Euler numbers of Hilbert schemes of points on simple surface singularities and quantum dimensions of standard modules of quantum affine algebras

Hiraku Nakajima

To Fukaya san on the occasion of his 60th birthday

Abstract We prove a recent conjecture by Gyenge, Némethi, and Szendrői giving a formula of the generating function of Euler numbers of Hilbert schemes of points $\mathrm{Hilb}^N(\mathbb{C}^2/\Gamma)$ on a simple singularity $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SL}(2)$. We deduce it from the claim that quantum dimensions of standard modules for the quantum affine algebra associated with Γ at $\zeta = \exp(\frac{2\pi\sqrt{-1}}{2(h^\vee+1)})$ are always 1, which is a special case of an earlier conjecture by Kuniba. Here $h^\vee$ is the dual Coxeter number. We also prove the claim, which was not known for E_7, E_8 before.

Introduction

In this article we prove a conjecture by Gyenge, Némethi, and Szendrői in [13] and [14] that gives a formula of the generating function of Euler numbers of Hilbert schemes of points $\mathrm{Hilb}^N(\mathbb{C}^2/\Gamma)$ on a simple singularity $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SL}(2)$. When Γ is of type A, Euler numbers were computed by Dijkgraaf and Sułkowski [9], and Toda [38]. The formula in [13] and [14] is given in a different form and makes sense for arbitrary Γ. The formula was proved for type D, as well as type A, in [14]. Our proof of the formula in type E is new.

The formula in [13] and [14] is written as a specialization of the character for the "Fock space," which is the tensor product of the basic representation of the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ corresponding to Γ, and the usual Fock space representation of the Heisenberg algebra. The Fock space was realized in a geometric way as the direct sum of homology groups of various connected components of $\mathrm{Hilb}^N(\mathbb{C}^2)^\Gamma$ for all N (see [31]). The character Z is well known (see, e.g., [17, Section 12]):

$$Z \equiv Z\big((e^{-\alpha_i})_{i=1}^n, e^{-\delta}\big) = \prod_{m=1}^{\infty} (1 - e^{-m\delta})^{-(n+1)} \sum_{\vec{m}\in\mathbb{Z}^n} e^{-\frac{(\vec{m},\mathbf{C}\vec{m})}{2}\delta} \prod_{i=1}^{n} e^{-m_i\alpha_i},$$

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where α_i is the ith simple root of $\mathfrak{g}_{\mathrm{fin}}$, the underlying finite-dimensional complex simple Lie algebra, δ is the positive primitive imaginary root, n is the rank of $\mathfrak{g}_{\mathrm{fin}}$, and $\mathbf{C}$ is the finite Cartan matrix for $\mathfrak{g}_{\mathrm{fin}}$. Here we omit the factor e^{Λ_0} for the 0th fundamental weight Λ_0. The well-known formula for the basic representation gives $(1-e^{-m\delta})^{-n}$ instead of the power $-(n+1)$. The additional -1 comes from the Heisenberg algebra.

Note also that we use $e^{-\alpha_i}$ $(i=1,\dots,n)$, $e^{-\delta}$ as variables for the character Z. This is possible as the 0th simple root α_0 of $\mathfrak{g}_{\mathrm{aff}}$ can be written by δ and α_i $(i=1,\dots,n)$ from the equation $\delta=\sum a_i\alpha_i$. See Section 1.1 for more details.

The conjectural formula in [13, Conjecture 2.1], which we will prove, is the following.

THEOREM 1

The generating function of Euler numbers of $\operatorname{Hilb}^N(\mathbb{C}^2/\Gamma)$

$$\sum_{n=0}^{\infty} e^{-N\delta}\chi\bigl(\operatorname{Hilb}^N(\mathbb{C}^2/\Gamma)\bigr)$$

is obtained from Z *by the substitution*

$$e^{-\alpha_i}=\exp\Bigl(\frac{2\pi\sqrt{-1}}{h^\vee+1}\Bigr)\quad (i=1,\dots,n),$$

where $h^\vee$ *is the dual Coxeter number.*

Let us sketch the strategy of our proof. In order to compute Euler numbers of $\operatorname{Hilb}^N(\mathbb{C}^2/\Gamma)$, we use a recent result of Craw, Gammelgaard, Gyenge, and Szendrői [6]: $\operatorname{Hilb}^N(\mathbb{C}^2/\Gamma)$ with the reduced scheme structure is isomorphic to a quiver variety $\mathfrak{M}_{\zeta^\bullet}(N\delta,\Lambda_0)$, where $\zeta^\bullet$ is a stability parameter

$$\zeta_i^\bullet=0\quad\text{for all } i\neq 0,\qquad \zeta_0^\bullet<0.$$

This stability parameter is different from the standard one ζ

$$\zeta_i<0\quad\text{for all } i=0,1,\dots,n,$$

which gives connected components $\mathfrak{M}_\zeta(\mathbf{v},\Lambda_0)$ of $\operatorname{Hilb}^N(\mathbb{C}^2)^\Gamma$. Here $\mathbf{v}$ is an isomorphism class of $\mathbb{C}[x,y]/I$ as a Γ-module for an ideal $I\subset\mathbb{C}[x,y]$ of the coordinate ring $\mathbb{C}[x,y]$ of $\mathbb{C}^2$, invariant under the Γ-action.

As $\zeta^\bullet$ lies in walls given by roots in the space of stability parameters, $\mathfrak{M}_{\zeta^\bullet}(N\delta,\Lambda_0)$ is singular in general. Nevertheless, it has a representation-theoretic meaning, as shown in [34]: the quiver variety $\mathfrak{M}_{\zeta^\bullet}(N\delta,\Lambda_0)$ is responsible for the restriction of $\mathfrak{g}_{\mathrm{aff}}$ to $\mathfrak{g}_{\mathrm{fin}}$. The essential geometric ingredient for this relation is a projective morphism

$$\pi_{\zeta^\bullet,\zeta}\colon \mathfrak{M}_\zeta(\mathbf{v},\Lambda_0)\to\mathfrak{M}_{\zeta^\bullet}(N\delta,\Lambda_0)=\operatorname{Hilb}^N(\mathbb{C}^2/\Gamma),\quad N=\mathbf{v}_0.$$

By a local description of singularities of quiver varieties (see [34, Section 2.7], which goes back to [27, Section 6]), fibers of $\pi_{\zeta^\bullet,\zeta}$ are isomorphic to Lagrangian subvarieties in quiver varieties $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)$ associated with the finite ADE quiver for $\mathfrak{g}_{\mathrm{fin}}$, the finite-dimensional complex simple Lie algebra underlying $\mathfrak{g}_{\mathrm{aff}}$. As a simple application of this result, we observe that

- Euler numbers of $\mathrm{Hilb}^N(\mathbb{C}^2/\Gamma)$ can be written in terms of Euler numbers of $\mathfrak{M}_\zeta(\mathbf{v},\Lambda_0)$ and quiver varieties $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)$ of the finite ADE type.

In fact, we obtain *more* data than we need: we have a stratification of $\mathrm{Hilb}^N(\mathbb{C}^2/\Gamma)$ so that $\pi_{\zeta^\bullet,\zeta}$ is a fiber bundle over each stratum. We can compute Euler numbers of all strata from Euler numbers of $\mathfrak{M}_\zeta(\mathbf{v},\Lambda_0)$ and quiver varieties $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)$ of the finite ADE type. See (1.8) for the precise formula.

There are several algorithms to compute Euler numbers of quiver varieties (see, e.g., [33] and [15]).[1] They are complicated and hard to use in practice. On the other hand, the conjectural formula in [13] above does *not* contain such a complicated algorithm. It is just given by a simple substitution. It means that we should have a drastic simplification if we take a *linear combination* of complicated Euler numbers of quiver varieties of finite ADE type. We do not need to compute Euler numbers of individual strata, as we only need their sum.

This simplification has a representation-theoretic origin. Let us explain it.

The above specialized character is called the *quantum dimension*:

$$\dim_q V = \operatorname{ch} V\big|_{e^{-\alpha_i}=\exp(\frac{2\pi\sqrt{-1}}{h^\vee+1})}.$$

Here V is a finite-dimensional representation V of $\mathfrak{g}_{\mathrm{fin}}$. It was introduced by Andersen [2, Definition 3.1] (see also Parshall–Wang [37]). We choose a specific root of unity $\exp(\frac{2\pi\sqrt{-1}}{2(h^\vee+1)})$, whereas these papers study more general roots of unity ζ. (The relation between the specialized character and quantum dimension will be recalled in Section 2.1.)

Therefore, the conjectural formula in [13] states that the generating function of Euler numbers of $\mathrm{Hilb}^N(\mathbb{C}^2/\Gamma)$ is given by the quantum dimension of the Fock space at $\exp(\frac{2\pi\sqrt{-1}}{2(h^\vee+1)})$, restricted from $\mathfrak{g}_{\mathrm{aff}}$ to $\mathfrak{g}_{\mathrm{fin}}$, just keeping track of $e^{-\delta}$.

A direct sum of homology groups (which are isomorphic to the complexified K-group) of Lagrangian subvarieties of quiver varieties of the finite ADE type carries a structure of a finite-dimensional representation of the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g}_{\mathrm{fin}})$. See [30]. It is called a *standard module* of $\mathbf{U}_q(\mathbf{L}\mathfrak{g}_{\mathrm{fin}})$.

It turns out that the above simplification is a consequence of the following representation-theoretic result of independent interest.

THEOREM 2

The quantum dimension of arbitrary standard modules of $\mathbf{U}_q(\mathbf{L}\mathfrak{g}_{\mathrm{fin}})$ *of type ADE at* $\zeta=\exp(\frac{2\pi\sqrt{-1}}{2(h^\vee+1)})$ *is equal to* 1.

[1] We make a distinction between an algorithm for a computation and an explicit computation as in [35]. Namely, a computation results in finitely many $\pm$, $\times$, integers and variables. We do not require that the final expression be readable by a human. An expression like $\sum_{i=1}^{2^{(2^{100})}} a_i$ with explicit a_i is an algorithm.

Since standard modules are tensor products of l-fundamental modules (see [39]), it is enough to prove this result for l-fundamental modules. The author was informed by Naoi that this result is a special case of a more general conjecture posed by Kuniba [21, Conjecture 2 (A.6a)] (see also Kuniba–Nakanishi–Suzuki [22, Conjecture 14.2]) formulated for Kirillov–Reshetikhin modules. (Recall that l-fundamental modules are the simplest examples of Kirillov–Reshetikhin modules.) It is not difficult to check the general conjecture for type A. The type D case was shown in [23], while the type E_6 case was shown in [11].

Although we only need the *simplest special* case of the more general conjecture, we could not find a proof of the relevant result for types E_7, E_8 in the literature. Therefore, we provide a proof for the reader's convenience. Fortunately, the necessary explicit computation for type E from the algorithm in [33] was already done in [35] by using a supercomputer. Alternatively, we could quote the computation by Kleber in [19], which assumed a fermionic formula conjectured at that time. The fermionic formula was proved later by Di Francesco and Kedem [8].

We also give proofs of the known cases A, D, E_6 for completeness. We encounter a new feature at E_7, E_8, which did not arise in other cases. Thus we expect new tools are necessary to attack the conjecture in [21] in full generality.

Because the general conjecture in [21] is about Kirillov–Reshetikhin modules, we expect that it should be studied in the framework of cluster algebras, as in [8]. Note also that Euler numbers of (graded) quiver varieties are understood in the context of cluster algebras in a recent work of Bittmann [3]. Thus the suggestion is compatible with our approach, though cluster algebras play no role in our work here.

The paper is organized as follows. In Section 1, we deduce Theorem 1 from Theorem 2 after recalling results from [34]. In Section 2, we prove Theorem 2 using a case-by-case analysis. In Section 3, we discuss an additional topic, the rational smoothness of $\mathrm{Hilb}^N(\mathbb{C}^2/\Gamma)$. It is rationally smooth if $N=2$, but not so in general.

1. Euler numbers of quiver varieties

1.1. Quiver varieties

Let Γ be a nontrivial finite subgroup of $\mathrm{SL}(2)$. We define the affine Dynkin diagram via the McKay correspondence (see, e.g., [18, Chapter 8] for details). Let $\{\rho_i\}_{i\in I}$ be the set of isomorphism classes of irreducible representations of Γ with the trivial representation ρ_0. We identify ρ_i with a vertex of a graph. We draw a_{ij} edges between ρ_i and ρ_j where $a_{ij} = \dim \mathrm{Hom}_\Gamma(\rho_i, \rho\otimes\rho_j) = \dim \mathrm{Hom}_\Gamma(\rho_j, \rho\otimes\rho_i)$, where ρ is the 2-dimensional representation of Γ given by the inclusion $\Gamma\subset \mathrm{SL}(2)$. Then the graph is an affine Dynkin diagram of type ADE. Let $\mathfrak{g}_{\mathrm{aff}}$ denote the corresponding affine Lie algebra, and let $\mathfrak{g}_{\mathrm{fin}}$ denote the underlying finite-dimensional complex simple Lie algebra corresponding to the Dynkin diagram obtained from the affine one by removing ρ_0. Let n be the rank of $\mathfrak{g}_{\mathrm{fin}}$, which is the number of vertices in I minus 1.

We use the convention of the root system for $\mathfrak{g}_{\mathrm{aff}}$ as in [17, Chapter 6 and Section 12.4]. Let α_i be the ith simple root of $\mathfrak{g}_{\mathrm{aff}}$ corresponding to ρ_i, and let δ be the primitive positive imaginary root of $\mathfrak{g}_{\mathrm{aff}}$. We have $\delta = \sum a_i\alpha_i$, and a_i is equal to the dimension of ρ_i. Let $\alpha_i^\vee$ be the ith simple coroot. We take the scaling element d satisfying

$$\langle \alpha_i, d\rangle = \delta_{i0}.$$

Then $\langle \alpha_i^\vee, d\rangle_{i\in I}$ is a base of $\mathfrak{h}_{\mathrm{aff}}$, the Cartan subalgebra of $\mathfrak{g}_{\mathrm{aff}}$. We define the fundamental weights Λ_i $(i \in I)$ by

$$\langle \Lambda_i, \alpha_j^\vee\rangle = \delta_{ij}, \qquad \langle \Lambda_i, d\rangle = 0.$$

Then $\langle \alpha_i, \Lambda_0\rangle_{i\in I}$ forms a base of $\mathfrak{h}_{\mathrm{aff}}^*$.

We choose an orientation Ω of edges in the affine Dynkin diagram and consider the corresponding affine quiver $Q = (I, \Omega)$. We take dimension vectors $\mathbf{w} = (\mathbf{w}_i)$ and $\mathbf{v} = (\mathbf{v}_i) \in \mathbb{Z}_{\geq 0}^I$, and consider quiver varieties $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})$, $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v},\mathbf{w})$, where ζ, $\zeta^\bullet$ are stability parameters such that

$$\zeta_i < 0 \quad \text{for all } i \in I, \qquad \zeta_i^\bullet = 0 \quad \text{for all } i \neq 0, \qquad \zeta_0^\bullet < 0.$$

(See [34, Section 2] for the definition of quiver varieties for these stability conditions.) Since $\zeta^\bullet$ lives in the boundary of a chamber containing ζ, we have a projective morphism (see [34, Section 2])

$$\pi_{\zeta^\bullet,\zeta}\colon \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{\zeta^\bullet}(\mathbf{v},\mathbf{w}).$$

In fact, this is an example studied in [34, Section 2.8] associated with a division $I = I^0 \sqcup I^+$ with $I^0 = I\setminus\{0\}$, $I^+ = \{0\}$.

We identify $\mathbf{v}$, $\mathbf{w}$ with weights of $\mathfrak{g}_{\mathrm{aff}}$ by

$$\mathbf{v} = \sum \mathbf{v}_i\alpha_i, \qquad \mathbf{w} = \sum \mathbf{w}_i\Lambda_i.$$

It is convenient to use a different convention for the dimension vector $\mathbf{v}$:

$$\mathbf{v} = m\delta + \sum_{i\in I^0} m_i\alpha_i, \quad \text{i.e., } m = \mathbf{v}_0, m_i = \mathbf{v}_i - \mathbf{v}_0 a_i. \tag{1.1}$$

Namely, we remove α_0 and use δ instead.

Let us take $\mathbf{w} = \Lambda_0$. Then an alternative description of $\mathfrak{M}_\zeta(\mathbf{v},\Lambda_0)$ is well known: it is the moduli space of Γ-invariant ideals I in $\mathbb{C}[x,y]$ such that $\mathbb{C}[x,y]/I$ is isomorphic to $\bigoplus \rho_i^{\oplus \mathbf{v}_i}$ as a Γ-module. Hence it is the Γ-fixed point component of Hilbert schemes I of points in the affine plane $\mathbb{C}^2$. The Euler number for $\mathfrak{M}_\zeta(\mathbf{v},\Lambda_0)$ is also known. It was given in [31]. Since it was stated without an explanation, let us explain how it is derived. We change the stability condition ζ to show that $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})$ is diffeomorphic to a moduli space of framed rank 1 torsion-free sheaves on the minimal resolution of $\mathbb{C}^2/\Gamma$. Then rank 1 torsion-free sheaves are ideal sheaves twisted by line bundles, hence Euler numbers are given by Göttsche formula for Hilbert schemes of points (see [12]). Moreover, this latter picture gives the Frenkel–Kac construction of the basic representation of $\mathfrak{g}_{\mathrm{aff}}$ (see, e.g., [17, Section 14.8]); hence we obtain the formula of Z in the Introduction. We have used the convention in (1.1).

We can define a structure of a representation of the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ on the direct sum of homology groups $\bigoplus_{\mathbf{v}} H_*(\mathfrak{M}_\zeta(\mathbf{v},\Lambda_0))$. See [31] and references therein. We can also construct a structure of a representation of the Heisenberg algebra commuting with $\mathfrak{g}_{\mathrm{aff}}$ by [28] and [29].

On the other hand, Craw, Gammelgaard, Gyenge, and Szendrői [6] recently proved that the Hilbert scheme $\mathrm{Hilb}^N(\mathbb{C}^2/\Gamma)$ of N points in $\mathbb{C}^2/\Gamma$ with the reduced scheme structure is isomorphic to $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v},\Lambda_0)$ with $\mathbf{v}=N\delta$.

1.2. Stratification

As explained in [34, Section 2], $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v},\mathbf{w})$ parameterizes S-equivalence classes of $\zeta^\bullet$-semistable framed representations of the preprojective algebra. Therefore, its points are represented by direct sum of $\zeta^\bullet$-stable representations. Under the above choice of $\zeta^\bullet$, we have one distinguished summand, giving a point in $\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w})$ for some $\mathbf{v}'\le\mathbf{v}$ (component-wise) such that $\mathbf{v}'_0=\mathbf{v}_0$, and other summands are simple representations S_i with $i\in I^0$ (see [34, Section 2.6]). Since multiplicities of S_i can be read off from the difference $\mathbf{v}-\mathbf{v}'$, we can regard $\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w})$ as a subset of $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v},\mathbf{w})$. By [34, Proposition 2.30], $\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w})\neq\emptyset$ if and only if $\mathbf{w}-\mathbf{v}'$ is an I^0-dominant weight appearing in the basic representation $V(\Lambda_0)$ of the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ associated with Γ. Here *I^0-dominant* means that $\langle\mathbf{w}-\mathbf{v}',\alpha_i^\vee\rangle\ge 0$ for $i\in I^0$. We thus have the stratification

$$\mathfrak{M}_{\zeta^\bullet}(\mathbf{v},\mathbf{w})=\bigsqcup_{\mathbf{v}'\text{ as above}}\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w}).$$

Moreover, the transversal slice to the stratum $\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w})$ is locally isomorphic to a quiver variety $\mathfrak{M}_0(\mathbf{v}^s,\mathbf{w}^s)$ around 0, associated with the finite ADE quiver $Q\setminus\{\rho_0\}$ such that dimension vectors are given by

$$\mathbf{v}^s=\mathbf{v}-\mathbf{v}',\qquad \mathbf{w}^s_i=\langle\mathbf{w}-\mathbf{v}',\alpha_i^\vee\rangle\quad i\in I^0.$$

Note that $\mathbf{v}-\mathbf{v}'$ has no 0th component as $\mathbf{v}'_0=\mathbf{v}_0$. Note also that $\mathbf{w}^s_i\ge 0$, as $\mathbf{w}-\mathbf{v}'$ is I^0-dominant. It is also known that the inverse image of the slice in $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})$ under $\pi_{\zeta^\bullet,\zeta}$ is locally isomorphic to $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)$ around $\mathfrak{L}(\mathbf{v}^s,\mathbf{w}^s)$, the inverse image of the origin 0 of $\mathfrak{M}_0(\mathbf{v}^s,\mathbf{w}^s)$ under the projective morphism $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)\to\mathfrak{M}_0(\mathbf{v}^s,\mathbf{w}^s)$. See [34, Section 2.7] and references therein for these claims on transversal slices. There is also an algebraic approach in [7].

For $\mathbf{w}=\Lambda_0$, we have

$$\mathbf{w}^s_i=-\sum_{j\in I^0}m'_j\langle\alpha_j,\alpha_i^\vee\rangle,\quad\text{where }\mathbf{v}'=m\delta+\sum_{j\in I^0}m'_j\alpha_j\text{ in the convention (1.1).}$$

Note that $\mathbf{v}=m\delta+\sum m_i\alpha_i$ and $\mathbf{v}'$ share the same m for the coefficient of δ, as $\mathbf{v}_0=\mathbf{v}'_0$. In particular,

$$\sum_{i\in I^0}\mathbf{w}^s_i\Lambda_i-\mathbf{v}^s_i\alpha_i=-\sum_{i\in I^0}m_i\alpha_i. \tag{1.2}$$

We also note that

(1.3) $$m'_i \leq 0 \quad (i \in I^0),$$

as $(m'_i)_{i\in I^0} = -(\mathbf{w}^s)_{i\in I^0}\mathbf{C}^{-1}$, and the inverse of the Cartan matrix $\mathbf{C}$ has positive entries. This was proved in [6, Proposition A.1] by a different method.

1.3. Examples

EXAMPLE 1.4

Let $\mathfrak{M}_{\zeta^\bullet}(N\delta,\Lambda_0) \cong \mathrm{Hilb}^N(\mathbb{C}^2/\Gamma)$. Consider a stratum containing framed representations corresponding to $N-1$ distinct points in $\mathbb{C}^2\setminus\{0\}/\Gamma$. (We add simple representations S_i as above.) We have $\mathbf{v}' = (N-1)\delta + \alpha_0 = N\delta - \sum_{i\in I^0} a_i\alpha_i$, and hence

$$\mathbf{v}^s = \sum_{i\in I^0} a_i\alpha_i, \qquad \mathbf{w}^s_i = \sum_{j\in I^0} C_{ij}a_j.$$

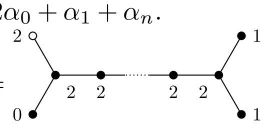
Note that $\mathbf{w}^s$ has entries 1 at vertices in the finite quiver I^0 which are connected to the 0-vertex in the affine quiver. In particular, $\mathfrak{M}_0(\mathbf{v}^s,\mathbf{w}^s)$ is $\mathbb{C}^2/\Gamma$, and $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)$ is its minimal resolution, the very first example of a quiver variety considered by Kronheimer [20], before the definition of quiver varieties was introduced.

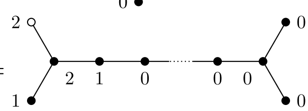
This is obvious, as the transversal slice is $\mathfrak{M}_{\zeta^\bullet}(\delta,\Lambda_0)\cong\mathbb{C}^2/\Gamma$, as we can ignore $N-1$ distinct points.

EXAMPLE 1.5 (see [40])

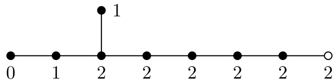
Consider $\mathfrak{M}_{\zeta^\bullet}(2\delta,\Lambda_0) = \mathrm{Hilb}^2(\mathbb{C}^2/\Gamma)$. There is only one 2-dimensional stratum, which is the $N=2$ case of Example 1.4. The 0-dimensional strata and the formal neighborhoods of fibers of $\pi_{\zeta^\bullet,\zeta}$ over them in $\mathfrak{M}_{\zeta^\bullet}(2\delta,\Lambda_0)$ were determined by Yamagishi [40]. He further identified the formal neighborhood with that of the intersection of the nilpotent cone for the complex simple Lie algebra $\mathfrak{g}_{\mathrm{fin}}$ and the Slodowy slice to a "sub-subregular" orbit. Let us recall his result in our notation.

There is only one 0-dimensional stratum for types A_n $(n\geq 3)$, E_6, E_7, E_8 while there are none for A_1, A_2, two for D_n $(n>4)$, and three for D_4. We can give corresponding vectors $\mathbf{v}'$ as follows:

- If Q is of type $A_1^{(1)}$ or $A_2^{(1)}$, then there is no such $\mathbf{v}'$.
- If Q is of type $A_n^{(1)}$ with $n>2$, then $\mathbf{v}' = 2\alpha_0+\alpha_1+\alpha_n$.
- If Q is of type $D_n^{(1)}$ with $n>4$, then $\mathbf{v}' =$ 

$= 2\alpha_0 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ or $\mathbf{v}' =$ 

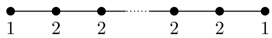
$= 2\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3$.
- If Q is of type $D_4^{(1)}$, then we have three possibilities: in addition to the second example in $D_n^{(1)}$ above, we have two others obtained by changing (α_1,α_3) to (α_1,α_4), (α_3,α_4) respectively.

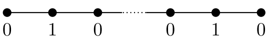
- If Q is of type $E_8^{(1)}$, then we have $\mathbf{v}' =$ 

. The cases $E_6^{(1)}$, $E_7^{(1)}$ are similar.

Here the 0th vertex is $\circ$, and the other vertices are $\bullet$.

Transversal slices are as follows:

- If Q is of type $A_n^{(1)}$ with $n > 2$, then $\mathbf{v}^s =$ 

, $\mathbf{w}^s =$ 

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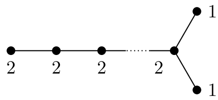
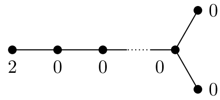
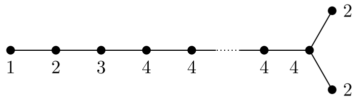
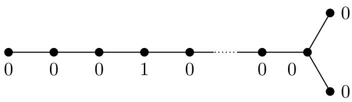
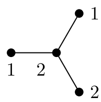
- If Q is of type $D_n^{(1)}$, then $\mathbf{v}^s =$ 

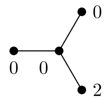
, $\mathbf{w}^s =$ 

in the first case, and $\mathbf{v}^s =$ 

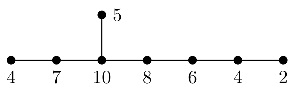
, $\mathbf{w}^s =$ 

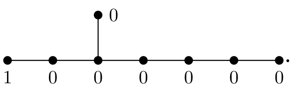
in the second case.

- If Q is of type $D_4^{(1)}$, then $\mathbf{v}^s =$ 

, $\mathbf{w}^s =$ 

, and two other cases are obtained by cyclic permutation.

- If Q is of type $E_8^{(1)}$, then $\mathbf{v}^s =$ 

, $\mathbf{w}^s =$ 

.

In particular, it means that the quiver variety $\mathfrak{M}_0(\mathbf{v}^s, \mathbf{w}^s)$ is isomorphic to the intersection of the Slodowy slice and the nilpotent cone in the formal neighborhood of the origin.

This result was known before for type A_n and the second case of type D_n. The type A_n case was proved in [27, Section 8] (see also [24]). The second case of type D_n was shown in [16]. In these cases, $\mathfrak{M}_0(\mathbf{v}^s, \mathbf{w}^s)$ itself is isomorphic to the intersection of the Slodowy slice and the nilpotent cone, not only in the formal neighborhood. The exceptional cases are conjectured, but not shown as far as the author knows.

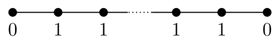
REMARK 1.6

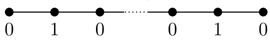
In cases for the above example, it is known that corresponding Coulomb branches of the quiver gauge theories (which are affine Grassmannian slices by [5]) are "next-to-minimal" nilpotent orbit closures in $\mathfrak{g}_{\mathrm{fin}}$ by [1].

EXAMPLE 1.7 (see [40])

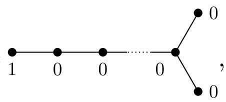
Let us consider $\mathbf{v} = \delta + \alpha_0$, $\mathbf{w} = \Lambda_0$, hence $\mathfrak{M}_{\zeta^\bullet}(\delta + \alpha_0, \Lambda_0)$. Note that this is different from $\mathfrak{M}_{\zeta^\bullet}(\delta, \Lambda_0) = \mathrm{Hilb}^1(\mathbb{C}^2/\Gamma) = \mathbb{C}^2/\Gamma$. Let us show that $\mathfrak{M}_{\zeta^\bullet}(\delta + \alpha_0, \Lambda_0)$ is *not* isomorphic to $\mathbb{C}^2/\Gamma$.

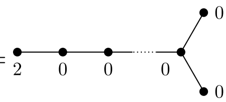
This $\mathfrak{M}_{\zeta^\bullet}(\delta + \alpha_0, \Lambda_0)$ appears as the union of 2- and 0-dimensional strata in $\mathfrak{M}_{\zeta^\bullet}(2\delta, \Lambda_0) = \mathrm{Hilb}^2(\mathbb{C}^2/\Gamma)$. Thus all possible strata are already determined as above: besides the open stratum for $\mathbf{v}' = \mathbf{v}$, we have a single 0-dimensional stratum for A_n $(n \geq 3)$, E_6, E_7, E_8, none for A_1, A_2, three for D_4, and two for D_n $(n > 4)$. The transversal slice $\mathfrak{M}_0(\mathbf{v}^s, \mathbf{w}^s)$ is given by the same $\mathbf{w}^s$ as above, and $\mathbf{v}^s$ is subtracting $\sum_{i \in I^0} a_i \alpha_i$ from the above example. Concretely, it is the following:

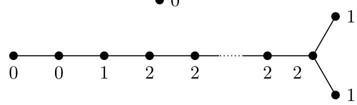
- For type $A_n^{(1)}$ with $n > 2$, $\mathbf{v}^s =$ 

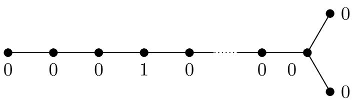
, $\mathbf{w}^s =$ 

.

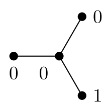
- For type $D_n^{(1)}$ with $n > 4$, $\mathbf{v}^s =$ 

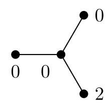
, $\mathbf{w}^s =$ 

in the first case, and $\mathbf{v}^s =$ 

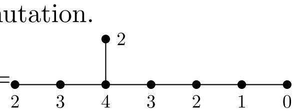
, $\mathbf{w}^s =$ 

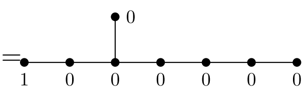
in the second case.

- For type $D_4^{(1)}$, $\mathbf{v}^s =$ 

, $\mathbf{w}^s =$ 

, and two other cases are obtained by cyclic permutation.

- For type $E_8^{(1)}$, $\mathbf{v}^s =$ 

, $\mathbf{w}^s =$ 

.

Let us consider $\pi_{\zeta^\bullet,\zeta} : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{\zeta^\bullet}(\mathbf{v}, \mathbf{w})$. The domain $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ is the minimal resolution of $\mathbb{C}^2/\Gamma$. From the above description of the transversal slice we immediately conclude the following:

- $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v}, \mathbf{w})$ is obtained from the minimal resolution of $\mathbb{C}^2/\Gamma$ by collapsing the $\mathbb{P}^1$'s whose vertices are not connected to the 0-vertex in the affine Dynkin diagram.

There are no such vertices for A_1, A_2. For type A_n $(n > 2)$, we collapse $n - 2$ $\mathbb{P}^1$'s corresponding to vertices except the leftmost and rightmost. For type D_n, we collapse $(n - 1)$ $\mathbb{P}^1$'s except the second one from the left, and produce singularities of type A_1 and D_{n-2}, where we understand that $D_2 = A_1 \times A_1$, $D_3 = A_3$. For type E_8, the seven $\mathbb{P}^1$'s except the rightmost one are collapsed.

1.4. Euler numbers

By Section 1.2, we can relate Euler numbers of $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$, $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v}', \mathbf{w})$, and $\mathfrak{M}_\zeta(\mathbf{v}^s, \mathbf{w}^s)$. Let $\chi(\ )$ denote the Euler number of a space. We have

$$\chi\big(\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})\big) = \sum_{\mathbf{v}'} \chi\big(\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w})\big)\chi\big(\mathfrak{L}_\zeta(\mathbf{v}^s,\mathbf{w}^s)\big)$$
$$= \sum_{\mathbf{v}'} \chi\big(\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w})\big)\chi\big(\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)\big).$$

It is known (see [27, Corollary 5.5]) that the central fiber $\mathfrak{L}_\zeta(\mathbf{v}^s,\mathbf{w}^s)$ of $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s) \to \mathfrak{M}_0(\mathbf{v}^s,\mathbf{w}^s)$ is homotopic to $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)$; hence the second equality follows.

From $\chi(\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}))$, $\chi(\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s))$, we compute $\chi(\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w}))$ recursively as

$$\chi\big(\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v},\mathbf{w})\big) = \chi\big(\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})\big) - \sum_{\mathbf{v}'\neq\mathbf{v}} \chi\big(\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w})\big)\chi\big(\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)\big). \tag{1.8}$$

Here we use $\mathbf{v}^s = 0$; hence $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)$ is a point for $\mathbf{v}' = \mathbf{v}$.

We take the generating function of Euler numbers as

$$\begin{aligned}&\sum_{\mathbf{v}} \chi\big(\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})\big)e^{-\mathbf{v}} \\ &\quad= \sum_{\mathbf{v}'} \chi\big(\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(\mathbf{v}',\mathbf{w})\big)e^{-\mathbf{v}'} \sum_{\mathbf{v}^s} \chi\big(\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}-\mathbf{v}'|_{I^0})\big)e^{-\mathbf{v}^s},\end{aligned} \tag{1.9}$$

where $\mathbf{w}-\mathbf{v}'|_{I^0} = \mathbf{w}^s = \sum_{i\in I^0}\langle \mathbf{w}-\mathbf{v}', \alpha_i^\vee\rangle \Lambda_i$.

We claim the following.

PROPOSITION 1.10

We have

$$\sum_{\mathbf{v}^s} \chi\big(\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)\big) \prod_{i\in I^0} e^{\mathbf{w}^s_i\Lambda_i - \mathbf{v}^s_i\alpha_i}\Big|_{e^{-\alpha_i}=\exp(\frac{2\pi\sqrt{-1}}{h^\vee+1})} = 1. \tag{1.11}$$

We take $\mathbf{w} = \Lambda_0$ and switch to the convention (1.1). Then the left-hand side of (1.9) is Z in the Introduction. By (1.2), (1.11) implies that

$$Z|_{e^{-\alpha_i}=\exp(\frac{2\pi\sqrt{-1}}{h^\vee+1})} = \sum_{m,\vec{m}'} \chi\Big(\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}\Big(m\delta + \sum_{i\in I^0} m'_i\alpha_i, \Lambda_0\Big)\Big)e^{-m\delta}.$$

By (1.3), we consider $\mathfrak{M}^{\mathrm{s}}_{\zeta^\bullet}(m\delta + \sum_{i\in I^0} m'_i\alpha_i, \Lambda_0)$ as a stratum of $\mathfrak{M}_{\zeta^\bullet}(m\delta,\Lambda_0) = \mathrm{Hilb}^m(\mathbb{C}^2/\Gamma)$. Therefore, the right-hand side of the above is the generating function of Euler numbers of $\mathrm{Hilb}^m(\mathbb{C}^2/\Gamma)$. Thus we have proved Theorem 1.

Recall that $\bigoplus_{\mathbf{v}^s} H_*(\mathfrak{L}_\zeta(\mathbf{v}^s,\mathbf{w}^s)) \cong \bigoplus_{\mathbf{v}^s} K(\mathfrak{L}_\zeta(\mathbf{v}^s,\mathbf{w}^s)) \otimes_{\mathbb{Z}} \mathbb{C}$ is the so-called *standard module* of the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g}_{\mathrm{fin}})$, specialized at $q=1$ (see [30]). The equality (1.11) means that its character (as a $\mathfrak{g}_{\mathrm{fin}}$-module), specialized at $e^{-\alpha_i} = \exp(\frac{2\pi\sqrt{-1}}{h^\vee+1})$, is equal to 1. As we mentioned in the Introduction, this specialization is the quantum dimension. Thus (1.11) follows from Theorem 2.

In order to prove (1.11), we may assume that $\mathbf{w}^s$ is a fundamental weight: there is a torus action on framing vector spaces, and the induced action on $\mathfrak{M}_\zeta(\mathbf{v}^s,\mathbf{w}^s)$. Let us suppose that $\mathbf{w}^s = \Lambda_i + \Lambda_j$ for $i,j\in I^0$ for simplicity. Then the torus fixed point set is

$$\bigsqcup_{\mathbf{v}^1+\mathbf{v}^2=\mathbf{v}^s} \mathfrak{M}_\zeta(\mathbf{v}^1,\Lambda_i)\times\mathfrak{M}_\zeta(\mathbf{v}^2,\Lambda_j).$$

As the Euler number is equal to the sum of Euler numbers of fixed points with respect to a torus action, (1.11) for $\mathbf{w}^s$ follows from (1.11) for Λ_i and Λ_j. This result is compatible with what we mentioned in the Introduction: standard modules are tensor products of l-fundamental modules (see [39]), while l-fundamental modules correspond to the case when $\mathbf{w}^s$ is a fundamental weight.

A standard module also depends on a spectral parameter, which is a specialization homomorphism $K_{\prod_{i\in I^0}\mathrm{GL}(\mathbf{w}_i^s)}(\mathrm{pt})\to\mathbb{C}$. But the restriction of a standard module to $\mathbf{U}_q(\mathfrak{g}_{\mathrm{fin}})$ is independent of the spectral parameter. Hence the spectral parameter is not relevant for Theorem 2.

REMARK 1.12

In order to prove Theorem 1, we need to check (1.11) only when $\mathbf{w}^s$ is contained in the root lattice. But Λ_i does not satisfy this condition in general, hence the above reduction cannot be performed among $\mathbf{w}^s$'s in the root lattice.

2. Quantum dimensions of standard modules

2.1. Quantum dimension

Let V be a finite-dimensional representation of $\mathfrak{g}_{\mathrm{fin}}$. The specialized character

$$\operatorname{ch} V|_{e^{-\alpha_i}=\exp(\frac{2\pi\sqrt{-1}}{h^\vee+1})}$$

is called the *quantum dimension* of V, denoted by $\dim_q V$, and was introduced by Andersen [2, Definition 3.1] (see also Parshall–Wang [37]) as $\operatorname{tr}(K^{2\rho})$ for a module of the quantized enveloping algebra $\mathbf{U}_q(\mathfrak{g}_{\mathrm{fin}})$ with q specialized at root of unity $\zeta=\exp(\frac{2\pi\sqrt{-1}}{2(h^\vee+1)})$. Here we denote the quantum variable by q following the convention in [30]. (It is denoted by v in [2].) Note that 2ρ, the sum of positive roots, in $K_{2\rho}$ should be understood as an element in the dual of the weight lattice by $\langle 2\rho,\lambda\rangle=(2\rho,\lambda)/(\alpha_0,\alpha_0)$ for a weight λ. Here α_0 is a short root (see [37, Lemma 1.1]). Then we have $\langle 2\rho,\alpha_i\rangle=2$ for type ADE, hence $\operatorname{tr}(K^{2\rho})$ at $q=\zeta$ is given by the specialization $e^{\alpha_i}=\zeta^2$. We also see that $\dim_q V$ is unchanged under the replacement $\zeta\mapsto\zeta^{-1}$ from the formula below. Hence the substitution $e^{-\alpha_i}=\exp(\frac{2\pi\sqrt{-1}}{h^\vee+1})$ corresponds to $q=\exp(\frac{2\pi\sqrt{-1}}{2(h^\vee+1)})$.

It was assumed that ζ is a primitive ℓth root of unity with odd ℓ in [2], but the definition still makes sense for our choice. By the Weyl character formula, we have (see [2, (3.2)], [37, Theorem 1.3])

$$\operatorname{ch} V|_{e^{-\alpha_i}=\exp(\frac{2\pi\sqrt{-1}}{h^\vee+1})}=\dim_q V=\prod_{\alpha\in\Delta_{\mathrm{fin}}^+}\frac{\zeta^{d_\alpha\langle\lambda+\rho,\alpha^\vee\rangle}-\zeta^{-d_\alpha\langle\lambda+\rho,\alpha^\vee\rangle}}{\zeta^{d_\alpha\langle\rho,\alpha^\vee\rangle}-\zeta^{-d_\alpha\langle\rho,\alpha^\vee\rangle}},$$

where λ is the highest weight of an irreducible representation $V=V(\lambda)$, Δ_{fin}^+ is the set of positive roots of $\mathfrak{g}_{\mathrm{fin}}$, $d_\alpha\in\{1,2,3\}$ is the square length of α divided by the length of a short root, and ρ is the half-sum of positive roots. (Since we are only considering type ADE, we have $d_\alpha = 1$ for any α.) We have $d_\alpha\langle\rho,\alpha^\vee\rangle \le \langle\rho,\theta\rangle = h^\vee$, hence the denominator does not vanish.

Let us introduce ζ-integers by

$$[k]_\zeta \stackrel{\text{def.}}{=} \frac{\zeta^k - \zeta^{-k}}{\zeta - \zeta^{-1}},$$

so that

$$\operatorname{ch} V|_{e^{-\alpha_i} = \exp(\frac{2\pi\sqrt{-1}}{h^\vee+1})} = \prod_{\alpha\in\Delta^+} \frac{[d_\alpha\langle\lambda+\rho,\alpha^\vee\rangle]_\zeta}{[d_\alpha\langle\rho,\alpha^\vee\rangle]_\zeta}. \tag{2.1}$$

We have

$$\big[2(h^\vee+1)+k\big]_\zeta = [k]_\zeta$$

as $\zeta^{2(h^\vee+1)} = 1$, as well as

$$[h^\vee+1-k]_\zeta = [k]_\zeta \tag{2.2}$$

as $\zeta^{h^\vee+1-k}\zeta^k = \zeta^{h^\vee+1} = -1$. In particular, we have $[h^\vee+1]_\zeta = 0$. The former is the usual analogy between roots of unity and characteristic $2(h^\vee+1)$. The latter is a new feature at an even root of unity.

2.2. Type A

Consider type A_{n-1}, that is, $\mathfrak{g}_{\text{fin}} = \mathfrak{sl}(n,\mathbb{C})$. We have $h^\vee = n$.

In type A_{n-1}, it is known that the kth l-fundamental module is the kth fundamental representation of $\mathrm{SL}(n)$. This follows from [32, Proposition 4.6], but it is also a consequence of the fact that the corresponding quiver varieties are either empty or points in type A_{n-1} (see, e.g., [26]), together with [30, Section 15].

The quantum dimension of the kth fundamental representation $V(\Lambda_k)$ is

$$\dim_q V(\Lambda_k) = \begin{bmatrix} n \\ k \end{bmatrix}_\zeta = \frac{[n]_\zeta[n-1]_\zeta\cdots[n-k+1]_\zeta}{[k]_\zeta[k-1]_\zeta\cdots[1]_\zeta}.$$

By (2.2), this is equal to 1, as $[n-i]_\zeta$ cancels with $[i+1]_\zeta$.

Hence Theorem 2 is proved for type A.

2.3. Type D

Consider type D_n, that is, $\mathfrak{g}_{\text{fin}} = \mathfrak{so}(2n,\mathbb{C})$. We have $h^\vee = 2n-2$.

It is known that the kth l-fundametal module for $1 \le k \le n-2$ is isomorphic as an $\mathfrak{so}(2n,\mathbb{C})$-module to

$$\textstyle\bigwedge^k(\mathbb{C}^{2n}) \oplus \bigwedge^{k-2}(\mathbb{C}^{2n}) \oplus \cdots, \tag{2.3}$$

where $\cdots$ ends as $\bigwedge^1(\mathbb{C}^{2n}) = \mathbb{C}^{2n}$ if k is odd, and $\bigwedge^0(\mathbb{C}^{2n}) = \mathbb{C}$ if k is even (see [32, Remark 5.9]). For $k = n-1$, n, the l-fundamental module is isomorphic to the kth fundamental representation of $\mathfrak{so}(2n,\mathbb{C})$.

As in [10], the positive roots are

$$\{L_i + L_j \mid i < j\} \sqcup \{L_i - L_j \mid i < j\},$$

and the simple roots are

$$\alpha_i = L_i - L_{i+1} \quad (i = 1, \ldots, n-1), \qquad \alpha_n = L_{n-1} + L_n$$

with the standard inner product $(L_i, L_j) = \delta_{ij}$. The fundamental weights are

$$\Lambda_i = L_1 + L_2 + \cdots + L_i \quad (i = 1, \ldots, n-2),$$
$$\Lambda_{n-1} = \frac{1}{2}(L_1 + L_2 + \cdots + L_{n-2} + L_{n-1} - L_n),$$
$$\Lambda_n = \frac{1}{2}(L_1 + L_2 + \cdots + L_{n-2} + L_{n-1} + L_n).$$

The Weyl vector ρ is

$$\rho = \sum_{i=1}^{n} (n-i) L_i.$$

We have $(\Lambda_k, L_1 + L_2) = 2$ unless $k = 1,\ n-1,\ n$. If this holds, then we have

$$\big[(\Lambda_k + \rho, L_1 + L_2)\big]_\zeta = [2n-1]_\zeta = 0.$$

Thus

$$\dim_q V(\Lambda_k) = 0 \quad \text{if } k \neq 1,\ n-1,\ n.$$

We have

$$\dim_q V(\Lambda_1) = \frac{[n]_\zeta [2n-2]_\zeta}{[1]_\zeta [n-1]_\zeta} = 1,$$

as $[2n-2]_\zeta = [1]_\zeta$, $[n]_\zeta = [n-1]_\zeta$ by (2.2).

Next we consider the case $\lambda = \Lambda_{n-1}$. We have

$$(\Lambda_{n-1}, L_i - L_j) = \begin{cases} 0 & \text{if } j \le n-1, \\ 1 & \text{if } j = n, \end{cases} \qquad (\Lambda_{n-1}, L_i + L_j) = \begin{cases} 1 & \text{if } j \le n-1, \\ 0 & \text{if } j = n. \end{cases}$$

These imply that

$$\dim_q V(\Lambda_{n-1}) = \frac{[2n-2]_\zeta [2n-4]_\zeta \cdots [4]_\zeta}{[n-1]_\zeta [n-2]_\zeta \cdots [3]_\zeta}$$
$$= \begin{cases} \frac{[2n-2]_\zeta [2n-4]_\zeta \cdots [n]_\zeta}{[n-1]_\zeta [n-3]_\zeta \cdots [3]_\zeta} & \text{if } n \text{ is even,} \\ \frac{[2n-2]_\zeta [2n-4]_\zeta \cdots [n+1]_\zeta}{[n-2]_\zeta [n-4]_\zeta \cdots [3]_\zeta} & \text{if } n \text{ is odd.} \end{cases}$$

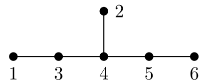
This is 1 by (2.2). Note that $[2n-2]_\zeta = [1]_\zeta = 1$. We also have $\dim_q V(\Lambda_n) = 1$ as we have a diagram automorphism $n \leftrightarrow n-1$, $i \leftrightarrow i$ $(i \neq n-1, n)$.

In summary,

$$\dim_q V(\Lambda_k) = \begin{cases} 1 & \text{if } k = 1, n-1, n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence Theorem 2 follows for $k = 1,\ n-1,\ n$. For $2 \le k \le n-2$, we substitute the above formula into (2.3). Then all summands except the last one have vanishing quantum dimensions. The last summand is $\bigwedge^1(\mathbb{C}^{2n}) = V(\Lambda_1)$ if k is odd, and the trivial representation $\bigwedge^0(\mathbb{C}^{2n})$ if k is even. Both have quantum dimension 1. Theorem 2 is proved for type D_n.

2.4. Type E_6

We have $h^\vee = 12$. The numbering of vertices is 

. We need to compute quantum dimensions of $V(\Lambda_k)$, as well as of $V(\Lambda_1+\Lambda_6)$ since it appears in standard representations when $\mathbf{w}^s$ is the 4th fundamental weight.

The positive roots are

$$\alpha_1, \dots, \alpha_6, \alpha_1+\alpha_3, \alpha_3+\alpha_4, \alpha_4+\alpha_5, \alpha_5+\alpha_6, \alpha_2+\alpha_4,$$
$$\alpha_1+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_4+\alpha_5, \alpha_3+\alpha_4+\alpha_5, \alpha_4+\alpha_5+\alpha_6,$$
$$\alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_3+\alpha_4+\alpha_5, \alpha_2+\alpha_3+\alpha_4+\alpha_5,$$
$$\alpha_2+\alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6,$$
$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$
$$\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+\alpha_5,$$
$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5, \alpha_2+\alpha_3 + 2\alpha_4+\alpha_5+\alpha_6,$$
$$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3 + 2\alpha_4+2\alpha_5+\alpha_6,$$
$$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,$$
$$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,$$
$$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,$$
$$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6.$$

Hence the denominator of (2.1) is

$$[1]_\zeta^6[2]_\zeta^5[3]_\zeta^5[4]_\zeta^5[5]_\zeta^4[6]_\zeta^3[7]_\zeta^3[8]_\zeta^2[9]_\zeta[10]_\zeta[11]_\zeta.$$

We compute the quantum dimension of the first fundamental representation as

$$\dim_q V(\Lambda_1) = \frac{[2]_\zeta[3]_\zeta[4]_\zeta[5]_\zeta^2[6]_\zeta^2[7]_\zeta^2[8]_\zeta^2[9]_\zeta^2[10]_\zeta[11]_\zeta[12]_\zeta}{[1]_\zeta[2]_\zeta[3]_\zeta[4]_\zeta^2[5]_\zeta^2[6]_\zeta^2[7]_\zeta^2[8]_\zeta^2[9]_\zeta[10]_\zeta[11]_\zeta} = \frac{[9]_\zeta[12]_\zeta}{[1]_\zeta[4]_\zeta}.$$

This is 1 by (2.2). We also have $\dim_q V(\Lambda_6) = 1$ by the diagram automorphism.

We have

$$(\Lambda_k+\rho, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6) = 13 \quad \text{for } k = 2,3,5,$$
$$(\Lambda_1+\Lambda_6+\rho, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6) = 13.$$

Therefore, $\dim_q V(\Lambda_k) = 0$ for $k=2,3,5$, $\dim_q V(\Lambda_1+\Lambda_6) = 0$. We also have

$$(\Lambda_4+\rho, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6) = 13,$$

hence $\dim_q V(\Lambda_4) = 0$.

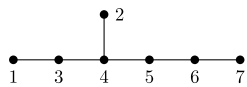
It is known that the kth fundamental module is isomorphic as a $\mathfrak{g}_{\mathrm{fin}}$-module to

$$\begin{cases} V(\Lambda_k) & \text{if } k=1,6, \\ V(\Lambda_2)\oplus V(0) & \text{if } k=2, \\ V(\Lambda_3)\oplus V(\Lambda_6) & \text{if } k=3, \\ V(\Lambda_5)\oplus V(\Lambda_1) & \text{if } k=5, \\ V(\Lambda_4)\oplus V(\Lambda_2)^{\oplus 2}\oplus V(\Lambda_1+\Lambda_6)\oplus V(0) & \text{if } k=4. \end{cases}$$

This can be given by using the algorithm in [33], as we did for E_8 in [35] with considerably less effort. Instead, the list can be found in [19], which assumed a fermionic formula conjectured at that time, and was proved later in [8] .

We substitute the above computation of quantum dimensions of various modules into the above combination. We find that the combination always has quantum dimension 1. Hence Theorem 2 is proved.

2.5. Type E_7

We have $h^\vee = 18$. The numbering of vertices is 

.

We calculate as above, using Sage:

$$\dim_q V(\Lambda_7) = \frac{[10]_\zeta[14]_\zeta[18]_\zeta}{[1]_\zeta[5]_\zeta[9]_\zeta} = 1$$

and

$$\dim_q V(\Lambda_k) = 0 \quad \text{if } k \neq 7.$$

We have a new pattern:

$$\dim_q V(2\Lambda_1) = \frac{[12]_\zeta[13]_\zeta[14]_\zeta[15]_\zeta[18]_\zeta[21]_\zeta}{[1]_\zeta[2]_\zeta[4]_\zeta[5]_\zeta[6]_\zeta[7]_\zeta} = \frac{[21]_\zeta}{[2]_\zeta}.$$

We use (2.2) for $k=-2$ this time: $[21]_\zeta = [-2]_\zeta$. This is $-[2]_\zeta$. Hence

$$\dim_q V(2\Lambda_1) = -1.$$

This is the first example of a representation whose quantum dimension is -1. Similarly, we have

$$\dim_q V(\Lambda_1+\Lambda_7) = \frac{[12]_\zeta[14]_\zeta[15]_\zeta[18]_\zeta[20]_\zeta}{[1]_\zeta^2[4]_\zeta[5]_\zeta[7]_\zeta} = -1.$$

On the other hand, we obtain

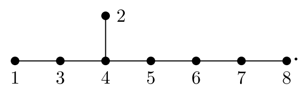
$$\dim_q V(2\Lambda_7) = \dim_q V(\Lambda_1+\Lambda_6) = \dim_q V(\Lambda_2+\Lambda_7) = 0.$$

It is known that the kth l-fundamental module is isomorphic as a $\mathfrak{g}_{\mathrm{fin}}$-module to

$$\begin{cases} V(\Lambda_k) & \text{if } k=7, \\ V(\Lambda_1)\oplus V(0) & \text{if } k=1, \\ V(\Lambda_2)\oplus V(\Lambda_7) & \text{if } k=2, \\ V(\Lambda_6)\oplus V(\Lambda_1)\oplus V(0) & \text{if } k=6, \\ V(\Lambda_3)\oplus V(\Lambda_6)\oplus V(\Lambda_1)^{\oplus 2}\oplus V(0) & \text{if } k=3, \\ V(\Lambda_5)\oplus V(\Lambda_2)^{\oplus 2}\oplus V(\Lambda_1+\Lambda_7)\oplus V(\Lambda_7)^{\oplus 2} & \text{if } k=5, \\ V(\Lambda_4)\oplus V(\Lambda_3)^{\oplus 3}\oplus V(\Lambda_2+\Lambda_7)^{\oplus 2}\oplus V(\Lambda_1+\Lambda_6)\oplus V(2\Lambda_1) & \\ \quad \oplus V(\Lambda_6)^{\oplus 4}\oplus V(\Lambda_1)^{\oplus 4}\oplus V(2\Lambda_7)\oplus V(0)^{\oplus 2} & \text{if } k=4. \end{cases}$$

This result can be checked by the computer program in [35] or by [19] with [8] as in type E_6. Substituting the above computation, we find that these combinations always have quantum dimension 1; for example, we have $\dim_q V(2\Lambda_1) + 2\dim_q V(0) = (-1)+2 = 1$ in the last case. Hence Theorem 2 is proved.

2.6. Type E_8

We have $h^\vee = 30$. The numbering of vertices is 

.

We have similar patterns:

$$\dim_q V(\Lambda_k) = 0 \quad \text{for any } k=1,\dots,8,$$

$$\dim_q V(2\Lambda_8) = \frac{[20]_\zeta[21]_\zeta[24]_\zeta[25]_\zeta[30]_\zeta[33]_\zeta}{[1]_\zeta[2]_\zeta[6]_\zeta[7]_\zeta[10]_\zeta[11]_\zeta} = -1,$$

$$\dim_q V(\Lambda_7+\Lambda_8) = \frac{[14]_\zeta[18]_\zeta[20]_\zeta[22]_\zeta[24]_\zeta[25]_\zeta[26]_\zeta[30]_\zeta[32]_\zeta[34]_\zeta}{[1]_\zeta^2[3]_\zeta[5]_\zeta[6]_\zeta[7]_\zeta[9]_\zeta[11]_\zeta[13]_\zeta[17]_\zeta} = 1,$$

$$\dim_q V(\Lambda_6+\Lambda_8) = \frac{[15]_\zeta[18]_\zeta[20]_\zeta[21]_\zeta[24]_\zeta[25]_\zeta[26]_\zeta[27]_\zeta[30]_\zeta[32]_\zeta[33]_\zeta[35]_\zeta}{[1]_\zeta^2[2]_\zeta[4]_\zeta^2[5]_\zeta[6]_\zeta[7]_\zeta[10]_\zeta[11]_\zeta[13]_\zeta[16]_\zeta}$$
$$= -1,$$

and

$$\begin{aligned}\dim_q V(\Lambda_1+\Lambda_8) &= \dim_q V(2\Lambda_1) = \dim_q V(\Lambda_2+\Lambda_8) = \dim_q V(\Lambda_1+\Lambda_7) \\ &= \dim_q V(\Lambda_1+2\Lambda_8) = \dim_q V(2\Lambda_7) \\ &= \dim_q V(3\Lambda_8) = \dim_q V(2\Lambda_1+\Lambda_8) \\ &= \dim_q V(\Lambda_3+\Lambda_8) = \dim_q V(\Lambda_2+\Lambda_7) = \dim_q V(\Lambda_1+\Lambda_6) \\ &= \dim_q V(\Lambda_1+\Lambda_2) = 0.\end{aligned}$$

The $\mathfrak{g}_{\mathrm{fin}}$-module structure of the kth l-fundamental module is known by [35] (see also [19], [8]). We omit *negligible* modules (i.e., those with $\dim_q = 0$) for brevity. We have

$$\begin{cases} V(0) & \text{if } k = 1,2,7,8, \\ V(2\Lambda_8) \oplus V(0)^{\oplus 2} & \text{if } k = 3,6, \\ V(2\Lambda_8)^{\oplus 5} \oplus V(\Lambda_7 + \Lambda_8)^{\oplus 2} \oplus V(0)^{\oplus 4} & \text{if } k = 5, \\ V(\Lambda_6 + \Lambda_8)^{\oplus 4} \oplus V(\Lambda_7 + \Lambda_8)^{\oplus 18} \oplus V(2\Lambda_8)^{\oplus 23} \oplus V(0)^{\oplus 10} & \text{if } k = 4. \end{cases}$$

Rather miraculously, we find that all have quantum dimension 1. For example, we calculate the result as $(-4) + 18 + (-23) + 10 = 1$ in the last case. Hence Theorem 2 is proved.

3. Rational smoothness

As mentioned in the Introduction, the original motivation behind the study of $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v}, \mathbf{w})$ in [34] was its relation between the restriction from $\mathfrak{g}_{\mathrm{aff}}$ to $\mathfrak{g}_{\mathrm{fin}}$. However, the Euler numbers of $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v}, \mathbf{w})$ did not play any role in [34]. Intersection cohomology groups of $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v}, \mathbf{w})$ appeared instead.

Therefore, we ask whether $\mathfrak{M}_{\zeta^\bullet}(\mathbf{v}, \mathbf{w})$ is a rational homology manifold, that is, its intersection cohomology complex is quasi-isomorphic to the constant sheaf. See [4, Section 1.4] for the original definition of a rational homology manifold and its equivalence to the present one. On an optimistic note, Borho and MacPherson [4, Section 2.3] showed that the nilpotent variety is a rational homology manifold. Also, $\mathfrak{M}_0(\mathbf{v}, \Lambda)$ is a symmetric power of $\mathbb{C}^2/\Gamma$, which has only finite quotient singularity, and hence is a rational homology manifold.

3.1. Example: Hilbert schemes of two points

Consider $\mathfrak{M}_{\zeta^\bullet}(2\delta, \Lambda_0) = \mathrm{Hilb}^2(\mathbb{C}^2/\Gamma)$. As we already mentioned in Example 1.5, Yamagishi showed that $\mathfrak{M}_{\zeta^\bullet}(2\delta, \Lambda_0)$ is locally isomorphic to the intersection of the nilpotent cone of $\mathfrak{g}_{\mathrm{fin}}$ and the Slodowy slice to a sub-subregular orbit around 0-dimensional strata. The transversal slice to the bigger stratum is $\mathbb{C}^2/\Gamma$. It is also a rational homology manifold. Combined with the rational smoothness of the nilpotent variety mentioned above, we obtain the following.

COROLLARY 3.1

$\mathfrak{M}_{\zeta^\bullet}(2\delta, \Lambda_0) = \mathrm{Hilb}^2(\mathbb{C}^2/\Gamma)$ *is a rational homology manifold.*

This result can be also proved from [34], together with [30] and some Euler number computation. See the argument below in a simpler situation.

3.2. Counterexample

Consider $A_1^{(1)}$ with $n = 4$, that is, $\mathrm{Hilb}^4(\mathbb{C}^2/(\mathbb{Z}/2)) = \mathfrak{M}_{\zeta^\bullet}(4\delta, \Lambda_0)$. We take $\mathbf{v}' = \underset{4}{\circ}\!\Longleftrightarrow\!\underset{2}{\bullet}$. Then $\mathbf{v}^s = 2\alpha_1$, $\mathbf{w}^s = 4\Lambda_1$. Hence the transversal slice is $\mathfrak{M}_0(\mathbf{v}^s, \mathbf{w}^s)$, which is the nilpotent orbit in $\mathfrak{sl}(4)$ of type (2^2). It is well known to be a nonrational homology manifold. For example, we can argue as follows. By [30, Theorem 15.1.1], $\dim i_0^! \mathrm{IC}(\mathfrak{M}_0(\mathbf{v}^s, \mathbf{w}^s))$ gives the multiplicities of the trivial representation in the standard module for $\mathbf{w}^s$. The latter is the 4th tensor power of the natural 2-dimensional representation of $\mathfrak{sl}(2)$. The multiplicity of the trivial representation is 2. Hence $\mathfrak{M}_0(\mathbf{v}^s, \mathbf{w}^s)$ is *not* a rational homology manifold, and neither is $\mathfrak{M}_{\zeta^\bullet}(4\delta, \Lambda_0)$.

Alternatively, we argue as follows. We realize this variety as the Coulomb branch of a quiver gauge theory of type A_3 with $\mathbf{v}'' = \underset{1}{\bullet}\!-\!\!-\!\underset{2}{\bullet}\!-\!\!-\!\underset{1}{\bullet}$, $\mathbf{w}'' = \underset{0}{\bullet}\!-\!\!-\!\underset{2}{\bullet}\!-\!\!-\!\underset{0}{\bullet}$. It is the closure of an $\mathrm{SL}(4)[[z]]$-orbit in the affine Grassmannian for $\mathrm{SL}(4)$, and hence the intersection cohomology is known by geometric Satake correspondence. We have

$$\dim i_0^! \mathrm{IC}\bigl(\mathcal{M}_C(\mathbf{v}'', \mathbf{w}'')\bigr) = \dim V_0(2\Lambda_2),$$

where i_0 is the embedding of the identity element to the affine Grassmannian. The right-hand side is the 0-weight space in the representation of $\mathrm{SL}(4)$ with the highest weight $2\Lambda_2$. This weight space is 2-dimensional.

In general, if $\mathfrak{M}_0(\mathbf{v}^s, \mathbf{w}^s)$ happens to be the closure of a $G[[z]]$-orbit in an affine Grassmannian $\mathrm{Gr}_G = G((z))/G[[z]]$ (e.g., it is true in type A by [25] or the combination of [36] and [5]), then it is a rational homology manifold if and only if weight spaces of the corresponding representation are all 1-dimensional.[2]

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