lemma A.2.3 implies that there is a locally free sheaf G_1 such that $\mathcal{M}_H^{G_1}(v)^{\mathrm{ss}} = \mathcal{M}_H^{G_1}(v)^s \neq \emptyset$. For a G with $\mathcal{M}_H^G(v)^{\mathrm{ss}} = \mathcal{M}_H^G(v)^s$, we use [Y2, Proposition 4.1], whose proof is similar to those of [Y3, Propositions 2.5, 2.7]. If $\mathcal{M}_H^G(v)^{\mathrm{ss}} \neq \mathcal{M}_H^G(v)^s$, then we can find a G' such that $c_1(G')/\operatorname{rk} G'$ is sufficiently close to $c_1(G)/\operatorname{rk} G$, $\mathcal{M}_H^{G'}(v)^{\mathrm{ss}} = \mathcal{M}_H^{G'}(v)^s \neq \emptyset$ and $\mathcal{M}_H^{G'}(v)^{\mathrm{ss}} \subset \mathcal{M}_H^G(v)^{\mathrm{ss}}$. Thus the claim also holds.

We next assume that $r = 0$. We take a line bundle G_1 on X such that $\langle v, v(G_1) \rangle \neq 0$ and set $v' := ve^{-c_1(G_1)}$. Then for a general H with respect to v', [Y5, Corollary 3.5] implies that $\mathcal{M}_H^{G_1}(v)^s \cong \mathcal{M}_H^{\mathcal{O}_X}(v')^s \neq 0$. Now we can use the same argument as in the case $r > 0$ to prove that $\mathcal{M}_H^G(v)^{\mathrm{ss}} \neq \emptyset$ for any G. □

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References

[BBH] C. Bartocci, U. Bruzzo, and D. Hernández Ruipérez, *A Fourier–Mukai transform for stable bundles on $K3$ surfaces*, J. Reine Angew. Math. **486** (1997), 1–16. MR 1450748.

[BS] A. Borel and J.-P. Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. France **86** (1958), 97–136. MR 0116022.

[Br1] T. Bridgeland, *Fourier–Mukai transforms for elliptic surfaces*, J. Reine Angew. Math. **498** (1998), 115–133. MR 1629929. DOI 10.1515/crll.1998.046.

[Br2] ———, *Equivalences of triangulated categories and Fourier–Mukai transforms*, Bull. London Math. Soc. **31** (1999), 25–34. MR 1651025. DOI 10.1112/S0024609398004998.

[Br3] ———, *Flops and derived categories*, Invent. Math. **147** (2002), 613-632. MR 1893007. DOI 10.1007/s002220100185.

[Br4] ———, *Stability conditions on $K3$ surfaces*, Duke Math. J. **141** (2008), 241–291. MR 2376815. DOI 10.1215/S0012-7094-08-14122-5.

[E] H. Esnault, *Reflexive modules on quotient surface singularities*, J. Reine Angew. Math. **362** (1985), 63–71. MR 0809966. DOI 10.1515/crll.1985.362.63.

[F1] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math **90** (1968), 511–521. MR 0237496.

[F2] ———, *Truncated Hilbert functors*, J. Reine Angew. Math. **234** (1969) 65–88. MR 0244268.

[H] D. Huybrechts, *Derived and abelian equivalence of $K3$ surfaces*, J. Algebraic Geom. **17** (2008), 375–400. MR 2369091. DOI 10.1090/S1056-3911-07-00481-X.

[Is1] A. Ishii, *On the moduli of reflexive sheaves on a surface with rational double points*, Math. Ann. **294** (1992), 125–150. MR 1180455. DOI 10.1007/BF01934318.

[K] A. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), 515–530. MR 1315461. DOI 10.1093/qmath/45.4.515.

[Ko] K. Kodaira, *On compact analytic surfaces, II*, Ann. of Math. (2) **77** (1963), 563–626.

[MW] K. Matsuki and R. Wentworth, *Mumford–Thaddeus principle on the moduli space of vector bundles on an algebraic surface*, Internat. J. Math. **8** (1997), 97–148. MR 1433203. DOI 10.1142/S0129167X97000068.

[Mu1] S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175. MR 0607081.

[Mu2] ———, "On the moduli space of bundles on $K3$ surfaces, I" in *Vector bundles on Algebraic Varieties (Bombay, 1984)*, Tata Inst. Fund Res. Stud. Math. **11**, Tata Inst. Fund Res., Bombay, 1987, 341–413. MR 0893604.

[Mu3] ———, "Duality of polarized $K3$ surfaces" in *New Trends in Algebraic Geometry (Warwick, 1996)*, London Math. Soc. Lecture Note Ser. **264**, Cambridge Univ. Press, Cambridge, 1999, 311–326. MR 1714828. DOI 10.1017/CBO9780511721540.012.

[N1] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras*, Duke Math. J. **76** (1994), 365–416. MR 1302318. DOI 10.1215/S0012-7094-94-07613-8.

[N2] ———, *Sheaves on ALE spaces and quiver varieties*, Mosc. Math. J. **7** (2007), 699–722. MR 2372210.

[NY1] H. Nakajima and K. Yoshioka, "Perverse coherent sheaves on blow-up, I: A quiver description" in *Exploring New Structures and Natural Constructions in Mathematical Physics*, Adv. Stud. Pure Math. **61**, Math. Soc. Japan, Tokyo, 2011, 349–386. MR 2867152.

[NY2] ———, *Perverse coherent sheaves on blow-up, II: Wall-crossing and Betti numbers formula*, J. Algebraic Geom. **20** (2011), 47–100. MR 2729275. DOI 10.1090/S1056-3911-10-00534-5.

[NY3] ———, *Perverse coherent shaves on blow-up, III: Blow-up formula from wall-crossing*, Kyoto J. of Math. **51** (2011), 263–335. MR 2793270. DOI 10.1215/21562261-1214366.

[OY] N. Onishi and K. Yoshioka, *Singularities on the 2-dimensional moduli spaces of stable sheaves on $K3$ surfaces*, Internat. J. Math. **14** (2003), 837–864. MR 2013148. DOI 10.1142/S0129167X03002022.

[O] D. Orlov, *Equivalences of derived categories and $K3$ surfaces*, J. Math. Sci. (New York) **84** (1997), 1361–1381. MR 1465519. DOI 10.1007/BF02399195.

[ST] P. Seidel and R. P. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), 37–108. MR 1831820. DOI 10.1215/S0012-7094-01-10812-0.

[S] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety, I*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 47–129. MR 1307297.

[T] Y. Toda, *Hilbert schemes of points via McKay correspondences*, preprint, arXiv:math/0508555v1 [math.AG].

[VB] M. Van den Bergh, *Three-dimensional flops and noncommutative rings*, Duke Math. J. **122** (2004), 423–455. MR 2057015. DOI 10.1215/S0012-7094-04-12231-6.

[Y1] K. Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. **321** (2001), 817–884. MR 1872531. DOI 10.1007/s002080100255.

[Y2] ———, *Twisted stability and Fourier–Mukai transform, I*, Compositio Math. **138** (2003), 261–288. MR 2019443. DOI 10.1023/A:1027304215606.

[Y3] ———, *Twisted stability and Fourier–Mukai transform, II*, Manuscripta Math. **110** (2003), 433–465. MR 1975096. DOI 10.1007/s00229-002-0340-6.

[Y4] ———, "Moduli of twisted sheaves on a projective variety" in *Moduli Spaces and Arithmetic Geometry*, Adv. Stud. Pure Math. **45**, Math. Soc. Japan, Tokyo, 2006, 1–30. MR 2306170.

[Y5] ———, *Stability and the Fourier–Mukai transform, II*, Compos. Math. **145** (2009), 112–142. MR 2480497. DOI 10.1112/S0010437X08003758.

[Y6] ———, "An action of a Lie algebra on the homology groups of moduli spaces of stable sheaves" in *Algebraic and Arithmetic Structures of Moduli Spaces (Sapporo, 2007)*, Adv. Stud. Pure Math. **58**, Math. Soc. Japan, Tokyo, 2010, 403–459. MR 2676164.

[Y7] ———, *Perverse coherent sheaves and Fourier–Mukai transforms on surfaces, II*, preprint.

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