Kenji Fukaya

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Kenji Fukaya is an extraordinary geometer who foresees important ideas and the directions of future research. He has proved fundamental results and stimulated research areas that have attracted the attention of many mathematicians. Fukaya is very vigorous and open to discussions on any research subjects in mathematics. He listens to people patiently. He prefers deep, original ideas rather than fashionable subjects.

Fukaya's early works were on Riemannian geometry. In the 1970s, M. Gromov introduced a new viewpoint to study Riemannian manifolds, namely, the Gromov–Hausdorff spaces, which revolutionized Riemannian geometry. Fukaya made many important contributions to understanding the structure of manifolds that converge in Gromov–Hausdorff topology and their limits. For example, with J. Cheeger and M. Gromov [3], he established a unified approach to collapsing theory with sectional curvature bounds. With T. Yamaguchi [20], he gave a satisfactory description of the structure of almost nonnegatively curved manifolds.

In the 1980s, while studying the Laplacian spectrum under Gromov–Hausdorff convergence, Fukaya introduced the notion of measured Gromov–Hausdorff convergence and obtained continuity results on the spectrum (see [7]). He was one of the first to realize the importance of this setting, and now many mathematicians are working within the framework of metric measure spaces.

Fukaya has made significant contributions to gauge theory, symplectic geometry, and mirror symmetry. In the mid-1980s, Floer initiated semi-infinite-dimensional Morse theory, which is now called *Floer theory*, in the contexts of the action functional associated with Lagrangian intersections (Lagrangian Floer theory) and the Chern–Simons functional for SU(2)-connections on integral homology spheres (instanton homology theory) (see [5], [6]).

The famous Atiyah–Floer conjecture states that instanton homology is isomorphic to the Floer theory of intersections of Lagrangian subspaces associated with handlebodies appearing in the Heegaard splitting of the homology 3-sphere. One may also study instanton homology theory for 3-dimensional manifolds with boundary, which are more general than handlebodies.

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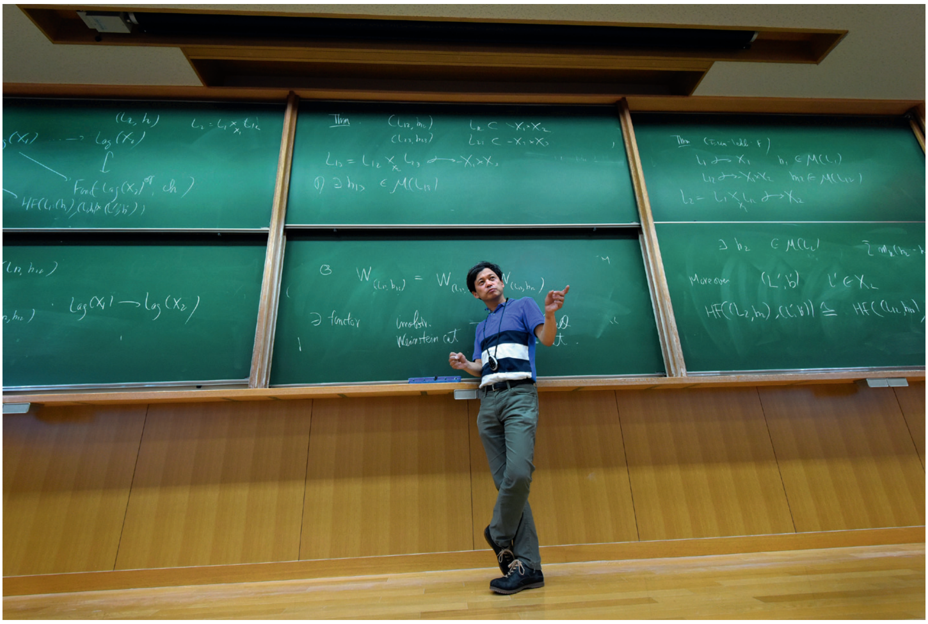

Kenji Fukaya, 17th Takagi Lectures, Kyoto University, June 2016.
Photo by Hiroaki Kohno.

Around 1990, Fukaya started working on gauge theory, in particular, instanton homology theory. His achievements include the development of instanton homology for closed oriented 3-manifolds and the establishment of the connected sum formula for instanton homology (see [8], [11]).

In 1992, at a conference at the University of Warwick, Donaldson suggested a categorical way to approach topological field theory that yields a description of instanton homology under gluing along a surface, not necessarily coming from a Heegaard splitting. Fukaya refined this approach and discovered a new mathematical structure, which is the Fukaya category of Lagrangians in a symplectic manifold (see [9], [10]). The Fukaya category has attracted much attention since Kontsevich [23] formulated the homological mirror symmetry conjecture as an equivalence between the (derived) Fukaya category on the A-side and the derived category of coherent sheaves on the B-side.

Gromov [21] invented the theory of pseudoholomorphic curves, which is the basis of modern symplectic geometry. To make the theory work on general symplectic manifolds, Fukaya and Ono [18] introduced the notion of Kuranishi structure and constructed virtual fundamental cycles/chains on moduli spaces of holomorphic curves and Floer trajectories in Hamiltonian Floer theory. Based on these, they defined Gromov–Witten invariants and proved the homology version of Arnold's conjecture for the number of fixed points of nondegenerate Hamiltonian diffeomorphisms.

Their machinery was further developed in joint works of Fukaya, Oh, Ohta, and Ono (see [16], [17]). They obtained the filtered A_∞-structure associated with

Lagrangian submanifolds and the deformation-obstruction theory for defining the Lagrangian Floer complex. This work led to the construction of the Fukaya category.

Recently, Fukaya [15] realized that this framework is indispensable to gauge theory in dimensions 2, 3, and 4. In particular, he came up with various important results, including a variant of the Atiyah–Floer conjecture (joint works with Deami and Lipyanskiy, [4] and others in preparation), 25 years after his discovery of the Fukaya category, which is the A_∞-structure arising from moduli spaces of holomorphic polygons. This manifests his far-reaching vision in mathematics.

Fukaya has been a great source of ideas, and he has generously presented them on various occasions. For example, around 2000, Fukaya started what he calls the "family Floer homology" project, which bridges Lagrangians on the A-side (symplectic side) and complexes of coherent sheaves on the B-side (complex side) (see [12]). This direction was further studied by Abouzaid [1]. Fukaya was also one of the first to predict how holomorphic disks on the A-side reflect to the "quantum" contribution to the complex structure on the mirror on the B-side around the same time (see [13]).

Fukaya has also made significant contributions to symplectic geometry. These include topological restrictions to Lagrangian submanifolds that are embedded in the standard symplectic vector space (see [14]; the argument was completed by Irie [22]) and major progress toward the "nearby Lagrangian conjecture" in a joint work with Seidel and Smith [19], which influenced subsequent work of Abouzaid and Kragh [2]. These are only some examples of his deep insight and achievements that significantly influenced many aspects of geometry.

This collection grew out of the conference "Fukaya 60: Geometry and Everything," which was held February 17–22, 2019, to celebrate Fukaya's 60th birthday. "Geometry and Everything" was the name of a working seminar that we used to have at the University of Tokyo in the early 1990s to discuss various topics in mathematics. Many young mathematicians, under the leadership of Kenji Fukaya and Mikio Furuta, attended the meetings, which would start in the late afternoon and last until late in the evening.

The conference was organized by Manabu Akaho (Tokyo Metropolitan University), Koji Fujiwara (Kyoto University), Tsuyoshi Kato (Kyoto University), Hiroshi Ohta (Nagoya University), and Kaoru Ono (RIMS, Kyoto University). The invited speakers were Mohammed Abouzaid (Columbia University), Denis Auroux (Harvard University), Paul Biran (ETH Zurich), Kai Cieliebak (Augsburg University), Aliakbar Daemi (SCGP), Yakov Eliashberg (Stanford University), Mark Gross (Cambridge University), Helmut Hofer (IAS), Ko Honda (UCLA), Shouhei Honda (Tohoku University), Kentaro Hori (Kavli IPMU), Kei Irie (University of Tokyo), Suguru Ishikawa (RIMS, Kyoto University), Dominic Joyce (Oxford University), Janko Latschev (Hamburg University), Ciprian Manolescu (UCLA), Mark McLean (Stony Brook University), Hiraku Nakajima (Kavli IPMU), Yong-Geun Oh (IBS-CGP, Postech), John Pardon (Princeton University), Paul Seidel (MIT), Nick Sheridan (University of Edinburgh), Ivan Smith (Cambridge University), Gang Tian (Peking University), and Aleksey Zinger (Stony Brook University).

See https://www.comp.tmu.ac.jp/pseudoholomorphic/FUKAYA60.html for more information about the conference.

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