

Eight flavors of cyclic homology

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Abstract We introduce eight “flavors” of cyclic homology of a mixed complex and study their properties. In particular, we determine their behavior with respect to Chen’s iterated integrals.

1. Introduction

Cyclic homology of an algebra was introduced in the mid-1980s by Tsygan in [20] and Connes in [9]. It can be seen as an algebraic counterpart to the S^1 -equivariant homology of a space with a circle action. Since then, cyclic homology has been generalized to *differential graded algebras (dgas)*, A_∞ -algebras, and beyond. Moreover, several other versions of cyclic homology have emerged: negative cyclic homology (see Jones [16]), periodic cyclic homology (see Goodwillie [12]), and completed negative cyclic homology (see Jones and Petrack [17]).

Cyclic homology is related to the homology of loop spaces in two different ways. First, for each connected space X , a suitable version of cyclic homology of singular chains on the based loop space of X (made a dga by the Pontryagin product) is isomorphic to the S^1 -equivariant homology of its free loop space LX . The second relation, which was the starting point of this research, goes back to the work by Chen in [5] and [6]. For a simply connected manifold X , Chen showed that the singular cohomology of its based loop space can be computed in terms of iterated integrals of differential forms. Getzler, Jones, and Petrack in [11] extended this result to the homology and the S^1 -equivariant homology of the free loop space LX .

The goal of the present article is to provide a systematic study of the different versions of cyclic homology and their relation to loop space homology via Chen’s iterated integrals. The natural setting is that of a *mixed complex*, introduced by Kassel in [18] (and popularized by Getzler, Jones, and Petrack in [11] under the name dg- Λ -module). This is a \mathbb{Z} -graded vector space C together with two anticommuting differentials δ, D of degrees $|\delta| = 1$ and $|D| = -1$. The two main examples are the Hochschild complex $(C(A), d_H, B)$ of a dga A with its Connes operator B and the singular cochain complex $(C^*(Y), d, P)$ of an S^1 -space with its contraction P by the circle action (see Section 2 for details).

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For a mixed complex (C, δ, D) , the map $\delta_u = \delta + uD$ defines a differential of degree 1 on the space $C[[u, u^{-1}]]$ of formal power series in a degree 2 variable u and its inverse. This complex has five subcomplexes $C[[u, u^{-1}]]$, $C[u, u^{-1}]$, $C[u, u^{-1}]$, $C[[u]]$, and $C[u]$, corresponding to power series that are polynomial in u^{-1} , and so on, and two quotient complexes $C[[u^{-1}]] = C[[u, u^{-1}]]/uC[[u]]$ and $C[u^{-1}] = C[u, u^{-1}]/uC[u]$. These give rise to eight versions (or “flavors”) of cyclic homology that we denote by $HC^{[[u, u^{-1}]}$ and so on.¹ The eight versions of cyclic homology are in general all different, and they are all invariant under homotopy equivalences of mixed complexes. However, only the three versions $HC^{[u^{-1}]}$, $HC^{[[u]]}$, and $HC^{[[u, u^{-1}]}$ are invariant under quasi-isomorphisms of mixed complexes (Proposition 2.3). These correspond to the positive, negative, and periodic cyclic homologies in [16], and we will refer to them as the *classical versions*.

For the Hochschild complex of a dga (or more generally of a cyclic cochain complex), each version of cyclic homology is either trivial or agrees with one of the three classical versions (Corollary 2.14). Moreover, the version $HC^{[u^{-1}]}$ agrees with *Connes’s version* HC^λ , defined as the d_H -homology of the Hochschild complex modulo cyclic permutations.

For the singular cochain complex of a smooth S^1 -space Y (such as the free loop space of a manifold), the version $HC^{[[u]]}$ of cyclic homology agrees with the singular cohomology $H_{S^1}^*(Y)$ of its Borel space $(Y \times ES^1)/S^1$ (see Jones [16]). The (nonclassical) version $HC^{[[u, u^{-1}]}$ satisfies fixed point localization (see Jones and Petrack [17]), whereas the version $HC^{[[u, u^{-1}]}$ for a free loop space $Y = LX$ depends only on the fundamental group of X (see Goodwillie [12]).

Consider now a manifold X with its de Rham dga $\Omega^*(X)$. According to Getzler, Jones, and Petrack in [11] (see also Proposition 3.1 below), Chen’s iterated integrals define a morphism of mixed complexes

$$I : (C(\Omega^*(X)), d_H, B) \rightarrow (C^*(LX), d, P).$$

For X simply connected this map is a quasi-isomorphism, so it induces isomorphisms on the three classical versions of cyclic homology. On the other five versions it does not induce an isomorphism in general. Their main result (Corollary 3.6) asserts that, for a simply connected manifold X , the cyclic (or Connes) variant I^λ of Chen’s iterated integral induces an isomorphism

$$I_*^\lambda : \overline{HC}_*^\lambda(\Omega^*(X)) \xrightarrow{\cong} H_{S^1}^*(LX, x_0)$$

between the reduced Connes version of cyclic homology of the de Rham complex and the S^1 -equivariant cohomology of LX relative to a fixed constant loop x_0 .

All the preceding results have counterparts for cyclic *cohomology*. For example (Corollary 4.11), for a simply connected manifold X the map J_λ adjoint to I^λ induces an isomorphism $J_{\lambda*} : H_*^{S^1}(LX, x_0) \xrightarrow{\cong} \overline{HC}_\lambda^*(\Omega^*(X))$.

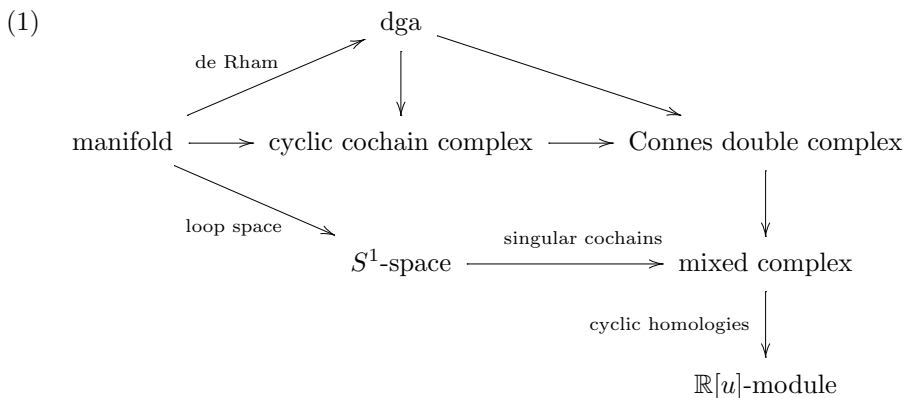
¹Note that the differential δ_u computing cyclic homology has degree +1; see Remark 2.1(c) for further discussion of our grading convention.

The motivation for this article comes from *string topology*. This term refers to algebraic structures on loop space homologies introduced by Chas and Sullivan in [4] and subsequent work. One of the puzzles in string topology concerns the appropriate versions of loop space homology on which these structures are defined. In the nonequivariant case, it has recently turned out that the Chas–Sullivan loop product and the Goresky–Hingston coproduct both naturally live on $H_*(LX)$ if the manifold X has vanishing Euler characteristic and on $H_*(LX, x_0)$ otherwise (see [8]).

In the equivariant case, Chas and Sullivan described in [3] an involutive Lie bialgebra structure on $H_*^{S^1}(LX, X)$, the S^1 -equivariant homology of the loop space LX relative to the subset $X \subset LX$ of constant loops. In [7] it was conjectured that a chain-level version of this structure exists on $HC_\lambda^*(\Omega^*(X))$, the Connes version of cyclic cohomology of the de Rham complex, so the question arose what this corresponds to on the loop space side. This question became more pressing when computations of examples showed that $HC_\lambda^*(\Omega^*(X))$ can be nontrivial in negative degrees and, thus, cannot correspond to any version of loop space homology. The solution is provided by Corollary 4.11: the negative degree part in $HC_\lambda^*(\Omega^*(X))$ only comes from the cohomology of a point, and after dividing this out (i.e., passing to reduced cohomology) it becomes isomorphic to $H_*^{S^1}(LX, x_0)$. In particular, this exhibits loop space homology relative to a point as the natural space supporting the involutive Lie bialgebra structure from [3].

2. Cyclic homology of mixed complexes

In this section we introduce the eight versions of cyclic homology associated to a mixed complex and discuss their properties. Moreover, we will establish the following diagram in which all maps are functors and the upper triangles commute, whereas the lower pentagon commutes for simply connected manifolds. (The map from dgas to Connes double complexes actually factors through A_∞ -algebras, but we will not discuss those here.)



2.1. Mixed complexes

DEFINITION 1

A *mixed complex* (C, δ, D) is a \mathbb{Z} -graded \mathbb{R} -vector space²

$$C = \bigoplus_{k \in \mathbb{Z}} C_k$$

with two linear maps $\delta, D : C \rightarrow C$ of degrees $|\delta| = 1$ and $|D| = -1$ satisfying

$$\delta^2 = 0, \quad D^2 = 0, \quad \delta D + D\delta = 0.$$

A *morphism* between mixed complexes is a linear map $f : C \rightarrow \tilde{C}$ of degree 0 satisfying

$$\tilde{\delta}f = f\delta, \quad \tilde{D}f = fD.$$

It is called a *quasi-isomorphism* if it induces an isomorphism on homology $H(C, \delta) \rightarrow H(\tilde{C}, \tilde{\delta})$. A *homotopy* between two morphisms $f, g : C \rightarrow \tilde{C}$ is a linear map $H : C \rightarrow \tilde{C}$ of degree -1 such that $\delta H + H\delta = f - g$ and $DH + HD = 0$. A morphism $f : C \rightarrow \tilde{C}$ is called a *homotopy equivalence* if there exists a morphism $g : \tilde{C} \rightarrow C$ such that fg and gf are homotopic to the identity. Every homotopy equivalence is a quasi-isomorphism but not vice versa.

Let u be a formal variable of degree $|u| = 2$. To a mixed complex (C, δ, D) we associate the cochain complex

$$C[[u, u^{-1}]] := \bigoplus_{k \in \mathbb{Z}} C_k[[u, u^{-1}]], \quad \delta_u := \delta + uD,$$

where $C_k[[u, u^{-1}]]$ denotes the space of formal power series $\sum_{i \in \mathbb{Z}} c_i u^i$ with $c_i \in C_{k-2i}$. We emphasize that $C[[u, u^{-1}]]$ is *not* the usual tensor product of C with $\mathbb{R}[[u, u^{-1}]]$. Note that δ_u has degree $+1$. This complex has seven sub/quotient complexes of interest, all equipped with the differential induced by δ_u :

$$(2) \quad \begin{aligned} & C[[u, u^{-1}]], \quad C[u, u^{-1}], \quad C[u, u^{-1}], \quad C[[u]], \quad C[u], \\ & C[[u^{-1}]] := C[[u, u^{-1}]]/uC[[u]] = C[u, u^{-1}]/uC[u], \\ & C[u^{-1}] := C[[u, u^{-1}]]/uC[[u]] = C[u, u^{-1}]/uC[u]. \end{aligned}$$

Here the complexes in the first line are the obvious subcomplexes of $C[[u, u^{-1}]]$, where $C[[u, u^{-1}]]$ denotes power series in u and polynomials in u^{-1} and so on, and the remaining two complexes are quotients. Note that

$$C[u, u^{-1}] = u^{-1}C[u], \quad C[[u, u^{-1}]] = u^{-1}C[[u]],$$

where the right-hand sides denote the localization of $C[u]$ (resp., $C[[u]]$) at the multiplicative set $\{1, u, u^2, \dots\}$. We denote the homology of $C[[u, u^{-1}]]$ with respect to δ_u by

²Most of this section works for modules over a commutative ring with unit instead of \mathbb{R} -vector spaces.

$$HC^{[[u, u^{-1}]]} := H(C[[u, u^{-1}]], \delta_u)$$

and similarly for the other versions with the obvious notation. By construction, all the chain complexes and thus also their homologies are modules over the polynomial ring $\mathbb{R}[u]$. Moreover, the versions $C[[u, u^{-1}]]$, $C[[u, u^{-1}]]$, $C[u, u^{-1}]$, $C[u, u^{-1}]$, and their cohomologies are modules over the larger ring $\mathbb{R}[u, u^{-1}]$ of Laurent polynomials. We will use the following names for some versions of cyclic homology:

- $HC^{[[u]]}$ *Borel version*,
- $HC^{[[u, u^{-1}]]}$ *Goodwillie version*,
- $HC^{[u^{-1}]}$ *nilpotent version*,
- $HC^{[[u, u^{-1}]]}$ *Jones–Petrack version*.

The first three of these versions will also be called the *classical versions*.

REMARK 2.1

(a) The explanations for the preceding names are the following (see later in this section for details): $HC^{[[u]]}$ applied to cochains on an S^1 -space yields the cohomology of its Borel construction; $HC^{[[u, u^{-1}]]}$ satisfies Goodwillie’s theorem: applied to cochains on a loop space LX it depends only on $\pi_1(X)$; $HC^{[u^{-1}]}$ is the version for which the action of u is nilpotent; $HC^{[[u, u^{-1}]]}$ applied to a smooth S^1 -space satisfies the fixed point localization theorem of Jones and Petrack.

(b) The notion of a mixed complex was introduced by Kassel in [18]. Its name reflects the fact that δ has degree $+1$ while D has degree -1 . Mixed complexes also appear in [11] under the name *dg- Λ -module*.

(c) Note that the differential δ_u computing cyclic homology has degree $+1$. This convention agrees with the one in [11], and it is related to the one in [18], [16], and [19] by replacing all degrees by their negatives. More precisely, if C_* is a mixed complex in our sense, then C_{-*} is a mixed complex in the sense of [18], [16], and [19] and their cyclic homology $H_*(C_{-*})$ equals our cyclic homology $H_{-*}(C_*)$ in negative degrees. Our grading convention avoids unnecessary minus signs in our main examples which arise from cochain complexes, such as the de Rham complex of a manifold or the singular cochain complex of its free loop space.

EXAMPLE 2.2

Consider the mixed complex with $C_k := \mathbb{R}$ in each degree $k \in \mathbb{Z}$ and trivial differentials $d = D = 0$. Then in each degree k and for each version $\{u, u^{-1}\}$ we have $HC_k^{\{u, u^{-1}\}} = C_k^{\{u, u^{-1}\}} = \mathbb{R}\{u, u^{-1}\}$, so all eight versions of cyclic homology are pairwise nonisomorphic as $\mathbb{R}[u]$ -modules.

Quasi-isomorphism invariance. A morphism f between mixed complexes (C, δ, D) and $(\tilde{C}, \tilde{\delta}, \tilde{D})$ induces homomorphisms f_* between all versions of homology defined above as modules over $\mathbb{R}[u]$ (resp., $\mathbb{R}[u, u^{-1}]$). Clearly f_* is an isomorphism if f is a homotopy equivalence. We say that a version of homology is

quasi-isomorphism invariant if the induced map f_* is an isomorphism whenever f is a quasi-isomorphism of mixed complexes.

PROPOSITION 2.3

The three classical versions $HC^{[[u]]}$, $HC^{[[u,u^{-1}]}$, and $HC^{[u^{-1}]}$ of cyclic homology are quasi-isomorphism invariants of mixed complexes, whereas the other five versions are not.

Proof

The quasi-isomorphism invariance of the three classical versions is proved in [16, Lemma 2.1] in the special case of cyclic chain complexes and in [21, Proposition 2.4] in the more general context of S^1 -complexes (cf. Remark 2.4 below). Examples 2.27, 2.28, and 2.29 below in conjunction with Lemma 2.30 show that the other five homology groups are not quasi-isomorphism invariant. \square

REMARK 2.4

Much of the preceding discussion can be generalized to S^1 -complexes as defined in [21]. This generalization is relevant if one wants to include symplectic homology in this framework, but will not be further discussed here.

Exact sequences. The eight versions of cyclic homology are connected by various exact sequences fitting into commuting diagrams.

PROPOSITION 2.5 ([15, Figure 1])

For every mixed complex (C, δ, D) there exists a commuting diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & \dots & & \dots & & \\
 & & HC_{*+1}^{[u^{-1}]} & \xrightarrow[\text{=}]{\text{id}} & HC_{*+1}^{[u^{-1}]} & & \\
 & & \downarrow D_0 & & \downarrow D_0 & & \\
 \dots HC_{*-2}^{[u]} & \xrightarrow{\cdot u} & HC_*^{[u]} & \xrightarrow{u=0} & H_*(C, \delta) & \xrightarrow{D_*} & HC_{*-1}^{[u]} \dots \\
 & = \downarrow \text{id} & \downarrow i_* & & \downarrow i_* & & = \downarrow \text{id} \\
 \dots HC_{*-2}^{[u]} & \xrightarrow{\cdot u} & HC_*^{[u, u^{-1}]} & \xrightarrow{p_*} & HC_*^{[u^{-1}]} & \xrightarrow{D_{0*}} & HC_{*-1}^{[u]} \dots \\
 & & \downarrow p_* \cdot u & & \downarrow \cdot u & & \\
 & & HC_{*+2}^{[u^{-1}]} & \xrightarrow[\text{=}]{\text{id}} & HC_{*+2}^{[u^{-1}]} & & \\
 & & \dots & & \dots & &
 \end{array}$$

and similarly for the $[u, u^{-1}]$, $[[u, u^{-1}]$, and $[[u, u^{-1}]$ versions. Here D_0 means D applied to the constant term in u , and the other maps are the obvious ones.

We will refer to the second vertical and the first horizontal sequences as the *Gysin* (or *Connes*) exact sequences and to the first vertical and the second horizontal

sequences (which are equivalent in view of the periodicity $HC_{*-2}^{[u, u^{-1}]} \cong HC_*^{[u, u^{-1}]}$) as the *tautological exact sequences*.

Proof

This diagram appears in [15, Figure 1]. It follows from the commuting square of short exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & uC[u] & \longrightarrow & C[u] & \longrightarrow & C[u]/uC[u] \longrightarrow 0 \\
 & & = \downarrow \text{id} & & \downarrow i & & \downarrow i \\
 0 & \longrightarrow & uC[u] & \longrightarrow & C[u, u^{-1}] & \xrightarrow{p} & C[u, u^{-1}]/uC[u] \longrightarrow 0 \\
 & & \downarrow & & \downarrow p & & \downarrow p \\
 & & 0 & \longrightarrow & C[u, u^{-1}]/C[u] & \xrightarrow[\cong]{\text{id}} & C[u, u^{-1}]/C[u] \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

together with the identifications $C[u]/uC[u] = C$, $C[u, u^{-1}]/uC[u] = C[u^{-1}]$, and $C[u, u^{-1}]/C[u] = u^{-1}C[u^{-1}]$. \square

Other homologies. Given a mixed complex (C, δ, D) , the chain complex (C, δ) has two natural subcomplexes $\text{im } D \subset \ker D \subset C$ which together with their quotient complexes fit into the commuting diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \ker D / \text{im } D & & \\
 & & 0 & & 0 & & \ker D / \text{im } D \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{im } D & \longrightarrow & (C, \delta) & \longrightarrow & C / \text{im } D \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \longrightarrow & \ker D & \longrightarrow & (C, \delta) & \longrightarrow & C / \ker D \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \ker D / \text{im } D & & 0 & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

On homology this yields the following commuting diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 H_{*-1}(\ker D / \operatorname{im} D) & & & & H_*(\ker D / \operatorname{im} D) & \xlongequal{\quad} & H_*(\ker D / \operatorname{im} D) \\
 \downarrow \delta_* & & & & \downarrow & & \downarrow \delta_* \\
 \cdots H_*(\operatorname{im} D) & \longrightarrow & H_*(C, \delta) & \longrightarrow & H_*(C / \operatorname{im} D) & \xrightarrow{\delta_*} & H_{*+1}(\operatorname{im} D) \cdots \\
 \downarrow & & \parallel & & \downarrow & & \downarrow \\
 \cdots H_*(\ker D) & \longrightarrow & H_*(C, \delta) & \longrightarrow & H_*(C / \ker D) & \xrightarrow{\delta_*} & H_{*+1}(\ker D) \cdots \\
 \downarrow & & & & \downarrow \delta_* & & \downarrow \\
 H_*(\ker D / \operatorname{im} D) & & & & H_{*+1}(\ker D / \operatorname{im} D) & \xlongequal{\quad} & H_{*+1}(\ker D / \operatorname{im} D)
 \end{array}$$

A morphism f of mixed complexes induces homomorphisms between all these homologies, which are isomorphisms if f is a homotopy equivalence. Note that the map D_0 in Proposition 2.5 naturally factors through chain maps (where D has degree -1)

$$(3) \quad D_0 : (C[[u^{-1}]], \delta_u) \xrightarrow{\pi_0} (C / \operatorname{im} D, \delta) \xrightarrow{D} (\operatorname{im} D, \delta) \xrightarrow{\iota_0} (C[u], \delta_u).$$

2.2. Cocyclic and cyclic objects

One source of mixed complexes are cyclic cochain complexes which we introduce in this section. For more background see [19, Section 6.1].

Cocyclic objects. A *cocyclic object* in some category is a sequence of objects C_n , $n \in \mathbb{N}_0$, with morphisms

- $\delta_i : C_{n-1} \rightarrow C_n$, $i = 0, \dots, n$ (faces),
- $\sigma_j : C_{n+1} \rightarrow C_n$, $j = 0, \dots, n$ (degeneracies),
- $\tau_n : C_n \rightarrow C_n$ (cyclic operators)

satisfying the following relations:

$$\begin{aligned}
 \delta_j \delta_i &= \delta_i \delta_{j-1} \quad \text{for } i < j, \\
 \sigma_j \sigma_i &= \sigma_i \sigma_{j+1} \quad \text{for } i \leq j, \\
 \sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & \text{for } i < j, \\ \operatorname{id} & \text{for } i = j \text{ or } i = j + 1, \\ \delta_{i-1} \sigma_j & \text{for } i > j + 1, \end{cases} \\
 \tau_n \delta_i &= \delta_{i-1} \tau_{n-1} \quad \text{for } 1 \leq i \leq n, & \tau_n \delta_0 &= \delta_n, \\
 \tau_n \sigma_i &= \sigma_{i-1} \tau_{n+1} \quad \text{for } 1 \leq i \leq n, & \tau_n \sigma_0 &= \sigma_n \tau_{n+1}^2, \\
 \tau_n^{n+1} &= \operatorname{id}.
 \end{aligned}$$

Forgetting τ_n we have a *cosimplicial object*, and forgetting the σ_j 's, we have a *pre-cocyclic object*.

EXAMPLE 2.6

A topological space X gives rise to a cocyclic space by setting $C_n(X) := X^{n+1} = X \times \cdots \times X$ ($n+1$ times) and

$$\delta_i(x_0, \dots, x_{n-1}) := (x_0, \dots, x_i, x_i, \dots, x_{n-1}) \quad \text{for } 0 \leq i \leq n-1,$$

$$\delta_n(x_0, \dots, x_{n-1}) := (x_0, \dots, x_{n-1}, x_0),$$

$$\sigma_j(x_0, \dots, x_{n+1}) := (x_0, \dots, \widehat{x}_i, \dots, x_{n+1}),$$

$$\tau_n(x_0, \dots, x_n) := (x_1, \dots, x_n, x_0).$$

For a subspace $Y \subset X$ we can define a cosimplicial space by $C_n(X, Y) := X^n \times Y$ and the same operations δ_i, σ_j (appropriately viewing elements in Y as elements in X via the inclusion).

Cyclic objects. Dualizing the notion of a cocyclic object, we obtain that of a cyclic object. A *cyclic object* in some category is a sequence of objects C_n , $n \in \mathbb{N}_0$, with morphisms

- $d_i : C_n \rightarrow C_{n-1}$, $i = 0, \dots, n$ (faces),
- $s_j : C_n \rightarrow C_{n+1}$, $j = 0, \dots, n$ (degeneracies),
- $t_n : C_n \rightarrow C_n$ (cyclic operators)

satisfying the following relations:

$$d_i d_j = d_{j-1} d_i \quad \text{for } i < j,$$

$$s_i s_j = s_{j+1} s_i \quad \text{for } i \leq j,$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j, \\ \text{id} & \text{for } i = j \text{ or } i = j + 1, \\ s_j d_{i-1} & \text{for } i > j + 1, \end{cases}$$

$$d_i t_n = t_{n-1} d_{i-1} \quad \text{for } 1 \leq i \leq n, \quad d_0 t_n = d_n,$$

$$s_i t_n = t_{n+1} s_{i-1} \quad \text{for } 1 \leq i \leq n, \quad s_0 t_n = t_{n+1}^2 s_n,$$

$$t_n^{n+1} = \text{id}.$$

Forgetting t_n we have a *simplicial object*, and forgetting the s_j 's, we have a *pre-cyclic object*. Note that if $F : A \rightarrow B$ is a contravariant functor, then a cocyclic object in the category A gives rise to a cyclic object $F(A)$ in the category B and vice versa.

The cyclic structure gives rise to the *extra degeneracy*

$$s_{n+1} := t_{n+1} s_n : C_n \rightarrow C_{n+1}$$

satisfying the following relations on C_n :

$$s_i s_{n+1} = s_{n+2} s_{i-1} \quad \text{for } 1 \leq i \leq n+1, \quad s_0 s_{n+1} = s_{n+2} s_{n+1},$$

$$d_i s_{n+1} = s_n d_{i-1} \quad \text{for } 1 \leq i \leq n, \quad d_{n+1} s_{n+1} = t_n, \quad d_0 s_{n+1} = \text{id},$$

$$s_{n+1} t_n = t_{n+1}^2 s_{n-1}.$$

EXAMPLE 2.7

For a manifold X denote by $C_n(\Omega^*(X)) := \Omega^*(X^{n+1})$ the dga of differential forms on the $(n + 1)$ -fold product of X . This becomes a cyclic object in the category of differential graded vector spaces (i.e., chain complexes) with the operations $d_i := \delta_i^*$, $s_j := \sigma_j^*$, and $t_n := \tau_n^*$ induced from those of Example 2.6. For a submanifold $Y \subset X$ we can define a simplicial chain complex by $C_n(\Omega^*(X), \Omega^*(Y)) := \Omega^*(X^n \times Y)$ and the same operations d_i, s_j (appropriately viewing forms on X as forms on Y via restriction). Note that $\Omega^*(Y)$ is an $\Omega^*(X)$ -bimodule.

EXAMPLE 2.8

A dga A over \mathbb{R} with unit 1 gives rise to a cyclic object in the category of differential graded \mathbb{R} -modules by setting $C_n(A) := A^{\otimes(n+1)}$ and

$$d_i(a_0 \mid \cdots \mid a_n) := (a_0 \mid \cdots \mid a_i a_{i+1} \mid \cdots \mid a_n) \quad \text{for } 0 \leq i \leq n - 1,$$

$$d_n(a_0 \mid \cdots \mid a_n) := (-1)^{\deg a_n (\deg a_0 + \cdots + \deg a_{n-1})} (a_n a_0 \mid a_1 \mid \cdots \mid a_{n-1}),$$

$$s_j(a_0 \mid \cdots \mid a_n) := (a_0 \mid \cdots \mid a_j \mid 1 \mid a_{j+1} \mid \cdots \mid a_n) \quad \text{for } 0 \leq i \leq n,$$

$$t_n(a_0 \mid \cdots \mid a_n) := (-1)^{\deg a_n (\deg a_0 + \cdots + \deg a_{n-1})} (a_n \mid a_0 \mid \cdots \mid a_{n-1}).$$

Here for t_n we use the convention in [19], which differs from the one in [16]. The extra degeneracy becomes

$$s_{n+1}(a_0 \mid \cdots \mid a_n) = (1 \mid a_0 \mid \cdots \mid a_n).$$

For a differential graded A -bimodule M we can define a simplicial differential graded R -module by $C_n(A, M) := A^{\otimes n} \otimes M$ and the same operations d_i, s_j (using the bimodule structure).

For the de Rham complex, the constructions in Examples 2.7 and 2.8 give rise to two cyclic cochain complexes $\Omega^*(X^{n+1})$ and $\Omega^*(X)^{\otimes n+1}$, which we will refer to as the *analytic* and *algebraic* versions, respectively. They are compatible in the following sense.

LEMMA 2.9

The exterior cross products

$$\phi_n : \Omega^*(X)^{\otimes n+1} \rightarrow \Omega^*(X^{n+1}), \quad (a_0 \mid \cdots \mid a_n) \mapsto a_0 \times \cdots \times a_n$$

define a morphism of cyclic cochain complexes.

Proof

Each ϕ_n is clearly a chain map with respect to the exterior derivative. Recall that

$$a_0 \times \cdots \times a_n = \pi_0^* a_0 \wedge \cdots \wedge \pi_n^* a_n$$

for the canonical projections $\pi_i : X_n \rightarrow X$, $i = 0, \dots, n$. Note that the π_i 's and π_i^* 's are compositions of the degeneracies in Examples 2.6 and 2.7, respectively. Using this, one deduces that ϕ is a map of cyclic cochain complexes. For example, the relations

$$\pi_{i-1} \tau_n = \pi_i \quad \text{for } 1 \leq i \leq n, \quad \pi_n \tau_n = \pi_0$$

imply the compatibility of ϕ with $t_n = \tau_n^*$:

$$\begin{aligned} \phi t_n(a_0 \mid \cdots \mid a_n) &= (-1)^{\deg a_n (\deg a_0 + \cdots + \deg a_{n-1})} a_n \times a_0 \times \cdots \times a_{n-1} \\ &= (-1)^{\deg a_n (\deg a_0 + \cdots + \deg a_{n-1})} \pi_0^* a_n \wedge \pi_1^* a_0 \wedge \cdots \wedge \pi_n^* a_{n-1} \\ &= (-1)^{\deg a_n (\deg a_0 + \cdots + \deg a_{n-1})} \tau_n^* (\pi_n^* a_n \wedge \pi_0^* a_0 \wedge \cdots \wedge \pi_{n-1}^* a_{n-1}) \\ &= \tau_n^* (\pi_0^* a_0 \wedge \cdots \wedge \pi_n^* a_n) \\ &= t_n \phi(a_0 \mid \cdots \mid a_n). \end{aligned} \quad \square$$

2.3. From cyclic cochain complexes to mixed complexes

Consider a cyclic cochain complex (C_n, d_i, s_j, t_n, d) , where (C_n, d) are \mathbb{Z} -graded cochain complexes for $n \in \mathbb{N}_0$ with differential of degree $+1$. Thus, the operations d_i, s_j , and t_n commute with d and satisfy the relations in Section 2.2. We define a \mathbb{Z} -graded \mathbb{R} -vector space $(C, \mid \cdot \mid)$ by

$$C := \bigoplus_{n \geq 0} C_n, \quad |c| := \deg(c) - n \quad \text{for } c \in C_n.$$

We define the following operations on homogeneous elements $c \in C_n$:

$$\begin{aligned} b'(c) &:= (-1)^{|c|+1} \sum_{i=0}^{n-1} (-1)^i d_i(c) \in C_{n-1}, \\ b(c) &:= (-1)^{|c|+1} \sum_{i=0}^n (-1)^i d_i(c) \in C_{n-1}, \\ (4) \quad t &:= (-1)^n t_n : C_n \rightarrow C_n, \\ N &:= 1 + t + \cdots + t^n : C_n \rightarrow C_n, \\ s(c) &:= (-1)^{|c|} s_{n+1}(c) \in C_{n+1}, \\ B &:= (1 - t) s N : C_n \rightarrow C_{n+1}. \end{aligned}$$

The operations have the following degrees with respect to the grading on C :

$$|d| = |b| = |b'| = 1, \quad |t| = |N| = 0, \quad |s| = |B| = -1.$$

Straightforward computations yield the following relations:

$$\begin{aligned}
 \ker(1-t) &= \operatorname{im} N, & \ker N &= \operatorname{im}(1-t), & (1-t)b' &= b(1-t), \\
 (5) \quad b'N &= Nb, & (b')^2 &= b^2 = db + bd = db' + b'd = 0, \\
 ds + sd &= 0, & b's + sb' &= 1, & bs + sb' &= (1-t).
 \end{aligned}$$

REMARK 2.10

The signs in (4) are chosen for compatibility under Chen’s iterated integral in Section 3. An alternative sign convention is given in [19] (in the case of a cyclic module):

$$\begin{aligned}
 \tilde{b}' &:= \sum_{i=0}^{n-1} (-1)^i d_i : C_n \rightarrow C_{n-1}, \\
 \tilde{b} &:= \sum_{i=0}^n (-1)^i d_i : C_n \rightarrow C_{n-1}, \\
 (6) \quad \tilde{t} &:= (-1)^n t_n : C_n \rightarrow C_n, \\
 \tilde{N} &:= 1 + t + \dots + t^n : C_n \rightarrow C_n, \\
 \tilde{s} &:= s_{n+1} : C_n \rightarrow C_{n+1}, \\
 \tilde{B} &:= (1 - \tilde{t})\tilde{s}\tilde{N} : C_n \rightarrow C_{n+1}.
 \end{aligned}$$

Note that $\tilde{t} = t$ and $\tilde{N} = N$ are unchanged. In this case we need to modify the differential d to make it anticommute with \tilde{b} , \tilde{b}' , and \tilde{s} , so we set

$$\tilde{d}(c) := (-1)^{|c|+1} d(c).$$

It is straightforward to check that with these signs the relations (5) continue to hold. Moreover, the two sign conventions are intertwined by the chain isomorphism $\Phi : (C, \tilde{d}) \rightarrow (C, d)$ defined on homogeneous elements by $\Phi(c) := (-1)^{|c|(|c|+1)/2} c$.

Consider the space $C[[\theta, \theta^{-1}]]$ of Laurent series in a formal variable θ of degree $|\theta| = 1$, as usual, understood in the graded sense so that elements of degree k are sums $\sum_{i \in \mathbb{Z}} c_i \theta^i$ with $|c_i| = k - i$. We view this as a double complex where powers of θ increase in the horizontal direction and the columns are copies of C (see Figure 1). We define a differential

$$\delta_\theta := \delta_{\text{ver}} + \delta_{\text{hor}} : C[[\theta, \theta^{-1}]] \rightarrow C[[\theta, \theta^{-1}]]$$

of degree 1 with

$$(7) \quad \begin{aligned}
 \delta_{\text{ver}}(c\theta^n) &:= \begin{cases} (d+b)(c)\theta^n & n \text{ even,} \\ -(d+b')(c)\theta^n & n \text{ odd,} \end{cases} \\
 \delta_{\text{hor}}(c\theta^n) &:= \begin{cases} N(c)\theta^{n+1} & n \text{ even,} \\ (1-t)(c)\theta^{n+1} & n \text{ odd.} \end{cases}
 \end{aligned}$$

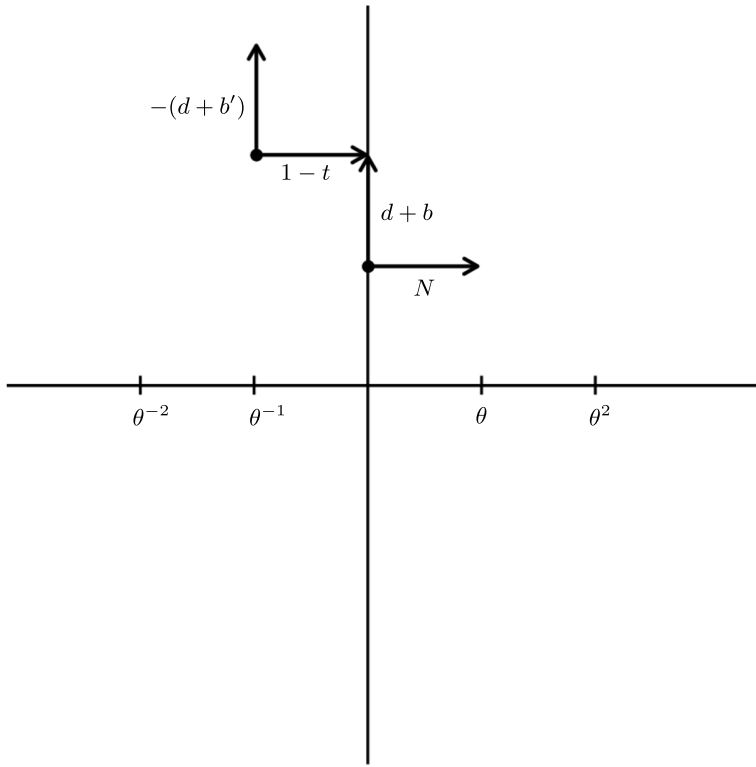


Figure 1. The double complex $C[[\theta, \theta^{-1}]]$.

The relations (5) imply that this is indeed a double complex. Its total differential δ_θ induces differentials on all seven sub- and quotient complexes $C[[\theta, \theta^{-1}]]$ and so on, defined as in (2) with u replaced by θ . The double complex $C[[\theta, \theta^{-1}]]$ has the following two key properties:

- (i) its odd columns are *contractible* with contracting homotopy $-s$, that is, $(d + b')s + s(d + b') = \text{id}$;
- (ii) its rows are exact (cf. [19, Theorem 2.1.5]).

LEMMA 2.11

Let (C_n, d_i, s_j, t_n, d) be a cyclic cochain complex. Then

$$(C, d_H := d + b, B)$$

is a mixed complex. Moreover, the chain map

$$k : C[[u, u^{-1}]] \longrightarrow C[[\theta, \theta^{-1}]], \quad \sum_j c_j u^j \mapsto \sum_j (c_j \theta^{2j} + s N c_j \theta^{2j+1})$$

induces an isomorphism on homology as an $\mathbb{R}[u, u^{-1}]$ -module, where u acts on $C[[\theta, \theta^{-1}]]$ as multiplication by θ^2 . A similar statement holds for all the other versions of homology. Moreover, these isomorphisms fit into a commuting diagram

$$(8) \quad \begin{array}{ccccccc} \cdots HC_{*-2}^{[u]} & \xrightarrow{\cdot u} & HC_*^{[u, u^{-1}]} & \xrightarrow{p_*} & HC_*^{[u^{-1}]} & \xrightarrow{B_0} & HC_{*-1}^{[u]} \cdots \\ \cong \downarrow k_* & & \cong \downarrow k_* & & \cong \downarrow k_* & & \cong \downarrow k_* \\ \cdots HC_{*-2}^{[\theta]} & \xrightarrow{\cdot \theta^2} & HC_*^{[\theta, \theta^{-1}]} & \xrightarrow{r_* p_*} & HC_*^{[\theta^{-1}]} & \xrightarrow{B_0} & HC_{*-1}^{[\theta]} \cdots \end{array}$$

and similarly for all other tautological and Connes exact sequences.

Proof

The proof is an easy adaptation of the arguments in [19, Section 2.1]: that $d + b$ and B are anticommuting differentials and k is a chain map follows by direct computation based on (5), and that k induces an isomorphism on homology follows from the contractibility of the odd columns of the double complex $C[[\theta, \theta^{-1}]]$ and [19, Lemma 2.1.6]. To derive the diagram (8), consider the commuting diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & uC[u] & \longrightarrow & C[u, u^{-1}] & \xrightarrow{p} & C[u^{-1}] \longrightarrow 0 \\ & & \downarrow k & & \downarrow k & & \downarrow q \\ 0 & \longrightarrow & \theta^2 C[\theta] & \longrightarrow & C[\theta, \theta^{-1}] & \xrightarrow{p} & \theta C[\theta^{-1}] \longrightarrow 0 \\ & & & & & & \downarrow r \\ & & & & & & C[\theta^{-1}] \end{array}$$

where q is the induced map between the quotients and r is the map dividing out the θ -column. Since that column is contractible, the induced map r_* on homology is an isomorphism, and we obtain on homology the diagram

$$\begin{array}{ccccccc} \cdots HC_{*-2}^{[u]} & \xrightarrow{\cdot u} & HC_*^{[u, u^{-1}]} & \xrightarrow{p_*} & HC_*^{[u^{-1}]} & \xrightarrow{B_0} & HC_{*-1}^{[u]} \cdots \\ \cong \downarrow k_* & & \cong \downarrow k_* & & \cong \downarrow q_* & & \cong \downarrow k_* \\ \cdots HC_{*-2}^{[\theta]} & \xrightarrow{\cdot \theta^2} & HC_*^{[\theta, \theta^{-1}]} & \xrightarrow{p_*} & H_*(\theta C[\theta^{-1}]) & \longrightarrow & HC_{*-1}^{[\theta]} \cdots \\ & & & & \cong \downarrow r_* & \nearrow B_0 & \\ & & & & HC_*^{[\theta^{-1}]} & & \end{array}$$

Using $r_* q_* = k_*$ we obtain (8). □

The homology $H(C, d_H := d + b)$ is the *Hochschild homology*, and the homologies $HC^{[[u, u^{-1}]}$ and so on are the various versions of *cyclic homology* of the cyclic cochain complex.

Connes's version of cyclic homology. Due to the relations (5) we have short exact sequences of cochain complexes

$$\begin{aligned} 0 \longrightarrow (\operatorname{im}(1-t), d+b) &\longrightarrow (C, d+b) \longrightarrow (C/\operatorname{im}(1-t), d+b) \longrightarrow 0, \\ 0 \longrightarrow (\operatorname{im} N, d+b') &\longrightarrow (C, d+b') \longrightarrow (C/\operatorname{im} N, d+b') \longrightarrow 0. \end{aligned}$$

The first one induces a long exact sequence

$$\begin{aligned} \cdots H_*(\operatorname{im}(1-t), d+b) &\longrightarrow H_*(C, d+b) \longrightarrow H_*(C/\operatorname{im}(1-t), d+b) \\ &\longrightarrow H_{*+1}(\operatorname{im}(1-t), d+b) \cdots \end{aligned}$$

and the second one, by the acyclicity of $(C, d+b')$, induces an isomorphism

$$H_*(C/\operatorname{im} N, d+b') \cong H_{*+1}(\operatorname{im} N, d+b').$$

Moreover, the chain isomorphisms $N : (C/\operatorname{im}(1-t), d+b) \xrightarrow{\cong} (\operatorname{im} N, d+b')$ and $1-t : (C/\operatorname{im} N, d+b') \xrightarrow{\cong} (\operatorname{im}(1-t), d+b)$ induce isomorphisms

$$\begin{aligned} N : H_*(C/\operatorname{im}(1-t), d+b) &\xrightarrow{\cong} H_*(\operatorname{im} N, d+b'), \\ 1-t : H_*(C/\operatorname{im} N, d+b') &\xrightarrow{\cong} H_*(\operatorname{im}(1-t), d+b). \end{aligned}$$

DEFINITION 2

Connes's version of cyclic homology of the cyclic cochain complex (C_n, d_i, s_j, t_n, d) is

$$\begin{aligned} HC_*^\lambda &:= H_{*+1}(C/\operatorname{im}(1-t), d+b) \cong H_{*+1}(\operatorname{im} N, d+b') \\ &\cong H_*(C/\operatorname{im} N, d+b') \cong H_*(\operatorname{im}(1-t), d+b). \end{aligned}$$

The following lemma identifies Connes's version of cyclic homology with two of the other versions.

LEMMA 2.12

Let (C_n, d_i, s_j, t_n, d) be a cyclic cochain complex. Then $HC_*^{[u, u^{-1}]} = HC_*^{[\theta, \theta^{-1}]} = 0$, and we have a commuting diagram

$$(9) \quad \begin{array}{ccc} HC_*^{[[u^{-1}]]} & \xrightarrow[\cong]{B_0} & HC_{*-1}^{[u]} \\ \cong \downarrow k_* & & \cong \downarrow k_* \\ HC_*^{[[\theta^{-1}]]} & \xrightarrow[\cong]{B_0} & HC_{*-1}^{[\theta]} \\ \cong \downarrow e_* & & \cong \uparrow ((1-t)s)_* \\ H_*(C/\operatorname{im}(1-t), d+b) & \xrightarrow[\cong]{N_*} & H_*(\operatorname{im} N, d+b') \end{array}$$

where the upper square arises from (8) and $e : C[[\theta^{-1}]] \rightarrow C/\text{im}(1-t)$ is the projection onto the 0th column.

Proof

Consider the double complex with exact rows $(C[\theta, \theta^{-1}], \delta_\theta)$. Standard zigzag arguments as in [19] (see also the proof of Lemma 2.13 below) show that its total homology vanishes and the map $e : C[[\theta^{-1}]] \rightarrow C/\text{im}(1-t)$ induces an isomorphism on homology. In view of (8) this yields the diagram (9), where the commutativity of the lower square follows from the definition of B . The maps B_0 are isomorphisms because the third terms in the corresponding tautological exact sequences vanish. Since N_* is an isomorphism, this implies that the map $((1-t)s)_*$ is an isomorphism as well. It is important to note, however, that these arguments fail for other versions of cyclic homology. \square

The following refinement of Lemma 2.12 will be crucial in our analysis here.

LEMMA 2.13

Let (C_n, d_i, s_j, t_n, d) be a cyclic cochain complex. Then the canonical inclusion $\phi : C[\theta^{-1}] \rightarrow C[[\theta^{-1}]]$ induces an isomorphism on homology

$$\phi_* : HC_*^{[\theta^{-1}]} \xrightarrow{\cong} HC_*^{[[\theta^{-1}]]}.$$

The same holds true for the inclusions $C[u^{-1}] \rightarrow C[[u^{-1}]]$, $C[u, u^{-1}] \rightarrow C[[u, u^{-1}]]$, and $C[[u, u^{-1}]] \rightarrow C[[[u, u^{-1}]]]$.

Proof

Consider the sequence of chain maps

$$(10) \quad (C[\theta^{-1}], \delta_\theta) \xrightarrow{\phi} (C[[\theta^{-1}]], \delta_\theta) \xrightarrow{e} (C/\text{im}(1-t), d+b).$$

Since e induces an isomorphism on homology by Lemma 2.12, it suffices to show that the composition $e\phi$ induces an isomorphism $(e\phi)_*$ on homology.

We call an element $c \in \theta^{-1}C[[\theta^{-1}]]$ a *resolution* of an element $x \in C$ if $\delta_\theta c = x$. (In particular, all nonzero powers of θ^{-1} cancel.) Since the rows of $C[[\theta^{-1}]]$ are exact, an element $x \in C$ admits a resolution if and only if $(d+b)x = 0$ and $x \in \text{im}(1-t) = \ker N$.

Claim. Every $x \in C$ with $(d+b)x = 0$ and $x \in \text{im}(1-t)$ admits a resolution c which is *polynomial* in θ^{-1} , that is, $c \in \theta^{-1}C[\theta^{-1}]$.

Let us assume the claim for the moment and finish the proof. We begin with the surjectivity of $(e\phi)_*$. Let $y \in C$ represent a closed element in $(C/\text{im}(1-t), d+b)$. Then $(d+b)y$ is $(d+b)$ -closed and lies in $\text{im}(1-t)$, so by the claim it has a polynomial resolution $c \in \theta^{-1}C[\theta^{-1}]$. Then

$$\delta_\theta(y-c) = Ny\theta + (d+b)y - \delta_\theta c = Ny\theta \equiv 0 \in C[\theta^{-1}] = C[\theta, \theta^{-1}/\theta C[\theta]],$$

so the element $y-c$ is closed in $C[\theta^{-1}]$. Since $e\phi(y-c) = y$, this proves the surjectivity of $(e\phi)_*$.

Next we show that $(e\phi)_*$ is injective. Since e_* is an isomorphism, it suffices to show that ϕ_* is injective. Let a closed element $c = \sum_{k=0}^n c_{-k}\theta^{-k} \in C[[\theta^{-1}]]$ represent an exact element in $C[[\theta^{-1}]]$, that is, $\delta_\theta a = c$ for some $a = \sum_{k=0}^\infty a_{-k}\theta^{-k} \in C[[\theta^{-1}]]$. We may assume that n is odd by allowing the last coefficient c_{-n} to be zero. Then $(d+b)a_{-n-1}$ is $(d+b)$ -closed and lies in $\text{im}(1-t)$, so by the claim it has a polynomial resolution $z = \sum_{k=1}^N z_{-k}\theta^{-k}$. Now

$$\tilde{a} := \sum_{k=0}^{n+1} a_{-k}\theta^{-k} - z\theta^{-n-1} \in C[[\theta^{-1}]]$$

satisfies

$$\delta_\theta \tilde{a} = c + (d+b)a_{-n-1}\theta^{-n-1} - \delta_\theta z\theta^{-n-1} = c,$$

so c is exact in $C[[\theta^{-1}]]$ and injectivity is proved.

It remains to prove the claim. By a slight abuse of notation, we will denote by δ_{ver} and δ_{hor} the maps $C \rightarrow C$ defined as in (7) but without the factors of θ . Recall the formula $c = \sum_{k=1}^\infty c_{-k}\theta^{-k}$ for a resolution of $x \in C$. We will write

$$c_{-1} = \delta_{\text{hor}}^{-1}(x) \quad \text{and} \quad c_{-(k+1)} = -\delta_{\text{hor}}^{-1}\delta_{\text{ver}}c_{-k} \quad \text{for } k \geq 1,$$

where by δ_{hor}^{-1} we mean some preimage under δ_{hor} . The main point of the proof is a certain special choice of the preimage. To explain it, recall that $C = \bigoplus_{n=0}^\infty C_n$. The maps t, N, d preserve C_n , whereas b and b' map C_{n+1} to C_n . In particular, δ_{hor} preserves C_n . Let us define the *weight* $w(c)$ of a nonzero element $c \in C$ as the smallest integer w such that $c \in \bigoplus_{n=0}^w C_n$, and set $w(0) := -1$. By the preceding discussion we can choose the preimages under δ_{hor} such that

$$(11) \quad w(c_{-k-1}) \leq w(c_{-k}) \quad \text{for all } k \geq 1.$$

This condition can be improved further. For this, note that the map $N : C_n \rightarrow C_n$ acts on its image as multiplication by $1+n$. Therefore, N^{-1} on the image of N can be taken to be multiplication by $(1+n)^{-1}$. In particular, it maps d -closed elements to d -closed ones. From this we can deduce the following strict (!) inequality:

$$(12) \quad w(c_{-k-2}) < w(c_{-k}) \quad \text{for } k \text{ odd and } c_{-k} \neq 0.$$

Indeed, for k odd we obtain from (11)

$$\begin{aligned} w(c_{-k-2}) &= w(\delta_{\text{hor}}^{-1}\delta_{\text{ver}}\delta_{\text{hor}}^{-1}\delta_{\text{ver}}c_{-k}) \leq w(\delta_{\text{ver}}\delta_{\text{hor}}^{-1}\delta_{\text{ver}}c_{-k}) \\ &= w((d+b)N^{-1}(d+b')c_{-k}). \end{aligned}$$

To compute the last term we write out

$$(d+b)N^{-1}(d+b')c_{-k} = dN^{-1}dc_{-k} + dN^{-1}b'c_{-k} + bN^{-1}dc_{-k} + bN^{-1}b'c_{-k},$$

where the first summand $dN^{-1}dc_{-k} = 0$ vanishes since N^{-1} maps closed elements to closed ones. All other summands have weight strictly less than that of c_{-k} due to the presence of either b or b' (or both), and inequality (12) follows.

Inequalities (11) and (12) imply that the weight $w(c_{-k})$ strictly decreases each time k increases by 2; hence, $c_{-k} = 0$ for all sufficiently large k , and the

resolution is polynomial. This proves the claim and, thus, the isomorphism $\phi_* : HC_*^{[\theta^{-1}]} \xrightarrow{\cong} HC_*^{[[\theta^{-1}]}$. The statement about the inclusion $C[u^{-1}] \rightarrow C[[u^{-1}]]$ follows from this and Lemma 2.11, and the statement about the other two inclusions follows from the tautological exact sequence in Proposition 2.5 and the five lemma. \square

The preceding two lemmas are summarized in the following result.

COROLLARY 2.14

For a cyclic cochain complex (C_n, d_i, s_j, t_n, d) the different versions of cyclic homology are given by

$$\begin{aligned} HC_*^{[u^{-1}]} &\cong HC_*^{[[u^{-1}]]} \cong HC_{*-1}^\lambda \cong HC_{*-1}^{[u]}, & HC_*^{[[u]]}, \\ HC_*^{[[u, u^{-1}]]} &\cong HC_*^{[[u, u^{-1}]]}, & HC_*^{[u, u^{-1}]} = HC_*^{[u, u^{-1}]} = 0. \end{aligned}$$

Behavior under maps of cyclic cochain complexes. According to Proposition 2.15, only the three classical versions of cyclic homology are invariant under quasi-isomorphisms of mixed complexes. By contrast, with respect to quasi-isomorphisms of cyclic cochain complexes we have the following result.

PROPOSITION 2.15

Hochschild homology and all versions of cyclic homology are invariant under quasi-isomorphisms of cyclic cochain complexes, that is, morphisms inducing an isomorphism on d -cohomology.

Proof

For Hochschild homology and the classical versions $HC^{[[u]]}$, $HC^{[[u, u^{-1}]}$, and $HC^{[u^{-1}]}$ this is an immediate consequence of Proposition 2.3. By Corollary 2.14, this covers all eight versions of cyclic homology. \square

One important application of this proposition concerns the analytic and algebraic cyclic cochain complexes in Examples 2.7 and 2.8 built from the de Rham algebra $\Omega^*(X)$ of a manifold X . Recall from Lemma 2.9 that the exterior cross products $\phi_n : \Omega^*(X)^{\otimes(n+1)} \rightarrow \Omega^*(X^{n+1})$ define a morphism of cyclic cochain complexes. Since ϕ_n induces an isomorphism on d -homology for all n by the Künneth formula, Proposition 2.15 implies the following.

COROLLARY 2.16 (Algebraic vs. analytic cyclic homology)

For a manifold X , the map of mixed complexes

$$\phi : \left(\bigoplus_{n \geq 0} \Omega^*(X)^{\otimes(n+1)}, d + b, B \right) \rightarrow \left(\bigoplus_{n \geq 0} \Omega^*(X^{n+1}), d + b, B \right)$$

induces isomorphisms on Hochschild homology and on all versions of cyclic homology.

2.4. From dgas to mixed complexes

Recall from Example 2.8 that each dga (A, d) canonically gives rise to a cyclic cochain complex with $C_n(A) = A^{\otimes(n+1)}$. By Lemma 2.11 it gives rise to a mixed complex $(C(A) = \bigoplus_{n \geq 0} C_n(A), d + b, B)$ and a double complex $(C[[\theta, \theta^{-1}]], \delta_\theta)$. The homology $HH(A) := H(C(A), d + b)$ is the *Hochschild homology* of (A, d) and the homologies $HC_{[[u, u^{-1}]]}(A)$ and so on are the various versions of the *cyclic homology* of (A, d) . In this section we will study these cyclic homologies in more detail.

Note that a morphism of dgas canonically induces a morphism of cyclic cochain complexes. A quasi-isomorphism of dgas induces a quasi-isomorphism of cyclic cochain complexes (by the Künneth formula and the exactness of the tensor functor on vector spaces), so by Proposition 2.15 it induces isomorphisms on Hochschild homology and all versions of cyclic homology. Moreover, by Corollary 2.14 all versions of cyclic homology of (A, d) are either zero or isomorphic to one of the three classical versions.

REMARK 2.17

The invariance of all versions of cyclic homology under quasi-isomorphisms of dgas implies, in particular, that the cyclic homology of a *commutative* dga with $H^0(A) = \mathbb{R}$ can be computed using its Sullivan minimal model.

Normalized and reduced cyclic homology of a dga. Consider a dga A with its associated cyclic cochain complex $(C_n(A) = A^{\otimes(n+1)}, d_i, s_j, t_n, d)$ as in Example 2.8. As in [19] we denote by $D_n(A) \subset C_n(A)$ the linear subspace spanned by words $(a_0 | a_1 | \dots | a_n)$ with $a_i = 1$ for some $i \geq 1$. (Such words are called *degenerate*.) So we get a short exact sequence

$$0 \rightarrow D_n(A) \rightarrow C_n(A) \rightarrow \overline{C}_n(A) \rightarrow 0,$$

where the quotient is given by

$$\overline{C}_n(A) = A \otimes \overline{A}^{\otimes n}, \quad \overline{A} = A/(\mathbb{R} \cdot 1).$$

Note that the subspaces $D_n(A) \subset C_n(A)$ define a simplicial subcomplex but not a cyclic one, that is, it is preserved under the operations d_i, s_j, d but not under t_n . Nonetheless, one easily checks that the direct sums $D(A) = \bigoplus_{n \geq 0} D_n(A)$ and $\overline{C}(A) = \bigoplus_{n \geq 0} \overline{C}_n(A)$ give rise to a short exact sequence of mixed complexes

$$0 \longrightarrow (D(A), d + b, B) \longrightarrow (C(A), d + b, B) \xrightarrow{p} (\overline{C}(A), d + b, \overline{B}) \longrightarrow 0$$

with the induced map $\overline{B} = sN$. The quotient $\overline{C}(A)$ is called the *normalized mixed complex*. It is a standard fact (see, e.g., [19, Proposition 1.15], which holds more generally for cyclic chain complexes) that $(D(A), d + b)$ has vanishing homology, so $p: C(A) \rightarrow \overline{C}(A)$ is a quasi-isomorphism of mixed complexes. We denote the resulting versions of *normalized cyclic homology* by $H\overline{C}^{[u, u^{-1}]}(A)$ and so on.

The normalized mixed complex $\overline{C}(A)$ of a dga A has the mixed subcomplex $\overline{C}(\mathbb{R})$ given by $\overline{C}_0(\mathbb{R}) = \mathbb{R}$ and $\overline{C}_n(\mathbb{R}) = 0$ for $n \geq 1$ with trivial operations.

Following [19], we define the *reduced mixed complex* $\overline{C}^{\text{red}}(A)$ by the short exact sequence

$$0 \longrightarrow \overline{C}(\mathbb{R}) \longrightarrow \overline{C}(A) \longrightarrow \overline{C}^{\text{red}}(A) \longrightarrow 0.$$

We denote the resulting versions of *reduced cyclic homology* by $\overline{HC}^{\{u, u^{-1}\}}(A)$ and so on. There is also a *reduced Connes version* $\overline{HC}^\lambda(A)$ defined as the $(d + b)$ -homology of the quotient $\overline{C}^\lambda(A)$ by all words containing 1 in some entry.

A dga A is called *augmented* if

$$A = \mathbb{R} \cdot 1 \oplus \overline{A}$$

for a dg ideal $\overline{A} \subset A$. For example, the de Rham complex $\Omega^*(X)$ of a connected manifold X is augmented with $\overline{A} = \Omega^*(X, x_0)$, the forms restricting to zero on a basepoint x_0 . If A is augmented, then the above exact sequence splits as a direct sum of mixed complexes

$$(13) \quad \overline{C}(A) = \overline{C}(\mathbb{R}) \oplus \overline{C}^{\text{red}}(A),$$

where $\overline{C}_0^{\text{red}}(A) = \overline{A}$ and $\overline{C}_n^{\text{red}}(A) = A \otimes \overline{A}^{\otimes n}$ for $n \geq 1$. It follows that each version $\{u, u^{-1}\}$ of normalized cyclic homology splits as

$$(14) \quad \overline{HC}^{\{u, u^{-1}\}}(A) = \overline{HC}^{\{u, u^{-1}\}}(\mathbb{R}) \oplus \overline{HC}^{\{u, u^{-1}\}}(A).$$

Note that for A augmented the reduced Connes version is given by $\overline{HC}^\lambda(A) = HC^\lambda(\overline{A})$, viewing \overline{A} as a nonunital dga.

PROPOSITION 2.18 (Normalized cyclic homology)

For a dga (A, d) , the projection $p : C(A) \rightarrow \overline{C}(A)$ induces isomorphisms from Hochschild homology $HH(A)$ and the classical versions $HC^{\llbracket u \rrbracket}(A)$, $HC^{\llbracket u, u^{-1} \rrbracket}(A)$, and $HC^{\llbracket u^{-1} \rrbracket}(A)$ of cyclic homology to the corresponding homologies of the normalized mixed complex. The same holds for the versions $HC^{\llbracket u^{-1} \rrbracket}(A)$ and $HC^{\llbracket u, u^{-1} \rrbracket}(A)$ if A is augmented. The remaining three versions of cyclic homology are in general not isomorphic to their normalized cyclic homology.

Proof

The first assertion follows directly from Proposition 2.3, because p is a quasi-isomorphism, and the last assertion follows from Example 2.27. For the second assertion, suppose that $A = \mathbb{R} \cdot 1 \oplus \overline{A}$ is augmented, and denote by $p : A \rightarrow \overline{A}$ the projection. By viewing \overline{A} as a nonunital dga, it has an associated Hochschild complex $(C(\overline{A}), d + b)$ and double complexes $(C(\overline{A})\{\theta, \theta^{-1}\}, \delta_\theta)$, where $\{\theta, \theta^{-1}\}$ stands for any of the eight versions $[\theta]$, $[\theta, \theta^{-1}]$, and so on. Note that the reduced cyclic complex has a canonical splitting as a vector space

$$\overline{C}^{\text{red}}(A) = C(\overline{A}) \oplus sC(\overline{A}),$$

where $sC(\overline{A})$ is generated by words with 1 in the 0th slot and elements from \overline{A} in all others. Now the main observation is that the map

$$r((x + sy)u^n) := x\theta^{2n} + y\theta^{2n-1}$$

defines a chain isomorphism

$$r : ((C(\overline{A}) \oplus sC(\overline{A}))\{u, u^{-1}\}, d + b + \overline{B}u) \xrightarrow{\cong} (C(\overline{A})\{\theta, \theta^{-1}\}, \delta_\theta).$$

Indeed, the map is clearly bijective, and the chain map property follows from

$$\begin{aligned} r(d + b + \overline{B}u)((x + sy)u^n) &= r((d + b)(x + sy)u^n + \overline{B}(x + sy)u^{n+1}) \\ &= r((dx + dsy + bx - sb'y + (1 - t)y)u^n + sNxu^{n+1}) \\ &= ((d + b)x + (1 - t)y)\theta^{2n} - (d + b')y\theta^{2n-1} + Nx\theta^{2n+1} \end{aligned}$$

and

$$\begin{aligned} \delta_\theta r((x + sy)u^n) &= \delta_\theta(x\theta^{2n} + y\theta^{2n-1}) \\ &= (d + b)x\theta^{2n} - (d + b')y\theta^{2n-1} + Nx\theta^{2n+1} + (1 - t)y\theta^{2n} \\ &= ((d + b)x + (1 - t)y)\theta^{2n} - (d + b')y\theta^{2n-1} + Nx\theta^{2n+1}. \end{aligned}$$

Combining this with the splitting (13), we obtain a chain isomorphism

$$\text{id} \oplus r : (\overline{C}(A)\{u, u^{-1}\}, d + b + \overline{B}u) \xrightarrow{\cong} \overline{C}(\mathbb{R})\{u, u^{-1}\} \oplus (C(\overline{A})\{\theta, \theta^{-1}\}, \delta_\theta).$$

The induced isomorphisms on homology for the $[u^{-1}]$ and $[[u^{-1}]]$ versions fit into the commuting diagram

$$\begin{array}{ccccc} HC_*^{[u^{-1}]}(A) & \xrightarrow[p_*]{\cong} & H\overline{C}_*^{[u^{-1}]}(A) & \xrightarrow[\cong]{(\text{id} \oplus r)_*} & \mathbb{R}[u^{-1}] \oplus HC_*^{[\theta^{-1}]}(\overline{A}) \\ \phi_* \downarrow \cong & & \phi_* \downarrow & & \text{id} \oplus \phi_* \downarrow \cong \\ HC_*^{[[u^{-1}]]}(A) & \xrightarrow[p_*]{} & H\overline{C}_*^{[[u^{-1}]]}(A) & \xrightarrow[\cong]{(\text{id} \oplus r)_*} & \mathbb{R}[u^{-1}] \oplus HC_*^{[[\theta^{-1}]]}(\overline{A}) \end{array}$$

Here the vertical maps ϕ_* come from Lemma 2.13, and the upper horizontal map p_* is an isomorphism by the first assertion above. It follows that the lower horizontal map p_* is an isomorphism as well. An analogous argument for the $[[u, u^{-1}]]$ and $[[u, u^{-1}]]$ versions concludes the proof. \square

COROLLARY 2.19 (Reduced cyclic homology)

For an augmented dga (A, d) we have a canonical splitting

$$HC_*^{[[u]]}(A) = H\overline{C}_*^{[[u]]}(\mathbb{R}) \oplus H\overline{C}_*^{[[u]]}(A),$$

and similarly for the $[[u, u^{-1}]]$, $[u^{-1}]$, $[[u^{-1}]]$, and $[[u, u^{-1}]]$ versions. Moreover, $H\overline{C}_*^{[[u, u^{-1}]]}(A) = 0$, and the connecting homomorphism in the corresponding tautological exact sequence splits into isomorphisms

$$H\overline{C}_{*+1}^{[[u^{-1}]]}(A) \cong H\overline{C}_*^\lambda(A) = HC_*^\lambda(\overline{A}) \cong H\overline{C}_*^{[u]}(A).$$

Proof

The first assertion follows immediately from the splitting (14) and Proposition 2.18. The second assertion follows from the following variant of the diagram in Lemma 2.12 in reduced cyclic homology:

$$\begin{array}{ccc}
 \overline{HC}_*^{[[u^{-1}]]}(A) & \xrightarrow{\overline{B}_0} & \overline{HC}_{*-1}^{[u]}(A) \\
 \cong \downarrow r_* & & \cong \downarrow r_* \\
 HC_*^{[[\theta^{-1}]]}(\overline{A}) & \xrightarrow[\cong]{\overline{B}_0} & HC_{*-1}^{[\theta]}(\overline{A}) \\
 \cong \downarrow e_* & & \cong \uparrow s_* \\
 H_*(C(\overline{A})/\text{im}(1-t), d+b) & \xrightarrow[\cong]{N_*} & H_*(\text{im } N, d+b')
 \end{array}$$

Here $\overline{B} = sN$, the maps r are the chain isomorphisms from the proof of Proposition 2.18, and the other maps are isomorphisms by Lemma 2.12 applied to the cyclic complex of \overline{A} . Since $H_{*+1}(C(\overline{A})/\text{im}(1-t), d+b) = HC_*^\lambda(\overline{A}) = \overline{HC}_*^\lambda(A)$, this proves the chain of isomorphisms. The vanishing of $\overline{HC}_*^{[[u, u^{-1}]]}(A)$ now follows from the tautological exact sequence. \square

2.5. From S^1 -spaces to mixed complexes

In this section we recall the mixed complexes arising from S^1 -spaces and their properties.

Topological S^1 -spaces. Let Y be a topological space with an S^1 -action $\phi : S^1 \times Y \rightarrow Y$. We make the singular cochain complex $(C^*(Y), d)$ a mixed complex by introducing the second differential $P : C^*(Y) \rightarrow C^{*-1}(Y)$ as follows. Let $\times : C_m(X) \otimes C_n(Y) \rightarrow C_{m+n}(X \times Y)$ be the Eilenberg–MacLane shuffle product, associating to $f : \Delta^m \rightarrow X$ and $g : \Delta^n \rightarrow Y$ the map $f \times g : \Delta^m \times \Delta^n \rightarrow X \times Y$, $(p, q) \mapsto (f(p), g(q))$ with a canonical subdivision of $\Delta^m \times \Delta^n$ into simplices (using shuffles of the variables; see, e.g., [14]). The shuffle product with the fundamental cycle $F_{S^1} : \Delta^1 \rightarrow S^1$ defines a map $C_*(Y) \rightarrow C_{*+1}(S^1 \times Y)$, $c \mapsto F_{S^1} \times c$, whose composition with $\phi_* : C_*(S^1 \times Y) \rightarrow C_*(Y)$ gives a map

$$(15) \quad Q : C_*(Y) \rightarrow C_{*+1}(Y), \quad c \mapsto \phi_*(F_{S^1} \times c).$$

The operator $P : C^*(Y) \rightarrow C^{*-1}(Y)$ is dual to Q , that is,

$$\langle P\alpha, c \rangle := \langle \alpha, \phi_*(F_{S^1} \times c) \rangle, \quad \alpha \in C^*(Y), c \in C_{*-1}(Y).$$

LEMMA 2.20

We have that $(C^(Y), d, P)$ is a mixed complex whose homology $H^*(Y, d)$ is the singular cohomology of Y .³*

³Throughout the present article, all singular (co)homology is with \mathbb{R} -coefficients.

Proof

To show $P^2 = 0$, we compute for $c \in C_*(Y)$ using the associativity of the shuffle product

$$Q^2c = \phi_*(F_{S^1} \times \phi_*(F_{S^1} \times c)) = \phi_*(\psi_*(F_{S^1} \times F_{S^1}) \times c),$$

where $\psi: S^1 \times S^1 \rightarrow S^1$ is the product $(\sigma, \tau) \mapsto \sigma + \tau$. Now by the definition of the shuffle product $F_{S^1} \times F_{S^1}$ is the difference of the singular simplices $f, g: \Delta^2 \rightarrow S^1 \times S^1$ given by $f(s, t) = (s, t)$ and $g(s, t) = (t, s)$. This shows that $\psi_*(F_{S^1} \times F_{S^1}) = 0 \in C_2(S^1)$; thus, $Q^2 = 0$ and therefore $P^2 = 0$. For the anticommutation of d and P note that

$$\partial(F_{S^1} \times c) = \partial F_{S^1} \times c - F_{S^1} \times \partial c = -F_{S^1} \times c,$$

because $\partial F_{S^1} = 0$. Applying ϕ_* and dualizing we find that $\partial Q + Q\partial = 0$ and $dP + Pd = 0$. The statement about cohomology is clear by definition. \square

We denote the cyclic homologies of the mixed complex $(C^*(Y), d, P)$ by $H_{[[u, u^{-1}]]}^*(Y)$ and so on.⁴

Relation to the Borel construction. According to the Borel construction, the S^1 -equivariant cohomology of a topological S^1 -space Y is defined as

$$H_{S^1}^*(Y) := H^*(Y \times_{S^1} ES^1),$$

where $ES^1 \rightarrow BS^1$ is the universal S^1 -bundle and $Y \times_{S^1} ES^1 = (Y \times ES^1)/S^1$ is the quotient by the diagonal circle action. In the following proposition the first isomorphism is shown in the discussion following Lemma 5.1 in [16], and the second isomorphism holds because the complex $C^*(Y)$ lives in nonnegative degrees.

PROPOSITION 2.21 ([16, Lemma 5.1])

For each topological S^1 -space Y we have canonical isomorphisms

$$H_{S^1}^*(Y) \xrightarrow{\cong} H_{[u]}^*(Y) \xrightarrow{\cong} H_{[[u]]}^*(Y).$$

Projection to a point. For an S^1 -space Y with a fixed point y_0 consider the inclusion and projection

$$\text{pt} \xrightarrow{\iota} Y \xrightarrow{\pi} \text{pt}, \quad \iota(\text{pt}) = y_0.$$

These maps are S^1 -equivariant and satisfy $\pi\iota = \text{id}$, so they induce morphisms of mixed complexes

$$(C^*(\text{pt}) = \mathbb{R}, d = 0, P = 0) \xrightarrow{\pi^*} (C^*(Y), d, P) \xrightarrow{\iota^*} (C^*(\text{pt}) = \mathbb{R}, d = 0, P = 0)$$

satisfying $\iota^*\pi^* = \text{id}$. These maps are functorial and the induced maps on cyclic homology are compatible with the Gysin and tautological sequences in the obvi-

⁴Here we use cohomological notation because $H_{[[u, u^{-1}]]}^*(Y)$ is the cyclic cohomology of the mixed complex $(C_{-*}(Y), \partial, Q)$ in the sense of Section 4.

ous way. For example, one tautological sequence yields the following commuting diagram, where all the vertical maps π^* are injective and we have surjective vertical maps ι^* in the other direction:

$$(16) \quad \begin{array}{ccccccc} H_{[[u, u^{-1}]}^*(Y) & \xrightarrow{P^*} & H_{[u^{-1}]}^*(Y) & \xrightarrow{P_0} & H_{[[u]]}^{*-1}(Y) & \xrightarrow{\cdot u} & H_{[[u, u^{-1}]}^{*+1}(Y) \\ \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* \\ \mathbb{R}[u, u^{-1}] & \xrightarrow{P^*} & \mathbb{R}[u^{-1}] & \xrightarrow{0} & \mathbb{R}[u] & \xrightarrow{\cdot u} & \mathbb{R}[u, u^{-1}] \end{array}$$

For a loop space $Y = LX$, the first group in this diagram can be explicitly computed.

THEOREM 2.22 ([12, Introduction])

For a path-connected space X , the group $H_{[[u, u^{-1}]}^*(LX)$ depends only on $\pi_1(X)$. In particular, for X simply connected the projection $LX \rightarrow \text{pt}$ to a point induces an isomorphism

$$H_{[[u, u^{-1}]}^*(LX) \cong H_{[[u, u^{-1}]}^*(\text{pt}) = \mathbb{R}[u, u^{-1}].$$

Smooth S^1 -spaces. Let now Y be a differentiable space in the sense of Chen in [6] equipped with a smooth circle action. For example, Y could be a finite-dimensional manifold or an infinite-dimensional Banach or Fréchet manifold. Our main example is the space $Y = LX$ of smooth maps $S^1 \rightarrow X$ into a finite-dimensional manifold X with the natural circle action $\phi(s, \gamma) := \gamma(\cdot + s)$ by reparameterization. Following Chen in [6], to such a *smooth S^1 -space* Y we associate a mixed complex $(\Omega^*(Y), d, P)$ as follows. Let $(\Omega^*(Y), d)$ be the space of differential forms on Y with exterior derivative. The inner product with the vector field v on Y generating the circle action gives a map $\iota : \Omega^*(Y) \rightarrow \Omega^{*-1}(Y)$. By Cartan’s formula, $d\iota + \iota d = L_v$ is the Lie derivative in the direction of v . Let $A : \Omega^*(Y) \rightarrow \Omega^*(Y)$ be the operator averaging over the circle action. In view of the relations $\iota A = A\iota$ and $dA = Ad$, Cartan’s formula implies that

$$P := A\iota : \Omega^*(Y) \rightarrow \Omega^{*-1}(Y)$$

satisfies $dP + Pd = 0$. The relation $\iota^2 = 0$ implies that $P^2 = 0$, so $(\Omega^*(Y), d, P)$ is indeed a mixed complex. It is shown in [6] that for “nice” smooth S^1 -spaces (such as loop spaces of manifolds) the homology of $(\Omega^*(Y), d)$ is isomorphic to the singular cohomology $H^*(Y; \mathbb{R})$.

To a smooth S^1 -space Y we can associate another canonical mixed complex $(\Omega_{S^1}^*(Y), d, \iota)$, where $\Omega_{S^1}^*(Y)$ denotes the space of S^1 -invariant forms.

PROPOSITION 2.23 ([17, proof of Theorem 3.1])

The inclusion $\Omega_{S^1}^*(Y) \hookrightarrow \Omega^*(Y)$ induces a homotopy equivalence of mixed complexes

$$(\Omega_{S^1}^*(Y), d, \iota) \rightarrow (\Omega^*(Y), d, P).$$

In particular, it induces isomorphisms on all versions of cyclic homology.

Fixed point free S^1 -actions. Let Y be a smooth S^1 -space without fixed points. Then there exists a connection form, that is, an invariant 1-form $\alpha \in \Omega_{S^1}^1(Y)$ satisfying $\alpha(v) = 1$ for the vector field v generating the action. The wedge product with α then defines a chain homotopy $H : \Omega_{S^1}^*(Y) \rightarrow \Omega_{S^1}^{*+1}(Y)$ satisfying

$$\iota H + H \iota = \text{id}.$$

Consider the tautological sequence

$$H_{[[u, u^{-1}]]}^{*+1}(Y) \xrightarrow{P_*} H_{[[u^{-1}]]}^{*+1}(Y) \xrightarrow{P_0} H_{[u]}^*(Y) \xrightarrow{\cdot u} H_{[u, u^{-1}]}^{*+2}(Y).$$

The map P_0 is the composition of the first two maps in

$$H_{[[u^{-1}]]}^{*+1}(Y) \longrightarrow H^*(\ker \iota) = H^*(Y/S^1) \longrightarrow H_{[u]}^*(Y) \xrightarrow{\cong} H_{[[u]]}^*(Y),$$

where the last map is an isomorphism because the complex $\Omega_{S^1}^*(Y)$ lives in nonnegative degrees. The homotopy formula for H shows that the double complex $(\Omega_{S^1}^*(Y)[u, u^{-1}], d + \iota u)$ for the computation of $H_{[[u, u^{-1}]]}^*(Y)$ has exact rows. Now a standard zigzag argument as in the proof of Lemma 2.12 yields the following.

LEMMA 2.24

Let Y be a smooth S^1 -space without fixed points. Then $H_{[[u, u^{-1}]]}^(Y) = 0$, and we have canonical isomorphisms*

$$H_{[[u^{-1}]]}^{*+1}(Y) \xrightarrow{\cong} H^*(Y/S^1) \xrightarrow{\cong} H_{[u]}^*(Y) \xrightarrow{\cong} H_{[[u]]}^*(Y).$$

REMARK 2.25

Consider a smooth S^1 -space Y (possibly with fixed points). Then the canonical projection $Y \times ES^1 \rightarrow Y$ induces a morphism of mixed complexes

$$(\Omega_{S^1}^*(Y), d, \iota) \rightarrow (\Omega_{S^1}^*(Y \times ES^1), d, \iota)$$

which induces an isomorphism on d -homology and, thus, on the $[[u]]$, $[u^{-1}]$, and $[[u, u^{-1}]$ versions of cyclic homology. Since $Y \times ES^1$ has no fixed points, applying Lemma 2.24 to it provides an alternative proof of Proposition 2.21 in the smooth case.

Fixed point localization. The version $H_{[[u, u^{-1}]]}^*(Y)$ satisfies the following fixed point localization theorem.

THEOREM 2.26 ([17, Theorem 2.1])

Let Y be a smooth S^1 -space whose fixed point set $F \subset Y$ is a smooth submanifold which has an S^1 -invariant tubular neighborhood. Then the inclusion $c : F \hookrightarrow Y$ induces an isomorphism

$$c^* : H_{[[u, u^{-1}]]}^*(Y) \xrightarrow{\cong} H^*(F) \otimes \mathbb{R}[u, u^{-1}].$$

In particular, for a manifold X the inclusion $c : X \hookrightarrow LX$ of the constant loops induces an isomorphism

$$c^* : H_{[[u, u^{-1}]]}^*(LX) \xrightarrow{\cong} H^*(X) \otimes \mathbb{R}[u, u^{-1}].$$

2.6. Examples

In this section we work out the different flavors of cyclic homology for some examples. In the case of a dga A we will use the equivalent sign convention (6), dropping the \sim 's, which spells out as

$$\begin{aligned} b'(a_0 | \cdots | a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 | \cdots | a_i a_{i+1} | \cdots | a_n), \\ b(a_0 | \cdots | a_n) &= b'(a_0 | \cdots | a_n) \\ &\quad + (-1)^{n+\deg a_n (\deg a_0 + \cdots + \deg a_{n-1})} (a_n a_0 | a_1 | \cdots | a_{n-1}), \\ t(a_0 | \cdots | a_n) &= (-1)^{n+\deg a_n (\deg a_0 + \cdots + \deg a_{n-1})} (a_n | a_0 | \cdots | a_{n-1}), \\ s(a_0 | \cdots | a_n) &= (1 | a_0 | \cdots | a_n). \end{aligned}$$

EXAMPLE 2.27 (de Rham complex of a point)

Consider the trivial dga ($A = \mathbb{R}, d = 0$) sitting in degree 0, which is the de Rham complex of a point. Its Hochschild complex $C(\mathbb{R})$ has the basis $1^n := (1 | \cdots | 1)$ of words with n 1's of degree $|1^n| = 1 - n$, for $n \geq 1$. From the definitions (and being careful about signs) we read off the operations

$$\begin{aligned} b(1^n) &= \begin{cases} 1^{n-1} & n \geq 3 \text{ odd,} \\ 0 & \text{else,} \end{cases} \\ t(1^n) &= (-1)^{n-1} 1^n, \\ N(1^n) &= \begin{cases} n 1^n & n \text{ odd,} \\ 0 & \text{else,} \end{cases} \\ sN(1^n) &= \begin{cases} n 1^{n+1} & n \text{ odd,} \\ 0 & \text{else,} \end{cases} \\ B(1^n) &= \begin{cases} 2n 1^{n+1} & n \text{ odd,} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

This gives us the mixed complex ($C = C(\mathbb{R}), b, B$). As shown in Figure 2, cycles in the complex ($C[[u, u^{-1}]], \delta_u = b + uB$) consist of zigzags in the lower half-plane starting on the horizontal axis and extending indefinitely to the right.

Thus, it makes no difference whether we consider polynomials or power series in u^{-1} . Using this, we read off the Hochschild and cyclic homology groups (where $[i]$ denotes degree shift by i)

$$\begin{aligned} H(C, \delta) &= \mathbb{R}, \\ HC^{[u]} &= \mathbb{R}[u^{-1}][-1] \text{ generated by } 1^2 \sim u1^4 \sim \cdots, \\ HC^{[u, u^{-1}]} &= HC^{[u, u^{-1}]} = 0, \end{aligned}$$

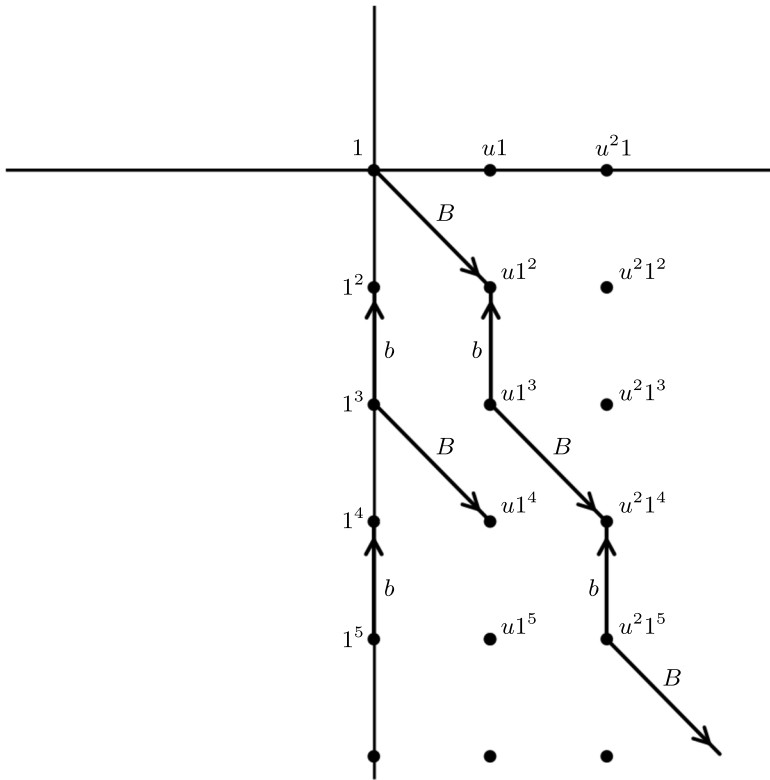


Figure 2. Cyclic complex of the trivial dga ($A = \mathbb{R}, d = 0$).

$$HC^{[u^{-1}]} = HC^{[[u^{-1}]]} = \mathbb{R}[u^{-1}]$$

generated by $u^{-k}1 + u^{-k+1}1^3 + \dots + 1^{2k+1}, k \geq 0$,

$$HC^{[[u]]} = \mathbb{R}[u] \text{ generated by } 1 + u1^3 + u^21^5 + \dots,$$

$$HC^{[[u, u^{-1}]]} = HC^{[[u, u^{-1}]]} = \mathbb{R}[u, u^{-1}] \text{ generated by } 1 + u1^3 + u^21^5 + \dots.$$

So we see two types of tautological sequences:

$$(17) \quad \begin{array}{ccccccc} 0 & \longrightarrow & HC_{*-2}^{[[u]]} & \xrightarrow{\cdot u} & HC_*^{[[u, u^{-1}]]} & \xrightarrow{p_*} & HC_*^{[u^{-1}]} \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \mathbb{R}[u][-2] & \xrightarrow{\cdot u} & \mathbb{R}[u, u^{-1}] & \xrightarrow{p_*} & \mathbb{R}[u^{-1}] \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc}
 HC_*^{[u, u^{-1}]} & \longrightarrow & HC_*^{[u^{-1}]} & \xrightarrow[\cong]{D_0} & HC_{*-1}^{[u]} & \longrightarrow & HC_{*-1}^{[u, u^{-1}]} \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 0 & \longrightarrow & \mathbb{R}[u^{-1}] \cdot 1 & \xrightarrow[\cong]{1 \mapsto 1^2} & \mathbb{R}[u^{-1}] \cdot 1^2 & \longrightarrow & 0
 \end{array}$$

The normalized complex is given by $\overline{C}_0(\mathbb{R}) = \mathbb{R}$ sitting in degree 0 and $\overline{C}_n(\mathbb{R}) = 0$ for $n \geq 1$, so the normalized cyclic homologies are

$$\begin{aligned}
 H\overline{C}^{[u]} &= H\overline{C}^{[[u]]} = \mathbb{R}[u], & H\overline{C}^{[u^{-1}]} &= H\overline{C}^{[[u^{-1}]]} = \mathbb{R}[u^{-1}], \\
 H\overline{C}^{[u, u^{-1}]} &= H\overline{C}^{[[u, u^{-1}]]} = H\overline{C}^{[u, u^{-1}]} = H\overline{C}^{[[u, u^{-1}]]} = \mathbb{R}[u, u^{-1}].
 \end{aligned}$$

EXAMPLE 2.28 (Singular cochains on a point)

Consider the trivial mixed complex $(C = \mathbb{R}, \delta = D = 0)$, which is the singular cochain complex of a point viewed as an S^1 -space. The homologies can be directly read off to be

$$\begin{aligned}
 H(C, \delta) &= \mathbb{R}, \\
 HC^{[u]} &= HC^{[[u]]} = \mathbb{R}[u], \\
 HC^{[u^{-1}]} &= HC^{[[u^{-1}]]} = \mathbb{R}[u^{-1}], \\
 HC^{[u, u^{-1}]} &= HC^{[u, u^{-1}]} = HC^{[[u, u^{-1}]]} = HC^{[[u, u^{-1}]]} = \mathbb{R}[u, u^{-1}].
 \end{aligned}$$

Here all tautological sequences look like (17) above.

EXAMPLE 2.29 (Singular cochains on ES^1)

Consider the mixed complex (C, δ, D) , where $C = \Lambda[\alpha, \beta]$ is the exterior algebra in two generators of degrees $|\alpha| = 1$ and $|\beta| = 2$ with the operations defined by the Leibniz rule and

$$\delta\alpha = \beta, \quad \delta\beta = 0, \quad D\alpha = 1, \quad D\beta = 0.$$

This is the Cartan–Weil model for the classifying space $ES^1 = S^\infty$ with its standard S^1 -action (see [1]). As shown in Figure 3, cycles in the complex $(C[[u, u^{-1}]], \delta_u = \delta + uD)$ consist of zigzags in the upper half-plane starting on the horizontal axis and extending indefinitely to the left.

Thus, it makes no difference whether we consider polynomials or power series in u . Using this, we read off the Hochschild and cyclic homology groups

$$\begin{aligned}
 H(C, \delta) &= \mathbb{R}, \\
 HC^{[[u]]} &= HC^{[u]} = \mathbb{R}[u] \quad \text{generated by } 1 \text{ (note that } \beta \sim u), \\
 HC^{[[u, u^{-1}]]} &= HC^{[u, u^{-1}]} = 0, \\
 HC^{[[u^{-1}]]} &= \mathbb{R}[u][1] \quad \text{generated by } \alpha(1 + u^{-1}\beta + u^{-2}\beta^2 + \dots),
 \end{aligned}$$

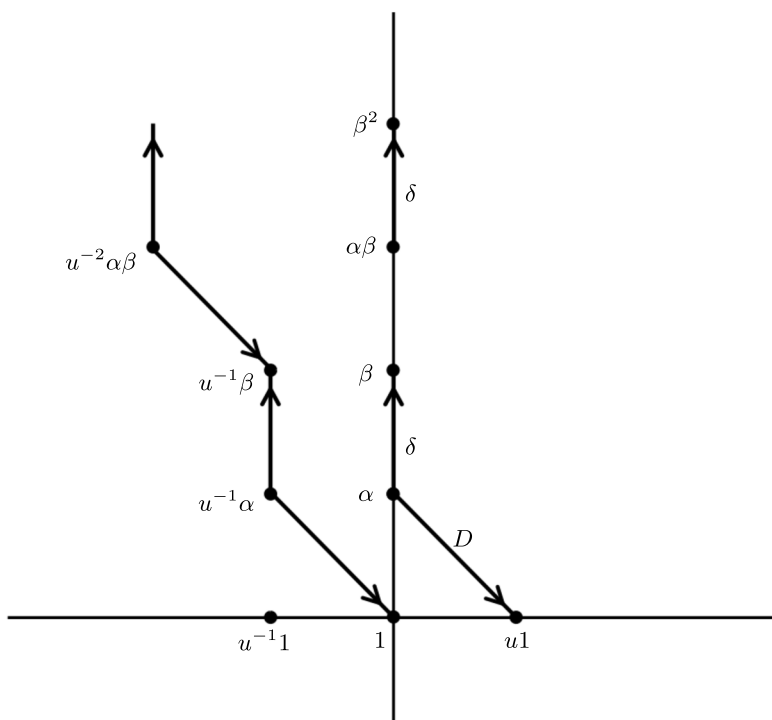


Figure 3. Cyclic complex of singular cochains on ES^1 .

$$HC[[u, u^{-1}]] = HC^{[u, u^{-1}]} = \mathbb{R}[u, u^{-1}] \text{ generated by } 1,$$

$$HC^{[u^{-1}]} = \mathbb{R}[u^{-1}] \text{ generated by } 1, u^{-1}, u^{-2}, \dots$$

So we see two types of tautological sequences:

$$\begin{array}{ccccccc} \xrightarrow{0} & HC_*^{[u]} & \xrightarrow{\cdot u} & HC_*^{[u, u^{-1}]} & \xrightarrow{p_*} & HC_*^{[u^{-1}]} & \xrightarrow{0} \\ & = \downarrow & & = \downarrow & & = \downarrow & \\ \xrightarrow{0} & \mathbb{R}[u][[-2]] & \xrightarrow{\cdot u} & \mathbb{R}[u, u^{-1}] & \xrightarrow{p_*} & \mathbb{R}[u^{-1}] & \xrightarrow{0} \end{array}$$

and

$$\begin{array}{ccccccc} HC_*^{[u, u^{-1}]} & \longrightarrow & HC_*^{[[u^{-1}]]} & \xrightarrow[\cong]{D_0} & HC_{*-1}^{[u]} & \longrightarrow & HC_{*+1}^{[u, u^{-1}]} \\ = \downarrow & & = \downarrow & & = \downarrow & & = \downarrow \\ 0 & \longrightarrow & \mathbb{R}[u^{-1}] \cdot \alpha & \xrightarrow[\cong]{\alpha \rightarrow 1} & \mathbb{R}[u^{-1}] \cdot 1 & \longrightarrow & 0 \end{array}$$

LEMMA 2.30

The mixed complexes in Examples 2.27, 2.28, and 2.29 are quasi-isomorphic.

Proof

The morphisms of mixed complexes

$$(C(\mathbb{R}), \delta = b, D = B) \rightarrow (\mathbb{R}, \delta = D = 0), \quad 1^n \mapsto \begin{cases} 1 & n = 1, \\ 0 & n \geq 2 \end{cases}$$

from Examples 2.27 to 2.28 and

$$(\mathbb{R}, \delta = D = 0) \rightarrow (\Lambda[\theta, \omega], \delta, D), \quad 1 \mapsto 1$$

from Examples 2.28 to 2.29 are quasi-isomorphisms, because all homologies $H(C, \delta)$ equal \mathbb{R} . □

Comparison of the homology groups in these three examples shows that the five remaining homologies in Proposition 2.3 are not quasi-isomorphic invariants of mixed complexes.

3. Chen’s iterated integral

In this section we study the behavior of the various flavors of cyclic homology under Chen’s iterated integral, introduced by K. T. Chen in [5] and [6].

3.1. Chen’s iterated integral as a map of mixed complexes

Let X be a connected oriented manifold, and let LX be its free loop space. For each $k \in \mathbb{N}$ the *Chen pairing*

$$(18) \quad \langle \cdot, \cdot \rangle : \Omega^{*+k}(X^{k+1}) \times C_*(LX) \longrightarrow \mathbb{R}$$

between differential forms on the $(k + 1)$ -fold product X^{k+1} and singular chains on LX is defined as follows. Recall the standard simplex

$$\Delta^k = \{(t_1, \dots, t_k) \in \mathbb{R}^k \mid 0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq 1\}, \quad \Delta^0 = \{0\},$$

with its face maps (parameterizing the boundary faces) $\delta_j : \Delta^{k-1} \longrightarrow \partial\Delta^k$ defined by

$$\begin{aligned} \delta_0(t_1, \dots, t_{k-1}) &= (0, t_1, \dots, t_{k-1}), & \delta_k(t_1, \dots, t_{k-1}) &= (t_1, \dots, t_{k-1}, 1), \\ \delta_j(t_1, \dots, t_{k-1}) &= (t_1, \dots, t_j, t_j, \dots, t_{k-1}), & j &= 1, \dots, k-1. \end{aligned}$$

We give Δ^k the induced orientation from \mathbb{R}^k . Then δ_0 is orientation-reversing, δ_1 is orientation-preserving, and in general δ_j changes orientation by $(-1)^{j+1}$. Consider a simplex B and a continuous map

$$f : B \longrightarrow LX.$$

For any $k \geq 0$ we define the evaluation map

$$ev_f : B \times \Delta^k \longrightarrow X^{k+1}, \quad ev_f(p, t_1, \dots, t_k) := (f(p)(0), f(p)(t_1), \dots, f(p)(t_k)).$$

Given $\omega \in \Omega^*(X^{k+1})$ and such a map f , we define the pairing (18) as

$$\langle \omega, f \rangle := \int_{B \times \Delta^k} ev_f^* \omega.$$

It gives rise to a degree-preserving linear map, *Chen's iterated integral*

$$(19) \quad I : \bigoplus_{n \geq 0} \Omega^{*+n}(X^{n+1}) \rightarrow C^*(LX), \quad (I\omega)(f) := \langle \omega, f \rangle.$$

Recall from Example 2.7 that the spaces $\Omega^*(X^{n+1})$ for $n \in \mathbb{N}_0$ form a cyclic cochain complex, so according to Lemma 2.11 they give rise to a mixed complex $(\bigoplus_{n \geq 0} \Omega^*(X^{n+1}), d_H = d + b, B)$. The other side of (19) carries the structure of a mixed complex $(C^*(LX), d, P)$ provided by Lemma 2.20.

PROPOSITION 3.1

Chen's iterated integral I defines a morphism of mixed complexes

$$I : \left(\bigoplus_{n \geq 0} \Omega^*(X^{n+1}), d_H, B \right) \longrightarrow (C^*(LX), d, -P).$$

Proof

Consider $\omega \in \Omega^*(X^{k+1})$ and $f : B \rightarrow LX$ as above. For compatibility with the differentials, we need to show

$$(20) \quad \langle d_H \omega, f \rangle = \langle \omega, \partial f \rangle.$$

Using Stokes's theorem, we rewrite the left-hand side as

$$\begin{aligned} \langle d_H \omega, f \rangle &= \int_{B \times \Delta^k} ev_f^* d\omega + \int_{B \times \Delta^{k-1}} ev_f^*(b\omega) \\ &= \int_{\partial(B \times \Delta^k)} ev_f^* \omega + \int_{B \times \Delta^{k-1}} ev_f^*(b\omega) \\ &= \int_{\partial B \times \Delta^k} ev_f^* \omega + (-1)^{\dim B} \int_{B \times \partial \Delta^k} ev_f^* \omega + \int_{B \times \Delta^{k-1}} ev_f^*(b\omega) \\ &= \langle \omega, \partial f \rangle + \left((-1)^{|\omega|+1} \int_{B \times \partial \Delta^k} ev_f^* \omega + \int_{B \times \Delta^{k-1}} ev_f^*(b\omega) \right). \end{aligned}$$

Here for the signs in the last step we have used that $\deg d\omega = \dim(B \times \Delta^k)$, and thus, $\dim B = \deg \omega + 1 - k = |\omega| + 1$. So for (20) it remains to show

$$(21) \quad \int_{B \times \Delta^{k-1}} ev_f^*(b\omega) = (-1)^{|\omega|} \int_{B \times \partial \Delta^k} ev_f^* \omega.$$

For this, observe that the face maps $\delta_j : X^k \rightarrow X^{k+1}$ from Example 2.6 and the face maps $\delta_j : \Delta^{k-1} \rightarrow \partial \Delta^k$ satisfy for each $j = 0, \dots, k$ the relation

$$ev_f \circ (\text{id} \times \delta_j) = \delta_j \circ ev_f : B \times \Delta^{k-1} \rightarrow X^{k+1}.$$

Using this and the fact that $\delta_j : \Delta^{k-1} \rightarrow \partial \Delta^k$ changes orientation by $(-1)^{j+1}$ we obtain

$$\int_{B \times \Delta^{k-1}} ev_f^* \delta_j^* \omega = \int_{B \times \Delta^{k-1}} (\text{id} \times \delta_j)^* ev_f^* \omega = (-1)^{j+1} \int_{B \times \delta_j(\Delta^{k-1})} ev_f^* \omega.$$

Now recall that $b\omega = \sum_{j=0}^k (-1)^{|\omega|+j+1} \delta_j^* \omega$ and $\partial\Delta^k = \bigcup_{j=0}^k \delta_j(\Delta^{k-1})$. Multiplying both sides of the last displayed equation with $(-1)^{j+|\omega|+1}$ and summing over $j = 0, \dots, k$ thus yields (21).

Next we show compatibility with the Batalin-Vilkovisky (BV) operators

$$(22) \quad (I \circ B)(\omega)(f) = (-P \circ I)(\omega)(f)$$

for $\omega \in \Omega^*(X^{k+1})$ and $f : B \rightarrow LX$. We first discuss the left-hand side of (22). Recall that $B\omega = (1-t)sN\omega = (-1)^{|\omega|}(1-t)\sigma_0^*N\omega$, where $\sigma_0 : X^{k+2} \rightarrow X^{k+1}$ is the projection forgetting the zeroth factor and $t = (-1)^{k+1}\tau_{k+1}^*$ with $\tau_{k+1} : X^{k+2} \rightarrow X^{k+2}$ given by $\tau_{k+1}(x_0, \dots, x_{k+1}) = (x_1, \dots, x_{k+1}, x_0)$. It follows that the form $tsN\omega \in \Omega^*(X^{k+2})$ is independent of the variable x_1 , so the contraction of its pullback $ev_f^*tsN\omega$ with the coordinate vector field ∂_{t_1} on Δ^{k+1} is zero, and therefore, the integrand $ev_f^*tsN\omega$ vanishes pointwise. This shows that $I(tsN\omega) = 0$, and thus,

$$(23) \quad (I \circ B)(\omega)(f) = \langle (1-t)sN\omega, f \rangle = (-1)^{|\omega|} \int_{B \times \Delta^{k+1}} \widehat{ev}_f^* N\omega$$

with $\widehat{ev}_f := \sigma_0 \circ ev_f : B \times \Delta^{k+1} \rightarrow X^{k+1}$. By the definition of ev_f and σ_0 we have

$$(24) \quad \widehat{ev}_f(p, t_1, \dots, t_{k+1}) = (f(p)(t_1), \dots, f(p)(t_{k+1})).$$

Let us now rewrite the right-hand side of (22). Recall that the pushforward $\phi_*(F_{S^1} \times f)$ is given by the map $\Delta^1 \times B \rightarrow LX$ sending (s, p) to the loop $f(p)(\cdot + s)$. Switching the order of Δ^1 and B , let us define the map $S^1 f : B \times S^1 \rightarrow LX$ by $S^1 f(p, s)(t) := f(p)(t + s)$. Since switching the order changes the orientation by $(-1)^{\dim B}$, we obtain

$$(25) \quad (P \circ I)(\omega)(f) = \langle \omega, \phi_*(F_{S^1} \times f) \rangle = (-1)^{\dim B} \int_{B \times (\Delta^1 \times \Delta^k)} ev_{S^1 f}^* \omega.$$

To proceed further we need to relate the two maps $ev_{S^1 f}$ and \widehat{ev}_f and their domains of definition. For this we split $\Delta^1 \times \Delta^k$ into the $k+1$ regions

$$R_i := \{(s, t_1, \dots, t_k) \mid t_{i-1} \leq 1-s \leq t_i\}, \quad i = 1, \dots, k+1,$$

where we stipulate that $t_0 = 0$ and $t_{k+1} = 1$. Then $\bigcup_{i=0}^k R_i = \Delta^1 \times \Delta^k$ with $R_i \cap R_j$ of measure zero for $i \neq j$. In R_i , the condition $t_{i-1} + s \leq 1 \leq t_i + s$ implies the ordering

$$0 \leq t_i + s - 1 \leq \dots \leq t_k + s - 1 \leq s \leq t_1 + s \leq \dots \leq t_{i-1} + s \leq 1,$$

so R_i is naturally diffeomorphic to Δ^{k+1} via the map $r_i : R_i \rightarrow \Delta^{k+1}$,

$$r_i(s, t_1, \dots, t_k) := (t_i + s - 1, \dots, t_k + s - 1, s, t_1 + s, \dots, t_{i-1} + s).$$

Note that r_i is orientation-preserving if and only if ik is even. Unwrapping the definition of the various maps we find that they satisfy the relation

$$ev_{S^1 f} = \tau_k^{-i} \circ \widehat{ev}_f \circ (\text{id} \times r_i),$$

with the permutation map τ_k from above. Using $t_k = \tau_k^*$ and the invariance of integration with respect to $\text{id} \times r_i$, we deduce that

$$\begin{aligned} (-1)^{\dim B} \int_{B \times R_i} \text{ev}_{S^1 f}^* \omega &= (-1)^{\dim B} \int_{B \times R_i} (\text{id} \times r_i)^* \widehat{\text{ev}}_f^* t_k^{-i} \omega \\ &= (-1)^{\dim B + ik} \int_{B \times \Delta^{k+1}} \widehat{\text{ev}}_f^* (t_k^{-i} \omega). \end{aligned}$$

Since $t = (-1)^k t_k$ satisfies $t^{k+1} = 1$, we can write

$$N = \sum_{j=0}^k t^j = \sum_{i=1}^k t^{k+1-i} = \sum_{i=1}^{k+1} t^{-i} = \sum_{i=1}^{k+1} (-1)^{ik} t_k^{-i}.$$

Summing up the previous displayed equation over $i = 1, \dots, k + 1$ and using (25) we therefore obtain

$$(P \circ I)(\omega)(f) = (-1)^{\dim B} \int_{B \times \Delta^{k+1}} \widehat{\text{ev}}_f^* (N\omega).$$

Now observe that $\dim B + k + 1 = \deg \omega$, and thus, $\dim B = |\omega| - 1$ since $\omega \in \Omega^*(X^{k+1})$. Comparing this with (23), we finish the proof of (22) and thus of Proposition 3.1. \square

3.2. Chen’s iterated integral on Connes’s version

On Connes’s version of cyclic homology, Chen’s iterated integral is induced by the *cyclic Chen pairing*

$$(26) \quad \langle \cdot, \cdot \rangle_{\text{cyc}} : \Omega^{*+k}(X^k) \times C_*(LX) \longrightarrow \mathbb{R}.$$

It is defined on $\omega \in \Omega^{*+k}(X^k)$ and $f : B \rightarrow LX$ by

$$\langle \omega, f \rangle_{\text{cyc}} := (-1)^{|\omega|} \int_{B \times \Delta_{\text{cyc}}^k} \widehat{\text{ev}}_f^* \omega,$$

where Δ_{cyc}^k denotes the space of tuples $(t_1, \dots, t_k) \in (S^1)^k$ in the cyclic order $t_1 \leq t_2 \leq \dots \leq t_k \leq t_1$ and $\widehat{\text{ev}}_f$ is defined in (24). The *Connes (or cyclic) version of Chen’s iterated integral* is the degree-preserving map

$$I^\lambda : \bigoplus_{n \geq 0} \Omega^{*+n}(X^n) \longrightarrow C^*(LX), \quad I^\lambda(\omega)(f) := \langle \omega, f \rangle_{\text{cyc}}.$$

LEMMA 3.2

(a) I^λ is given by the composition

$$I^\lambda : \bigoplus_{n \geq 0} \Omega^{*+n}(X^n) \xrightarrow{B} \bigoplus_{n \geq 0} \Omega^{*+n}(X^{n+1}) \xrightarrow{I} C^*(LX).$$

In particular, $I^\lambda = I \circ B : (C_*(\Omega(X)), d + b) \longrightarrow (C^*(LX), d)$ is a chain map.

(b) I^λ vanishes on $\text{im}(1 - t)$ and thus descends to $C_*^\lambda(\Omega(X)) = C_{*+1}(\Omega(X)) / \text{im}(1 - t)$.

(c) The following diagram commutes, where π_0 denotes the quotient projection of the constant term in u^{-1} :

$$\begin{array}{ccccc}
 C_{*+1}(\Omega(X))[u^{-1}] & \xrightarrow{\pi_0} & C_*^\lambda(\Omega(X)) & \xrightarrow{B} & C_*(\Omega(X))[[u]] \\
 \downarrow I & & \downarrow I^\lambda & & \downarrow I \\
 C^{*+1}(LX)[u^{-1}] & \xrightarrow{-P\pi_0} & C^*(LX) & \xrightarrow{\iota_0} & C^*(LX)[[u]]
 \end{array}$$

(d) For smooth maps $f : B \rightarrow LX$ and $\sigma : B \rightarrow S^1$ define $f_\sigma : B \rightarrow LX$ by $f_\sigma(p)(t) := f(p)(t + \sigma(p))$. Then for each $\omega \in \Omega^*(X^k)$ we have

$$\langle I^\lambda \omega, f_\sigma \rangle = \langle I^\lambda \omega, f \rangle.$$

Proof

For part (a) we decompose

$$\Delta_{\text{cyc}}^n = \bigcup_{i=1}^n \Delta_i^n, \quad \Delta_i^n := \{(t_1, \dots, t_n) \mid t_i \leq 0 \leq t_{i+1}\},$$

where each Δ_i^n is isomorphic to the standard simplex via the permutation map $\tau_{n-1}^i : \Delta^n \xrightarrow{\cong} \Delta_i^n$. Now for $\omega \in \Omega^{*+n}(X^n)$ and $f : B \rightarrow LX$ we compute

$$\begin{aligned}
 \langle \omega, f \rangle_{\text{cyc}} &= (-1)^{|\omega|} \int_{B \times \Delta_{\text{cyc}}^n} \widehat{ev}_f^* \omega = (-1)^{|\omega|} \sum_{i=1}^n (-1)^{(n-1)i} \int_{B \times \Delta_i^n} \widehat{ev}_f^* (\tau_{n-1}^i)^* \omega \\
 &= (-1)^{|\omega|} \int_{B \times \Delta^n} \widehat{ev}_f^* \left(\sum_{i=1}^n t^i \omega \right) = (-1)^{|\omega|} \int_{B \times \Delta^n} \widehat{ev}_f^* (N\omega) = \langle B\omega, f \rangle,
 \end{aligned}$$

where the last equality follows from (23). This proves $I^\lambda = I \circ B$, and this is a chain map because I (by Proposition 3.1) and B are chain maps.

Part (b) follows from $B(1-t) = 0$, which holds because $B = (1-t)sN$ and $N(1-t) = 0$. In part (c) the commutativity of the first square follows by applying π_0 to $I^\lambda = I \circ B = -P \circ I$ (which holds by Proposition 3.1), and the commutativity of the second square follows from part (a). For part (d), note that $ev_{f_\sigma} = ev_f \circ \rho$ with the orientation-preserving diffeomorphism

$$\rho : B \times \Delta_{\text{cyc}}^k \rightarrow B \times \Delta_{\text{cyc}}^k, \quad (p, t_1, \dots, t_k) \mapsto (p, t_1 + \sigma(p), \dots, t_k + \sigma(p)),$$

so part (d) follows from the invariance of integration under diffeomorphisms. \square

Consider the following diagram:

$$\begin{array}{ccc}
 & LX \times ES^1 & \xrightarrow{pr_1} LX \\
 \tilde{g} \nearrow & \downarrow \pi & \\
 B & \xrightarrow{g} & LX \times_{S^1} ES^1
 \end{array}$$

Using this, we define

$$\langle \bar{I}^\lambda \omega, g \rangle := \langle I^\lambda \omega, pr_1 \circ \tilde{g} \rangle$$

for $\omega \in \Omega^*(X^k)$ and $\tilde{g}: B \rightarrow LX \times ES^1$ any lift of $g: B \rightarrow LX \times_{S^1} ES^1$, which exists for each simplex B because $\pi: LX \times ES^1 \rightarrow LX \times_{S^1} ES^1$ is a circle bundle. Any other lift of g is of the form $\tilde{g}_\sigma(p) = \sigma(p) \cdot \tilde{g}(p)$ for a smooth map $\sigma: B \rightarrow S^1$ (where \cdot denotes the circle action on $LX \times ES^1$), and applying Lemma 3.2 to $pr_1 \circ \tilde{g}_\sigma = (pr_1 \circ \tilde{g})_\sigma$, we see that the definition does not depend on the lift \tilde{g} and defines a chain map

$$\bar{I}^\lambda: (C_*^\lambda(\Omega(X)), d+b) \longrightarrow (C^*(LX \times_{S^1} ES^1), d).$$

Note that the homology of the right-hand side is the equivariant cohomology $H_{S^1}^*(LX) = H^*(LX \times_{S^1} ES^1, d)$ of LX defined via the Borel construction. Passing to homology in Lemma 3.2(c) we thus obtain the following commuting diagram, where the maps π_{0*} and ι_{0*} are isomorphisms by Lemma 2.12 and Proposition 2.21, respectively:

$$(27) \quad \begin{array}{ccccc} HC_{*+1}^{[u^{-1}]}(\Omega(X)) & \xrightarrow[\cong]{\pi_{0*}} & HC_*^\lambda(\Omega(X)) & \xrightarrow{B_*} & HC_*^{[[u]]}(\Omega(X)) \\ \downarrow I_* & & \downarrow I_*^\lambda & & \downarrow I_* \\ H_{[u^{-1}]}^{*+1}(LX) & \xrightarrow{-P_{0*}} & H_{S^1}^*(LX) & \xrightarrow[\cong]{\iota_{0*}} & H_{[[u]]}^*(LX) \end{array}$$

3.3. Chen’s iterated integral for simply connected manifolds

In this section we assume in addition that X is simply connected. This has two important consequences. First, by Goodwillie’s Theorem 2.22 the projection $LX \rightarrow \text{pt}$ to a point induces an isomorphism $H_{[[u, u^{-1}]]}^*(LX) \cong \mathbb{R}[u, u^{-1}]$. For the second consequence, recall that composition of the map I from Proposition 3.1 with the morphism ϕ from Corollary 2.16 yields a morphism of mixed complexes (still denoted I) $I: (\bigoplus_{n \geq 0} \Omega^*(X)^{\otimes n+1}, d_H, B) \rightarrow (C^*(LX), d, -P)$. The following theorem is proved in [11] and attributed to Chen (see also Jones [16]).

THEOREM 3.3 ([11, Theorem 3.1])

Let X be a simply connected manifold. Then Chen’s iterated integral

$$I: (C_*(\Omega(X)), d_H, B) \longrightarrow (C^*(LX), d, -P)$$

induces an isomorphism on Hochschild homology and, hence, defines a quasi-isomorphism of mixed complexes.

COROLLARY 3.4

In the situation of Theorem 3.3, Chen’s iterated integral I induces isomorphisms on the classical $[[u]]$, $[[u, u^{-1}]]$, and $[u^{-1}]$ versions of cyclic homology. On the other five versions it does not induce an isomorphism in general.

Proof

The first assertion follows directly from Theorem 3.3 combined with Proposition 2.3. For the second assertion, a comparison of Examples 2.27 and 2.28 shows that I does not induce isomorphisms on the $[u]$, $[u, u^{-1}]$, and $[u, u^{-1}]$ versions

for $X = \{\text{pt}\}$. The remaining two versions fit into the commuting diagram of tautological sequences

$$\begin{array}{ccccc}
 \cdots HC_*^{[[u]]}(\Omega(X)) & \longrightarrow & HC_*^{[[u, u^{-1}]]}(\Omega(X)) & \longrightarrow & HC_*^{[[u^{-1}]]}(\Omega(X)) \cdots \\
 \cong \downarrow I_* & & \downarrow I_* & & \downarrow I_* \\
 \cdots H_{[[u]]}^*(LX) & \longrightarrow & H_{[[u, u^{-1}]]}^*(LX) & \longrightarrow & H_{[[u^{-1}]]}^*(LX) \cdots
 \end{array}$$

Now $HC_*^{[[u, u^{-1}]]}(\Omega(X)) \cong HC_*^{[[u, u^{-1}]]}(\Omega(X)) \cong H_{[[u, u^{-1}]]}^*(LX) \cong \mathbb{R}[u, u^{-1}]$, where the first isomorphism follows from Corollary 2.14, the second one is given by I_* , and the last one follows from Goodwillie’s Theorem 2.22. On the other hand, by Theorem 2.26 we have $H_{[[u, u^{-1}]]}^*(LX) \cong H^*(X) \otimes \mathbb{R}[u, u^{-1}]$. For X noncontractible these two $\mathbb{R}[u]$ -modules differ; hence, the middle map I_* in the diagram is not an isomorphism. It follows from the five lemma that the right map I_* is not an isomorphism either. \square

The following theorem, which is the main result of this paper, describes the behavior of Connes’s version under Chen’s iterated integral.

THEOREM 3.5

Let X be a simply connected manifold. Then the kernel and image of the iterated integral $I_*^\lambda : HC_*^\lambda(\Omega(X)) \rightarrow H_{S^1}^*(LX)$ are given by

$$\begin{aligned}
 \ker I_*^\lambda &= \text{span}\{[1], [1^3], [1^5], \dots\} \cong \mathbb{R}[u^{-1}], \\
 \text{im } I_*^\lambda &= \ker(\iota^* : H_{S^1}^*(LX) \rightarrow \mathbb{R}[u]) \cong H_{S^1}^*(LX) / \mathbb{R}[u].
 \end{aligned}$$

Proof

Consider the commuting diagram

$$\begin{array}{ccccccc}
 & & HC_*^\lambda(\Omega(X)) & \xrightarrow{I_*^\lambda} & H_{S^1}^*(LX) & & \\
 & & \uparrow \cong & & \cong \downarrow \iota_{0*} & & \\
 H_{[[u, u^{-1}]]}^{*+1}(LX) & \xrightarrow{p_*} & H_{[[u^{-1}]]}^{*+1}(LX) & \xrightarrow{-P_{0*}} & H_{[[u]]}^*(LX) & \xrightarrow{\cdot u} & H_{[[u, u^{-1}]]}^{*+2}(LX) \\
 \downarrow \cong & & \uparrow \pi^* & & \downarrow \iota^* & & \downarrow \cong \\
 \mathbb{R}[u, u^{-1}] & \xrightarrow{p_*} & \mathbb{R}[u^{-1}] & \xrightarrow{0} & \mathbb{R}[u] & \xrightarrow{\cdot u} & \mathbb{R}[u, u^{-1}]
 \end{array}$$

Here the lower two rows follow from (16) for $Y = LX$, where π^* is injective, ι^* is surjective, and the outer vertical maps are isomorphisms by Goodwillie’s Theorem 2.22. The upper square follows from diagram (27), where I_* is an isomorphism by Corollary 3.4 and we have abbreviated $\iota_0 P_0$ as P_0 . We read off

$$\begin{aligned} \ker P_{0*} &= \pi^* \mathbb{R}[u^{-1}] = \text{span}\{u^{-k}1 \mid k \in \mathbb{N}_0\}, \\ \text{im } P_{0*} &= \ker(\iota^* : H_{[[u]]}^{*-1}(LX) \rightarrow \mathbb{R}[u]), \end{aligned}$$

and therefore,

$$\begin{aligned} \ker I_*^\lambda &= \pi_{0*} I_*^{-1} \ker P_{0*} \\ &= \pi_{0*} \text{span}\{u^{-k}1 + u^{-k+1}1^3 + \dots + 1^{2k+1} \mid k \in \mathbb{N}_0\} \\ &= \text{span}\{[1], [1^3], [1^5], \dots\}, \\ \text{im } I_*^\lambda &= \text{im } P_{0*} = \ker(\iota^* : H_{S^1}^{*-1}(LX) \rightarrow \mathbb{R}[u]). \end{aligned}$$

Here the description of $I_*^{-1} \ker P_{0*}$ and its image under π_{0*} follow from Example 2.27. □

COROLLARY 3.6

Let X be a simply connected manifold. Denote by $\overline{HC}_*^\lambda(\Omega(X))$ the reduced Connes version of cyclic homology of the de Rham complex of X , and denote by $H_{S^1}^*(LX, x_0)$ the S^1 -equivariant cohomology of LX relative to a fixed constant loop x_0 . Then Chen’s iterated integral induces an isomorphism

$$I_*^\lambda : \overline{HC}_*^\lambda(\Omega(X)) \xrightarrow{\cong} H_{S^1}^*(LX, x_0).$$

Proof

By passing to reduced homologies, the commuting diagram in the proof of Theorem 3.5 simplifies to

$$\begin{array}{ccccc} \overline{HC}_*^\lambda(\Omega(X)) & \xrightarrow{I_*^\lambda} & H_{S^1}^*(LX, x_0) & & \\ \pi_{0*} I_*^{-1} \uparrow \cong & & \cong \downarrow \iota_{0*} & & \\ 0 \longrightarrow & H_{[u^{-1}]}^{*+1}(LX, x_0) & \xrightarrow[-\cong]{-P_{0*}} & H_{[[u]]}^*(LX, x_0) & \longrightarrow 0 \end{array}$$

and the corollary follows. □

3.4. Computations for spheres

For a simply connected manifold X , the S^1 -equivariant cohomology of LX can be computed via minimal models as follows (see, e.g., [2]). Let $(M = \Lambda[x_1, x_2, \dots], d)$ be the minimal model of X . To it we associate a mixed complex (LM, d, B) by setting $LM := \Lambda[x_1, \bar{x}_1, x_2, \bar{x}_2, \dots]$ with new generators \bar{x}_i of degrees $\text{deg } \bar{x}_i = \text{deg } x_i - 1$, defining B as the derivation satisfying $Bx_i = \bar{x}_i$ and $B\bar{x}_i = 0$, and extending d from M to LM by the requirement $dB + Bd = 0$, that is, by defining $d\bar{x}_i := -B(dx_i)$. Then $(LM[u], d_u = d + uB)$ is the minimal model for the Borel space $LX \times_{S^1} ES^1$, so it computes $H_{S^1}^*(LX)$.

Recall that a manifold X is *formal over* \mathbb{R} (in the sense of rational homotopy theory) if its de Rham dga $\Omega^*(X)$ is connected to its cohomology $H^*(X)$ by a

zigzag of quasi-isomorphisms (cf. [13]). By Proposition 2.15, all versions of cyclic homology of $\Omega^*(X)$ can then be computed using the dga (with trivial differential) $H^*(X)$.

EXAMPLE 3.7 (Odd-dimensional spheres)

The sphere S^n with $n \geq 3$ odd has minimal model $\Lambda[a]$ with $\deg a = n$ and $da = 0$. So the minimal model of $LS^n \times_{S^1} ES^1$ is $\Lambda[a, \bar{a}, u]$ with differential $d_u \bar{a} = 0$ and $d_u a = u\bar{a}$. It follows that

$$H_{S^1}^*(LS^n) = \Lambda[\bar{a}, u] / \langle u\bar{a} \rangle = \bar{a}\mathbb{R}[\bar{a}] \oplus \mathbb{R}[u],$$

where u acts trivially on the first summand. Relative to a point $x_0 \in S^n$ this becomes

$$H_{S^1}^*(LS^n, x_0) = \bar{a}\mathbb{R}[\bar{a}], \quad \deg \bar{a} = n - 1.$$

Let us now compute the reduced Connes version of cyclic homology of $\Omega^*(S^n)$. Since S^n is formal, we can compute this from its cohomology $H^*(S^n) = \mathbb{R}1 \oplus \mathbb{R}v$, where $\deg v = n$, or rather the reduced cohomology $\overline{H}^*(S^n) = \mathbb{R}v$. For $k \geq 1$ denote by v^k the word with k letters v . Since $t(v^k) = (-1)^{(k-1)+n^2(k-1)}v^k = v^k$, each v^k survives in the quotient by cyclic permutation, and by the definition of \overline{HC}_*^λ (with trivial Hochschild differential) we get

$$\overline{HC}_*^\lambda(\Omega^*(S^n)) = HC_*^\lambda(\overline{H}^*(S^n)) = v\mathbb{R}[v], \quad |v| = n - 1.$$

This is compatible with Theorem 3.5 if I_*^λ sends v to \bar{a} .

EXAMPLE 3.8 (Even-dimensional spheres)

The sphere S^n with $n \geq 2$ even has minimal model $\Lambda[a, b]$ with $\deg a = n$, $\deg b = 2n - 1$, $da = 0$, and $db = a^2$. So the minimal model of $LS^n \times_{S^1} ES^1$ is $\Lambda[a, \bar{a}, b, \bar{b}, u]$ with differential $d_u a = u\bar{a}$, $d_u b = a^2 + u\bar{b}$, $d\bar{a} = 0$, and $d\bar{b} = -2a\bar{a}$. It follows (cf. [2]) that

$$H_{S^1}^*(LS^n) = \bar{a}(\Lambda[\bar{a}, u] / \langle u, 2a, a^2 + u\bar{b} \rangle) \oplus \mathbb{R}[u] = \bar{a}\mathbb{R}[\bar{b}] \oplus \mathbb{R}[u],$$

where u acts trivially on the first summand. Relative to a point $x_0 \in S^n$ this becomes

$$H_{S^1}^*(LS^n, x_0) = \bar{a}\mathbb{R}[\bar{b}], \quad \deg \bar{a} = n - 1, \deg \bar{b} = 2n - 2.$$

Again, we can compute the reduced Connes version of cyclic homology of $\Omega^*(S^n)$ from $\overline{H}^*(S^n) = \mathbb{R}v$, where $\deg v = n$. Since $t(v^k) = (-1)^{(k-1)+n^2(k-1)}v^k = (-1)^{k-1}v^k$, the word v^k survives in the quotient by cyclic permutation if and only if k is odd, and by the definition of \overline{HC}_*^λ (with trivial Hochschild differential) we get

$$\overline{HC}_*^\lambda(\Omega^*(S^n)) = HC_*^\lambda(\overline{H}^*(S^n)) = v\mathbb{R}[v^2], \quad |v| = n - 1.$$

This is compatible with Theorem 3.5 if I_*^λ sends v^{2i+1} to $\bar{a}\bar{b}^i$.

4. Duality

4.1. Generalities

As before, all complexes are over the ground field \mathbb{R} . Let $(C = \bigoplus_{k \in \mathbb{Z}} C_k, \delta)$ be a chain complex. We dualize it in such a way that the result is a chain complex as well, that is,

$$(C^\vee := \bigoplus_{k \in \mathbb{Z}} \text{Hom}(C_{-k}, \mathbb{R}), \delta^*)$$

Suppose now that (C_j, δ_j) , $j = 1, 2$, are two chain complexes and that

$$\langle \cdot, \cdot \rangle : C_1 \otimes C_2 \longrightarrow \mathbb{R}$$

is a bilinear pairing of degree 0 such that the differentials are mutual adjoints with respect to the pairing

$$\langle \delta_1 x, y \rangle = \langle x, \delta_2 y \rangle.$$

The pairing naturally gives rise to two maps

$$f_{12} : C_1 \longrightarrow C_2^\vee, \quad f_{21} : C_2 \longrightarrow C_1^\vee$$

by $f_{12}(x) := \langle x, \cdot \rangle$ and $f_{21}(y) := \langle \cdot, y \rangle$. Since the differentials are adjoints, the maps f_{12} and f_{21} are chain maps. Denoting by $\iota_j : C_j \hookrightarrow (C_j^\vee)^\vee$, $j = 1, 2$, the canonical embeddings into the second dual, we then have the following equalities of chain maps:

$$(28) \quad f_{21} = f_{12}^* \circ \iota_2 \quad \text{and} \quad f_{12} = f_{21}^* \circ \iota_1.$$

We say that a graded vector space $E = \bigoplus_{k \in \mathbb{Z}} E_k$ is *graded finite-dimensional* if each E_k is finite-dimensional.

LEMMA 4.1

In the above setting, assume that the homologies of C_1 and C_2 are graded finite-dimensional. Then f_{12} is a quasi-isomorphism if and only if f_{21} is.

Proof

For each chain complex C over \mathbb{R} we have a commuting diagram

$$\begin{CD} H(C) @>H\iota>> H((C^\vee)^\vee) \\ @VV\iota_HV @VV\cong V \\ (H(C)^\vee)^\vee @>\cong>> H(C^\vee)^\vee \end{CD}$$

where $H\iota$ is the map on homology induced by the canonical embedding $\iota : C \hookrightarrow (C^\vee)^\vee$, ι_H is the canonical embedding for $H(C)$, and the two isomorphisms come from the universal coefficient theorems. If $H(C)$ is graded finite-dimensional, then the map ι_H is an isomorphism (because finite-dimensional spaces are reflexive) and, hence, so is $H\iota$. This shows that in the situation of the lemma both

canonical embeddings ι_1 and ι_2 are quasi-isomorphisms. In view of (28) this implies the lemma, recalling that the dual of a quasi-isomorphism is again a quasi-isomorphism. \square

4.2. Duality of mixed complexes

We now generalize the preceding discussion to mixed complexes. The dual mixed complex to (C, δ, D) is defined as (C^\vee, δ^*, D^*) . Suppose now that (C_j, δ_j, D_j) , $j = 1, 2$, are two mixed complexes and

$$\langle \cdot, \cdot \rangle : C_1 \otimes C_2 \longrightarrow \mathbb{R}$$

is a bilinear pairing of degree 0 such that both differentials are mutual adjoints with respect to the pairing

$$\langle \delta_1 x, y \rangle = \langle x, \delta_2 y \rangle \quad \text{and} \quad \langle D_1 x, y \rangle = \langle x, D_2 y \rangle.$$

This implies that the maps f_{12} and f_{21} are morphisms of mixed complexes, and Lemma 4.1 yields the following result.

COROLLARY 4.2

In the above setting assume that the homologies $H(C_1, \delta_1)$ and $H(C_2, \delta_2)$ are graded finite-dimensional. Then f_{12} is a quasi-isomorphism of mixed complexes if and only if f_{21} is.

Now let (C, δ, D) be a mixed complex. We want to investigate the relation between the total complex of its dual and the dual of its total complex. For concreteness, let us consider the version $C[u^{-1}]$. We define a degree 0 pairing

$$(29) \quad \langle \cdot, \cdot \rangle : C^\vee[[u]]^{-k} \otimes C[u^{-1}]_k \rightarrow \mathbb{R}, \quad \langle \phi, c \rangle := \sum_{i \geq 0} \phi_i(c_{-i}),$$

where $\phi = \sum_{i \geq 0} \phi_i u^i$ with $\phi_i \in (C^\vee)^{-k-2i} = \text{Hom}(C_{k+2i}, \mathbb{R})$ and $c = \sum_{i \geq 0} c_{-i} u^{-i}$ with $c_{-i} \in C_{k+2i}$. Note that the sum $\sum_{i \geq 0} \phi_i(c_{-i})$ is finite because only finitely many c_{-i} 's are nonzero. Direct computation yields the following.

LEMMA 4.3

The pairing (29) induces via $\iota(\phi)(c) = \langle \phi, c \rangle$ a chain isomorphism

$$\iota : (C^\vee[[u]], \delta^* + uD^*) \xrightarrow{\cong} (C[u^{-1}]^\vee, (\delta + uD)^*)$$

respecting the $\mathbb{R}[u]$ -module structures with $|u| = 2$ on both sides. Similarly, we obtain the chain isomorphisms

$$C^\vee[[u^{-1}]] \cong C[u]^\vee \quad \text{and} \quad C^\vee[[u, u^{-1}]] \cong C[u, u^{-1}]^\vee.$$

Finally, note that for a mixed complex (C, δ, D) and its dual we have a commuting diagram of chain maps (with respect to δ^*)

$$\begin{array}{ccccc}
 \mathrm{im} D^* & \xrightarrow{\cong} & (\mathrm{im} D)^\vee & \xrightarrow[D^*]{\cong} & (C/\ker D)^\vee \\
 \downarrow & & & & \downarrow \\
 \ker D^* & \xrightarrow[\phi]{\cong} & & & (C/\mathrm{im} D)^\vee
 \end{array}$$

where the maps D^* and ϕ have degree -1 . On homology this yields

$$(30) \quad \begin{array}{ccccc}
 H(\mathrm{im} D^*) & \xrightarrow{\cong} & H(\mathrm{im} D)^\vee & \xrightarrow[(D^*)_*]{\cong} & H(C/\ker D)^\vee \\
 \downarrow & & & & \downarrow \\
 H(\ker D^*) & \xrightarrow[\phi_*]{\cong} & & & H(C/\mathrm{im} D)^\vee
 \end{array}$$

4.3. Equivariant homology of S^1 -spaces

Let Y be a topological S^1 -space. It was shown in the proof of Lemma 2.20 that

$$(C_{-*}(Y), \partial, Q)$$

is a mixed complex, where $(C_*(Y), \partial)$ is the singular chain complex and Q is the map (15) induced by the circle action. Note that we grade the singular chains negatively to give ∂ and Q degrees 1 and -1 , respectively. The homology of this mixed complex is the (negatively graded) singular homology $H_{-*}(Y)$, and its dual is the mixed complex $(C^*(Y), d, P)$ in Lemma 2.20. We denote the cyclic homology of $(C_{-*}(Y), \partial, Q)$ by $H_{-*}^{[[u]]}(Y)$ and so on.

Lemma 5.1 in [16] and the fact that $C_{-*}(Y)$ lives in nonpositive degrees imply the following dual version of Proposition 2.21.

PROPOSITION 4.4 ([16, Lemma 5.1])

For each topological S^1 -space Y we have canonical isomorphisms

$$H_{-*}^{S^1}(Y) \cong H_{-*}^{[u^{-1}]}(Y) \cong H_{-*}^{[[u^{-1}]]}(Y).$$

As in Section 2.5, for an S^1 -space Y with a fixed point y_0 the inclusion and projection $\mathrm{pt} \xrightarrow{\iota} Y \xrightarrow{\pi} \mathrm{pt}$, $\iota(\mathrm{pt}) = y_0$, induce the following commuting diagram, where all the vertical maps ι_* are injective and we have surjective vertical maps π_* in the other direction:

$$(31) \quad \begin{array}{ccccccc}
 H_{-*}^{[[u, u^{-1}]]}(Y) & \xrightarrow{p_*} & H_{-*}^{[u^{-1}]}(Y) & \xrightarrow{Q_0} & H_{-*}^{[[u]]}(Y) & \xrightarrow{\cdot u} & H_{-*}^{[[u, u^{-1}]]}(Y) \\
 \uparrow \iota_* & & \uparrow \iota_* & & \uparrow \iota_* & & \uparrow \iota_* \\
 \mathbb{R}[u, u^{-1}] & \xrightarrow{p_*} & \mathbb{R}[u^{-1}] & \xrightarrow{0} & \mathbb{R}[u] & \xrightarrow{\cdot u} & \mathbb{R}[u, u^{-1}]
 \end{array}$$

REMARK 4.5

Lemma 2.24 has the following dual version: If Y is a smooth S^1 -space without fixed points, then $H_{-*}^{[u, u^{-1}]}(Y) = 0$ and we have canonical isomorphisms

$$H_{-*}^{[u^{-1}]}(Y) \xrightarrow{\cong} H_{-*+1}^{[[u^{-1}]]}(Y) \xrightarrow{\cong} H_{-*}(Y/S^1) \xrightarrow{\cong} H_{-*}^{[u]}(Y).$$

For Y with fixed points, applying this to $Y \times ES^1$ provides an alternative proof of Proposition 4.4 in the smooth case.

4.4. Finiteness

In this section we prove two finiteness results on homology.

LEMMA 4.6

Let (A, d) be a dga whose cohomology $H^*(A)$ is graded finite-dimensional. Assume in addition that $\dim H^0(A) = 1$ and $\dim H^1(A) = 0$. Then the Hochschild homology $HH(A)$ is graded finite-dimensional.

Proof

Consider the word length filtration on the Hochschild complex. This filtration is bounded from below and exhaustive; therefore, the corresponding spectral sequence converges to $HH(A)$. It is enough to show that graded finite dimensionality holds for the second page. The first page computes to $E_1^{p,q} = H^p(A^{\otimes(q+1)}, d)$, and the second page computes to

$$E_2^{p,q} = H^q(E_1, b)^p = H^q(E_1/D(A), b)^p,$$

where q denotes the word length degree, p is the degree in A , and $D(A)$ is the acyclic subcomplex generated by words with 1 in some positive slot considered in Section 2.4. We will show that the desired finite dimensionality holds even before we take the homology with respect to b . Fix some degree $k = p - q$ for the chain complex E_1/D , and write out the degree k part of the complex,

$$(E_1/D)^k = \bigoplus_{p-q=k} (E_1/D)^{p,q}.$$

Since $H^1(A) = 0$ and we have factored out $D(A)$, we have $(E_1/D)^{p,q} = 0$ for $p < 2q$. So the sum runs over the p 's that satisfy $p \geq 2q$; in other words, $k = p - q \geq q$. This leaves us with only finitely many options for q . Therefore, we have only finitely many nonzero summands in $(E_1/D)^k$ and thus $\dim(E_1/D)^k < \infty$. \square

Note that Lemma 4.6 applies in particular to $A = \Omega^*(X)$ for a simply connected manifold X .

LEMMA 4.7

If X is a simply connected manifold, then $H_*(LX)$ and $H_*^S(LX)$ are graded finite-dimensional.

Proof

Consider the Sullivan minimal model $M = (\Lambda[x_1, x_2, \dots], d)$ for X , where $\Lambda[x_1, x_2, \dots]$ is the free graded commutative algebra on generators x_i of degrees $\deg x_i \geq 2$. Moreover, by [10, Proposition 12.2] there are only finitely many generators x_i of any given degree. Then the minimal model for LX is $LM = (\Lambda[x_1, \bar{x}_1, x_2, \bar{x}_2, \dots], \delta)$, with $\deg \bar{x}_i = \deg x_i - 1$ and a suitable differential δ . Since all generators of LM have strictly positive degrees, LM is graded finite-dimensional and, hence, so is its homology $H_*(LX)$. The graded finite dimensionality of $H_*^{S^1}(LX)$ follows by the same argument from its minimal model $(\Lambda[x_1, \bar{x}_1, x_2, \bar{x}_2, \dots, u], \delta_u)$, with $\deg u = 2$ and a suitable differential δ_u . \square

4.5. Cyclic cohomology

Consider a mixed complex (C, δ, D) and its dual mixed complex (C^\vee, δ^*, D^*) . The *cyclic cohomology* of C is defined as

$$HC_{[u, u^{-1}]}^* := H(C^\vee)_{-*}^{[u, u^{-1}]},$$

and similarly for the other seven versions. Lemma 4.3 and the universal coefficient theorem yield

$$(32) \quad HC_{[[u]]}^k = H(C^\vee)_{-k}^{[[u]]} \cong ((HC^{[u^{-1}]})^\vee)_{-k} = \text{Hom}(HC_k^{[u^{-1}]}, \mathbb{R}),$$

and similarly for the other two versions in Lemma 4.3. Thus, results about polynomial versions of cyclic homology dualize to results about the corresponding power series versions of cyclic cohomology.

Consider now a cyclic cochain complex (C_n, d_i, s_j, t_n, d) with its associated mixed complex $(C := \bigoplus_{n \geq 0} C_n, d + b, B)$. Its *Connes's version of cyclic cohomology* is defined as

$$\begin{aligned} HC_\lambda^* &:= H_{-*}((C/\text{im}(1-t))^\vee, d^* + b^*) = H_{-*}(\ker(1-t)^*, d^* + b^*) \\ &\cong H_{-*}(C^\vee/\text{im}(1-t^*), d^* + b^*), \end{aligned}$$

where the last isomorphism is induced by the inverse of the chain isomorphism $N^* : C^\vee/\text{im}(1-t^*) \xrightarrow{\cong} \text{im } N^* = \ker(1-t^*) \subset C^\vee$. Recall from Corollary 2.14 the series of canonical isomorphisms

$$(33) \quad HC_*^{[u^{-1}]} \cong HC_*^{[[u^{-1}]]} \cong HC_{*-1}^\lambda \cong HC_{*-1}^{[u]}.$$

In view of (32), dualizing the first, third, and fourth terms yields the isomorphisms

$$HC_{[[u^{-1}]]}^{*-1} \cong HC_\lambda^{*-1} \cong HC_{[[u]]}^*.$$

Moreover, the proof of Lemma 2.12 (which uses only the exactness of the rows in the θ double complex) carries over to C^\vee to yield the isomorphisms

$$HC_{[[u^{-1}]]}^{*-1} \cong HC_\lambda^{*-1} \cong HC_{[u]}^*.$$

Combining these, we have proved the following.

LEMMA 4.8

For a cyclic cochain complex (C_n, d_i, s_j, t_n, d) , the canonical maps on cyclic cohomology give the series of isomorphisms

$$HC_{[[u^{-1}]]}^{*-1} \cong HC_{\lambda}^{*-1} \cong HC_{[u]}^* \cong HC_{[[u]]}^*.$$

Note that this series of isomorphisms differs from (33) in the degrees and by the appearance of the $[u]$ rather than the $[u^{-1}]$ version.

4.6. Chen’s iterated integral on cyclic cohomology

Let X be a manifold, and let $\Omega(X)$ be its de Rham dga. Recall the mixed complexes $(C_*(\Omega(X)), d + b, B)$ from Corollary 2.16 and $(C_{-*}(LX), \partial, Q)$ from Section 4.3 for $Y = LX$. Let

$$\langle \cdot, \cdot \rangle : C_*(\Omega(X)) \otimes C_{-*}(LX) \rightarrow \mathbb{R}$$

be the Chen pairing (18) from Section 3. By the proof of Proposition 3.1 this pairing respects the structures of mixed complexes, so it induces two maps of mixed complexes: Chen’s iterated integral $I : C_*(\Omega(X)) \rightarrow C^*(LX)$ and its adjoint

$$J : (C_{-*}(LX), \partial, -Q) \rightarrow (C^{-*}(\Omega(X)), d^* + b^*, B^*).$$

Similarly, the cyclic Chen pairing $\langle \cdot, \cdot \rangle_{\text{cyc}}$ defined in (26) induces two chain maps: Connes’s version of Chen’s iterated integral I^λ and its adjoint

$$J_\lambda : (C_{-*+1}(LX), \partial) \rightarrow (C^{-*}(\Omega(X)), d^* + b^*).$$

Lemma 3.2 and the discussion following it dualize to the following.

LEMMA 4.9

(a) J_λ is given by the composition of chain maps

$$J_\lambda : C_{-*+1}(LX) \xrightarrow{J} C^{-*+1}(\Omega(X)) \xrightarrow{B^*} C^{-*}(\Omega(X)).$$

(b) J_λ lands in $(C_*(\Omega(X))/\text{im}(1 - t))^\vee = \ker(1 - t^*) = C_\lambda^{-*+1}(\Omega(X))$ and induces a map

$$\bar{J}_\lambda : C_{-*}(LX \times_{S^1} ES^1) \rightarrow \ker(1 - t^*).$$

(c) The following diagram commutes:

$$\begin{array}{ccccc} C_{[u^{-1}]}^{-*+1}(\Omega(X)) & \xrightarrow{B^* \pi_0} & C_\lambda^{-*+1}(\Omega(X)) & \xrightarrow{\iota_0} & C_{[[u]]}^{-*}(\Omega(X)) \\ \uparrow J & & \uparrow J_\lambda & & \uparrow J \\ C_{- * + 1}^{[u^{-1}]}(LX) & \xrightarrow{\pi_0} & C_{- * + 1}(LX) & \xrightarrow{-\iota_0 Q} & C_{- * }^{[[u]]}(LX) \end{array}$$

The map \bar{J}_λ induces a map on homology which we denote by

$$J_{\lambda*} : H_*^{S^1}(LX) \rightarrow HC_\lambda^*(\Omega(X)).$$

Passing to homology in Lemma 4.9(c) we obtain the following commuting diagram, where the maps B_* and P_{0*} are isomorphisms by Lemma 4.8 and Proposition 4.4, respectively:

$$(34) \quad \begin{array}{ccccc} HC_{[u^{-1}]}^{-*+1}(\Omega(X)) & \xrightarrow{B_*^* \pi_{0*}} & HC_{\lambda}^{-*+1}(\Omega(X)) & \xrightarrow[\cong]{\iota_{0*}} & HC_{[[u]]}^{-*}(\Omega(X)) \\ \uparrow J_* & & \uparrow J_{\lambda*} & & \uparrow J_* \\ H_{-*+1}^{[u^{-1}]}(LX) & \xrightarrow[\cong]{\pi_{0*}} & H_{-*+1}^{S^1}(LX) & \xrightarrow{-\iota_{0*} Q_*} & H_{-*}^{[[u]]}(LX) \end{array}$$

The simply connected case. Assume now in addition that X is simply connected. Then by Theorem 3.3, Chen’s iterated integral $I : C_*(\Omega(X)) \rightarrow C^*(LX)$ is a quasi-isomorphism of mixed complexes. Since the homologies $HH_*(\Omega(X))$ (by Lemma 4.6) and $H_*(LX)$ (by Lemma 4.7) are graded finite-dimensional, Corollary 4.2 applied to the Chen pairing yields the following.

COROLLARY 4.10

Let X be a simply connected manifold. Then the dual Chen iterated integral $J : C_{-}(LX) \rightarrow C^{-*}(\Omega(X))$ defines a quasi-isomorphism of mixed complexes and, therefore, induces isomorphisms on the $[[u]]$, $[[u, u^{-1}]]$, and $[u^{-1}]$ versions of cyclic cohomology.*

As in the proof of Theorem 3.5, we obtain a commuting diagram

$$\begin{array}{ccccccc} & & H_*^{S^1}(LM) & \xrightarrow{J_{\lambda*}} & HC_{\lambda}^*(\Omega(X)) & & \\ & & \uparrow \cong & & \cong \downarrow \iota_{0*} J_*^{-1} & & \\ H_*^{[[u, u^{-1}]}(LX) & \xrightarrow{p_*} & H_*^{[u^{-1}]}(LX) & \xrightarrow{-\iota_{0*} Q_{0*}} & H_{*-1}^{[[u]]}(LX) & \xrightarrow{\cdot u} & H_{*+1}^{[[u, u^{-1}]}(LX) \\ \downarrow \cong & & \uparrow \iota_* & & \downarrow \pi_* & & \downarrow \cong \\ \mathbb{R}[u, u^{-1}] & \xrightarrow{p_*} & \mathbb{R}[u^{-1}] & \xrightarrow{0} & \mathbb{R}[u] & \xrightarrow{\cdot u} & \mathbb{R}[u, u^{-1}] \end{array}$$

Here the lower two rows follow from (31) for $Y = LX$, where ι_* is injective, π_* is surjective, and the outer vertical maps are isomorphisms by the dual version of Goodwillie’s Theorem 2.22. The upper square follows from diagram (34), where J_* is an isomorphism by Corollary 4.10. We read off

$$\begin{aligned} \ker J_{\lambda*} &= \pi_{0*} \iota_* \mathbb{R}[u^{-1}], \\ \text{im } J_{\lambda*} &= \ker(\pi_* \iota_{0*} J_*^{-1} : HC_{\lambda}^*(\Omega(X)) \rightarrow \mathbb{R}[u]), \end{aligned}$$

and passing to reduced homologies as in the proof of Corollary 3.6, we conclude the following.

COROLLARY 4.11

Let X be a simply connected manifold. Denote by $\overline{HC}_{\lambda}^(\Omega(X))$ the reduced Connes*

version of cyclic cohomology of the de Rham complex of X , and denote by $H_*^{S^1}(LX, x_0)$ the S^1 -equivariant homology of LX relative to a fixed constant loop x_0 . Then Chen's iterated integral induces an isomorphism

$$J_{\lambda*} : H_*^{S^1}(LX, x_0) \xrightarrow{\cong} \overline{HC}_\lambda^*(\Omega(X)).$$

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