WDVV-type relations for Welschinger invariants: Applications

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Abstract We first recall Solomon's relations for Welschinger invariants counting real curves in real symplectic fourfolds and the *Witten–Dijkgraaf–Verlinde–Verlinde (WDVV)*-style relations for Welschinger invariants counting real curves in real symplectic sixfolds with some symmetry. We then explicitly demonstrate that, in some important cases (projective spaces with standard conjugations, real blowups of the projective plane, and two- and threefold products of the one-dimensional projective space with two involutions each), these relations provide complete recursions determining all Welschinger invariants from basic input. We include extensive tables of Welschinger invariants in low degrees obtained from these recursions with *Mathematica*. These invariants provide lower bounds for counts of real rational curves, including with curve insertions in smooth algebraic threefolds.

1. Introduction

Let (X,ω,ϕ) be a compact, connected, real symplectic manifold of (real) dimension $2n$. The fixed locus X^ϕ of the antisymplectic involution ϕ on X is then a Lagrangian submanifold of (X,ω). Let

$$H_2^\phi(X)=\big\{B\in H_2(X;\mathbb{Z}):\phi_*B=-B\big\},$$

$$H^*(X)_\pm^\phi=\big\{\mu\in H^*(X;\mathbb{R}):\phi^*\mu=\pm\mu\big\},$$

$$\mathfrak{d}:H_2(X;\mathbb{Z})\longrightarrow H_2^\phi(X),\qquad \mathfrak{d}(B)=B-\phi_*(B).$$

We denote by $\mathcal{J}_\omega$ the space of ω-compatible (or -tamed) almost-complex structures J on X, and we denote by $\mathcal{J}_\omega^\phi\subset\mathcal{J}_\omega$ the subspace of almost-complex structures J such that $\phi^*J=-J$. Let

$$c_1(X,\omega)\equiv c_1(TX,J)\in H^2(X)$$

be the first Chern class of TX with respect to some $J\in\mathcal{J}_\omega$; it is independent of such a choice. Define

$$\ell_\omega:H_2(X;\mathbb{Z})\longrightarrow\mathbb{Z},\qquad \ell_\omega(B)=\big\langle c_1(X,\omega),B\big\rangle+n-3,$$

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with $\langle\cdot,\cdot\rangle$ denoting the usual pairing between cohomology and homology classes. The paradigmatic example of a real symplectic manifold is the complex projective space $\mathbb{P}^n$ with the Fubini–Study symplectic form and the standard conjugation

$$\tau_n: \mathbb{P}^n \longrightarrow \mathbb{P}^n, \qquad \tau_n\big([Z_0, Z_1, \ldots, Z_n]\big) = [\overline{Z_0}, \overline{Z_1}, \ldots, \overline{Z_n}].$$

For $J \in \mathcal{J}_\omega$ and $B \in H_2(X;\mathbb{Z})$, a subset $C \subset X$ is a *genus* 0 (or *rational*) *irreducible J-holomorphic degree B curve* if there exists a simple (not multiply covered) J-holomorphic map

$$u: \mathbb{P}^1 \longrightarrow X \quad \text{s.t. } C = u(\mathbb{P}^1), u_*[\mathbb{P}^1] = B. \tag{1}$$

If $J \in \mathcal{J}_\omega^\phi$, then such a curve C is *real* if in addition $\phi(C) = C$; if so, then $B \in H_2^\phi(X)$.

Invariant signed counts of real rational J-holomorphic curves in compact real symplectic fourfolds and sixfolds, now known as *Welschinger invariants*, were defined in [19] and [20] and interpreted in terms of moduli spaces of J-holomorphic maps from the two-disk in [17]. An adaptation of the interpretation of [17] in terms of real J-holomorphic maps from the Riemann sphere $\mathbb{P}^1$ appeared later in [9] and was reformulated in terms of degrees of relatively oriented pseudocycles in [5] and [6]. The moduli-theoretic perspectives on Welschinger invariants lead to the WDVV-type relations for them established in [5] and [6].

Our article is organized as follows. We recall the relevant versions of definitions of Welschinger invariants of real symplectic fourfolds and sixfolds in Sections 2 and 6, respectively. We conclude both sections with statements of WDVV-type relations for Welschinger invariants in the respective settings (see Theorems 1 and 3). In the case of $(\mathbb{P}^2, \tau_2)$, the two relations of Theorem 1 restrict to the two relations of [18, p. 13], which determine all Welschinger invariants of $(\mathbb{P}^2, \tau_2)$ from basic input (see Section 3). We explicitly demonstrate in Section 5 that the first relation of Theorem 1 and elementary geometric considerations reduce all Welschinger invariants of real blowups $(\mathbb{P}^2_{r,s}, \tau_{r,s})$ of $(\mathbb{P}^2, \tau_2)$ to the standard Gromov–Witten invariants of blowups $\mathbb{P}^2_k$ of $\mathbb{P}^2$ and Welschinger invariants of $(\mathbb{P}^2, \tau_2)$. In Section 4, we apply Theorem 1 to $\mathbb{P}^1 \times \mathbb{P}^1$ with two natural orientation-reversing involutions and show that it determines all Welschinger invariants from basic input in these cases as well.

In Sections 7 and 8, we explicitly show that the relations of Theorem 3 similarly determine all Welschinger invariants of $(\mathbb{P}^3, \tau_3)$ and $(\mathbb{P}^1)^3$ with two distinct conjugations from some basic inputs. However, new nuances arise in these settings. First, Welschinger invariants of a real symplectic sixfold (X, ω, ϕ) depend on the choice of a Spin-structure on X^ϕ in all perspectives on these invariants. Second, the WDVV-type relations of Theorem 3 for these invariants involve signed counts of real curves through constraints of dimension 2. Such counts are not part of Welschinger's original definition in [20]; they arise from the moduli-theoretic perspective on Welschinger invariants introduced in [17] and the use of a symmetry of (X, ω, ϕ) introduced in [6]. Finally, the relations of Theorem 3 determine Welschinger invariants of $(\mathbb{P}^3, \tau_3)$ from a single input, as shown in [1, Section 4.1.4]; the sign of this input depends on the choice of a Spin-structure on the fixed locus $\mathbb{RP}^3 \subset \mathbb{P}^3$ of τ_3. On the other hand, the determination of Welschinger invariants of $(\mathbb{P}^1)^3$ with either of the two natural conjugations ϕ requires a basic input beyond those that can be directly changed by changing the Spin-structure on the fixed locus of ϕ. We determine this input via a reduction to $(\mathbb{P}^1)^2$. The tables provide low-degree Welschinger invariants based on the formulas in Sections 3–5, 7, and 8; the *Mathematica* programs implementing these formulas are available from the Wolfram Foundation's *Notebook Archive*.

2. Solomon's relations for real symplectic fourfolds

Let (X,ω,ϕ) be a compact, real symplectic fourfold. For $B,B' \in H_2(X;\mathbb{Z})$, we denote by $B\cdot_X B' \in \mathbb{Z}$ the homology intersection product of B with B', and we denote by $B^2 \in \mathbb{Z}$ the self-intersection number of B. Since ϕ is an antisymplectic involution, $\mathfrak{d}(B)^2 \in 2\mathbb{Z}$ for every $B \in H_2(X;\mathbb{Z})$. Thus, $B^2/2$ in the first expression on the right-hand side of (SWDVV1) in Theorem 1 below is an integer whenever the sum in this expression is nontrivial.

If $B \in H_2(X;\mathbb{Z})$, $\ell_\omega(B) \geq 0$, and $J \in \mathcal{J}_\omega$, then there are only finitely many rational irreducible J-holomorphic degree B curves C passing through $\ell_\omega(B)$ points in general position. The number of such curves counted with appropriate signs is independent of the choices of J and the points. This is the standard (complex) genus 0 degree B *Gromov–Witten* (or *GW-*) *invariant* of (X,ω) with $\ell_\omega(B)$ point insertions; we denote it by N_B^X. If $\ell_\omega(B) < 0$, then we set $N_B^X = 0$. By the standard WDVV relation,

$$\langle H_1H_2,X\rangle N_B^X = \sum_{\substack{B_1,B_2\in H_2(X;\mathbb{Z})\\ B_1+B_2=B\\ B_1,B_2\neq 0}} \langle H_1,B_1\rangle\bigg(\langle H_2,B_2\rangle\binom{\ell_\omega(B)-3}{\ell_\omega(B_1)-1} - \langle H_2,B_1\rangle\binom{\ell_\omega(B)-3}{\ell_\omega(B_1)}\bigg)N_{B_1}^X N_{B_2}^X \tag{2}$$

for all $H_1,H_2 \in H^2(X;\mathbb{Q})$. Formally, this relation follows from [16, (1.7)] with $H_\alpha = H_1$, $H_\beta = H_2$, and H_γ, H_δ being the Poincaré dual of the point class and the divisor relation for GW-invariants. More directly, it is an immediate consequence of [16, Theorem 7.2] with $A = B$, $g = 0$, $k = 4$, $l = \ell_\omega(B) - 3$, $\alpha_1 = H_1$, $\alpha_2 = H_2$, $\alpha_3,\alpha_4,\beta_1,\ldots,\beta_l$ being the Poincaré duals of the point class, and $\mathcal{C}$ being the connected reducible rational curve with four marked points so that either the marked points 1, 2 lie on the same irreducible component of $\mathcal{C}$ or the marked points 1, 3 do.

Suppose that $J \in \mathcal{J}_\omega^\phi$, that $C \subset X$ is a real curve, and that u is as in (1). A point $x \in C$ is a *simple node* if

$$\big|u^{-1}(x)\big| = 2 \qquad \text{and} \qquad \bigoplus_{z\in u^{-1}(x)} \operatorname{Im} \mathrm{d}_z u = T_xX.$$

We denote by $\delta_E(C)$ the number of nodes of C that are isolated points of $C\cap X^\phi$.

Suppose in addition that the fixed locus X^ϕ of (X,ω,ϕ) is connected. Let $B \in H_2(X;\mathbb{Z})$ and $l \in \mathbb{Z}^{\geq 0}$ be such that

$$k \equiv \ell_\omega(B) - 2l \in \mathbb{Z}^{\geq 0}. \tag{3}$$

For a generic $J \in \mathcal{J}_\omega^\phi$, there are then only finitely many rational, irreducible, real J-holomorphic degree B curves $C \subset X$ passing through k points in X^ϕ and l points in $X - X^\phi$ in general position. According to [19, Theorem 0.1], the sum

$$N_{B,l}^\phi \equiv \sum_C (-1)^{\delta_E(C)}$$

over the set of these curves is independent of the choices of J and the points. If the number k in (3) is negative, then we set $N_{B,l}^\phi = 0$.

The theorem below follows immediately from [5, Theorem 1.1], which established similar relations for signed counts of real J-holomorphic maps as in [17], [9], and [7, Theorem 13.1], which compares the signs of these counts with the signs in Welschinger's definition in [19]. For $B \in H_2(X;\mathbb{Z})$ and $l \in \mathbb{Z}$, let

$$\langle B \rangle_l = \begin{cases} 1 & \text{if } 2l = \ell_\omega(B) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 1

Suppose that (X,ω,ϕ) is a compact real symplectic fourfold with connected fixed locus X^ϕ, $H_1, H_2 \in H^2(X)_-^\phi$, $B \in H_2(X)$, and $l \in \mathbb{Z}^{\geq 0}$.

(SWDVV1) *If $l \geq 1$ and $\ell_\omega(B) - 2l \geq 1$ (i.e., $k \geq 1$), then*

$$\begin{aligned}
&\langle H_1 H_2, X \rangle N_{B,l}^\phi \\
&\quad = (-1)^{l + B^2/2} 2^{l-3} \langle B \rangle_l \langle H_1, B \rangle \langle H_2, B \rangle \sum_{\substack{B' \in H_2(X) \\ \mathfrak{d}(B') = B}} N_{B'}^X \\
&\qquad + \sum_{\substack{B_0, B' \in H_2(X) - \{0\} \\ B_0 + \mathfrak{d}(B') = B}} (-1)^{\ell_\omega(B') + \mathfrak{d}(B')^2/2} 2^{\ell_\omega(B')} (B_0 \cdot_X B') \langle H_1, B' \rangle \langle H_2, B' \rangle \\
&\qquad \times \binom{l-1}{\ell_\omega(B')} N_{B'}^X N_{B_0, l-1-\ell_\omega(B')}^\phi \\
&\qquad + \sum_{\substack{B_1, B_2 \in H_2(X) - \{0\} \\ B_1 + B_2 = B \\ l_1 + l_2 = l-1, l_1, l_2 \geq 0}} \langle H_1, B_1 \rangle \binom{l-1}{l_1} \left(\langle H_2, B_2 \rangle \binom{\ell_\omega(B) - 2l - 1}{\ell_\omega(B_1) - 2l_1 - 1} \right. \\
&\qquad \left. - \langle H_2, B_1 \rangle \binom{\ell_\omega(B) - 2l - 1}{\ell_\omega(B_1) - 2l_1} \right) N_{B_1, l_1}^\phi N_{B_2, l_2}^\phi.
\end{aligned}$$

(SWDVV2) *If $l \geq 2$, then*

$$\begin{aligned}
&\langle H_1 H_2, X \rangle N_{B,l}^\phi \\
&\quad = - \sum_{\substack{B_0, B' \in H_2(X) - \{0\} \\ B_0 + \mathfrak{d}(B') = B}} (-1)^{\ell_\omega(B') + \mathfrak{d}(B')^2/2} 2^{\ell_\omega(B')-1} (B_0 \cdot_X B') \langle H_1, B' \rangle
\end{aligned}$$

$$\times\left(\langle H_2,B_0\rangle\binom{l-2}{\ell_\omega(B')-1}-2\langle H_2,B'\rangle\binom{l-2}{\ell_\omega(B')}\right)N^X_{B'}N^\phi_{B_0,l-1-\ell_\omega(B')}$$

$$+\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\ B_1+B_2=B\\ l_1+l_2=l-2,l_1,l_2\ge 0}}\langle H_2,B_1\rangle\binom{l-2}{l_1}\left(\langle H_1,B_2\rangle\binom{\ell_\omega(B)-2l}{\ell_\omega(B_1)-2l_1-1}\right.$$

$$\left.-\langle H_1,B_1\rangle\binom{\ell_\omega(B)-2l}{\ell_\omega(B_1)-2l_1}\right)N^\phi_{B_1,l_1}N^\phi_{B_2,l_2+1}.$$

REMARK 1

Theorem 1 extends to compact real symplectic fourfolds (X,ω,ϕ) with disconnected fixed loci X^ϕ and applies with finer notions of the curve degree B (see Theorem 1.1 and Remark 1.3 in [5]). This makes no difference for the examples of Sections 3–5. Theorem 1.1 in [5] contains another WDVV-type relation; it involves three divisor insertions. Since it is not needed for the present purposes, we do not state it here.

3. Projective plane $\mathbb{P}^2$

The fixed locus of the standard conjugation

$$\tau_2:\mathbb{P}^2\longrightarrow\mathbb{P}^2,\qquad \tau_2\big([Z_0,Z_1,Z_2]\big)=[\overline{Z_0},\overline{Z_1},\overline{Z_2}],$$

on the complex projective plane is the real projective plane $\mathbb{RP}^2$. The group $H_2^{\tau_2}(\mathbb{P}^2)=H_2(\mathbb{P}^2;\mathbb{Z})$ is identified with $\mathbb{Z}$ via the standard generator $L=[\mathbb{P}^1]$.

For $d\in\mathbb{Z}^+$ and $l\in\mathbb{Z}^{\ge 0}$, we set

$$N_d^{\mathbb{P}^2}=N_{dL}^{\mathbb{P}^2},\qquad N_{d,l}^{\tau_2}=N_{dL,l}^{\tau_2},\qquad \langle d\rangle_l=\langle dL\rangle_l=\begin{cases}1 & \text{if } 2l=3d-2,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, $d/2$ in the identity [18, (10)] below is an integer whenever $\langle d\rangle_l\neq 0$. Since there is a unique line $\mathbb{P}^1$ through every pair of points in $\mathbb{P}^2$ and $\delta_E(\mathbb{P}^1)=0$,

$$N_1^{\mathbb{P}^2},N_{1,0}^{\tau_2},N_{1,1}^{\tau_2}=1.$$

Since there is a unique conic C through five general points in $\mathbb{P}^2$ and $\delta_E(C)=0$,

$$N_2^{\mathbb{P}^2},N_{2,0}^{\tau_2},N_{2,1}^{\tau_2},N_{2,2}^{\tau_2}=1.$$

Kontsevich's recursion for the standard Gromov–Witten invariants of $\mathbb{P}^2$ can be written as

$$N_d^{\mathbb{P}^2}=\frac{1}{6(d-1)}\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}}\left(d_1d_2-2\frac{(d_1-d_2)^2}{3d-2}\right)\binom{3d-2}{3d_1-1}d_1d_2N_{d_1}^{\mathbb{P}^2}N_{d_2}^{\mathbb{P}^2}.$$

This particular version of Kontsevich's recursion is obtained by symmetrizing [16, (10.4)] in d_1, d_2, which in turn is obtained from (2) by taking H_1, H_2 to be the Poincaré dual of L. However, the only property of this recursion needed for the purposes of the present section is that it determines all numbers $N_d^{\mathbb{P}^2}$ from the single input $N_1^{\mathbb{P}^2}=1$. This is immediate from either [16, (10.4)] or its version stated above.

Taking $H_1,H_2\in H^2(\mathbb{P}^2)$ to be the Poincaré duals of L in the two formulas of Theorem 1, we obtain:

[18, (10)] If $d\ge 2$, $l\ge 1$, and $3d-2l\ge 2$ (i.e., $k\ge 1$), then

$$\begin{aligned}N_{d,l}^{\tau_2}=&-(-2)^{3d/2-4}\langle d\rangle_l d^2 N_{d/2}^{\mathbb{P}^2}+\sum_{\substack{d_0+2d'=d\\ d_0,d'\ge 1}}(-2)^{3d'-1}d_0d'^3\binom{l-1}{3d'-1}N_{d'}^{\mathbb{P}^2}N_{d_0,l-3d'}^{\tau_2}\\&+\sum_{\substack{d_1+d_2=d\\ l_1+l_2=l-1\\ d_1,d_2\ge 1,l_1,l_2\ge 0}}\binom{l-1}{l_1}\left(d_1d_2\binom{3d-2l-2}{3d_1-2l_1-2}\right.\\&\left.-d_1^2\binom{3d-2l-2}{3d_1-2l_1-1}\right)N_{d_1,l_1}^{\tau_2}N_{d_2,l_2}^{\tau_2}.\end{aligned}$$

[18, (9)] If $d\ge 2$ and $l\ge 2$, then

$$\begin{aligned}N_{d,l}^{\tau_2}=&\sum_{\substack{d_0+2d'=d\\ d_0,d'\ge 1}}(-2)^{3d'-2}d_0d'^2\left(d_0\binom{l-2}{3d'-2}-2d'\binom{l-2}{3d'-1}\right)N_{d'}^{\mathbb{P}^2}N_{d_0,l-3d'}^{\tau_2}\\&+\sum_{\substack{d_1+d_2=d\\ l_1+l_2=l-2\\ d_1,d_2\ge 1,l_1,l_2\ge 0}}\binom{l-2}{l_1}\left(d_1d_2\binom{3d-2l-1}{3d_1-2l_1-2}\right.\\&\left.-d_1^2\binom{3d-2l-1}{3d_1-2l_1-1}\right)N_{d_1,l_1}^{\tau_2}N_{d_2,l_2+1}^{\tau_2}.\end{aligned}$$

The $(d,l)=(2,2)$ cases of these recursions reduce to

$$N_{2,2}^{\tau_2}=2N_1^{\mathbb{P}^2}-N_{1,0}^{\tau_2}N_{1,1}^{\tau_2},\qquad N_{2,2}^{\tau_2}=N_{1,0}^{\tau_2}N_{1,1}^{\tau_2}.$$

These recursions for $N_{d+1,2}^{\tau_2}$ with $d\ge 2$ give

$$\begin{aligned}N_{d+1,2}^{\tau_2}=&\sum_{\substack{d_1+d_2=d+1\\ d_1,d_2\ge 1}}\left(\left(d_1d_2\binom{3d-3}{3d_1-2}-d_1^2\binom{3d-3}{3d_1-1}\right)\right.\\&\left.+\left(d_1d_2\binom{3d-3}{3d_1-2}-d_2^2\binom{3d-3}{3d_1-3}\right)\right)N_{d_1,0}^{\tau_2}N_{d_2,1}^{\tau_2},\\ N_{d+1,2}^{\tau_2}=&\sum_{\substack{d_1+d_2=d+1\\ d_1,d_2\ge 1}}\left(d_1d_2\left(\binom{3d-3}{3d_1-3}+\binom{3d-3}{3d_1-2}\right)\right.\\&\left.-d_1^2\left(\binom{3d-3}{3d_1-2}+\binom{3d-3}{3d_1-1}\right)\right)N_{d_1,0}^{\tau_2}N_{d_2,1}^{\tau_2}.\end{aligned}$$

The first equation above is obtained from [18, (10)] by interchanging d_1 and d_2 in the $(l_1,l_2)=(1,0)$ summand of the last sum, while the second equation is

Table 1. The counts $N_d^{\mathbb{P}^2}$ of complex genus 0 degree d curves in $\mathbb{P}^2$ through $3d-1$ points (the $\mathbb{C}$ line) and Welschinger invariant counts $N_{d,l}^{\tau_2}$ of real genus 0 degree d curves in $\mathbb{P}^2$ through l conjugate pairs of points and $3d-1-2l$ real points.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$\mathbb{C}$	1	1	12	620	87304	26312976	14616808192	13525751027392
$l=0$	1	1	8	240	18264	2845440	792731520	359935488000
$l=1$	1	1	6	144	9096	1209600	293758272	118173265920
$l=2$		1	4	80	4272	490368	104600448	37486448640
$l=3$			2	40	1872	188544	35670576	11463469056
$l=4$			0	16	744	67968	11579712	3367084032
$l=5$				0	248	22400	3538080	944056320
$l=6$					64	6400	995904	249999360
$l=7$					64	1536	248976	61424640
$l=8$						1024	54272	13643776
$l=9$							11776	2705408
$l=10$							−14336	499712
$l=11$								−280576

obtained from [18, (9)] via the binomial identity

$$\binom{a}{b}=\binom{a-1}{b-1}+\binom{a-1}{b} \tag{4}$$

with $a=3d-2$ and $b=3d_1-2, 3d_1-1$. Taking the difference between the two expressions for $N_{d+1,2}^{\tau_2}$ above, as suggested at the end of [18], and using $N_{1,1}^{\tau_2}=1$, we obtain

$$N_{d,0}^{\tau_2}=\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}}\left(d_1\binom{3d-3}{3d_1-2}-(d_2+1)\binom{3d-3}{3d_1-3}\right)N_{d_1,0}^{\tau_2}N_{d_2+1,1}^{\tau_2}\qquad \forall d\ge 2.$$

The three recursions above determine all numbers $N_{d,l}^{\tau_2}$ with $d\ge 2$. The low-degree numbers obtained from these recursions and shown in Table 1 agree with [12, Corollary 6] and [2, Section 7.3].

4. Quadric surfaces $\mathbb{P}^1\times\mathbb{P}^1$

Let $\tau:\mathbb{P}^1\longrightarrow\mathbb{P}^1$ be the standard conjugation on $\mathbb{P}^1$; its fixed locus is $\mathbb{RP}^1\approx S^1$. The fixed loci of the antiholomorphic involutions

$$\tau_{1,1},\tau_{1,1}':\mathbb{P}^1\times\mathbb{P}^1\longrightarrow\mathbb{P}^1\times\mathbb{P}^1,$$
$$\tau_{1,1}(z_1,z_2)=\big(\tau(z_1),\tau(z_2)\big),\qquad \tau_{1,1}'(z_1,z_2)=\big(\tau(z_2),\tau(z_1)\big),$$

are the two-torus $\mathbb{RP}^1\times\mathbb{RP}^1$ and the antidiagonal two-sphere

$$(\mathbb{P}^1\times\mathbb{P}^1)^{\tau_{1,1}'}=\big\{\big(z,\tau_1(z)\big):z\in\mathbb{P}^1\big\}\approx\mathbb{P}^1,$$

respectively. The groups $H_2^{\tau_{1,1}}(\mathbb{P}^1\times\mathbb{P}^1)=H_2(\mathbb{P}^1\times\mathbb{P}^1)$ are identified with $\mathbb{Z}\oplus\mathbb{Z}$ via the standard generators

$$L_1\equiv[\mathbb{P}^1\times\mathrm{pt}]\qquad\text{and}\qquad L_2\equiv[\mathrm{pt}\times\mathbb{P}^1].$$

The group $H_2^{\tau_{1,1}'}(\mathbb{P}^1\times\mathbb{P}^1)$ is the diagonal subgroup

$$\{d_1L_1+d_2L_2:d_1,d_2\in\mathbb{Z}^2,d_1=d_2\}\approx\mathbb{Z}\subset\mathbb{Z}\times\mathbb{Z}.$$

For $a,b\in\mathbb{Z}^{\ge0}$ with $(a,b)\neq(0,0)$, $d\in\mathbb{Z}^+$, and $l\in\mathbb{Z}^{\ge0}$, we set

$$N_{a,b}^{\mathbb{P}^1\times\mathbb{P}^1}=N_{aL_1+bL_2}^{\mathbb{P}^1\times\mathbb{P}^1},\qquad N_d^{\mathbb{P}^1\times\mathbb{P}^1}=\sum_{\substack{a+b=d\\a,b\ge0}}N_{a,b}^{\mathbb{P}^1\times\mathbb{P}^1},$$

$$N_{(a,b),l}^{\tau_{1,1}}=N_{aL_1+bL_2,l}^{\tau_{1,1}},\qquad N_{d,l}^{\tau_{1,1}'}=N_{dL_1+dL_2,l}^{\tau_{1,1}'},$$

$$\langle a,b\rangle_l=\begin{cases}1 & \text{if } a,b\in2\mathbb{Z}, l=a+b-1,\\ 0 & \text{otherwise,}\end{cases}\qquad \langle d\rangle_l=\begin{cases}1 & \text{if } l=2d-1,\\ 0 & \text{otherwise.}\end{cases}$$

By symmetry,

$$N_{a,b}^{\mathbb{P}^1\times\mathbb{P}^1}=N_{b,a}^{\mathbb{P}^1\times\mathbb{P}^1},\qquad N_{(a,b),l}^{\tau_{1,1}}=N_{(b,a),l}^{\tau_{1,1}}. \tag{5}$$

An irreducible degree $(1,b)$-curve C in $\mathbb{P}^1\times\mathbb{P}^1$ is the graph of a ratio of two degree b polynomials on $\mathbb{C}$, and thus, $\delta_E(C)=0$. Since every such rational function is determined by its values at $2b+1$ points,

$$N_{1,b}^{\mathbb{P}^1\times\mathbb{P}^1},N_{(1,b),0}^{\tau_{1,1}},\ldots,N_{(1,b),b}^{\tau_{1,1}},N_{1,0}^{\tau_{1,1}'},N_{1,1}^{\tau_{1,1}'}=1\quad\forall b\in\mathbb{Z}^{\ge0}. \tag{6}$$

Since a degree $(a,0)$-curve in $\mathbb{P}^1\times\mathbb{P}^1$ is a degree a cover of a horizontal section,

$$N_{a,0}^{\mathbb{P}^1\times\mathbb{P}^1},N_{(a,0),l}^{\tau_{1,1}}=0\quad\forall a\ge2. \tag{7}$$

The complex WDVV recursions for $\mathbb{P}^1\times\mathbb{P}^1$ can be written as

$$N_{0,0}\equiv0,\qquad N_{1,0},N_{0,1}=1,$$

$$\begin{aligned}N_{a,b}=\frac{1}{2}\sum_{\substack{a_1+a_2=a\\b_1+b_2=b\\a_1,a_2,b_1,b_2\ge0}}(a_1b_2+a_2b_1)(a_1+b_1)\bigg((a_2+b_2)\binom{2a+2b-4}{2a_1+2b_1-2}\\ -(a_1+b_1)\binom{2a+2b-4}{2a_1+2b_1-1}\bigg)N_{a_1,b_1}N_{a_2,b_2}.\end{aligned}$$

The last equation is obtained by taking H_1, H_2 in (2) to be the Poincaré dual of L_1+L_2.

4.1. The twisted involution

Let H_1, H_2 be the Poincaré dual $(L_1+L_2)/\sqrt{2}$. For $B\equiv dL_1+dL_2$ and $B'\equiv aL_1+bL_2$ with $\mathfrak{d}(B')=d'L_1+d'L_2$, we have

$$\ell_\omega(B)=4d-1,\qquad \ell_\omega(B')=2d'-1,$$

$$\langle H_i,B\rangle=\sqrt{2}d,\qquad \langle H_i,B\rangle=d'/\sqrt{2},\qquad B\cdot_X B'=dd'.$$

The two formulas of Theorem 1 with $B\equiv dL_1+dL_2$ thus give

$(\tau'_{1,1}1)$ If $d\geq 2$, $l\geq 1$, and $2d-l\geq 1$ (i.e., $k\geq 1$), then

$$\begin{aligned}N^{\tau'_{1,1}}_{d,l}&=-(-1)^d2^{l-2}\langle d\rangle_l d^2 N^{\mathbb{P}^1\times\mathbb{P}^1}_d\\&\quad-\sum_{\substack{d_0+d'=d\\d_0,d'\geq 1}}(-1)^{d'}2^{2d'-2}d_0d'^3\binom{l-1}{2d'-1}N^{\mathbb{P}^1\times\mathbb{P}^1}_{d'}N^{\tau'_{1,1}}_{d_0,l-2d'}\\&\quad+2\sum_{\substack{d_1+d_2=d\\l_1+l_2=l-1\\d_1,d_2\geq 1,l_1,l_2\geq 0}}\binom{l-1}{l_1}\left(d_1d_2\binom{4d-2l-2}{4d_1-2l_1-2}\right.\\&\quad\left.-d_1^2\binom{4d-2l-2}{4d_1-2l_1-1}\right)N^{\tau'_{1,1}}_{d_1,l_1}N^{\tau'_{1,1}}_{d_2,l_2}.\end{aligned}$$

$(\tau'_{1,1}2)$ If $d\geq 2$ and $l\geq 2$, then

$$\begin{aligned}N^{\tau'_{1,1}}_{d,l}=&\sum_{\substack{d_0+d'=d\\d_0,d'\geq 1}}(-1)^{d'}2^{2d'-2}d_0d'^2\left(d_0\binom{l-2}{2d'-2}-d'\binom{l-2}{2d'-1}\right)N^{\mathbb{P}^1\times\mathbb{P}^1}_{d'}N^{\tau'_{1,1}}_{d_0,l-2d'}\\&+2\sum_{\substack{d_1+d_2=d\\l_1+l_2=l-2\\d_1,d_2\geq 1,l_1,l_2\geq 0}}\binom{l-2}{l_1}\left(d_1d_2\binom{4d-2l-1}{4d_1-2l_1-2}\right.\\&\left.-d_1^2\binom{4d-2l-1}{4d_1-2l_1-1}\right)N^{\tau'_{1,1}}_{d_1,l_1}N^{\tau'_{1,1}}_{d_2,l_2+1}.\end{aligned}$$

Suppose that $d\geq 2$. Interchanging d_1 and d_2 in the $(l_1,l_2)=(1,0)$ summand of the last sum in the recursion $(\tau'_{1,1}1)$ for $N^{\tau'_{1,1}}_{d+1,2}$, we obtain

$$\begin{aligned}N^{\tau'_{1,1}}_{d+1,2}&=2dN^{\tau'_{1,1}}_{d,0}\\&\quad+2\sum_{\substack{d_1+d_2=d+1\\d_1,d_2\geq 1}}\left(\left(d_1d_2\binom{4d-2}{4d_1-2}-d_1^2\binom{4d-2}{4d_1-1}\right)\right.\\&\quad\left.+\left(d_1d_2\binom{4d-2}{4d_1-2}-d_2^2\binom{4d-2}{4d_1-3}\right)\right)N^{\tau'_{1,1}}_{d_1,0}N^{\tau'_{1,1}}_{d_2,1}.\end{aligned}$$

Applying the binomial identity (4) with $a=4d-1$ and $b=4d_1-2,4d_1-1$ to the recursion $(\tau'_{1,1}2)$ for $N^{\tau'_{1,1}}_{d+1,2}$, we obtain

$$\begin{aligned}N^{\tau'_{1,1}}_{d+1,2}&=-2d^2N^{\tau'_{1,1}}_{d,0}\\&\quad+2\sum_{\substack{d_1+d_2=d+1\\d_1,d_2\geq 1}}\left(d_1d_2\left(\binom{4d-2}{4d_1-3}+\binom{4d-2}{4d_1-2}\right)\right.\\&\quad\left.-d_1^2\left(\binom{4d-2}{4d_1-2}+\binom{4d-2}{4d_1-1}\right)\right)N^{\tau'_{1,1}}_{d_1,0}N^{\tau'_{1,1}}_{d_2,1}.\end{aligned}$$

Table 2. The counts $N_{d,d}^{\mathbb{P}^1\times\mathbb{P}^1}$ of complex genus 0 bidegree (d,d) curves in $\mathbb{P}^1\times\mathbb{P}^1$ through $4d-1$ points (the $\mathbb{C}$ line) and Welschinger invariant counts $N_{d,l}^{\tau_{1,1}'}$ of real genus 0 bidegree (d,d) curves in $\mathbb{P}^1\times\mathbb{P}^1$ through l conjugate pairs of points and $4d-1-2l$ real points.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$\mathbb{C}$	1	12	3510	6508640	43628131782	780252921765888	3081423619442642332
$l=0$	1	6	576	294336	493848576	2079965454336	18546841177030656
$l=1$	1	4	288	116352	160966656	576148930560	4464575005261824
$l=2$		2	128	42624	49582080	152559783936	1035147394547712
$l=3$		0	48	14208	14303232	38410813440	230362111475712
$l=4$			16	4224	3821568	9135710208	48998058983424
$l=5$			16	1152	938496	2039070720	9915262009344
$l=6$				320	215040	425871360	1901347799040
$l=7$				-256	47872	83951616	345169133568
$l=8$					10496	15949824	59646984192
$l=9$					26880	2998272	9935069184
$l=10$						630784	1624670208
$l=11$						-2637824	270151680
$l=12$							42536960
$l=13$							500240384

Taking the difference between the last two equations and using $N_{1,1}^{\tau_{1,1}'}=1$, we obtain

$$N_{d,0}^{\tau_{1,1}'}=\frac{1}{2(d-1)}\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}}\left(d_1\binom{4d-2}{4d_1-2}-(d_2+1)\binom{4d-2}{4d_1-3}\right)N_{d_1,0}^{\tau_{1,1}'}N_{d_2+1,1}^{\tau_{1,1}'}$$

$$\forall d\ge2.$$

Since $k=4d-1-2l$ is not zero in this case, the first and third recursions above determine all numbers $N_{d,l}^{\tau_{1,1}'}$ with $d\ge2$. The low-degree numbers obtained from these recursions and shown in Table 2 agree with [21, Corollary 3.18] and [14, p. 586].

4.2. The product involution

Taking $H_1,H_2\in H^2(\mathbb{P}^1\times\mathbb{P}^1)$ to be the Poincaré duals of L_1 and L_2, respectively, and $B=aL_1+bL_2$ in the first formula of Theorem 1, we obtain:

($\tau_{1,1}1a$) If $a,b\ge0$, $l\ge1$, and $(a+b)-l\ge1$ (i.e., $k\ge1$), then

$$\begin{aligned}N_{(a,b),l}^{\tau_{1,1}}=&-2^{l-3}ab\langle a,b\rangle_l N_{a/2,b/2}^{\mathbb{P}^1\times\mathbb{P}^1}\\ &-\sum_{\substack{a_0+2a'=a\\ b_0+2b'=b\\ a_0,b_0,a',b'\ge0\\ (a_0,b_0),(a',b')\neq(0,0)}}2^{2(a'+b')-1}a'b'(a_0b'+b_0a')\binom{l-1}{2(a'+b')-1}\\ &\times N_{a',b'}^{\mathbb{P}^1\times\mathbb{P}^1}N_{(a_0,b_0),l-2(a'+b')}^{\tau_{1,1}}\\ &+\sum_{\substack{a_1+a_2=a\\ b_1+b_2=b\\ a_i,b_i\ge0,(a_i,b_i)\neq(0,0)\\ l_1+l_2=l-1,l_1,l_2\ge0}}b_1\binom{l-1}{l_1}\left(a_2\binom{2(a+b)-2l-2}{2(a_1+b_1)-2l_1-2}\right.\end{aligned}$$

Table 3. The counts $N_{a,b}^{\mathbb{P}^1\times\mathbb{P}^1}$ of complex genus 0 bidegree (a,b) curves in $\mathbb{P}^1\times\mathbb{P}^1$ through $2(a+b)-1$ points (the $\mathbb{C}$ line) and Welschinger invariant counts $N_{(a,b),l}^{\tau_{1,1}}$ of real genus 0 bidegree (a,b) curves in $\mathbb{P}^1\times\mathbb{P}^1$ through l conjugate pairs of points and $2(a+b)-1-2l$ real points.

	$(2,2)$	$(2,3)$	$(3,3)$	$(2,4)$	$(3,4)$	$(4,4)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$
$\mathbb{C}$	12	96	3510	640	87544	6508640	3840	1763415	348005120	43628131782
$l=0$	8	48	1086	256	18424	819200	1280	268575	28312064	2082934630
$l=1$	6	32	606	160	9256	360896	768	125855	11406848	756290790
$l=2$	4	20	318	96	4432	152192	448	56831	4428160	265412198
$l=3$	2	12	158	56	2032	61568	256	24831	1659264	90118886
$l=4$		8	78	32	904	24064	144	10559	602496	29678982
$l=5$			46	16	408	9280	80	4415	213888	9532294
$l=6$					224	3712	48	1887	75776	3020358
$l=7$						1536		991	28160	965958
$l=8$									13056	327974
$l=9$										142758

$$-a_1\binom{2(a+b)-2l-2}{2(a_1+b_1)-2l_1-1}\Bigg)\Bigg)N_{(a_1,b_1),l_1}^{\tau_{1,1}}N_{(a_2,b_2),l_2}^{\tau_{1,1}}.$$

Taking instead $H_1=H_2$ to be the Poincaré dual of L_1, $B=aL_1+(b+1)L_2$, and $l=1$ in the first formula of Theorem 1, we obtain:

$(\tau_{1,1}1b)$ If $a\geq 2$ and $b\geq 1$, then

$$N_{(a,b),0}^{\tau_{1,1}}=\frac{1}{2(a-1)}\sum_{\substack{a_1+a_2=a\\ b_1+b_2=b+1\\ a_i,b_i\geq 1}}\left(b_1b_2\binom{2(a+b)-2}{2(a_1+b_1)-2}-b_1^2\binom{2(a+b)-2}{2(a_1+b_1)-1}\right)$$
$$\times N_{(a_1,b_1),0}^{\tau_{1,1}}N_{(a_2,b_2),0}^{\tau_{1,1}}.$$

We note that the first two lines in the first formula of Theorem 1 vanish in this case. The third line reduces to a sum as above, but over all (a_i,b_i) satisfying

$$a_1+a_2=a,\qquad b_1+b_2=b+1,\qquad a_i,b_i\geq 0,\qquad (a_i,b_i)\neq(0,0).$$

The terms with $(a_i=0,b_i\geq 2)$ and $(a_i\geq 2,b_i=0)$ vanish by the second statements in (5) and (7). It is immediate that the terms with $(a_i,b_i)=(1,0)$ also vanish. The terms with $(a_i,b_i)=(0,1)$ contribute

$$\big(b-\big(2(a+b)-2\big)\big)N_{(0,1),0}^{\tau_{1,1}}N_{(a,b),0}^{\tau_{1,1}}+(b-0)N_{(a,b),0}^{\tau_{1,1}}N_{(0,1),0}^{\tau_{1,1}}=-2(a-1)N_{(a,b),0}^{\tau_{1,1}}.$$

Moving this term to the left-hand side gives $(\tau_{1,1}1b)$.

Since $k=2(a+b)-1-2l$ is not zero in this case, these two recursions determine all numbers $N_{(a,b),l}^{\tau_{1,1}}$ with $a\geq 2$. The low-degree numbers obtained from these recursions and shown in Table 3 agree with [14, p. 586].

5. Real blowups of $\mathbb{P}^2$

For $k\in\mathbb{Z}^{\geq 0}$, we denote by $\mathbb{P}^2_k$ the blowup of $\mathbb{P}^2$ at a generic collection of k points. For $r,s\in\mathbb{Z}^{\geq 0}$, we denote by

$$\mathbb{P}^2_{r,s}\approx\mathbb{P}^2_{r+2s}$$

the blowup of $\mathbb{P}^2$ at r real points and s conjugate pairs of points in $(\mathbb{P}^2,\tau_2)$. The involution τ_2 induces an orientation-reversing involution $\tau_{r,s}$ on $\mathbb{P}^2_{r,s}$ so that the blowdown map $\mathbb{P}^2_{r,s}\longrightarrow\mathbb{P}^2$ intertwines the two involutions. The fixed locus

$(\mathbb{P}^2_{r,s})^{\tau_{r,s}}$ of $\tau_{r,s}$ is the real blowup of $\mathbb{RP}^2$ at r points. Before formulating two recursions for Welschinger invariants of $(\mathbb{P}^2_{k,s},\tau_{r,s})$ obtained from the first relation of Theorem 1 in Section 5.2, we recall a recursion of [11, Theorem 3.6] for the complex GW-invariants of $\mathbb{P}^2_k$ in Section 5.1.

5.1. The complex case

Let $[k]=\{1,\ldots,k\}$. We denote by $L\in H_2(\mathbb{P}^2_k;\mathbb{Z})$ the pullback of the line class L in $\mathbb{P}^2$. For each $i\in[k]$, let $E_i\in H_2(\mathbb{P}^2_k;\mathbb{Z})$ be the class of the exceptional divisor for the ith blowup point. We note that

$$L^2=1, \qquad E_i^2=-1, \qquad L\cdot_{\mathbb{P}^2_k}E_i=0, \qquad E_i\cdot_{\mathbb{P}^2_k}E_j=0 \quad \forall i\neq j.$$

The classes L and $E_1,\ldots,E_k$ freely generate $H_2(\mathbb{P}^2_k;\mathbb{Z})$. If a class

$$B_{(d,(c_1,\ldots,c_k))}\equiv dL-c_1E_1-\cdots-c_kE_k\in H_2(\mathbb{P}^2_k;\mathbb{Z}) \tag{8}$$

is different from all of the E_i's, contains an irreducible curve, and satisfies $\ell_\omega(B_{(d,(c_1,\ldots,c_k))})\geq 0$, then

$$\begin{gathered} d\in\mathbb{Z}^+, \qquad c_i\in\mathbb{Z}^{\geq 0} \quad \forall i\in[k],\\ \sum_{i\in I}c_i\leq d'd \quad \forall I\subset\{1,\ldots,k\}, \qquad 2|I|\leq d'(d'+3), \qquad d'\leq d,\\ \sum_{i=1}^k c_i\leq 3d-1, \qquad \sum_{i=1}^k\binom{c_i}{2}\leq\binom{d-1}{2}. \end{gathered} \tag{9}$$

The restrictions on the first line above are obtained by intersecting (8) with the homology classes

$$L, \qquad E_i, \qquad \text{and} \qquad d'L-\sum_{i\in I}E_i,$$

respectively; all of these classes contain irreducible curves. The first condition on the second line is obtained from

$$c_1(\mathbb{P}^2_k)=3L-\sum_{i=1}^k E_i.$$

The last condition in (9) follows from arithmetic genus considerations for the image of a curve in the homology class (8) under the projection to $\mathbb{P}^2$.

For $k\in\mathbb{Z}^{\geq 0}$, let

$$\mathcal{H}_k=\mathbb{Z}^+\times(\mathbb{Z}^{\geq 0})^k.$$

For an element $v\equiv(d,(c_1,\ldots,c_k))$ of $\mathcal{H}_k$, we define

$$\begin{gathered} N_v^k=N_{B_v}^{\mathbb{P}^2_k}, \qquad \ell(v)=\ell_\omega(B_v)=3d-1-\sum_{i=1}^k c_i, \qquad \check{v}=d,\\ v0=\big(d,(c_1,\ldots,c_k,0)\big), \qquad v1=\big(d,(c_1,\ldots,c_k,1)\big),\\ \mathcal{P}(v)=\big\{(v_1,v_2)\in\mathcal{H}_k\times\mathcal{H}_k: v_1+v_2=v\big\}. \end{gathered}$$

If in addition $i\in[k]$, then define

$$v-e_i=\big(d,(c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_k)\big),\qquad \langle v,e_i\rangle=c_i.$$

If $v'\equiv(d',(c_1',\ldots,c_k'))$ is another element of $\mathcal{H}_k$, let

$$\langle v,v'\rangle=B_v\cdot_{\mathbb{P}^2_k}B_{v'}=dd'-\sum_{i=1}^{k}c_ic_i'.$$

The (complex) GW-invariants N_v^k vanish unless v satisfies all conditions in (9). These numbers are preserved by the permutations of the elements of the k-tuple part of v. Furthermore,

$$N_{v0}^{k+1}=N_v^k\quad\forall v\in\mathcal{H}_k,\qquad N_{v1}^{k+1}=N_v^k\quad\forall v\in\mathcal{H}_k\text{ s.t. }\ell(v)>0,$$

$$N_{E_i}^{\mathbb{P}^2_k}=1\quad\forall i\in[k],\qquad N^0_{(d,())}=N_d^{\mathbb{P}^2}\quad\forall d\in\mathbb{Z}^+.$$

Along with these identities, the next recursion reduces all numbers N_v^k with $v\in\mathcal{H}_k$ and $k\in\mathbb{Z}^{\ge0}$ to the numbers $N_d^{\mathbb{P}^2}$.

[11, R(i)] If $\ell(v)\ge0$ and $i\in[k]$, then

$$\begin{aligned}\check{v}^2\langle v,e_i\rangle N_v^k &\\ &=\big(\check{v}^2-(\langle v,e_i\rangle-1)^2\big)N^k_{v-e_i}\\ &\quad+\sum_{(v_1,v_2)\in\mathcal{P}(v-e_i)}\langle v_1,v_2\rangle\big(\check{v}_1\check{v}_2\langle v_1,e_i\rangle\langle v_2,e_i\rangle-\check{v}_1^2\langle v_2,e_i\rangle^2\big)\binom{\ell(v)}{\ell(v_1)}N^k_{v_1}N^k_{v_2}.\end{aligned}$$

5.2. The real case

For $r,s\in\mathbb{Z}^{\ge0}$, let

$$E_i^{\mathbb{R}}\in H_2(\mathbb{P}^2_{r,s};\mathbb{Z})\quad\forall i\in[r]\qquad\text{and}\qquad E_i^+,E_i^-\in H_2(\mathbb{P}^2_{r,s};\mathbb{Z})\quad\forall i\in[s]\tag{10}$$

be the classes of the exceptional divisors corresponding to the real blowup points and to the conjugate pairs of blowup points, respectively. The subgroup $H_2^{\tau_{r,s}}(\mathbb{P}^2_{r,s})$ of $H_2(\mathbb{P}^2_{r,s};\mathbb{Z})$ is freely generated by the classes

$$L,\qquad E_1^{\mathbb{R}},\ldots,E_r^{\mathbb{R}},\qquad\text{and}\qquad E_1^{\mathbb{C}}\equiv E_1^++E_1^-,\ldots,E_s^{\mathbb{C}}\equiv E_s^++E_s^-.$$

For $r,s\in\mathbb{Z}^{\ge0}$, define

$$\mathfrak{d}:\mathcal{H}_{r+2s}\longrightarrow\mathcal{H}_{r,s}\equiv\mathbb{Z}^+\times(\mathbb{Z}^{\ge0})^r\times(\mathbb{Z}^{\ge0})^s,$$

$$\mathfrak{d}\big(d,(c_1,\ldots,c_{r+2s})\big)=\big(2d,(2c_1,\ldots,2c_r),(c_{r+1}+c_{r+2},\ldots,c_{r+2s-1}+c_{r+2s})\big).$$

For an element

$$v\equiv(d,\mathbf{a},\mathbf{b})\equiv\big(d,(a_1,\ldots,a_r),(b_1,\ldots,b_s)\big)\tag{11}$$

of $\mathcal{H}_{r,s}$ and $l\in\mathbb{Z}^{\ge0}$, we define

$$B_v=dL-\sum_{i=1}^{r}a_iE_i^{\mathbb{R}}-\sum_{i=1}^{s}b_iE_i^{\mathbb{C}},\qquad N_{v,l}^{r,s}=N^{\tau_{r,s}}_{B_v,l},$$

$$\ell(v)=\ell_\omega(B_v)3d-1-\sum_{i=1}^r a_i-2\sum_{i=1}^s b_i,$$

$$\check v=d,\qquad |v|_{\mathbb C}=\sum_{i=1}^s b_i,\qquad v+\check e=(d+1,\mathbf a,\mathbf b),$$

$$\mathcal P_{\mathbb R}(v)=\big\{(v_1,v_2)\in\mathcal H_{r,s}\times\mathcal H_{r,s}:v_1+v_2=v\big\},$$

$$\mathcal P_{\mathbb C}(v)=\big\{(v_0,v')\in\mathcal H_{r,s}\times\mathcal H_{r+2s}:v_0+\mathfrak d(v')=v\big\}.$$

In particular, $\mathfrak d(B_{v'})=B_{\mathfrak d(v')}$ for $v'\in\mathcal H_{r+2s}$. If in addition $\epsilon\in\{0,1\}$, $i\in[r]$, and $j\in[s]$, let

$$v\epsilon_{\mathbb R}=\big(d,(a_1,\ldots,a_r,\epsilon),\mathbf b\big),\qquad v\epsilon_{\mathbb C}=\big(d,\mathbf a,(b_1,\ldots,b_s,\epsilon)\big),$$

$$\langle v,e_i^{\mathbb R}\rangle=a_i,\qquad \langle v,e_j^{\mathbb C}\rangle=2b_j.$$

Define

$$\langle\cdot,\cdot\rangle:\mathcal H_{r,s}\times\mathcal H_{r+2s}\longrightarrow\mathbb Z,$$

$$\big\langle\big(d_0,(a_1,\ldots,a_r),(b_1,\ldots,b_s)\big),\big(d',(c_1,\ldots,c_{r+2s})\big)\big\rangle$$

$$=d_0d'-\sum_{i=1}^r a_ic_i-\sum_{i=1}^s b_i(c_{r+2i-1}+c_{r+2i}).$$

For an element $v'\equiv(d,(c_1,\ldots,c_{r+2s}))$ of $\mathcal H_{r+2s}$, $i\in[r]$, and $j\in[s]$, let

$$|v'|_{[r]}=d+\sum_{i=1}^r c_i,\qquad \langle v',e_i^{\mathbb R}\rangle=c_i,\qquad \langle v',e_j^{\mathbb C}\rangle=c_{r+2j-1}+c_{r+2j}.$$

For $a,b\in\mathbb Z$, let $\delta_{a,b}=1$ if $a=b$ and $\delta_{a,b}=0$ otherwise. We note that

$$2B_v^2/2\cong|v|_{\mathbb C}\mod 2\quad\forall v\in\mathfrak d(\mathcal H_{r+2s})\subset\mathcal H_{r,s},$$

$$\ell_\omega(B_{v'})+\mathfrak d(B_{v'})^2/2\cong 1+|v'|_{[r]}\mod 2\quad\forall v'\in\mathcal H_{r+2s}.$$

Welschinger invariants $N_{v,l}^{r,s}$ with $v\in\mathcal H_{r,s}$ as above vanish unless v satisfies all conditions in (9) with

$$(c_1,\ldots,c_k)=(a_1,\ldots,a_r,b_1,b_1,\ldots,b_s,b_s).$$

These numbers are preserved by the permutations of the elements of the tuples $\mathbf a$ and $\mathbf b$. Furthermore,

$$N_{v0_{\mathbb R},l}^{r+1,s},N_{v0_{\mathbb C},l}^{r,s+1}=N_{v,l}^{r,s}\quad\forall v\in\mathcal H_{r,s},l\in\mathbb Z^{\ge0},$$

$$N_{E_i^{\mathbb R},0}^{\tau_{r,s}}=1\quad\forall i\in[r],\qquad N_{E_i^{\mathbb C},0}^{\tau_{r,s}}=0\quad\forall i\in[s].$$

Let $\mathbb F=\mathbb R$ and $i\in[r]$ or $\mathbb F=\mathbb C$ and $i\in[s]$. Taking $H_1,H_2\in H^2(\mathbb P^2_{r+2s})$ to be the Poincaré duals of L and $E_i^{\mathbb F}$, respectively, $B=B_{v+\check e}$, and $l=1$ in the first formula of Theorem 1, we obtain:

($\tau_{r,s}1a$) If $\ell(v)\ge 0$ and either $\mathbb{F}=\mathbb{R}$ and $i\in[r]$ or $\mathbb{F}=\mathbb{C}$ and $i\in[s]$, then

$$\begin{aligned}
&\big(\check v-\ell(v)\big)\langle v,e_i^{\mathbb{F}}\rangle N_{v,0}^{r,s}\\
&\quad=-\frac{(-1)^{|v|_{\mathbb{C}}}\delta_{\ell(v),0}}{4}(\check v+1)\langle v,e_i^{\mathbb{F}}\rangle\sum_{\substack{v'\in\mathcal{H}_{r+2s}\\ \mathfrak{d}(v')=v+\check e}}N_{v'}^{r+2s}\\
&\qquad-\sum_{\substack{(v_0,v')\in\mathcal{P}_{\mathbb{C}}(v+\check e)\\ \ell(v')=0}}(-1)^{|v'|_{[r]}}\langle v_0,v'\rangle\check v'\langle v',e_i^{\mathbb{F}}\rangle N_{v'}^{r+2s}N_{v_0,0}^{r,s}\\
&\qquad+\sum_{\substack{(v_1,v_2)\in\mathcal{P}_{\mathbb{R}}(v+\check e)\\ v_1,v_2\neq v}}\check v_1\left(\langle v_2,e_i^{\mathbb{F}}\rangle\binom{\ell(v)}{\ell(v_1)-1}-\langle v_1,e_i^{\mathbb{F}}\rangle\binom{\ell(v)}{\ell(v_1)}\right)N_{v_1,0}^{r,s}N_{v_2,0}^{r,s}.
\end{aligned}$$

The first term on the right-hand side above is obtained directly from the first term on the right-hand side of (SWDVV1) with the summands $B'=B_{v'}$. The terms with $B_0=E_j^{\mathbb{R}},E_j^{\mathbb{C}}$ for some j or with $B'=E_j^{\mathbb{R}},E_j^{\mathbb{C}}$ for some j do not contribute to the sum on the second line of (SWDVV1), because of the conditions $\ell_\omega(B')\le l-1$ and $\ell(B)\ge 3$ in the first case and because of the factor $\langle H_1,B'\rangle$ in the second case. Thus, only the summands $(B_0,B')=(B_{v_0},B_{v'})$ with $(v_0,v')\in\mathcal{P}_{\mathbb{C}}(v+\check e)$ contribute to the sum on the second line of (SWDVV1); this yields the second line in ($\tau_{r,s}1a$) above. The terms with $B_1=E_j^{\mathbb{R}},E_j^{\mathbb{C}}$ for some j or with $B_2=E_j^{\mathbb{R}},E_j^{\mathbb{C}}$ for some j do not contribute to the sum on the third line of (SWDVV1), because of the factor $\langle H_1,B_1\rangle$ in the first case and the condition $\ell_\omega(B_1)-2l_1\le\ell_\omega(B)-2l$ in the second case. Thus, only the summands $(B_1,B_2)=(B_{v_1},B_{v_2})$ with $(v_1,v_2)\in\mathcal{P}_{\mathbb{R}}(v+\check e)$ contribute to the sum on the third line of (SWDVV1); this yields the third line in ($\tau_{r,s}1a$) above, but without the restriction $v_1,v_2\neq v$. The terms with $v_i=v$ contribute

$$\begin{aligned}
&\check v\big(0-\langle v,e_i^{\mathbb{F}}\rangle\big)N_{v,0}^{r,s}N_{(1,\mathbf{0},\mathbf{0}),0}^{r,s}+\big(\langle v,e_i^{\mathbb{F}}\rangle\ell(v)-0\big)N_{(1,\mathbf{0},\mathbf{0}),0}^{r,s}N_{v,0}^{r,s}\\
&\qquad=-\big(\check v-\ell(v)\big)\langle v,e_i^{\mathbb{F}}\rangle N_{v,0}^{r,s}.
\end{aligned}$$

Moving this term to the left-hand side gives ($\tau_{r,s}1a$).

Taking $H_1,H_2\in H^2(\mathbb{P}^2_{r+2s})$ to be the Poincaré duals of $L-E_i^{\mathbb{R}}$ with $i\in[r]$ instead, $B=B_{v-2e_i^{\mathbb{R}}}$, and $l=1$ in the first formula of Theorem 1, we obtain:

($\tau_{r,s}1b$) If $\ell(v)\ge 1$, $i\in[r]$, and $\langle v,e_i^{\mathbb{R}}\rangle\ge 2$, then

$$\begin{aligned}
\langle v,e_i^{\mathbb{R}}\rangle N_{v,0}^{r,s}&=N_{v-e_i^{\mathbb{R}},0}^{r,s}+\frac{(-1)^{|v|_{\mathbb{C}}}\delta_{\ell(v),1}}{4}\big(\check v-\langle v,e_i^{\mathbb{R}}\rangle+2\big)^2\sum_{\substack{v'\in\mathcal{H}_{r+2s}\\ \mathfrak{d}(v')=v-2e_i^{\mathbb{R}}}}N_{v'}^{r+2s}\\
&\quad+\sum_{\substack{(v_0,v')\in\mathcal{P}_{\mathbb{C}}(v-2e_i^{\mathbb{R}})\\ \ell(v')=0}}(-1)^{|v'|_{[r]}}\langle v_0,v'\rangle\big(\check v'-\langle v',e_i^{\mathbb{R}}\rangle\big)^2N_{v'}^{r+2s}N_{v_0,0}^{r,s}
\end{aligned}$$

$$-\sum_{(v_1,v_2)\in\mathcal{P}_{\mathbb{R}}(v-2e_i^{\mathbb{R}})} (\check{v}_1-\langle v_1,e_i^{\mathbb{R}}\rangle)\Bigg(\big(\check{v}_2-\langle v_2,e_i^{\mathbb{R}}\rangle\big)\binom{\ell(v)-1}{\ell(v_1)-1}$$
$$-\big(\check{v}_1-\langle v_1,e_i^{\mathbb{R}}\rangle\big)\binom{\ell(v)-1}{\ell(v_1)}\Bigg)N^{r,s}_{v_1,0}N^{r,s}_{v_2,0}.$$

The *negative* of the first term on the right-hand side of (SWDVV1) with the summands $B'=B_{v'}$ is the second term on the right-hand side above. The terms with $B_0=E_j^{\mathbb{R}},E_j^{\mathbb{C}}$ for some j do not contribute to the sum on the second line of (SWDVV1) because of the condition $\ell_\omega(B')\le l-1$ and $\ell(B)\ge 3$. The terms with $B'=E_j^{\mathbb{R}},E_j^{\mathbb{C}}$ for some j other than $B'=E_i^{\mathbb{R}}$ do not contribute to this sum either because of the factors $\langle H_i,B'\rangle$. The term $(B_0,B')=(B_v,E_i^{\mathbb{R}})$ contributes

$$(-1)^{0+2}\langle v,e_i^{\mathbb{R}}\rangle N^{\mathbb{P}^2_{r+2s}}_{E_i^{\mathbb{R}}}N^{r,s}_{v,0}=\langle v,e_i^{\mathbb{R}}\rangle N^{r,s}_{v,0};$$

we move this term to the left-hand side in $(\tau_{r,s}1b)$. The remaining contributions to the sum on the second line of (SWDVV1) arise from the summands $(B_0,B')=(B_{v_0},B_{v'})$ with $(v_0,v')\in\mathcal{P}_{\mathbb{C}}(v-2e_i^{\mathbb{R}})$; the *negative* of this contribution is the second line in $(\tau_{r,s}1b)$ above. The terms with $B_2=E_j^{\mathbb{R}},E_j^{\mathbb{C}}$ for some j do not contribute to the sum on the third line of (SWDVV1) because of the condition $\ell_\omega(B_1)-2l_1\le\ell_\omega(B)-2l$. The terms $B_1=E_j^{\mathbb{R}},E_j^{\mathbb{C}}$ for some j other than $B_1=E_i^{\mathbb{R}}$ do not contribute to this sum either because of the factor $\langle H_1,B_1\rangle$. The term $(B_1,B_2)=(E_i^{\mathbb{R}},B_{v-e_i^{\mathbb{R}}})$ contributes

$$(0-1)N^{\tau_{r,s}}_{E_i^{\mathbb{R}},0}N^{r,s}_{v-e_i^{\mathbb{R}},0}=-N^{r,s}_{v-e_i^{\mathbb{R}},0};$$

the *negative* of this term is the first term on the right-hand side of $(\tau_{r,s}1b)$. The remaining contributions to the sum on the third line of (SWDVV1) arise from the summands $(B_1,B_2)=(B_{v_1},B_{v_2})$ with $(v_1,v_2)\in\mathcal{P}_{\mathbb{R}}(v-2e_i^{\mathbb{R}})$; the *negative* of this contribution is the third line in $(\tau_{r,s}1b)$ above.

Let $v\in\mathcal{H}_{r,s}$ be as in (11) with $\ell(v)\ge 0$. If $a_i=0$ for all $i\in[r]$ and $b_j=0$ for all $j\in[s]$, then the numbers

$$N^{r,s}_{v,l}=N^{0,0}_{(d,(),()),l}=N^{\tau_2}_{d,l}$$

are computable from the recursions for Welschinger invariants of $(\mathbb{P}^2,\tau_2)$ provided by Section 3. Since

$$N^{r,s+1}_{v1_{\mathbb{C}},l}=N^{r,s}_{v,l+1}\quad\forall v\in\mathcal{H}_{r,s},l\in\mathbb{Z}^{\ge 0},$$

it suffices in general to determine Welschinger invariants $N^{r,s}_{v,0}$ with real point insertions only. Since

$$N^{r,s}_{v'1_{\mathbb{R}},l}=N^{r-1,s}_{v',l}\quad\forall v'\in\mathcal{H}_{r-1,s},l\in\mathbb{Z}^{\ge 0}\text{ with }\ell(v')>2l,$$

we can also assume that $a_i\ge 2$ for all $i\in[r]$.

Thus, it is sufficient to determine Welschinger invariants $N^{r,s}_{v,0}$ with v as in (11) so that $a_i\ge 2$ for all $i\in[r]$. If $r\ge 1$, then $N^{r,s}_{v,0}$ is reduced to the numbers $N^{r,s}_{v',0}$ with $|v'|<|v|$ by the relation $(\tau_{r,s}1b)$ if $\ell(v)>0$ and by the $\mathbb{F}=\mathbb{R}$ case of $(\tau_{r,s}1a)$ if $\ell(v)=0$. If $r=0$, then $\check{v}\ne\ell(v)$. If in addition $b_j\ne 0$ for some $j\in[s]$,

Table 4. The counts $N^1_{(d,(a))}$ of complex genus 0 degree $dL-aE$ curves in $\mathbb{P}^2_1$ through $3d-1-a$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{1,0}_{(d,(a),()),l}$ of real genus 0 degree $dL-aE$ curves in $\mathbb{P}^2_{1,0}$ through l conjugate pairs of points and $3d-1-a-2l$ real points.

d,a	3,2	4,2	4,3	5,2	5,3	5,4	6,2	6,3	6,4	6,5	7,2
$\mathbb{C}$	1	96	1	18132	640	1	6506400	401172	3840	1	4059366000
$l=0$	1	48	1	4584	256	1	817920	71360	1280	1	249486624
$l=1$	1	32	1	2412	160	1	359616	34512	768	1	94578912
$l=2$	1	20	1	1200	96	1	150912	16000	448	1	34464936
$l=3$	1	12	1	564	56	1	60288	7136	256	1	12045432
$l=4$		8	1	248	32	1	22784	3072	144	1	4020816
$l=5$				92	16	1	8000	1264	80	1	1271088
$l=6$				0			2432	448	48	1	373464
$l=7$							256	0			97352
$l=8$											21248
$l=9$											13056

Table 5. The counts $N^2_{(d,\mathbf{a})}$ of complex genus 0 degree $B_{(d,\mathbf{a})}$ curves in $\mathbb{P}^2_2$ through $3d-1-|\mathbf{a}|$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{2,0}_{(d,\mathbf{a},()),l}$ of real genus 0 degree $B_{(d,\mathbf{a},())}$ curves in $\mathbb{P}^2_{2,0}$ through l conjugate pairs of points and $3d-1-|\mathbf{a}|-2l$ real points.

$d,\mathbf{a}$	$4,(\mathbf{2^2})$	$5,(\mathbf{2^2})$	$5,(3,2)$	$6,(\mathbf{2^2})$	$6,(3,2)$	$6,(\mathbf{3^2})$	$6,(4,2)$	$7,(\mathbf{2^2})$
$\mathbb{C}$	12	3510	96	1558272	87544	3510	640	1108152240
$l=0$	8	1086	48	229152	18424	1086	256	77453856
$l=1$	6	606	32	104352	9256	606	160	30056988
$l=2$	4	318	20	45312	4432	318	96	11209752
$l=3$	2	158	12	18752	2032	158	56	4012308
$l=4$		78	8	7392	904	78	32	1374864
$l=5$		46		2784	408	46	16	448812
$l=6$				1088	224			138056
$l=7$								38052
$l=8$								4096

Table 6. The counts $N^2_{(d,(b,b))}$ of complex genus 0 degree $dL-b(E_1+E_2)$ curves in $\mathbb{P}^2_2$ through $3d-1-2b$ points (the $\mathbb{C}$ line) and Welschinger invariant counts $N^{0,1}_{(d,(),(b)),l}$ of real genus 0 degree $dL-bE^{\mathbb{C}}_1$ curves in $\mathbb{P}^2_{0,1}$ through l conjugate pairs of points and $3d-1-2b-2l$ real points.

d,b	4,2	5,2	6,2	6,3	7,2	7,3	8,2	8,3
$\mathbb{C}$	12	3510	1558272	3510	1108152240	6508640	1219053648960	12330654896
$l=0$	6	576	88992	576	22823424	294336	9282332160	166440960
$l=1$	4	288	36864	288	8162688	116352	2933701632	55692288
$l=2$	2	128	14208	128	2769408	42624	888970752	17639424
$l=3$	0	48	4992	48	882432	14208	256790016	5240832
$l=4$		16	1536	16	259200	4224	70027008	1441536
$l=5$		16	416	16	68128	1152	17742336	361728
$l=6$			288		15936	320	4084992	83584
$l=7$					3616	-256	848384	18688
$l=8$					-4096		163840	-8192
$l=9$							-86016	

then the $(\mathbb{F},i)=(\mathbb{C},j)$ case of $(\tau_{r,s}1a)$ expresses $N^{r,s}_{v,0}$ in terms of the numbers $N^{r,s}_{v',0}$ with $|v'|<|v|$.

Welschinger invariants $N^{r,s}_{v,l}$ of $(\mathbb{P}^2_{r,s},\tau_{r,s})$ obtained from the above recursions and shown in Tables 4–18 agree with [3, Table 4], [4, Corollary 3.2], [14, pp. 585–586], and [13, Example 17]. Other numbers obtained from these recursions agree with [3, Tables 6 and 9] and [15, Section 2.3].

Table 7. The counts $N^3_{(d,\mathbf{a})}$ of complex genus 0 degree $B_{(d,\mathbf{a})}$ curves in $\mathbb{P}^2_3$ through $\ell(d,\mathbf{a})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{3,0}_{(d,\mathbf{a},()),l}$ of real genus 0 degree $B_{(d,\mathbf{a},())}$ curves in $\mathbb{P}^2_{3,0}$ through l conjugate pairs of points and $\ell(d,\mathbf{a})-2l$ real points.

$d,\mathbf{a}$	$5,(\mathbf{2}^3)$	$5,(3,\mathbf{2}^2)$	$6,(\mathbf{2}^3)$	$6,(3,\mathbf{2}^2)$	$6,(\mathbf{3}^2,2)$	$6,(\mathbf{3}^3)$	$6,(4,\mathbf{2}^2)$	$7,(\mathbf{2}^3)$	$7,(3,\mathbf{2}^2)$	$7,(\mathbf{3}^2,2)$
$\mathbb{C}$	620	12	359640	18132	620	12	96	296849546	23133696	1558272
$l=0$	240	8	62400	4584	240	8	48	23698434	2481632	229152
$l=1$	144	6	29520	2412	144	6	32	9423618	1052448	104352
$l=2$	80	4	13280	1200	80	4	20	3598722	428032	45312
$l=3$	40	2	5680	564	40	2	12	1318722	167040	18752
$l=4$	16		2304	248	16	0	8	463026	62624	7392
$l=5$			848	92				155378	22624	2784
$l=6$								50002	8128	1088
$l=7$								16978		

Table 8. The counts $N^3_{(d,\mathbf{abb})}$ of complex genus 0 degree $B_{(d,\mathbf{abb})}$ curves in $\mathbb{P}^2_3$ through $\ell(d,\mathbf{abb})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{1,1}_{(d,\mathbf{a},\mathbf{b}),l}$ of real genus 0 degree $B_{(d,\mathbf{a},\mathbf{b})}$ curves in $\mathbb{P}^2_{1,1}$ through l conjugate pairs of points and $\ell(d,\mathbf{abb})-2l$ real points.

$d,\mathbf{ab}$	$4,(2),(2)$	$5,(2),(2)$	$5,(3),(2)$	$6,(2),(2)$	$6,(3),(2)$	$6,(2),(3)$	$6,(3),(3)$	$6,(4),(2)$	$7,(2),(2)$
$\mathbb{C}$	1	620	12	359640	18132	620	12	96	296849546
$l=0$	1	144	8	26064	2412	144	8	48	7330368
$l=1$	1	80	6	11328	1200	80	6	32	2696640
$l=2$	1	40	4	4608	564	40	4	20	943488
$l=3$		16	2	1728	248	16	2	12	311616
$l=4$		0		560	92	0	0	8	95536
$l=5$				64	0				26096
$l=6$									6160
$l=7$									3856

Table 9. The counts $N^4_{(d,\mathbf{a})}$ of complex genus 0 degree $B_{(d,\mathbf{a})}$ curves in $\mathbb{P}^2_4$ through $\ell(d,\mathbf{a})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{4,0}_{(d,\mathbf{a},()),l}$ of real genus 0 degree $B_{(d,\mathbf{a})}$ curves in $\mathbb{P}^2_{4,0}$ through l conjugate pairs of points and $\ell(d,\mathbf{a})-2l$ real points.

$d,\mathbf{a}$	$5,(\mathbf{2}^4)$	$6,(\mathbf{2}^4)$	$6,(3,\mathbf{2}^3)$	$6,(\mathbf{3}^2,\mathbf{2}^2)$	$6,(4,\mathbf{2}^3)$	$7,(\mathbf{2}^4)$	$7,(3,\mathbf{2}^3)$	$7,(\mathbf{3}^2,\mathbf{2}^2)$	$7,(\mathbf{3}^3,2)$	$7,(\mathbf{3}^4)$
$\mathbb{C}$	96	79416	3510	96	12	77866800	5739856	359640	18132	640
$l=0$	48	16440	1086	48	8	7137408	714592	62400	4584	256
$l=1$	32	8120	606	32	6	2912448	312208	29520	2412	160
$l=2$	20	3800	318	20	4	1140000	130496	13280	1200	96
$l=3$	12	1688	158	12	2	427848	52208	5680	564	56
$l=4$		728	78			153824	20000	2304	248	32
$l=5$						52688	7248	848		
$l=6$						16512				

Table 10. The counts $N^4_{(d,\mathbf{abb})}$ of complex genus 0 degree $B_{(d,\mathbf{abb})}$ curves in $\mathbb{P}^2_4$ through $\ell(d,\mathbf{abb})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{2,1}_{(d,\mathbf{a},\mathbf{b}),l}$ of real genus 0 degree $B_{(d,\mathbf{a},\mathbf{b})}$ curves in $\mathbb{P}^2_{2,1}$ through l conjugate pairs of points and $\ell(d,\mathbf{abb})-2l$ real points.

$d,\mathbf{a},\mathbf{b}$	$5,(\mathbf{2}^2),(2)$	$5,(3,2),(2)$	$6,(\mathbf{2}^2),(2)$	$6,(3,2),(2)$	$6,(\mathbf{3}^2),(2)$	$6,(\mathbf{2}^2),(3)$	$6,(3,2),(3)$	$7,(\mathbf{2}^2),(2)$
$\mathbb{C}$	96	1	79416	3510	96	96	1	77866800
$l=0$	32	1	7368	606	32	32	1	2316864
$l=1$	20	1	3360	318	20	20	1	876576
$l=2$	12	1	1440	158	12	12	1	315936
$l=3$	8		584	78	8	8	1	108040
$l=4$			248	46				34720
$l=5$								9968
$l=6$								1152

Table 11. The counts $N^4_{(d,\mathbf{bb})}$ of complex genus 0 degree $B_{(d,\mathbf{bb})}$ curves in $\mathbb{P}^2_4$ through $\ell(d,\mathbf{bb})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{0,2}_{(d,(),\mathbf{b}),l}$ of real genus 0 degree $B_{(d,\mathbf{bb})}$ curves in $\mathbb{P}^2_{0,2}$ through l conjugate pairs of points and $\ell(d,\mathbf{bb})-2l$ real points.

$d,\mathbf{b}$	$5,(\mathbf{2}^2)$	$6,(\mathbf{2}^2)$	$6,(3,2)$	$7,(\mathbf{2}^2)$	$7,(3,2)$	$7,(\mathbf{3}^2)$	$8,(\mathbf{2}^2)$	$8,(3,2)$	$8,(\mathbf{3}^2)$
$\mathbb{C}$	96	79416	96	77866800	359640	640	105128477280	939726048	5739856
$l=0$	16	2640	16	625728	8352	64	227372544	4232448	65920
$l=1$	8	1024	8	210240	3072	32	67970688	1322112	22528
$l=2$	4	352	4	65184	1024	16	19175808	383424	7168
$l=3$	4	112	4	18248	320	8	5035008	102208	2176
$l=4$		80		4640	96	0	1208320	25344	640
$l=5$				1104	-64		264192	5984	-128
$l=6$				-1152			53376	-2304	
$l=7$							-25984		

Table 12. The counts $N^5_{(d,\mathbf{a})}$ of complex genus 0 degree $B_{(d,\mathbf{a})}$ curves in $\mathbb{P}^2_5$ through $\ell(d,\mathbf{a})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{5,0}_{(d,\mathbf{a},()),l}$ of real genus 0 degree $B_{(d,\mathbf{a})}$ curves in $\mathbb{P}^2_{5,0}$ through l conjugate pairs of points and $\ell(d,\mathbf{a})-2l$ real points.

$d,\mathbf{a}$	$5,(\mathbf{2}^5)$	$6,(\mathbf{2}^5)$	$6,(3,\mathbf{2}^4)$	$6,(\mathbf{3}^2,\mathbf{2}^3)$	$7,(\mathbf{2}^5)$	$7,(3,\mathbf{2}^4)$	$7,(\mathbf{3}^2,\mathbf{2}^3)$	$7,(\mathbf{3}^3,\mathbf{2}^2)$	$7,(\mathbf{3}^4,2)$
$\mathbb{C}$	12	16608	620	12	19948176	1380648	79416	3510	96
$l=0$	8	4160	240	8	2112480	201192	16440	1086	48
$l=1$	6	2160	144	6	886224	90856	8120	606	32
$l=2$	4	1056	80	4	356076	39144	3800	318	20
$l=3$		480	40		137012	16104	1688	158	12
$l=4$					50568	6376	728		
$l=5$					18088				

Table 13. The counts $N^5_{(d,\mathbf{abb})}$ of complex genus 0 degree $B_{(d,\mathbf{abb})}$ curves in $\mathbb{P}^2_5$ through $\ell(d,\mathbf{abb})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{3,1}_{(d,\mathbf{a},\mathbf{b}),l}$ of real genus 0 degree $B_{(d,\mathbf{a},\mathbf{b})}$ curves in $\mathbb{P}^2_{3,1}$ through l conjugate pairs of points and $\ell(d,\mathbf{abb})-2l$ real points.

$d,\mathbf{a},\mathbf{b}$	$5,(\mathbf{2}^3),(2)$	$6,(\mathbf{2}^3),(2)$	$6,(3,\mathbf{2}^2),(2)$	$6,(\mathbf{3}^2,2),(2)$	$6,(\mathbf{2}^3),(3)$	$7,(\mathbf{2}^3),(2)$
$\mathbb{C}$	12	16608	620	12	12	19948176
$l=0$	6	2004	144	6	6	720144
$l=1$	4	960	80	4	4	280320
$l=2$	2	428	40	2	2	103948
$l=3$		168	16			36660
$l=4$						12376
$l=5$						4408

Table 14. The counts $N^5_{(d,\mathbf{abb})}$ of complex genus 0 degree $B_{(d,\mathbf{abb})}$ curves in $\mathbb{P}^2_5$ through $\ell(d,\mathbf{abb})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{1,2}_{(d,\mathbf{a},\mathbf{b}),l}$ of real genus 0 degree $B_{(d,\mathbf{a},\mathbf{b})}$ curves in $\mathbb{P}^2_{1,2}$ through l conjugate pairs of points and $\ell(d,\mathbf{abb})-2l$ real points.

$d,\mathbf{a},\mathbf{b}$	$5,(2),(\mathbf{2}^2)$	$6,(2),(\mathbf{2}^2)$	$6,(3),(\mathbf{2}^2)$	$6,(2),(3,\mathbf{2})$	$6,(4),(\mathbf{2}^2)$	$7,(2),(\mathbf{2}^2)$
$\mathbb{C}$	12	16608	620	12	1	19948176
$l=0$	4	808	80	4	1	207744
$l=1$	2	336	40	2	1	72528
$l=2$	0	120	16	0	1	23468
$l=3$		16	0			6804
$l=4$						1768
$l=5$						1128

Table 15. The counts $N^6_{(d,\mathbf{a})}$ of complex genus 0 degree $B_{(d,\mathbf{a})}$ curves in $\mathbb{P}^2_6$ through $\ell(d,\mathbf{a})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{6,0}_{(d,\mathbf{a},()),l}$ of real genus 0 degree $B_{(d,\mathbf{a},())}$ curves in $\mathbb{P}^2_{6,0}$ through l conjugate pairs of points and $\ell(d,\mathbf{a})-2l$ real points.

$d,\mathbf{a}$	$6,(\mathbf{2}^6)$	$6,(3,\mathbf{2}^5)$	$7,(\mathbf{2}^6)$	$7,(3,\mathbf{2}^5)$	$7,(\mathbf{3}^2,\mathbf{2}^4)$	$7,(\mathbf{3}^3,\mathbf{2}^3)$	$7,(\mathbf{3}^4,\mathbf{2}^2)$	$7,(4,\mathbf{2}^5)$	$7,(4,3,\mathbf{2}^4)$
$\mathbb{C}$	3240	96	4974460	320160	16608	620	12	3510	96
$l=0$	1000	48	613128	55168	4160	240	8	1086	48
$l=1$	552	32	265074	25856	2160	144	6	606	32
$l=2$	288	20	109532	11520	1056	80	4	318	20
$l=3$			43222	4864	480			158	
$l=4$			16240						

Table 16. The counts $N^6_{(d,\mathbf{abb})}$ of complex genus 0 degree $B_{(d,\mathbf{abb})}$ curves in $\mathbb{P}^2_6$ through $\ell(d,\mathbf{abb})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{4,1}_{(d,\mathbf{a},\mathbf{b}),l}$ of real genus 0 degree $B_{(d,\mathbf{a},\mathbf{b})}$ curves in $\mathbb{P}^2_{4,1}$ through l conjugate pairs of points and $\ell(d,\mathbf{abb})-2l$ real points.

$d,\mathbf{a},\mathbf{b}$	$6,(\mathbf{2}^4),(2)$	$6,(3,\mathbf{2}^3),(2)$	$7,(\mathbf{2}^4),(2)$	$7,(3,\mathbf{2}^3),(2)$	$7,(\mathbf{3}^2,\mathbf{2}^2),(2)$	$7,(\mathbf{2}^4),(3)$	$7,(\mathbf{3}^3,2),(2)$
$\mathbb{C}$	3240	96	4974460	320160	16608	16608	620
$l=0$	522	32	219912	22768	2004	2004	144
$l=1$	266	20	88186	9904	960	960	80
$l=2$	130	12	33644	4080	428	428	40
$l=3$			12142	1568	168	168	
$l=4$			3984				

Table 17. The counts $N^6_{(d,\mathbf{abb})}$ of complex genus 0 degree $B_{(d,\mathbf{abb})}$ curves in $\mathbb{P}^2_6$ through $\ell(d,\mathbf{abb})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{2,2}_{(d,\mathbf{a},\mathbf{b}),l}$ of real genus 0 degree $B_{(d,\mathbf{a},\mathbf{b})}$ curves in $\mathbb{P}^2_{2,2}$ through l conjugate pairs of points and $\ell(d,\mathbf{abb})-2l$ real points.

$d,\mathbf{a},\mathbf{b}$	$6,(\mathbf{2}^2),(\mathbf{2}^2)$	$6,(3,2),(\mathbf{2}^2)$	$7,(\mathbf{2}^2),(\mathbf{2}^2)$	$7,(3,2),(\mathbf{2}^2)$	$7,(\mathbf{3}^2),(\mathbf{2}^2)$	$7,(\mathbf{2}^2),(3,2)$	$7,(3,2),(3,2)$
$\mathbb{C}$	3240	96	4974460	320160	16608	16608	620
$l=0$	236	20	67608	8320	864	808	80
$l=1$	108	12	24530	3344	392	336	40
$l=2$	52	8	8332	1280	176	120	16
$l=3$			2518	432	72	16	
$l=4$			320				

Table 18. The counts $N^6_{(d,\mathbf{bb})}$ of complex genus 0 degree $B_{(d,\mathbf{bb})}$ curves in $\mathbb{P}^2_6$ through $\ell(d,\mathbf{bb})$ points (the $\mathbb{C}$ line) and Welschinger invariants $N^{0,3}_{(d,(),\mathbf{b}),l}$ of real genus 0 degree $B_{(d,(),\mathbf{b})}$ curves in $\mathbb{P}^2_{0,3}$ through l conjugate pairs of points and $\ell(d,\mathbf{bb})-2l$ real points.

$d,\mathbf{b}$	$5,(\mathbf{2}^3)$	$6,(\mathbf{2}^3)$	$6,(3,\mathbf{2}^2)$	$7,(\mathbf{2}^3)$	$7,(3,\mathbf{2}^2)$	$7,(\mathbf{3}^2,2)$
$\mathbb{C}$	1	3240	1	4974460	16608	12
$l=0$	1	78	1	15864	244	4
$l=1$	1	30	1	4794	88	2
$l=2$		22		1340	28	0
$l=3$				334	-16	
$l=4$				-320		

6. WDVV-type relations for real symplectic sixfolds

We now take (X,ω,ϕ) to be a compact, real symplectic sixfold with connected, orientable fixed locus X^ϕ. An *automorphism* of (X,ω,ϕ) is a diffeomorphism ψ of X such that

$$\psi^*\omega=\omega \qquad \text{and} \qquad \psi\circ\phi=\phi\circ\psi.$$

We call such an automorphism an *averager* if ψ is an involution which acts trivially on $H_2^\phi(X)$ and restricts to an orientation-reversing diffeomorphism of X^ϕ. As explained in [6, Section 2.5], these conditions imply that the natural homomorphisms

$$(12)\qquad \begin{aligned} \iota_*: H_2(X-X^\phi;\mathbb{Z}) &\longrightarrow H_2(X;\mathbb{Z}) \qquad \text{and} \\ r: H^4(X,X^\phi;\mathbb{R}) &\longrightarrow H^4(X;\mathbb{R}) \end{aligned}$$

restrict to isomorphisms

$$\iota_*^\psi:\big\{B\in H_2(X-\check{X}^\phi;\mathbb{Z}):\phi_*B=-B,\psi_*B=B\big\}\xrightarrow{\approx} H_2^\phi(X) \qquad \text{and}$$

$$r_\psi:\big\{\mu\in H^4(X,\check{X}^\phi;\mathbb{R}):\phi^*\mu=\mu,\psi^*\mu=\mu\big\}\xrightarrow{\approx} H^4(X)_+^\phi,$$

respectively. For a homogeneous element μ of $H^{2*}(X)$, let

$$(13)\qquad \widetilde{\mu}=\begin{cases} r_\psi^{-1}(\mu) & \text{if } \mu\in H^4(X)_+^\phi, \\ 0 & \text{if } \mu\in H^4(X)_-^\phi, \\ \mu & \text{if } \mu\notin H^4(X). \end{cases}$$

Since X^ϕ is an orientable, three-dimensional manifold, its tangent bundle is trivializable and thus admits a Spin-structure $\mathfrak{s}$ for any choice of orientation $\mathfrak{o}$ in X^ϕ. We call such a pair $\mathfrak{os} \equiv (\mathfrak{o}, \mathfrak{s})$ an *OSpin-structure* on X^ϕ. There is an associated OSpin-structure $\overline{\mathfrak{os}} \equiv (\overline{\mathfrak{o}}, \overline{\mathfrak{s}})$ for the opposite orientation $\overline{\mathfrak{o}}$ of $\mathfrak{o}$ on X^ϕ (see the SpinPin 3 property in [7, Section 1.2]). If $\mathfrak{os}'$ is another OSpin-structure on X^ϕ and $b \in H_1(X^\phi; \mathbb{Z}_2)$, then we write $\mathfrak{os}|_b = \mathfrak{os}'|_b$ if $\alpha^*\mathfrak{os} = \alpha^*\mathfrak{os}'$ for any loop $\alpha : S^1 \longrightarrow X^\phi$ representing b. The group $H^1(X^\phi; \mathbb{Z}_2)$ acts on the set of OSpin-structures on X^ϕ associated with a fixed orientation $\mathfrak{o}$ freely and transitively (see the SpinPin 2 property in [7, Section 1.2]). If $\eta \in H^1(X^\phi; \mathbb{Z}_2)$ and $\mathfrak{os}' = \eta \cdot \mathfrak{os}$, then $\mathfrak{os}|_b = \mathfrak{os}'|_b$ if and only if $\langle \eta, b \rangle = 0$. For $B \in H_2(X; \mathbb{Z})$ and a tuple $(\mu_1, \ldots, \mu_l)$ of homogeneous elements of $H^{2*}(X)$ and $H^{2*}(X, X^\phi)$, let

$$k = k_B(\mu_1, \ldots, \mu_l) \equiv \frac{1}{2}\Big(\ell_\omega(B) + 2l - \sum_{i=1}^{l} \deg \mu_i\Big). \tag{14}$$

Under certain conditions on B and $\mu_1, \ldots, \mu_l$, an OSpin-structure $\mathfrak{os}$ on X^ϕ determines an *open GW-invariant*

$$\langle \mu_1, \ldots, \mu_l \rangle_B^{\phi, \mathfrak{os}} \in \mathbb{R} \tag{15}$$

of (X, ω, ϕ) enumerating real irreducible degree B J-holomorphic curves $C \subset X$ that meet X^ϕ and pass through generic representatives for the Poincaré duals of $\mu_1, \ldots, \mu_l$ and through k points in X^ϕ. These conditions are recalled in the next paragraph. If such curves exist, then $\ell_\omega(B)$ is even and thus $k \in \mathbb{Z}$. The number (15) is defined to be 0 if $k < 0$.

Invariant signed counts (15) were first defined in [20] under the assumptions that

$$\ell_\omega(B) > 0, \qquad \mu_i \in H^2(X; \mathbb{Z}) \cup H^6(X; \mathbb{Z}) \quad \forall i; \tag{16}$$

that is, each μ_i represents a Poincaré dual of a "complex" hypersurface or a point, and either

$$k > 0 \qquad \text{or} \qquad B \notin \mathfrak{d}\big(H_2(X; \mathbb{Z})\big) \subset H_2^\phi(X). \tag{17}$$

The interpretation of these counts in terms of J-holomorphic maps from disks in [17] dropped the first restriction in (16) and allowed insertions $\mu_i \in H^4(X, X^\phi; \mathbb{R})$.

As shown in [6], the restriction (17) can be dropped if (X, ω, ϕ) admits an averager ψ. With the notation as in (13), we then define

$$\langle \mu_1, \ldots, \mu_l \rangle_{B;\psi}^{\phi, \mathfrak{os}} = \langle \widetilde{\mu}_1, \ldots, \widetilde{\mu}_l \rangle_B^{\phi, \mathfrak{os}} \tag{18}$$

for all $\mu_1, \ldots, \mu_l \in H^{2*}(X; \mathbb{R})$. These numbers satisfy the usual divisor relation; that is,

$$\langle \mu, \mu_1, \ldots, \mu_l \rangle_{B;\psi}^{\phi, \mathfrak{os}} = \langle \mu, B \rangle \langle \mu_1, \ldots, \mu_l \rangle_{B;\psi}^{\phi, \mathfrak{os}} \quad \forall \mu \in H^2(X; \mathbb{R}). \tag{19}$$

In general, the numbers (18) depend on the ψ-invariant subspace of $H^4(X, X^\phi; \mathbb{R})$ or equivalently on the ψ-invariant subspace of $H_2(X - X^\phi; \mathbb{R})$. However, they do not depend on the choice of ψ which acts trivially on $H^2(X; \mathbb{R})$ if the subspace of $H^4(X;\mathbb{R})$ spanned by the cup products of the elements of $H^2(X;\mathbb{R})$ contains $H^4(X)_+^\phi$. This is the case in all three examples of Sections 7 and 8.

We denote by

$$\partial_{X^\phi;\mathbb{Z}_2}: H_2(X,X^\phi;\mathbb{Z}) \longrightarrow H_1(X^\phi;\mathbb{Z}) \longrightarrow H_1(X^\phi;\mathbb{Z}_2)$$

the composition of the boundary homomorphism of the relative exact sequence for the pair (X,X^ϕ) with the mod 2 reduction of the coefficients. Let

$$\mathfrak{d}_{X^\phi}: H_2(X,X^\phi;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})$$

be the homomorphism obtained by gluing each continuous map $f:(\Sigma,\partial\Sigma)\longrightarrow(X,X^\phi)$ from a bordered surface with the map $\phi\circ f$ from Σ with the opposite orientation along $\partial\Sigma$. By Proposition 1.3 and Remark 1.4 in [6], the numbers (18) satisfy the following vanishing property.

THEOREM 2

Suppose that (X,ω,ϕ) is a compact real symplectic sixfold with connected fixed locus X^ϕ, that ψ is an averager for (X,ω,ϕ), that $\mathfrak{os}$ is an OSpin-structure on X^ϕ, that $B\in H_2^\phi(X)$, that $\mu_1,\ldots,\mu_l\in H^{2*}(X)$, and that $k\in\mathbb{Z}$ is as in (14). The numbers (18) vanish if any of the following conditions hold:

(a) $\mu_i\in H^2(X)_+^\phi\oplus H^4(X)_-^\phi$ *for some* i,
(b) $k\in 2\mathbb{Z}$ *and* $\partial_{X^\phi;\mathbb{Z}_2}(B')=0$ *for every* $B'\in\mathfrak{d}_{X^\phi}^{-1}(B)$,
(c) $k\in 2\mathbb{Z}$ *and* $(\psi^*\mathfrak{os})|_b=\overline{\mathfrak{os}}|_b$ *for every* $b\in\partial_{X^\phi;\mathbb{Z}_2}(\mathfrak{d}_{X^\phi}^{-1}(B))$,
(d) $k\notin 2\mathbb{Z}$ *and* $(\psi^*\mathfrak{os})|_b\neq\overline{\mathfrak{os}}|_b$ *for every* $b\in\partial_{X^\phi;\mathbb{Z}_2}(\mathfrak{d}_{X^\phi}^{-1}(B))$.

Choose a basis $\mu_1^\star,\ldots,\mu_N^\star$ for

$$H^0(X)\oplus H^2(X)_-^\phi\oplus H^4(X)_+^\phi\oplus H^6(X) \tag{20}$$

consisting of homogeneous elements. Let $(g_{ij})_{i,j}$ be the $N\times N$-matrix given by

$$g_{ij}=\langle\mu_i^\star\mu_j^\star,[X]\rangle,$$

and let $(g^{ij})_{ij}$ be its inverse. For $\mu_1,\ldots,\mu_l\in H^{2*}(X;\mathbb{R})$, define

$$\langle\mu_1,\ldots,\mu_l\rangle_{0;\psi}^{\phi,\mathfrak{os}}=\begin{cases}\langle\mu_1,[\mathrm{pt}]\rangle & \text{if } l=1,\\ 0 & \text{otherwise.}\end{cases}$$

We denote by $\langle\mu_1,\ldots,\mu_l\rangle_B^X\in\mathbb{R}$ the (complex) GW-invariant of (X,ω) enumerating rational degree B J-holomorphic curves $C\subset X$ through generic representatives of the Poincaré duals of $\mu_1,\ldots,\mu_l$. If in addition $I\subset\{1,2,\ldots,l\}$, then let μ_I be the $|I|$-tuple consisting of the entries of $\mu\equiv(\mu_1,\ldots,\mu_l)$ indexed by I. For $i,j=2,\ldots,l$, we define

$$2\mathcal{P}(l)=\big\{(I,J):\{1,2,\ldots,l\}=I\sqcup J,1\in I\big\},\qquad \mathcal{P}_{i;}(l)=\big\{(I,J)\in\mathcal{P}(l):i\in I\big\},$$

$$\mathcal{P}_{;j}(l)=\big\{(I,J)\in\mathcal{P}(l):j\in J\big\},\qquad \mathcal{P}_{i;j}(l)=\mathcal{P}_{i;}(l)\cap\mathcal{P}_{;j}(l).$$

Let $[N]=\{1,2,\ldots,N\}$.

Theorem 3 below is precisely equivalent to [6, Theorem 1.5]. The latter is a pair of differential equations satisfied by generating functions for the numbers (18); the former restates it in terms of the coefficients of these power series. In fact, the relations proved in [6, Section 4.2] are for the coefficients of these power series and establish the relations of Theorem 3.

THEOREM 3

Let (X,ω,ϕ), ψ, $\mathfrak{os}$, B, *and* $\mu\equiv(\mu_1,\ldots,\mu_l)$ *be as in Theorem 2 with*

$$k\equiv\frac{1}{2}\Big(\ell_\omega(B)+2l-\sum_{i=1}^{l}\deg\mu_i\Big)-1\geq 0.$$

($\mathbb{R}$WDVV1) *If* $l\geq 2$ *and* $k\geq 1$, *then*

$$\begin{aligned}
&\sum_{\substack{B_0,B'\in H_2(X;\mathbb{Z})\\ B_0+\mathfrak{d}(B')=B\\ (I,J)\in\mathcal{P}_{2;}(l)}} 2^{|I|-2}\sum_{i,j\in[N]}\langle\mu_I,\mu_i^{\bigstar}\rangle_{B'}^{X}g^{ij}\langle\mu_j^{\bigstar},\mu_J\rangle_{B_0;\psi}^{\phi,\mathfrak{os}}\\
&\quad+\sum_{\substack{B_1,B_2\in H_2^{\phi}(X)\\ B_1+B_2=B\\ (I,J)\in\mathcal{P}_{2;}(l)}}\binom{k-1}{k_{B_1}(\mu_I)}\langle\mu_I\rangle_{B_1;\psi}^{\phi,\mathfrak{os}}\langle\mu_J\rangle_{B_2;\psi}^{\phi,\mathfrak{os}}\\
&\quad=\sum_{\substack{B_1,B_2\in H_2^{\phi}(X)\\ B_1+B_2=B\\ (I,J)\in\mathcal{P}_{;2}(l)}}\binom{k-1}{k_{B_1}(\mu_I)-1}\langle\mu_I\rangle_{B_1;\psi}^{\phi,\mathfrak{os}}\langle\mu_J\rangle_{B_2;\psi}^{\phi,\mathfrak{os}}.
\end{aligned}$$

($\mathbb{R}$WDVV2) *If* $l\geq 3$, *then*

$$\begin{aligned}
&\sum_{\substack{B_0,B'\in H_2(X;\mathbb{Z})\\ B_0+\mathfrak{d}(B')=B\\ (I,J)\in\mathcal{P}_{2;3}(l)}} 2^{|I|-2}\sum_{i,j\in[N]}\langle\mu_I,\mu_i^{\bigstar}\rangle_{B'}^{X}g^{ij}\langle\mu_j^{\bigstar},\mu_J\rangle_{B_0;\psi}^{\phi,\mathfrak{os}}\\
&\quad+\sum_{\substack{B_1,B_2\in H_2^{\phi}(X)\\ B_1+B_2=B\\ (I,J)\in\mathcal{P}_{2;3}(l)}}\binom{k}{k_{B_1}(\mu_I)}\langle\mu_I\rangle_{B_1;\psi}^{\phi,\mathfrak{os}}\langle\mu_J\rangle_{B_2;\psi}^{\phi,\mathfrak{os}}\\
&\quad=\sum_{\substack{B_0,B'\in H_2(X;\mathbb{Z})\\ B_0+\mathfrak{d}(B')=B\\ (I,J)\in\mathcal{P}_{3;2}(l)}} 2^{|I|-2}\sum_{i,j\in[N]}\langle\mu_I,\mu_i^{\bigstar}\rangle_{B'}^{X}g^{ij}\langle\mu_j^{\bigstar},\mu_J\rangle_{B_0;\psi}^{\phi,\mathfrak{os}}\\
&\quad+\sum_{\substack{B_1,B_2\in H_2^{\phi}(X)\\ B_1+B_2=B\\ (I,J)\in\mathcal{P}_{3;2}(l)}}\binom{k}{k_{B_1}(\mu_I)}\langle\mu_I\rangle_{B_1;\psi}^{\phi,\mathfrak{os}}\langle\mu_J\rangle_{B_2;\psi}^{\phi,\mathfrak{os}}.
\end{aligned}$$

Let $\mathcal{N}X^\phi$ denote the normal bundle of X^ϕ in X. An orientation $\mathfrak{o}$ on X^ϕ determines an orientation on a unit sphere $S(\mathcal{N}_pX^\phi)$ in the fiber of $\mathcal{N}X^\phi$ over any $p\in X^\phi$. As explained in Section 2.5 of [6], the first homomorphism in (12) is surjective with the kernel generated by the homology class $[S(\mathcal{N}_pX^\phi)]_{X-X^\phi}$ of $S(\mathcal{N}_pX^\phi)$ in $X-X^\phi$; the second homomorphism in (12) is related to the first by the Poincaré duality. According to [6, Proposition 2.1],

$$\big\langle \mu_1,\ldots,\mu_l,\mathrm{PD}_{X,X^\phi}\big(\big[S(\mathcal{N}_pX^\phi)\big]_{X-X^\phi}\big)\big\rangle_B^{\phi,\mathfrak{os}}=2\langle\mu_1,\ldots,\mu_l\rangle_B^{\phi,\mathfrak{os}} \tag{21}$$

for all $B\in H_2^\phi(X)$ and $\mu_i\in H^2(X;\mathbb{R})\oplus H^4(X,X^\phi;\mathbb{R})\oplus H^6(X;\mathbb{R})$; that is, an insertion of $S(\mathcal{N}_pX^\phi)$ can be traded for a real point insertion. Thus, the invariants (18) with $\mu_i\in H^{2*}(X;\mathbb{R})$ determine the invariants (15) with $\mu_i\in H^2(X;\mathbb{R})\oplus H^4(X,X^\phi;\mathbb{R})\oplus H^6(X;\mathbb{R})$.

REMARK 2

Theorems 2 and 3 extend to compact real symplectic sixfolds (X,ω,ϕ) with disconnected fixed loci X^ϕ admitting finite groups of symmetries with certain properties and apply with finer notions of the curve degree B (see Proposition 1.3, Remark 1.4, and Theorem 1.5 in [6]). This makes no difference for the examples of Sections 7 and 8.

REMARK 3

By [7, Theorem 13.2], the real curve counts (15) defined in [20] intrinsically differ from those defined in [6] based on the moduli space considerations of [17] and [9] by $(-1)^{\binom{k_B(\mu_1,\ldots,\ldots,\mu_l)}{2}}$. The analogues of the two equations of Theorem 3 for the real curve counts (15) as defined in [20] would thus involve multiplying the summands in the (B_1,B_2) sums by (-1) to the power of

$$\begin{aligned}\binom{k_{B_1}(\mu_I)+k_{B_2}(\mu_J)-1}{2}-\binom{k_{B_1}(\mu_I)}{2}-\binom{k_{B_2}(\mu_J)}{2}\\ =k_{B_1}(\mu_I)k_{B_2}(\mu_J)-\big(k_{B_1}(\mu_I)+k_{B_2}(\mu_J)\big)+1.\end{aligned}$$

7. The projective space $\mathbb{P}^3$

The fixed locus of the standard conjugation

$$\tau_3:\mathbb{P}^3\longrightarrow\mathbb{P}^3,\qquad \tau_3\big([Z_0,Z_1,Z_2,Z_3]\big)=[\overline{Z_0},\overline{Z_1},\overline{Z_2},\overline{Z_3}],$$

is the real projective space $\mathbb{RP}^3$. The group $H_2^{\tau_3}(\mathbb{P}^3)=H_2(\mathbb{P}^3;\mathbb{Z})$ is identified with $\mathbb{Z}$ via the standard generator $L=[\mathbb{P}^1]$. An averager ψ in this case is a reflection about a τ_3-invariant complex hyperplane such as

$$\psi_3:\mathbb{P}^3\longrightarrow\mathbb{P}^3,\qquad \psi_3\big([Z_0,Z_1,Z_2,Z_3]\big)=[Z_0,Z_1,Z_2,-Z_3].$$

Each of the two orientations on $\mathbb{RP}^3$ is compatible with two Spin-structures. A canonical OSpin-structure $\mathfrak{os}_0$ on $\mathbb{RP}^3$ is specified in [10, Section 2.2]. It is straightforward to see that

$$\partial_{\mathbb{RP}^3;\mathbb{Z}_2}(B') = 0 \quad \text{if } \mathfrak{d}_{\mathbb{RP}^3}(B') \in 2\mathbb{Z}L, \qquad \text{and} \tag{22}$$
$$\psi^*\mathfrak{os}_0|_{\partial_{\mathbb{RP}^3;\mathbb{Z}_2}(B')} \neq \overline{\mathfrak{os}_0}|_{\partial_{\mathbb{RP}^3;\mathbb{Z}_2}(B')} \quad \text{if } \mathfrak{d}_{\mathbb{RP}^3}(B') \notin 2\mathbb{Z}L.$$

By the transitivity of the $H^1(\mathbb{RP}^3;\mathbb{Z}_2)$-action on the set of OSpin-structures on $\mathbb{RP}^3$, the last inequality holds for all OSpin-structures on $\mathbb{RP}^3$.

Let $H \in H^2(\mathbb{P}^3)$ denote the hyperplane class. For $d \in \mathbb{Z}^+$ and a tuple $\mathbf{m} \equiv (m_1, \ldots, m_l)$ of nonnegative integers, define

$$2k_d(\mathbf{m}) = k_{dL}\big((H^{m_i})_{i\in I}\big) = 2d + l - \sum_{i=1}^{l} m_i,$$

$$\langle d\rangle_{\mathbf{m}} = \begin{cases} 1 & \text{if } d \in 2\mathbb{Z}, k_d(\mathbf{m}) = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \mathbf{m}\rangle_d = \langle H^{m_1}, \ldots, H^{m_l}\rangle_{dL}^{\mathbb{P}^3}, \qquad \langle \mathbf{m}\rangle_d^{\tau_3} = \langle H^{m_1}, \ldots, H^{m_l}\rangle_{dL;\psi_3}^{\tau_3,\mathfrak{os}_0}.$$

If in addition $I \subset [l]$, let $\mathbf{m}_I$ be the $|I|$-tuple consisting of the entries of $\mathbf{m}$ indexed by I.

Since there is a unique line $\mathbb{P}^1$ passing through every pair of points in $\mathbb{P}^3$ and the orientation conventions of [6] yield the opposite sign of [8, Example 6.3],

$$\langle 3,3\rangle_1 = 1, \qquad \langle 3\rangle_1^{\tau_3} = -1. \tag{23}$$

By (22) and Theorems 2(c) and 2(d),

$$\langle \mathbf{m}\rangle_d^{\tau_3} = 0 \qquad \text{if } d \equiv k_d(\mathbf{m}) \mod 2.$$

The $\mathbb{P}^3$ case of [16, Theorem 10.4] gives

$$\begin{aligned}\langle m_1, m_2, m_3+1, m_4, \ldots, m_l\rangle_d &\\ = \langle m_1, m_2+1, m_3, m_4, \ldots, m_l\rangle_d + d\langle m_1+m_3, m_2, m_4, \ldots, m_l\rangle_d &\\ + \sum_{\substack{d_1,d_2\in\mathbb{Z}^+\\ d_1+d_2=d}} \Big\{ \sum_{(I,J)\in\mathcal{P}_{3;2}(l)} - \sum_{(I,J)\in\mathcal{P}_{2;3}(l)} \Big\} d_2 \sum_{\substack{i,j\in\mathbb{Z}^+\\ i+j=3}} \langle \mathbf{m}_I, i\rangle_{d_1}\langle j, \mathbf{m}_J\rangle_{d_2}&\end{aligned}$$

whenever $m_1, m_2 \geq 2$. Along with the divisor relation, the above identity with

$$m_1 \geq m_2 \geq m_3+1 \geq m_4 \geq \cdots \geq m_l \geq 2$$

recursively determines all complex GW-invariants $\langle \mathbf{m}\rangle_d$ of $\mathbb{P}^3$.

For the purposes of applying Theorem 3 in this case, we replace the set $[N]$ indexing a basis for (20) by the set $\{0,1,2,3\}$ and take

$$(\mu_0^\star, \mu_1^\star, \mu_2^\star \mu_3^\star) = (1, H, H^2, H^3).$$

The sums over $[N]$ then become sums over $i, j \in \mathbb{Z}^{\geq 0}$ with $i+j=3$ and with g^{ij} dropped. We apply Theorem 3 with $B = dL$, l replaced by $l+1$, and

$$(\mu_1, \ldots, \mu_{l+1}) = (H, H^{m_1}, H^{m_2}, \ldots, H^{m_l}).$$

($\tau_3 1$) If $d \ge 1$, $l \ge 1$, and $\mathbf{m} \equiv (m_1, \ldots, m_l) \in (\mathbb{Z}^+)^l$ with $k_d(\mathbf{m}) \ge 2$, then

$$\begin{aligned}
&\langle m_1+1, m_2, \ldots, m_l \rangle_d^{\tau_3} \\
&= -2^{l-2} \langle d \rangle_{\mathbf{m}} d \langle m_1, m_2, \ldots, m_l, 3 \rangle_{d/2} \\
&\quad - \sum_{\substack{d_0, d' \in \mathbb{Z}^+ \\ d_0 + 2d' = d}} d' \sum_{(I,J) \in \mathcal{P}(l)} 2^{|I|-1} \sum_{\substack{i,j \in \mathbb{Z}^+ \\ i+j=3}} \langle \mathbf{m}_I, i \rangle_{d'} \langle j, \mathbf{m}_J \rangle_{d_0}^{\tau_3} \\
&\quad + \sum_{\substack{d_1, d_2 \in \mathbb{Z}^+ \\ d_1 + d_2 = d}} \sum_{(I,J) \in \mathcal{P}(l)} \left(d_2 \binom{k_d(\mathbf{m}) - 2}{k_{d_2}(\mathbf{m}_J) - 1} - d_1 \binom{k_d(\mathbf{m}) - 2}{k_{d_1}(\mathbf{m}_I)} \right) \\
&\quad \times \langle \mathbf{m}_I \rangle_{d_1}^{\tau_3} \langle \mathbf{m}_J \rangle_{d_2}^{\tau_3}.
\end{aligned}$$

($\tau_3 2$) If $d \ge 1$, $l \ge 2$, and $\mathbf{m} \equiv (m_1, \ldots, m_l) \in (\mathbb{Z}^+)^l$ with $k_d(\mathbf{m}) \ge 1$, then

$$\begin{aligned}
&\langle m_1, m_2+1, m_3, \ldots, m_l \rangle_d^{\tau_3} \\
&= \langle m_1+1, m_2, m_3, \ldots, m_l \rangle_d^{\tau_3} \\
&\quad + \sum_{\substack{d_0, d' \in \mathbb{Z}^+ \\ d_0 + 2d' = d}} d' \sum_{(I,J) \in \mathcal{P}_{;2}(l)} \sum_{\substack{i,j \in \mathbb{Z}^+ \\ i+j=3}} \big(2^{|I|-1} \langle \mathbf{m}_I, i \rangle_{d'} \langle j, \mathbf{m}_J \rangle_{d_0}^{\tau_3} \\
&\quad - 2^{|J|-1} \langle \mathbf{m}_J, j \rangle_{d'} \langle i, \mathbf{m}_I \rangle_{d_0}^{\tau_3} \big) \\
&\quad + \sum_{\substack{d_1, d_2 \in \mathbb{Z}^+ \\ d_1 + d_2 = d}} \sum_{(I,J) \in \mathcal{P}_{;2}(l)} \left(d_1 \binom{k_d(\mathbf{m}) - 1}{k_{d_1}(\mathbf{m}_I)} - d_2 \binom{k_d(\mathbf{m}) - 1}{k_{d_2}(\mathbf{m}_J)} \right) \langle \mathbf{m}_I \rangle_{d_1}^{\tau_3} \langle \mathbf{m}_J \rangle_{d_2}^{\tau_3}.
\end{aligned}$$

Proof of ($\tau_3 1$) and ($\tau_3 2$)
In both cases, $k = k_d(\mathbf{m}) - 1$. Since all components of the tuples μ_I and μ_J in Theorem 3 now have positive cohomology degrees, the B_1, B_2 sums reduce to sums over $(B_1, B_2) = (d_1 L, d_2 L)$ with $d_1, d_2 \in \mathbb{Z}^+$ and $d_1 + d_2 = d$. The divisor relation (19) gives

$$\begin{aligned}
\langle \mu_I \rangle_{d_1 L; \psi_3}^{\tau_3, \mathfrak{os}_0} \langle \mu_J \rangle_{d_2 L; \psi_3}^{\tau_3, \mathfrak{os}_0} &= d_1 \big\langle (m_{i-1})_{i \in I - \{1\}} \big\rangle_{d_1}^{\tau_3} \big\langle (m_{j-1})_{j \in J} \big\rangle_{d_2}^{\tau_3} \\
&\forall (I,J) \in \mathcal{P}(l+1).
\end{aligned} \tag{24}$$

Thus, the four (B_1, B_2) sums reduce to sums over (d_1, d_2) and (I,J) with $I \sqcup J = [l]$. Interchanging the roles of (d_1, I) and (d_2, J) in the (B_1, B_2) sum on the right-hand side of ($\mathbb{R}$WDVV1) (resp., ($\mathbb{R}$WDVV2)) and subtracting the (B_1, B_2) sum on the left-hand side from the resulting expression (resp., the resulting expression from the (B_1, B_2) sum on the left-hand side), we obtain the last double sum in ($\tau_3 1$) (resp., ($\tau_3 2$)).

The invariants $\langle \cdot \rangle_{B'}^X$ with $B' \neq 0$ and $\langle \cdot \rangle_{B_0; \psi}^{\phi, \mathfrak{os}}$ with $B_0 \neq 0$ vanish on tuples of insertions that contain a component in $H^0(X; \mathbb{R})$. Thus, only the pairs $i, j \in [3]$ with $i, j \in \mathbb{Z}^+$ and $i + j = 3$ contribute to the $(B_0, B') = (d_0 L, d' L)$ summands with $d_0, d' \neq 0$. The complex divisor relation gives

$$\langle \mu_I, \mu_i^\bigstar \rangle_{d'L}^{\mathbb{P}^3} \langle \mu_j^\bigstar, \mu_J \rangle_{d_0 L;\psi_3}^{\tau_3,\mathfrak{os}_0} = d' \big\langle (m_{i'-1})_{i' \in I-\{1\}}, i \big\rangle_{d'} \big\langle j, (m_{j'-1})_{j' \in J} \big\rangle_{d_0}^{\tau_3} \quad \forall d' \in \mathbb{Z}^+, (I,J) \in \mathcal{P}(l+1). \tag{25}$$

The *negative* of the sum of the (B_0, B')-terms in ($\mathbb{R}$WDVV1) with $B_0, B' \neq 0$ is thus the double sum on the second line of ($\tau_3 1$). Interchanging the roles of (I,i) and (J,j) in the (B', B_0)-terms on the right-hand side of ($\mathbb{R}$WDVV2) with $B_0, B' \neq 0$ and subtracting them from the corresponding terms on the left-hand side, we obtain the double sum on the second line of ($\tau_3 2$).

It remains to consider the $(B_0, B') = (d_0 L, d'L)$ summands in Theorem 3 with either $d' = 0$ or $d_0 = 0$. For $(I,J) \in \mathcal{P}(l+1)$,

$$\langle \mu_I, \mu_i^\bigstar \rangle_0^{\mathbb{P}^3} = \begin{cases} 1 & \text{if } |I| = 2, 1 + |(m_{i'-1})_{i' \in I-\{1\}}| + i = 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \mu_j^\bigstar, \mu_J \rangle_{0;\psi_3}^{\tau_3,\mathfrak{os}_0} = \begin{cases} 1 & \text{if } J = \emptyset, j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $(B_0, J, j) = (0, \emptyset, 0)$, then the corresponding $(B', I, i) = ((d/2)L, [l+1], 3)$ factor vanishes unless $d \in 2\mathbb{Z}$ and $k_d(\mathbf{m}) = 2$. Along with the $d' = d/2$ case of (25), this implies that the *negative* of the only possibly nonzero $B_0 = 0$ summand in ($\mathbb{R}$WDVV1) is the first term on the right-hand side of ($\tau_3 1$). All $B_0 = 0$ summands in ($\mathbb{R}$WDVV2) vanish, because $J \neq \emptyset$ there. If $B' = 0$, $I = \{1, i'\}$ with $i' \neq 1$, and $m_{i'} + i = 2$, then

$$\langle \mu_j^\bigstar, \mu_J \rangle_{B_0;\psi_3}^{\tau_3,\mathfrak{os}_0} = \big\langle m_{i'} + 1, (m_{j-1})_{j \in J} \big\rangle_d^{\tau_3}.$$

The only possibly nonzero $B' = 0$ summands in ($\mathbb{R}$WDVV1), on the left-hand side of ($\mathbb{R}$WDVV2), and on the right-hand side of ($\mathbb{R}$WDVV2), are thus the left-hand side of ($\tau_3 1$), the first term on the right-hand side of ($\tau_3 2$), and the left-hand side of ($\tau_3 2$), respectively. □

For $d = 1$, all double sums in ($\tau_3 1$) and ($\tau_3 2$) vanish. From ($\tau_3 1$) with $(l, \mathbf{m}) = (1, (1))$ and ($\tau_3 2$) with $(l, \mathbf{m}) = (2, (2, 1))$, we thus obtain

$$\langle 2 \rangle_1^{\tau_3} = 0 \qquad \text{and} \qquad \langle 2, 2 \rangle_1^{\tau_3} = \langle 3, 1 \rangle_1^{\tau_3} = -1, \tag{26}$$

respectively; the last equality above follows from the real divisor relation (19) and the second statement in (23). For $d = 2$, the (d_0, d') double sums in ($\tau_3 1$) and ($\tau_3 2$) vanish. From ($\tau_3 1$) with $(l, \mathbf{m}) = (2, (2, 2))$ and ($\tau_3 2$) with $(l, \mathbf{m}) = (2, (3, 1))$, we thus obtain

$$\langle 3, 2 \rangle_2^{\tau_3} = -2\langle 2, 2, 3 \rangle_1 + \langle 2 \rangle_1^{\tau_3} \langle 2 \rangle_1^{\tau_3} - \langle 2, 2 \rangle_1^{\tau_3} \langle \rangle_1^{\tau_3} \qquad \text{and}$$
$$\langle 3, 2 \rangle_2^{\tau_3} = \langle 3 \rangle_1^{\tau_3} \langle 1 \rangle_1^{\tau_3} = -\langle \rangle_1^{\tau_3}, \tag{27}$$

respectively; the last equality above follows from the second statement in (23) and the real divisor relation (19). Along with $\langle 2, 2, 3 \rangle_1 = 1$ (the number of lines through two lines and a point in $\mathbb{P}^3$) and (26), (27) gives

$$\langle\rangle_1^{\tau_3} = 1.$$

A similar computation appears in [1, Section 4.1.4].

Let $d \geq 2$. The recursion ($\tau_3 1$) for $\langle 3,2\rangle_{d+1}^{\tau_3}$ gives

$$\langle 3,2\rangle_{d+1}^{\tau_3} = \sum_{\substack{d_1,d_2\in\mathbb{Z}^+\\ d_1+d_2=d+1}} \Bigg(\bigg(d_1\binom{2d-2}{2d_1-2} - d_2\binom{2d-2}{2d_2-1}\bigg)\langle 2\rangle_{d_1}^{\tau_3}\langle 2\rangle_{d_2}^{\tau_3} + \bigg(d_2\binom{2d-2}{2d_2-1} - d_1\binom{2d-2}{2d_1-2}\bigg)\langle 2,2\rangle_{d_1}^{\tau_3}\langle\rangle_{d_2}^{\tau_3}\Bigg);$$

the expression on the first line above is obtained by interchanging the roles of d_1 and d_2 for $(I,J) = (\{1\},\{2\})$. We note that

$$d_1\binom{2d-2}{2d_1-2} - d_2\binom{2d-2}{2d_2-1} = \frac{(2d-2)!(d_1(2d_2-1) - d_2(2d_1-2))}{(2d_1-2)!(2d_2-1)!} = \frac{2d_2-d_1}{2d_2-1}\binom{2d-2}{2d_1-2}.$$

The recursion ($\tau_3 2$) for $\langle 3,2\rangle_{d+1}^{\tau_3}$ and the divisor relation (19) give

$$\langle 3,2\rangle_{d+1}^{\tau_3} = \sum_{\substack{d_1,d_2\in\mathbb{Z}^+\\ d_1+d_2=d+1}} \bigg(d_1\binom{2d-1}{2d_1-2} - d_2\binom{2d-1}{2d_2}\bigg)d_2\langle 3\rangle_{d_1}^{\tau_3}\langle\rangle_{d_2}^{\tau_3} = \sum_{\substack{d_1,d_2\in\mathbb{Z}^+\\ d_1+d_2=d+1}} \binom{2d-1}{2d_1-2}d_2\langle 3\rangle_{d_1}^{\tau_3}\langle\rangle_{d_2}^{\tau_3}.$$

Subtracting the first equation for $\langle 3,2\rangle_{d+1}^{\tau_3}$ above from the second one and using the second statement in (23) and (26), we obtain

$$\langle\rangle_d^{\tau_3} = \frac{1}{d+1}\sum_{\substack{d_1,d_2\in\mathbb{Z}^+\\ d_1+d_2=d}} \Bigg(d_2\binom{2d-1}{2d_1}\langle 3\rangle_{d_1+1}^{\tau_3}\langle\rangle_{d_2}^{\tau_3} + \frac{2d_2-d_1-1}{2d_2-1}\binom{2d-2}{2d_1}\Big(\langle 2,2\rangle_{d_1+1}^{\tau_3}\langle\rangle_{d_2}^{\tau_3} - \langle 2\rangle_{d_1+1}^{\tau_3}\langle 2\rangle_{d_2}^{\tau_3}\Big)\Bigg)$$

for all $d \geq 2$.

Along with the last relation above and the real divisor relation (19), ($\tau_3 1$) and ($\tau_3 2$) determine all numbers $\langle \mathbf{m}\rangle_d^{\tau_3}$ with $d \geq 2$. The low-degree numbers obtained from these recursions are as in [1, Table 4.2.2]. Along with (21), the numbers $\langle \mathbf{m}\rangle_d^{\tau_3}$ determine the invariants (15) of $(\mathbb{P}^3, \tau_3)$ with all possible insertions μ_i in $H^6(\mathbb{P}^3;\mathbb{R})$ and $H^4(\mathbb{P}^3, \mathbb{RP}^3;\mathbb{R})$. Since only two elements of $H_2(\mathbb{P}^3 - \mathbb{RP}^3;\mathbb{Z})$ can be represented by a linearly embedded $\mathbb{P}^1 \subset \mathbb{P}^3$, the invariants (15) provide lower bounds for the counts of real holomorphic curves in $(\mathbb{P}^3, \tau_3)$ through line and point constraints (see [6, Table 1]).

8. The sixfold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Let τ, $\tau_{1,1}$, $\tau'_{1,1}$ be the conjugations of Section 4. The fixed loci of the conjugations

$$\phi_3 \equiv \tau_{1,1} \times \tau, \phi'_3 \equiv \tau'_{1,1} \times \tau : (\mathbb{P}^1)^3 \longrightarrow (\mathbb{P}^1)^3$$

are the three-torus $(\mathbb{RP}^1)^3 \approx (S^1)^3$ and

$$(\mathbb{P}^1 \times \mathbb{P}^1)^{\tau'_{1,1}} \times (\mathbb{P}^1)^\tau \approx \mathbb{P}^1 \times S^1,$$

respectively. The groups $H_2^{\phi_3}((\mathbb{P}^1)^3) = H_2((\mathbb{P}^1)^3)$ are identified with $\mathbb{Z}^3$ via the standard generators

$$L_1 \equiv [\mathbb{P}^1 \times \mathrm{pt} \times \mathrm{pt}], \qquad L_2 \equiv [\mathrm{pt} \times \mathbb{P}^1 \times \mathrm{pt}], \qquad \text{and} \qquad L_3 \equiv [\mathrm{pt} \times \mathrm{pt} \times \mathbb{P}^1].$$

The group $H_2^{\phi'_3}((\mathbb{P}^1)^3)$ is identified with $\mathbb{Z}^2$ via the generators $L_{1,2} \equiv L_1 + L_2$ and L_3.

An averager ψ for both conjugations is given by

$$\psi : (\mathbb{P}^1)^3 \longrightarrow (\mathbb{P}^1)^3,$$
$$\psi\big([X_1, Y_1], [X_2, Y_2], [X_3, Y_3]\big) = \big([X_1, Y_1], [X_2, Y_2], [X_3, -Y_3]\big).$$

Each of the two orientations on $(S^1)^3$ and $\mathbb{P}^1 \times S^1$ is compatible with eight Spin-structures and two Spin-structures, respectively. Both fixed loci have a canonical OSpin-structure $\mathfrak{os}_0$ obtained by trivializing each factor of TS^1 separately. It is immediate that

$$\psi^* \mathfrak{os}_0 = \overline{\mathfrak{os}_0}. \tag{28}$$

Denote by $e_1, e_2, e_3 \in \mathbb{Z}^3$ the standard basis elements, by $\mathbf{1} \in \mathbb{Z}^3$ their sum, and denote by $H_1, H_2, H_3 \in H^2((\mathbb{P}^1)^3; \mathbb{R})$ the Poincaré duals of the complex submanifolds

$$\mathrm{pt} \times \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \mathrm{pt} \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{pt} \subset (\mathbb{P}^1)^3,$$

respectively. For an element $m = (a, b, c)$ of $\mathbb{Z}^3$, define

$$|m| = a + b + c, \qquad H^m = H_1^a H_2^b H_3^c \in H^{2*}\big((\mathbb{P}^1)^3; \mathbb{R}\big),$$
$$mL = aL_1 + bL_2 + cL_3 \in H_2\big((\mathbb{P}^1)^3; \mathbb{Z}\big).$$

For an element $d \equiv (d_1, d_2, d_3)$ of $(\mathbb{Z}^{\ge 0})^3$, a tuple $\mathbf{m} \equiv (m_1, \ldots, m_l)$ of elements of $(\mathbb{Z}^{\ge 0})^3$, and $\phi = \phi_3, \phi'_3$, let

$$k_d(\mathbf{m}) = k_{dL}\big((H^{m_i})_{i \in [l]}\big) = |d| + l - \sum_{i=1}^{l} |m_i|,$$
$$\langle \mathbf{m} \rangle_d = \langle H^{m_1}, \ldots, H^{m_l} \rangle_{dL}^{(\mathbb{P}^1)^3}, \qquad \langle \mathbf{m} \rangle_d^\phi = \langle H^{m_1}, \ldots, H^{m_l} \rangle_{dL;\psi}^{\phi, \mathfrak{os}_0}.$$

If in addition $I \subset [l]$, let $\mathbf{m}_I$ be the $|I|$-tuple consisting of the entries of $\mathbf{m}$ indexed by I. For $a_1, a_2, a_3, b \in \mathbb{Z}^{\ge 0}$, let

$$\mathbf{m}_{a_1a_2a_3b} = (\underbrace{\mathbf{1}-e_1,\ldots,\mathbf{1}-e_1}_{a_1},\underbrace{\mathbf{1}-e_2,\ldots,\mathbf{1}-e_2}_{a_2},\underbrace{\mathbf{1}-e_3,\ldots,\mathbf{1}-e_3}_{a_3},\underbrace{\mathbf{1},\ldots,\mathbf{1}}_{b}),$$

$$N_{d;a_1a_2a_3b} = \langle \mathbf{m}_{a_1a_2a_3b}\rangle_d, \qquad N^{\phi}_{d;a_1a_2a_3b} = \langle \mathbf{m}_{a_1a_2a_3b}\rangle^{\phi}_d.$$

The complex GW-invariants $N_{d;a_1a_2a_3b}$ of $(\mathbb{P}^1)^3$ are preserved by simultaneous permutations of the components of the triples d and (a_1,a_2,a_3). Since a degree $d_1L_1+d_2L_2$ curve is contained in $\mathbb{P}^1\times\mathbb{P}^1\times \mathrm{pt}$ for some $\mathrm{pt}\in\mathbb{P}^1$,

$$(29)\quad N_{(d_1,d_2,0);a_1a_2a_3b} = \begin{cases} 0 & \text{if } a_1+a_2+b\neq 1,\\ N^{\mathbb{P}^1\times\mathbb{P}^1}_{d_1,d_2} & \text{if } (a_1,a_2,a_3,b)=(0,0,2(d_1+d_2)-2,1),\\ d_2N^{\mathbb{P}^1\times\mathbb{P}^1}_{d_1,d_2} & \text{if } (a_1,a_2,a_3,b)=(1,0,2(d_1+d_2)-1,0).\end{cases}$$

Since an irreducible degree $d_1L_1+d_2L_2+L_3$ curve is the graph of a holomorphic map from $\mathbb{P}^1$ to $\mathbb{P}^1\times\mathbb{P}^1$,

$$(30)\quad N_{(d_1,d_2,1);a_1a_2a_3b} = \begin{cases} 0 & \text{if } a_1+a_2+b<3,\\ d_1^{a_2}d_2^{a_1}N^{\mathbb{P}^1\times\mathbb{P}^1}_{d_1,d_2} & \text{if } a_1+a_2+b=3, a_3+b=2(d_1+d_2)-1.\end{cases}$$

By (28) and Theorem 2(c),

$$(31)\qquad \langle\mathbf{m}\rangle^{\phi}_d=0 \quad \forall\phi=\phi_3,\phi_3', d\in(\mathbb{Z}^{\ge 0})^3, \mathbf{m}\in\big((\mathbb{Z}^{\ge0})^3\big)^l \text{ with } k_d(\mathbf{m})\in 2\mathbb{Z}.$$

This implies that

$$\langle H^{m_1},\ldots,H^{m_l}\rangle^{\phi,\mathfrak{os}_0}_{dL;\psi} = \langle H^{m_1},\ldots,H^{m_l}\rangle^{\phi,\overline{\mathfrak{os}}_0}_{dL;\psi};$$

that is, these invariants do not depend on an a priori choice of orientation of $((\mathbb{P}^1)^3)^{\phi}$. Furthermore, the invariants $N^{\phi_3}_{d;a_1a_2a_3b}$ are preserved by simultaneous permutations of the components of the triples d and (a_1,a_2,a_3). By the same geometric reasoning as in the complex case, the absolute values of the invariants $N^{\phi}_{d;a_1a_2a_3b}$ of $((\mathbb{P}^1)^3,\phi)$ and the invariants $N^{\varphi}_{(d_1,d_2),l}$ of $(\mathbb{P}^1)^2$ with the conjugation $\varphi=\tau_{1,1},\tau_{1,1}'$ corresponding to $\phi\equiv\varphi\times\tau$ satisfy the analogues of (29) and (30). We describe the relative signs between these invariants in the next paragraph.

By [17], an OSpin-structure $\mathfrak{os}$ on the fixed locus $(\mathbb{P}^1\times\mathbb{P}^1)^{\varphi}$ determines signed counts $N^{\varphi,\mathfrak{os}}_{(d_1,d_2),l}$ of real rational J-holomorphic curves in $(\mathbb{P}^1\times\mathbb{P}^1,\varphi)$ with $|N^{\varphi,\mathfrak{os}}_{(d_1,d_2),l}|=|N^{\varphi}_{(d_1,d_2),l}|$. Similarly to the $(\mathbb{P}^1)^3$ case, there is a natural OSpin-structure $\mathfrak{os}_0$ on $(\mathbb{P}^1\times\mathbb{P}^1)^{\varphi}$; the invariants $N^{\varphi,\mathfrak{os}_0}_{(d_1,d_2),l}$ do not depend on an a priori choice of orientation of $(\mathbb{P}^1\times\mathbb{P}^1)^{\varphi}$. By Theorem 13.1 and Examples 13.4 and 13.5 in [7],

$$(32)\quad \begin{aligned} N^{\tau_{1,1},\mathfrak{os}_0}_{(d_1,d_2),l} &= (-1)^{d_1+d_2+l-1}N^{\tau_{1,1}}_{(d_1,d_2),l} \qquad \text{and}\\ N^{\tau_{1,1}',\mathfrak{os}_0}_{(d,d),l} &= (-1)^{d+l-1}N^{\tau_{1,1}'}_{d,l}.\end{aligned}$$

By [7, Theorem 12.2] and the real divisor relation (19), the invariants satisfy the exact analogues of (29) and (30). The implications of the last two statements for the two involutions ϕ are stated in the respective Sections 8.1 and 8.2.

Similarly to the $\mathbb{P}^3$ case, the standard WDVV recursion for the GW-invariants $\langle \mathbf{m} \rangle_d$ gives

$$\begin{aligned}
&\langle m_1, m_2, m_3+e_r, m_4, \ldots, m_l \rangle_d \\
&\quad = \langle m_1, m_2+e_r, m_3, m_4, \ldots, m_l \rangle_d + d_r \langle m_1+m_3, m_2, m_4, \ldots, m_l \rangle_d \\
&\qquad + \sum_{\substack{d',d'' \in (\mathbb{Z}^{\geq 0})^3-\{0\} \\ d'+d''=d}} \bigg\{ \sum_{(I,J)\in \mathcal{P}_{3;2}(l)} - \sum_{(I,J)\in \mathcal{P}_{2;3}(l)} \bigg\} \\
&\qquad \times d''_r \sum_{\substack{i,j \in (\mathbb{Z}^{\geq 0})^3-\{0\} \\ i+j=\mathbf{1}}} \langle \mathbf{m}_I, i \rangle_{d'} \langle j, \mathbf{m}_J \rangle_{d''}
\end{aligned}$$

whenever $|m_1|, |m_2| \geq 2$ and $r=1,2,3$. Formally, this relation follows from [16, (1.7)] with $H_\alpha = H^{m_1}$, $H_\beta = H^{m_2}$, $H_\gamma = H^{m_3}$, and $H_\delta = H^{e_r}$ and the divisor relation for GW-invariants. More directly, it is an immediate consequence of [16, Theorem 7.2] with $A=dL$, $g=0$, $k=4$, l replaced by $l-3$,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4; \beta_1, \ldots, \beta_{l-3}) = (H^{m_1}, H^{m_2}, H^{m_3}, H^{e_r}; H^{m_4}, \ldots, H^{m_l}),$$

and $\mathcal{C}$ being the connected reducible rational curve with four marked points so that either the marked points 1, 2 lie on the same irreducible component of $\mathcal{C}$ or the marked points 1, 3 do. Along with the divisor relation, the above identity with

$$|m_1| \geq |m_2| \geq |m_3|+1 \geq |m_4| \geq \cdots \geq |m_l| \geq 2$$

recursively determines all complex GW-invariants $\langle \mathbf{m} \rangle_d$ of $(\mathbb{P}^1)^3$.

8.1. The product involution

By the sentence immediately after (32) and the first identity in (32),

$$\begin{aligned}
&N^{\phi_3,\mathfrak{os}_0}_{(d_1,d_2,0);a_1a_2a_3b} \\
&\qquad = \begin{cases} N^{\tau_{1,1}}_{(d_1,d_2),d_1+d_2-1} & \text{if } a_1,a_2,b=0, a_3=d_1+d_2-1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

$$\begin{aligned}
&N^{\phi_3,\mathfrak{os}_0}_{(d_1,d_2,1);a_1a_2a_3b} \\
&\qquad = \begin{cases} 0 & \text{if } d_1+d_2+a_1+a_2 \leq a_3+1, \\ -N^{\tau_{1,1}}_{(d_1,d_2),d_1+d_2-2} & \text{if } a_1,a_2,b=0, a_3=d_1+d_2-2, \\ N^{\tau_{1,1}}_{(d_1,d_2),d_1+d_2-1} & \text{if } a_1,a_2=0, b=1, a_3=d_1+d_2-2, \\ d_2 N^{\tau_{1,1}}_{(d_1,d_2),d_1+d_2-1} & \text{if } a_1=1, a_2,b=0, a_3=d_1+d_2-1. \end{cases}
\end{aligned} \tag{33}$$

Along with (6), this gives

$$\langle\rangle^{\phi_3}_{e_r} = 1 \quad \forall r=1,2,3, \qquad \langle\rangle^{\phi_3}_{e_1+e_2+e_3} = -1. \tag{34}$$

For $d \in (\mathbb{Z}^{\geq 0})^3$ and $\mathbf{m} \in ((\mathbb{Z}^{\geq 0})^3)^l$, let

$$\langle d \rangle_{\mathbf{m}} = \begin{cases} 1 & \text{if } k_d(\mathbf{m}) = 2, d \in (2\mathbb{Z})^3, \\ 0 & \text{otherwise.} \end{cases}$$

For the purposes of applying the first recursion of Theorem 3 in this case, we replace the set $[N]$ indexing a basis for (20) by the set $\{0,1\}^3$ and take

$$\mu_i^{\star} = H^i \quad \forall i \in \{0,1\}^3.$$

The sum over $[N]$ then becomes a sum over $i, j \in (\mathbb{Z}^{\geq 0})^3$ with $i + j = \mathbf{1}$ and with g^{ij} dropped. We apply Theorem 3 with $B = dL$, l replaced by $l+1$, and

$$(\mu_1, \ldots, \mu_{l+1}) = (H_r, H^{m_1}, H^{m_2}, \ldots, H^{m_l}).$$

The exact same reasoning as for the recursion ($\tau_3 1$) in Section 7, with the divisor relations (24) and (25) replaced by

$$\langle \mu_I \rangle_{d'L;\psi}^{\phi_3,\mathfrak{os}_0} \langle \mu_J \rangle_{d''L;\psi}^{\phi_3,\mathfrak{os}_0} = d'_r \big\langle (m_{i-1})_{i \in I - \{1\}} \big\rangle_{d'}^{\phi_3} \big\langle (m_{j-1})_{j \in J} \big\rangle_{d''}^{\phi_3}$$

$$\langle \mu_I, \mu_i^{\star} \rangle_{d'L}^{(\mathbb{P}^1)^3} \langle \mu_j^{\star}, \mu_J \rangle_{d''L;\psi}^{\phi_3,\mathfrak{os}_0} = d'_r \big\langle (m_{i'-1})_{i' \in I - \{1\}}, i \big\rangle_{d'} \big\langle j, (m_{j'-1})_{j' \in J} \big\rangle_{d''}^{\phi_3}$$

for all $(I, J) \in \mathcal{P}(l+1)$ and $d' \in (\mathbb{Z}^{\geq 0})^3 - \{0\}$, yields the following.

($\phi_3 1a$) If $d \in (\mathbb{Z}^{\geq 0})^3 - \{0\}$, $l \geq 1$, $\mathbf{m} \equiv (m_1, \ldots, m_l) \in ((\mathbb{Z}^{\geq 0})^3 - \{0\})^l$ with $k_d(\mathbf{m}) \geq 2$, and $r = 1, 2, 3$, then

$$\begin{aligned} &\langle m_1 + e_r, m_2, \ldots, m_l \rangle_d^{\phi_3} \\ &\quad = -2^{l-2} \langle d \rangle_{\mathbf{m}} d_r \langle m_1, m_2, \ldots, m_l, \mathbf{1} \rangle_{d/2} \\ &\qquad - \sum_{\substack{d', d'' \in (\mathbb{Z}^{\geq 0})^3 - \{0\} \\ d'' + 2d' = d}} d'_r \sum_{(I,J) \in \mathcal{P}(l)} 2^{|I|-1} \sum_{\substack{i, j \in (\mathbb{Z}^{\geq 0})^3 - \{0\} \\ i + j = \mathbf{1}}} \langle \mathbf{m}_I, i \rangle_{d'} \langle j, \mathbf{m}_J \rangle_{d''}^{\phi_3} \\ &\qquad + \sum_{\substack{d', d'' \in (\mathbb{Z}^{\geq 0})^3 - \{0\} \\ d' + d'' = d}} \sum_{(I,J) \in \mathcal{P}(l)} \left(d''_r \binom{k_d(\mathbf{m}) - 2}{k_{d''}(\mathbf{m}_J) - 1} - d'_r \binom{k_d(\mathbf{m}) - 2}{k_{d'}(\mathbf{m}_I)} \right) \\ &\qquad \times \langle \mathbf{m}_I \rangle_{d'}^{\phi_3} \langle \mathbf{m}_J \rangle_{d''}^{\phi_3}. \end{aligned}$$

Replacing d in ($\phi_3 1a$) by $d + e_r$ with $d \neq 0, e_r$ and taking $(l, m_1) = (1, e_r)$, we obtain

$$0 = \sum_{\substack{d', d'' \in (\mathbb{Z}^{\geq 0})^3 \\ d' + d'' = d + e_r}} \left(d''_r \binom{|d| - 1}{|d''| - 1} - d'_r \binom{|d| - 1}{|d'|} \right) \langle e_r \rangle_{d'}^{\phi_3} \langle \rangle_{d''}^{\phi_3}.$$

By (19), $\langle e_r \rangle_{d'}^{\phi_3} = d'_r \langle \rangle_{d'}^{\phi_3}$. Using the first statement in (34), we thus obtain the following.

Table 19. The counts $N_{(2,2,2);2\mathbf{a}}$, with $\mathbf{a}\equiv(a_1,a_2,a_3)$ of complex genus 0 degree $(2,2,2)$ curves in $(\mathbb{P}^1)^3$ through $2a_1$, $2a_2$, $2a_3$ generic representatives for the standard line classes L_1, L_2, L_3 and $6-|\mathbf{a}|$ general points (the $\mathbb{C}$ line) and the signed count $N^{\phi_3}_{(2,2,2);\mathbf{a}b}$ of real genus 0 degree $(2,2,2)$ curves in $(\mathbb{P}^1)^3$ through a_1, a_2, a_3 conjugate pairs of generic representatives for L_1, L_2, L_3, b conjugate pairs of points and $6-|\mathbf{a}|-2b$ real points.

(a_1,a_2,a_3)	$(1,0,0)$	$(1,1,1)$	$(2,1,0)$	$(3,0,0)$	$(2,2,1)$	$(3,1,1)$	$(3,2,0)$	$(4,1,0)$	$(5,0,0)$
$\mathbb{C}$	12	110	48	0	672	192	192	0	0
$b=0$	6	-12	-8	0	16	8	8	0	0
$b=1$	-4	6	4	0					
$b=2$	2								

($\phi_3 1b$) If $r=1,2,3$ and $d\in(\mathbb{Z}^{\ge 0})^3-\{0,e_r\}$, then

$$\big(|d|-1-2d_r\big)\langle\rangle^{\phi_3}_d=\sum_{\substack{d',d''\in(\mathbb{Z}^{\ge 0})^3\\ |d'|,|d''|\ge 2\\ d'+d''=d+e_r}} d'_r\left(d''_r\binom{|d|-1}{|d''|-1}-d'_r\binom{|d|-1}{|d'|}\right)\langle\rangle^{\phi_3}_{d'}\langle\rangle^{\phi_3}_{d''}.$$

Along with the $\phi=\phi_3$ case of (31) and (34), ($\phi_3 1a$) and ($\phi_3 1b$) determine all real GW-invariants $\langle\mathbf{m}\rangle^{\phi_3}_d$ of $((\mathbb{P}^1)^3,\phi_3)$; some of them are shown in Table 19.

In light of (31), changing the sign of all invariants $\langle\mathbf{m}\rangle^{\phi_3}_d$ with $k_d(\mathbf{m})\equiv 3 \mod 4$ would not invalidate either of the relations of Theorem 3 for $((\mathbb{P}^1)^3,\phi_3)$. Thus, the last identity in (34) is not redundant. Since only one element of the preimage of each L_i under the first homomorphism in (12) can be represented by a holomorphic curve, the invariants $\langle\mathbf{m}\rangle^{\phi_3}_d$ provide lower bounds for the counts of real holomorphic curves in $((\mathbb{P}^1)^3,\phi_3)$ through line and point constraints.

8.2. The twisted involution

By the sentence immediately after (32) and the second identity in (32),

$$N^{\phi'_3,\mathfrak{os}_0}_{(d,d,0);a_1a_2a_3b}=\begin{cases}(-1)^dN^{\tau'_{1,1}}_{d,2d-1} & \text{if } a_1,a_2,b=0, a_3=2d-1,\\ 0 & \text{otherwise,}\end{cases}$$

(35)
$$N^{\phi'_3,\mathfrak{os}_0}_{(d,d,1);a_1a_2a_3b}=\begin{cases}0 & \text{if } 2d+a_1+a_2\le a_3+1,\\ (-1)^{d-1}N^{\tau'_{1,1}}_{d;2d-2} & \text{if } a_1,a_2,b=0, a_3=2d-2,\\ (-1)^dN^{\tau'_{1,1}}_{d,2d-1} & \text{if } a_1,a_2=0, b=1, a_3=2d-2,\\ (-1)^d dN^{\tau'_{1,1}}_{d,2d-1} & \text{if } a_1=1, a_2,b=0, a_3=2d-1.\end{cases}$$

Along with (6), this gives

(36)
$$\langle\rangle^{\phi'_3}_{e_3}=1,\qquad \langle H_1H_2\rangle^{\phi'_3}_{e_1+e_2}=-1,\qquad \langle\rangle^{\phi'_3}_{e_1+e_2+e_3}=1.$$

Let $e_1,e_2\in\mathbb{Z}^2$ be the standard basis elements, and let

$$H'_1=\frac{1}{2}(H_1+H_2),\qquad H'_2=H_3.$$

Thus, H'^2_1, $H'_1H'_2$, and $H'^2_1H'_2$ are the Poincaré duals of $L_3/2$, $(L_1+L_2)/2$, and pt/2, respectively. For elements $d=(a,b)$ and $m=(r,s)$ of $(\mathbb{Z}^{\ge 0})^2$, define

$$|d|=a+b,\qquad d_1=a,\qquad d_2=b,\qquad H'^m=H'^r_1H'^s_2.$$

If in addition $\mathbf{m}\equiv(m_1,\dots,m_l)$ is an element of $((\mathbb{Z}^{\ge 0})^2)^l$, let

$$k_d(\mathbf{m}) = 2a+b+l-\sum_{i=1}^{l}|m_i|, \qquad \langle \mathbf{m}\rangle_d^{\phi_3'} = \langle H'^{m_1},\ldots,H'^{m_l}\rangle_{a(L_1+L_2)+bL_3;\psi}^{\phi_3',\mathfrak{os}_0},$$

$$\langle d\rangle_{\mathbf{m}} = \begin{cases} 1 & \text{if } k_d(\mathbf{m})=2,\\ 0 & \text{otherwise,}\end{cases}$$

$$\langle \mathbf{m}\rangle_d = \begin{cases} \sum_{\substack{a'+a''=a\\ a',a''\ge 0}} \langle H'^{m_1},\ldots,H'^{m_l}\rangle_{a'L_1+a''L_2+(b/2)L_3}^{(\mathbb{P}^1)^3} & \text{if } b\in 2\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}$$

For the purposes of applying the first recursion of Theorem 3 in this case, we replace the set $[N]$ indexing a basis for (20) by $\{0,1,2\}\times\{0,1\}$, and we take

$$\mu_i^\bigstar = H^i \quad \forall i\in\{0,1,2\}\times\{0,1\}.$$

The sum over $[N]$ then becomes a sum over $i,j\in\{0,1,2\}\times\{0,1\}$ with $i+j=(2,1)$ and with g^{ij} replaced by 2. We apply Theorem 3 with $B=d_1(L_1+L_2)+d_2L_3$, l replaced by $l+1$, and

$$(\mu_1,\ldots,\mu_{l+1}) = (H'_r, H'^{m_1}, H'^{m_2},\ldots,H'^{m_l}).$$

($\phi_3'1a$) If $d\in(\mathbb{Z}^{\ge 0})^2-\{0\}$, $l\ge 1$, $\mathbf{m}\equiv(m_1,\ldots,m_l)\in((\mathbb{Z}^{\ge 0})^2-\{0\})^l$ with $k_d(\mathbf{m})\ge 2$, and $r=1,2$, then

$$\begin{aligned}
&\langle m_1+e_r, m_2,\ldots,m_l\rangle_d^{\phi_3'}\\
&\quad = -2^{l-1}\langle d\rangle_{\mathbf{m}} d_r \big\langle m_1,m_2,\ldots,m_l,(2,1)\big\rangle_d\\
&\qquad - \sum_{\substack{d',d''\in(\mathbb{Z}^{\ge 0})^2-\{0\}\\ d'+d''=d}} d'_r \sum_{(I,J)\in\mathcal{P}(l)} 2^{|I|-1} \sum_{\substack{i,j\in(\mathbb{Z}^{\ge 0})^2-\{0\}\\ i+j=(2,1)}} \langle \mathbf{m}_I, i\rangle_{d'}\langle j,\mathbf{m}_J\rangle_{d''}^{\phi_3'}\\
&\qquad + \sum_{\substack{d',d''\in(\mathbb{Z}^{\ge 0})^2-\{0\}\\ d'+d''=d}} \sum_{(I,J)\in\mathcal{P}(l)} \left(d''_r\binom{k_d(\mathbf{m})-2}{k_{d''}(\mathbf{m}_J)-1} - d'_r\binom{k_d(\mathbf{m})-2}{k_{d'}(\mathbf{m}_I)}\right)\\
&\qquad \times \langle \mathbf{m}_I\rangle_{d'}^{\phi_3'}\langle \mathbf{m}_J\rangle_{d''}^{\phi_3'}.
\end{aligned}$$

The derivation is almost the same as that of the recursion ($\tau_3 1$) in Section 7. However, the divisor relations (24) and (25) are replaced by

$$\begin{aligned}
&\langle \mu_I\rangle_{d_1'(L_1+L_2)+d_2'L_3;\psi}^{\phi_3',\mathfrak{os}_0} \langle \mu_J\rangle_{d_1''(L_1+L_2)+d_2''L_3;\psi}^{\phi_3',\mathfrak{os}_0}\\
&\qquad = d'_r \big\langle (m_{i-1})_{i\in I-\{1\}}\big\rangle_{d'}^{\phi_3'} \big\langle (m_{j-1})_{j\in J}\big\rangle_{d''}^{\phi_3'} \qquad \text{and}\\
&\sum_{\substack{a'+a''=d_1'\\ a',a''\ge 0}} \langle \mu_I,\mu_i^\bigstar\rangle_{a'L_1+a''L_2+(d_2'/2)L_3}^{(\mathbb{P}^1)^3} \langle \mu_j^\bigstar,\mu_J\rangle_{d''L;\psi}^{\phi_3',\mathfrak{os}_0}\\
&\qquad = \frac{d'_r}{2}\big\langle (m_{i'-1})_{i'\in I-\{1\}}, i\big\rangle_{d'} \big\langle j, (m_{j'-1})_{j'\in J}\big\rangle_{d''}^{\phi_3'}
\end{aligned}$$

for all $(I,J)\in\mathcal{P}(l+1)$ and $d'\in(\mathbb{Z}^{\geq 0})^2-\{0\}$. Furthermore, the $\langle\mu_I,\mu_i^{\star}\rangle_0^{\mathbb{P}^3}$ statement in the last paragraph of the proof of $(\tau_3 1)$ and $(\tau_3 2)$ is replaced by

$$\langle\mu_I,\mu_i^{\star}\rangle_0^{(\mathbb{P}^1)^3}=\begin{cases}1/2 & \text{if } I=\{1,i'\} \text{ with } i'\neq 1, e_r+m_{i'-1}+i=(2,1),\\ 0 & \text{otherwise.}\end{cases}$$

The resulting $1/2$ is offset by $g^{ij}\in\{0,2\}$ in this case.

Let $d\equiv(a,b)\in(\mathbb{Z}^{\geq 0})^2-\{0,e_2\}$. Replacing d in $(\phi_3'1a)$ by $d+e_2$ and taking $r=2$ and $(l,m_1)=(1,e_2)$, we obtain

$$0=\sum_{\substack{a'+a''=a\\ b'+b''=b+1\\ (a',b'),(a'',b'')\neq(0,0)}}\left(b''\binom{2a+b-1}{2a''+b''-1}-b'\binom{2a+b-1}{2a'+b'}\right)\langle e_2\rangle^{\phi_3'}_{(a',b')}\langle\rangle^{\phi_3'}_{(a'',b'')}.$$

By (19), $\langle e_2\rangle^{\phi_3'}_{(a',b')}=b'\langle\rangle^{\phi_3'}_{(a',b')}$. By the second case of the first statement in (35), $\langle\rangle^{\phi_3'}_{(a',0)}=0$ for all a'. It is immediate that $\langle\rangle^{\phi_3'}_{(0,b')}=0$ for all $b'\geq 2$. Using the first statement in (36), we thus obtain the following.

$(\phi_3'1b)$ If $(a,b)\in(\mathbb{Z}^{\geq 0})^2-\{0,e_2\}$, then

$$(2a-1-b)\langle\rangle^{\phi_3'}_{(a,b)}$$
$$=\sum_{\substack{a'+a''=a\\ b'+b''=b+1\\ a',b',a'',b''\geq 1}}b'\left(b''\binom{2a+b-1}{2a''+b''-1}-b'\binom{2a+b-1}{2a'+b'}\right)\langle\rangle^{\phi_3'}_{(a',b')}\langle\rangle^{\phi_3'}_{(a'',b'')}.$$

Replacing d in $(\phi_3'1a)$ instead by $d+e_1$ and taking $r=1$ and $(l,m_1)=(1,2e_1)$, we obtain

$$0=-\big(\langle 2e_1,2e_1\rangle_{e_1}\langle e_2\rangle^{\phi_3'}_{(a,b)}+\langle 2e_1,e_1+e_2\rangle_{e_1}\langle e_1\rangle^{\phi_3'}_{(a,b)}\big)$$
$$+\sum_{\substack{a'+a''=a+1\\ b'+b''=b\\ (a',b'),(a'',b'')\neq 0}}\left(a''\binom{2a+b-1}{2a''+b''-1}-a'\binom{2a+b-1}{2a'+b'-1}\right)\langle 2e_1\rangle^{\phi_3'}_{(a',b')}\langle\rangle^{\phi_3'}_{(a'',b'')}.$$

By (29), $\langle 2e_1,2e_1\rangle_{e_1}=0$ and $\langle 2e_1,e_1+e_2\rangle_{e_1}=1/2$. By (19), $\langle e_1\rangle^{\phi_3'}_{(a,b)}=a\langle\rangle^{\phi_3'}_{(a,b)}$. By the second case of the first statement in (35), $\langle\rangle^{\phi_3'}_{(a'',0)}=0$ for all a''. By the first equation in (35) and the second statement in (36), $\langle 2e_1\rangle^{\phi_3'}_{(a',0)}=-\delta_{a,1}/2$. It is immediate that $\langle 2e_1\rangle^{\phi_3'}_{(0,b')}=0$ for all b' and $\langle\rangle^{\phi_3'}_{(0,b'')}=0$ for all $b''\geq 2$. We thus obtain the following.

Table 20. The counts $N_{(2,2,2);(a_1,a_1,2a_2)}$ of complex genus 0 degree $(2,2,2)$ curves in $(\mathbb{P}^1)^3$ through a_1, a_1, $2a_2$ generic representatives for the standard line classes L_1, L_2, L_3 and $6-a_1-a_2$ general points (the $\mathbb{C}$ line) and the signed count $N^{\phi_3'}_{(2,2,2);a_1 0 a_2 b}$ of real genus 0 degree $(2,2,2)$ curves in $(\mathbb{P}^1)^3$ through a_1 conjugate pairs of generic representatives for L_1, L_2, a_2 conjugate pairs of generic representatives for L_3, b conjugate pairs of points and $6-a_1-a_2-2b$ real points.

(a_1,a_2)	$(1,0)$	$(0,1)$	$(2,1)$	$(1,2)$	$(3,0)$	$(0,3)$	$(3,2)$	$(2,3)$	$(4,1)$	$(1,4)$	$(5,0)$	$(0,5)$
$\mathbb{C}$	16	12	110	48	64	0	788	192	672	0	256	0
$b=0$	4	4	-4	-4	0	0	0	0	-4	0	-4	0
$b=1$	-2	-2	0	0	-2	0						
$b=2$	0	0										

($\phi_3' 1c$) If $(a,b)\in(\mathbb{Z}^{\ge 0})^2-\{0\}$, then

$$\frac{b-1}{2}\langle\rangle^{\phi_3'}_{(a,b)}=\sum_{\substack{a'+a''=a+1\\ b'+b''=b\\ a',a'',b',b''\ge 1}}\left(a'\binom{2a+b-1}{2a'+b'-1}-a''\binom{2a+b-1}{2a''+b''-1}\right)\times\langle 2e_1\rangle^{\phi_3'}_{(a',b')}\langle\rangle^{\phi_3'}_{(a'',b'')}.$$

Along with the $\phi=\phi_3'$ case of (31) and (34), the three recursions above determine all real GW-invariants $\langle\mathbf{m}\rangle^{\phi_3'}_d$; some of them are shown in Table 20.

For the same reasons as for the product involution of Section 8.1, the last identity in (36) is not redundant. Since only one element of the preimage L_1+L_2 and L_3 under the first homomorphism in (12) can be represented by a holomorphic curve, the invariants $\langle\mathbf{m}\rangle^{\phi_3'}_d$ provide lower bounds for the counts of real curves in $((\mathbb{P}^1)^3,\phi_3')$ through line and point constraints.

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