Stabilized convex symplectic manifolds are Weinstein

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Abstract We show that a stabilized convex symplectic (or Liouville) manifold with the homotopy type of a half-dimensional CW-complex is symplectomorphic to a flexible Weinstein manifold.

1. Introduction

1.1. Convex symplectic manifolds

Recall that a primitive λ of a symplectic form ω, $d\lambda = \omega$, is called a *Liouville form*, and the vector field Z which is ω-dual to λ, $\iota(Z)\omega = \lambda$, is called a *Liouville vector field*. The equation $\iota(Z)\omega = \lambda$ is equivalent to the equation $L_Z\omega = \omega$, where L_Z is the Lie derivative. That is, Liouville vector fields are *conformally symplectic*.

An open symplectic manifold (V, ω) with an exact symplectic form ω is called *symplectically convex* (see [4]), or *Liouville*, if there exists a Liouville form λ such that the corresponding Liouville vector field Z is complete and there exists an exhaustion $\bigcup_{j=1}^{\infty} V_j$, $V_j \subset V_{j+1}$, by compact domains V_j with smooth boundaries ∂V_j such that Z is outward transverse to ∂V_j. The domains V_j with this property are called *Liouville domains*.

Given a Liouville domain (V_1, λ), the attractor $\operatorname{Core}(V_1, \lambda) := \bigcap_{t>0} Z^{-t}(V_1)$ of the field $-Z$ is called the *core* of the Liouville domain. For a convex symplectic manifold $V = \bigcup_{j=1}^{\infty} V_j$ with a fixed Liouville form λ, we define its core as $\operatorname{Core}(V, \lambda) := \bigcup_{j=1}^{\infty} \operatorname{Core}(V_j, \lambda)$. Equivalently, we can define $\operatorname{Core}(V, \lambda)$ as

$$\operatorname{Core}(V, \lambda) = \bigcup_{K \subset V, \text{compact}} \bigcap_{t>0} Z^{-t}(K),$$

and this definition shows the independence of $\operatorname{Core}(V, \lambda)$ from the choice of exhausting Liouville domains. Of course, the core does depend on the choice of the Liouville form λ.

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An important class of convex symplectic manifolds is formed by convex symplectic manifolds of finite type, or as they are also called *convex symplectic manifolds with cylindrical ends*. One says that (V, ω) is a convex symplectic manifold of *finite type* if it admits a Liouville form λ with a Liouville vector field Z and a compact Liouville subdomain $V_1 \subset V$, that is, a domain with boundary ∂V_1 transverse to Z, such that each point of $V \setminus V_1$ belongs to a Z trajectory originating from a point of ∂V_1. The manifold (V, ω) can be identified with the *completion* of the Liouville domain V_1, that is, attaching to V_1 the cylindrical end $([0,\infty) \times \partial V_1, d(e^s(\lambda|_{\partial V_1})))$, where s is the coordinate corresponding to the factor $[0, \infty)$.

Note that the definition of a convex symplectic manifold of finite type fits into the definition of a general convex symplectic manifold by taking the translates $V_n := Z^{n-1}(V_1)$, $n \geq 1$, as the required exhausting sequence. For a finite type (V, λ), its core is compact: $\mathrm{Core}(V, \lambda) = \mathrm{Core}(V_1, \lambda)$, and conversely finite-type convex symplectic manifolds can be characterized among convex symplectic manifolds as those which have a compact core for some choice of the Liouville form.

By fixing a cylindrical end structure, a contact structure is induced on the ideal boundary $\partial_\infty V \cong \partial V_1$. However, this contact boundary is not determined by the symplectomorphism type of V. In fact, as was shown by Sylvain Courte in [2], even the diffeomorphism type of $\partial_\infty V$ can depend on the choice of the cylindrical end structure on a given convex symplectic manifold of finite type.

1.2. Weinstein manifolds

We say that a convex symplectic manifold V is of *Weinstein type*, or simply *Weinstein*, if the corresponding Liouville vector field Z can be chosen to admit a Lyapunov function $\phi\colon V \to \mathbb{R}$ which is Morse or generalized Morse (i.e., possibly with death-birth singularities). The Lyapunov condition means that $|d\phi(Z)| \geq c\|Z\|^2$ for a positive function $c > 0$ and some choice of a Riemannian metric on V. We note that the Lyapunov function ϕ can always be modified to be *exhausting* (i.e., proper and bounded below) and constant on boundaries ∂V_j of domains V_j implied by the definition of symplectic convexity. The core of a Weinstein manifold (V, λ) is stratified by Z-stable manifolds of zeros of Z, which are isotropic (see [4], [1]). Hence, the critical points of a Lyapunov Morse function for a Liouville field have index at most $n = \frac{1}{2}\dim V$, and therefore, any Weinstein manifold admits an exhausting Morse function with critical points of index at most n; that is, it has *Morse type at most* n and, in particular, is homotopy equivalent to an n-dimensional CW-complex.

Not every convex symplectic manifold is Weinstein. Indeed, it may have Morse type greater than n. The first example of this type, a 4-dimensional convex symplectic manifold of Morse type 3, was constructed by Dusa McDuff in [12]. More examples were constructed in [8], [13], and [10].

The product $(V, \omega) \times (V', \omega')$ of two symplectically convex manifolds is symplectically convex, and the product of two Weinstein manifolds is Weinstein. If

$(V', \omega') = (\mathbb{R}^{2k}, \omega_{\mathrm{st}})$, then the product $(V, \omega) \times (\mathbb{R}^{2k}, \omega_{\mathrm{st}})$ is called the *stabilization*, or *k-stabilization*, of (V, ω).

1.3. Main results

We prove the following theorem.

THEOREM 1.1

Let V be a $(2n-2)$-dimensional convex symplectic manifold of Morse type at most n. Then its 1-stabilization X is Weinstein, and moreover, if $n \geq 3$, then it is flexible Weinstein (see Section 2 below for the definition and discussion of flexibility). In particular, the 1-stabilization of McDuff's example in [12] is Weinstein.

REMARK 1.2

(1) It was proved in [4] that, for any two tangentially homotopy equivalent convex symplectic manifolds, their 2-stabilizations are symplectomorphic. Moreover, it was shown that for Weinstein manifolds 1-stabilization is sufficient. This implies that a 2-stabilization of a $(2n-2)$-dimensional convex symplectic manifold of Morse type at most n is Weinstein. The improvement in this paper became possible thanks to the development of the theory of flexible Weinstein manifolds (see [1]).

(2) Even if V is of finite type, we do not know whether the ideal contact boundary of its stabilization is isomorphic to the ideal contact boundary of the corresponding flexible Weinstein manifold.

Theorem 1.1 is a corollary of a more general theorem which we formulate below.

Given a manifold V and a closed subset $A \subset V$, we say that each point of A *has an access to infinity* if every compact subset $B \subset A$ has an arbitrarily small open neighborhood $U \supset B$ such that each connected component C of $V \setminus U$ is not compact.

For instance, if V is a noncompact connected manifold and $A \subset V$ is a closed (as a subset) submanifold of V of codimension greater than 1, then each point of A has an access to infinity. For codimension 1 connected submanifolds, the condition is violated only for compact submanifolds homological to 0.

If V is a symplectic manifold and $A \subset V$ is a locally closed subset, then we say that A *admits a symplectic extension of positive codimension* if for each point $a \in A$ there exist a neighborhood $U_a \ni a$ in V and a closed (as a subset) symplectic submanifold $\Sigma_a \subset U_a$ of positive codimension such that $U_a \cap A \subset \Sigma_a$, and each point of $U_a \cap A$ has an access to infinity in Σ_a.

THEOREM 1.3

Let (X, ω) be a $2n$-dimensional, $n \geq 3$, convex symplectic manifold of Morse type at most n. Suppose that, for an appropriate choice of a Liouville form λ, its core $C := \mathrm{Core}(X, \lambda)$ can be presented as a finite or countable union $C = \bigcup_{i \geq 1} C_i$ of *disjoint sets C_i which admits a symplectic extension of positive codimension and such that $\bigcup_{i\le j} C_i$ is compact for all $j \ge 1$. Then X is symplectomorphic to a flexible Weinstein manifold.*

The core of any stabilized convex symplectic manifold clearly admits a symplectic extension of positive codimension, and hence, Theorem 1.1 is a special case of Theorem 1.3.

Here is another corollary of Theorem 1.3. We say that a submanifold A of a symplectic manifold V is *nowhere coisotropic* if each tangent plane $T_xA \subset T_xV$ is not coisotropic, that is, $(T_xA)^{\perp_\omega} \not\subseteq T_xA$.

THEOREM 1.4

Let X be a $2n$-dimensional, $n \ge 3$, convex symplectic manifold of Morse type at most n. Suppose that, for an appropriate choice of a Liouville form λ, the core of X admits a stratification $\mathrm{Core}(X,\lambda) = \bigcup_{i\ge 1} S_i$ of codimension at least 3 such that each stratum is nowhere coisotropic. Then the convex symplectic manifold X is Weinstein and, moreover, flexible Weinstein.

We use above the term stratification in a weak sense. A *stratified* closed subset $A \subset V$ is a closed set presented as a finite or countable union of locally closed submanifolds A_i, called *strata*, $A := \bigcup_{i\ge 1} A_i$ such that all unions $\bigcup_{i\le j} A_i$ are compact.

Proof of Theorem 1.4

It is sufficient to prove that each stratum S_i admits a symplectic extension of positive codimension. Note that any non-coisotropic subspace A of codimension at least 3 in a symplectic vector space (B,ω) is contained in a symplectic subspace $C \subset B$ such that $\dim A < \dim C < \dim B$. Indeed, take a vector $v \in A^{\perp_\omega} \setminus A$, and consider its ω-orthogonal complement subspace $v^{\perp_\omega} \subset B$. Note that $A \subset v^{\perp_\omega}$. Then any codimension 1 subspace $C \subset v^{\perp_\omega}$ which is transverse to v and contains A is a codimension 2 symplectic subspace of B. Given $x \in S_i$, we therefore can find a $(2n-2)$-dimensional symplectic subspace C_x so that $T_xS_i \subset C_x \subset T_xX$. Let us choose a complementary subspace $\theta_x \subset C_x$ such that $\theta_x \oplus T_xS_i = C_x$, and extend it continuously to a field θ of planes transverse to S_i on a neighborhood U_x of x in S_i. If the neighborhood U_x is small enough, then the space $C_y := \mathrm{Span}(\theta_y, T_yS_i)$ is symplectic for each $y \in U_x$, and so is the codimension 2 symplectic hypersurface Σ containing U_x and tangent to the plane field θ. □

Organization

In Sections 2 and 3, we review Weinstein flexibility (see [1], [7], [5]) and Gromov's h-principle for exact symplectic embeddings of open symplectic manifolds (see [9], [6]), respectively. Section 4 is a remark on the existence of a Liouville homotopy and a symplectomorphism. With the help of the above tools, we first prove Theorem 1.1 in Section 5 to make the main ideas more transparent, and then we prove Theorem 1.3 in Section 6.

2. Recollection of Weinstein flexibility

2.1. Loose Legendrian knots

We recall that an $(n-1)$-dimensional submanifold Λ of a $(2n-1)$-dimensional contact manifold (M,ξ) is called *Legendrian* if it is tangent to ξ. The contact plane field ξ carries a canonical conformal symplectic structure, and tangent planes to a Legendrian are Lagrangian subspaces of ξ for that conformal structure. A formal Legendrian submanifold $\Lambda \to M$ is an $(n-1)$-dimensional smooth submanifold together with a homotopy of its tangent planes to a field of Lagrangian subspaces of ξ.

In [14], Emmy Murphy introduced a class of so-called *loose* Legendrian submanifolds in contact manifolds of dimension at least 5 for which a formal Legendrian isotopy between Legendrian embeddings yields a genuine Legendrian isotopy. We define this class below. We begin with an operation of *stabilization* of a Legendrian submanifold which was first introduced in [3] (see also [14], [1]).

In $\mathbb{R}^{2n-1}$, $n \geq 3$, with the standard contact form $\alpha = dz - \sum_{i=1}^{n-1} y_i\, dx_i$ consider a Legendrian submanifold Λ_0 with the front $F_0 = \{z^2 = x_1^3\}$, that is, $\Lambda_0 = \{z^2 = x_1^3, 4y_1^2 = 9x_1, y_2 = \cdots = y_{n-1} = 0\}$. Let $\mathbb{R}^{n-1} = \{z = 0, y = 0\}$ be the x-coordinate subspace, and let $\mathbb{R}^{n-1}_+ = \{x_1 > 0\} \cap \mathbb{R}^{n-1}$. Choose open domains U and $U' \Subset U$ with smooth boundaries, and let $\theta\colon \mathbb{R}^{n-1}_+ \to \mathbb{R}$ be a function supported in U such that $U' = \{x \in \mathbb{R}^{n-1}_+ \mid \theta(x) > 2x_1^{\frac{3}{2}}\}$. Let $\Lambda_U \subset \mathbb{R}^{2n-1}$ be a Legendrian submanifold whose front is obtained from F_0 by replacing the branch $z = -x_1^{\frac{3}{2}}$ by the graph $z = -x_1^{\frac{3}{2}} + \theta(x)$. The Legendrian submanifold Λ_U is called the *U-stabilization* of Λ_0. Given any Legendrian submanifold Λ in a contact manifold (M,ξ), one can find Darboux coordinates in an arbitrarily small neighborhood U of a point $a \in \Lambda$ such that the pair $(U, \Lambda \cap U)$ is contactomorphic to $(\mathbb{R}^{2n-1}, \Lambda_0)$. Hence, the U-stabilization operation can be performed on Λ in a neighborhood $U \ni a$. We will keep the notation Λ_U for the stabilized Legendrian. As was shown in [3], the Legendrians Λ and Λ_U are always smoothly isotopic, and if $\chi(U) = 0$, then they are *formally Legendrian isotopic*.

A connected Legendrian submanifold $\Lambda \subset M$ is called *loose* if it can be *destabilized*, that is, it is Legendrian isotopic to a stabilization of another Legendrian knot. A possibly disconnected Legendrian is called loose if each of its components is loose in the complement of the others.

THEOREM 2.1 ([14, Theorem 1.3])

For any contact manifold of dimension at least 5*, the inclusion of the space of loose Legendrian embeddings into the space of formal Legendrian embeddings is a homotopy equivalence.*

2.2. Flexible Weinstein manifolds

The notion of *Weinstein flexibility*, introduced in [1], is based on the theory of loose Legendrians. Let (V,λ,ϕ) be a $2n$-dimensional Weinstein manifold. Consider its partition into *elementary cobordisms*: $V = W_1 \cup \cdots \cup W_m \cup \cdots$, where $W_i = \phi^{-1}([c_{i-1}, c_i])$ for regular values c_i of ϕ separating the critical values a_i of ϕ, that is, $c_0 < a_0 < c_1 < a_1 < c_2 < \cdots$. Each cobordism W_i deformation retracts onto the union of the stable disks (with respect to the Liouville vector field Z) of its critical points and the stable disk of an index k critical point of value a_i intersects the level set $M_i = \phi^{-1}(c_i)$ in the $(k-1)$-dimensional isotropic *attaching sphere* for the contact structure $\{\lambda|_{M_i} = 0\}$.

The Weinstein structure (V,λ,ϕ) is called *flexible* if for each cobordism W_i the attaching Legendrian spheres of critical points of index n on the level a_i form a loose Legendrian link in the contact level set M_i. In particular, subcritical Weinstein manifolds (i.e., those for which ϕ has no critical points of index n) are flexible.

The following h-principle-type result clarifies the term "flexible."

THEOREM 2.2 ([1, Sections 13.1])

For Weinstein structures on a fixed manifold or domain V of dimension $2n \geq 6$, the following statements hold.

(1) *(Existence) Given a nondegenerate 2-form η and an exhausting Morse function $\phi\colon V \to \mathbb{R}$ without critical points of index greater than n, there exists a flexible Weinstein structure (λ,ϕ) (with the same function ϕ) such that η and $d\lambda$ are homotopic as nondegenerate 2-forms.*

(2) *(Homotopy) Two flexible Weinstein structures (λ_0,ϕ_0) and (λ_1,ϕ_1) are Weinstein homotopic if and only if $d\lambda_0$ and $d\lambda_1$ are homotopic as nondegenerate 2-forms.*

(3) *(Morse–Smale theory for Lyapunov functions) Given a flexible Weinstein structure (λ,ϕ) and any Morse function $\psi\colon V \to \mathbb{R}$ without critical points of index greater than n, there exists a Weinstein homotopy (λ_t,ϕ_t) with $(\lambda_0,\phi_0) = (\lambda,\phi)$ and $\phi_1 = \psi$.*

The definition of flexibility naturally extends to *Weinstein cobordisms*.

It is important to point out that the flexibility property is not invariant under Weinstein homotopy (see [15]). When calling a symplectic manifold flexible Weinstein, we always mean the existence of a flexible Weinstein structure for the given symplectic form.

2.3. Symplectic embeddings of flexible Weinstein manifolds

For two symplectic manifolds (W,ω) and (X,η), a *formal symplectic embedding* of W into X is a smooth embedding $f\colon W \to X$ together with a homotopy $\Phi_t\colon TW \to f^*TX$, $t \in [0,1]$, of injective bundle homomorphisms such that $\Phi_0 = df$ and $\Phi_1^*\eta = \omega$. Any genuine symplectic embedding $f\colon X \to W$ can be considered formal by setting $\Phi_t \equiv df$.

A symplectic embedding $f\colon (W,d\lambda)\to(X,d\mu)$ between two exact symplectic manifolds with fixed Liouville forms is called *exact* if $f^*\mu=\lambda+dH$ for some function H on W. Note that if W is compact, then given an exact symplectic isotopy $f_t\colon (W,d\lambda)\to(X,d\mu)$ there exists an ambient Hamiltonian isotopy $F_t\colon X\to X$ such that $F_t|_{f_0(W)}=f_t$, $t\in[0,1]$. Hence, we will refer in this paper to an exact symplectic isotopy as a *Hamiltonian* isotopy.//

THEOREM 2.3 ([7, Corollary 6.3])

Let (W,λ,ϕ) be a $2n$-dimensional Weinstein domain, and let W_0 be its Weinstein subdomain. Suppose that the Weinstein cobordism $(W\setminus \operatorname{Int}W_0,\lambda,\phi)$ is flexible. Let $(X,d\mu)$ be a convex symplectic manifold of the same dimension $2n$ such that the Liouville vector field Z dual to μ is forward complete.

(1) *Any formal symplectic embedding $f\colon (W,d\lambda)\to(X,d\mu)$ which is a genuine exact symplectic embedding on W_0 is formally isotopic relative to W_0 (rel. W_0) to a genuine exact symplectic embedding.*

(2) *Any two exact symplectic embeddings $f_0,f_1\colon (W,d\lambda)\to(X,d\mu)$ which coincide on W_0 and are formally isotopic rel. W_0 can be connected by a Hamiltonian isotopy $f_t\colon (W,d\lambda)\to(X,d\mu)$, $t\in[0,1]$, fixed on W_0.*

The nonparametric part (1) is proved in [7]. The parametric part (2) will appear in [5].

3. h-Principle for symplectic embeddings

In this section, we review Gromov's h-principle for symplectic embeddings of open symplectic manifolds (see [9]; see also [6]). We continue to use in this paper the term "symplectic" rather than *isometric* as in [9, Section 3.4.2(B)] or *isosymplectic* as in [6, Theorem 12.1.1].

Recall that given a manifold V and a closed subset $A\subset V$ we say that each point of A *has an access to infinity* if every compact subset $B\subset A$ has an arbitrarily small open neighborhood $U\supset B$ such that each connected component C of $V\setminus U$ is not compact. Slightly reformulating Gromov's h-principle for symplectic embeddings from [9], we have the following theorem. We use below Gromov's notation $\mathcal{O}p\,A$ for an unspecified neighborhood of a closed subset A.

THEOREM 3.1 ([9, Section 3.4.2(B)], see also [6, Theorem 12.1.1])

Let (W,ω) and (X,η) be symplectic manifolds of dimension $2n$ and $2m$, respectively. Suppose that X is an open manifold and that $m<n$. Then the following statements hold.

(1) *For a formal symplectic embedding (φ,Φ_t) of (X,η) into (W,ω), there exists a symplectic embedding $f\colon (X,\eta)\to(W,\omega)$ formally isotopic to (φ,Φ_t).*

(2) *Any symplectic embeddings $f_0,f_1\colon (X,\eta)\to(W,\omega)$ which are formally isotopic can be connected by a symplectic isotopy $f_t\colon (X,\eta)\to(W,\omega)$, $t\in[0,1]$.*

(3) *Let $A \subset X$ be a closed subset such that each of its points has an access to infinity, and let (φ, Φ_t) be a formal symplectic embedding of (X, η) into (W, ω) which is a genuine symplectic embedding on $\mathcal{O}p\, A$. Then, there exists a symplectic embedding $f\colon (X, \eta) \to (W, \omega)$ formally isotopic to (φ, Φ_t) rel. A.*

(4) *Let A be as in (3). Then for any two symplectic embeddings $f_0, f_1\colon X \to W$ which coincide on $\mathcal{O}p\, A$ and are formally isotopic rel. A, there exists a symplectic isotopy $f_t\colon (X, \eta) \to (W, \omega)$, $t \in [0, 1]$, fixed on $\mathcal{O}p\, A$.*

If symplectic forms $\omega = d\lambda$ and $\eta = d\mu$ are exact, then one can talk about *exact* symplectic embeddings $f\colon X \to W$ which satisfy the condition $f^*\lambda = \mu + dH$ (see Section 2.3 above).

PROPOSITION 3.2
Let $(X, \eta = d\mu)$ be a convex symplectic manifold. Then for any symplectomorphism $f_0\colon (X, \eta) \to (X, \eta)$, there exists a symplectic diffeotopy $f_t\colon (X, \eta) \to (X, \eta)$ such that f_1 is exact, that is, $f_1^*\mu = \mu + dH$ for a smooth function $H\colon X \to \mathbb{R}$.

For the case when (X, η) is a finite-type convex symplectic manifold, this was proven in [1, Lemma 11.2]. To prove the statement in the general case we will need the following two lemmas.

LEMMA 3.3
Let $((0, \infty) \times \Sigma, \omega = d(s\alpha))$ be the symplectization of a contact manifold $(\Sigma, \xi = \ker \alpha)$ with a fixed contact form α. Let R be the Reeb vector field of α (i.e., $\iota(R)\, d\alpha = 0$ and $\alpha(R) = 1$). For a closed 1-form θ on Σ, denote $\widehat{\theta} := \pi^*\theta$, and let Y be the symplectic vector field ω-dual to $\widehat{\theta}$, that is, $\iota(Y)\omega = \widehat{\theta}$. Here $\pi\colon (0, \infty) \times \Sigma \to \Sigma$ is the projection to the second factor. Then one has the equality $ds(Y) = \theta(R)$.

Proof
Let us write $Y = aR + b\frac{\partial}{\partial s} + Y_\xi$, where $Y_\xi \in \xi$ and $a, b\colon (0, \infty) \times \Sigma \to \mathbb{R}$. Then

$$\begin{aligned}\theta(R) &= \big(\iota(Y)(ds \wedge \alpha + s\, d\alpha)\big)(R) = ds \wedge \alpha\Big(aR + b\frac{\partial}{\partial s} + Y_\xi, R\Big) + s\, d\alpha(Y, R) = b \\ &= ds(Y).\end{aligned}$$

□

LEMMA 3.4
Let $(X, d\mu)$ be a convex symplectic manifold, and let Z be the Liouville field corresponding to μ. Then there exists an exhaustion $X = \bigcup_{j=1}^\infty X_j$ by compact domains with smooth boundaries transverse to Z such that the following condition is satisfied: for any $j \geq 1$ one has

$$Z^{\ln(1+2T_j)}(\partial X_j) \subset \operatorname{Int} X_{j+1}, \quad j \geq 1, \tag{1}$$

where $T_j := \max\{1, \max_{x \in \partial X_j} |\theta(R_j(x))|\}$ and R_j is the Reeb vector field of the contact form $\mu|_{\partial X_j}$.

Proof
We begin with any exhaustion $\bigcup_{j=1}^{\infty} X_j^0$ by compact domains with smooth boundaries transverse to Z and then inductively modify it to ensure the property (1) by using the following procedure. Set $X_1 := X_1^0$ and $T_1 := \max\{1, \max_{x\in\partial X_1} |\theta(R_1(x))|\}$, where R_1 is the Reeb vector field of the contact form $\mu|_{\partial X_1}$. Define $X_j^1 := Z^{\ln(1+3T_1)}(X_j^0)$, $j \geq 2$, and denote $X_2 := X_2^1$ and $T_2 := \max\{1, \max_{x\in\partial X_2} |\theta(R_2(x))|\}$, where R_2 is the Reeb vector field of the contact form $\mu|_{\partial X_2}$. Define $X_j^2 := Z^{\ln(1+3T_2)}(X_j^1)$, $j \geq 3$, and denote $X_3 := X_3^2$. Continuing this process we construct the required exhaustion. □

Proof of Proposition 3.2
We have $f_0^*\mu = \mu - \theta$ for a closed 1-form θ. Let Z be the Liouville field corresponding to the Liouville form μ. Choose an exhaustion $X = \bigcup_{j=1}^{\infty} X_j$ which satisfies property (1).

Consider disjoint domains

$$U_j := \bigcup_{t\in[0,\ln(1+2T_j)]} Z^t(\partial X_j) \subset X,$$

and set $\alpha_j := \mu|_{\partial X_j}$, $j \geq 1$. These domains can be identified with the domains $[1, 1+2T_j] \times \partial X_j$ in the symplectizations $((0,\infty) \times \partial X_j, d(s\alpha_j))$ of the contact manifolds $(\partial X_j, \ker \alpha_j)$ via Liouville isomorphisms

$$\phi_j \colon \big([1, 1+2T_j] \times \partial X_j, s\alpha_j\big) \to (U_j, \mu) : (s,x) \mapsto Z^{\ln s}(x),$$

where $s \in [1, 1+2T_j]$ and $x \in \partial X_j$. Let θ_j be the closed 1-form on U_j defined by the formula

$$\theta_j := (\phi_j)_* \pi_j^*(\theta|_{\partial X_j}),$$

where $\pi_j \colon [1, 1+2T_j] \times \partial X_j \to \partial X_j$ is the projection to the second factor. Note that $\theta|_{U_j} - \theta_j = dH_j$ for a smooth function $H_j \colon U_j \to \mathbb{R}$. Let $\delta_j \colon U_j \to \mathbb{R}$ be a cutoff function supported in U_j and equal to 1 on $\widehat{U}_j := \phi_j([1+T_j/2, 1+3T_j/2] \times \partial X_j)$. Then the closed 1-form

$$\widehat{\theta} = \theta - dG, \quad \text{where } G = \Big(\sum_{j=1}^{\infty} \delta_j H_j\Big),$$

coincides with θ_j on $\widehat{U}_j$ for all $j \geq 1$. We claim that the symplectic vector field Y which is η-dual to $\widehat{\theta}$ is complete, that is, its flow Y^t is defined for all time $t \in \mathbb{R}$. Indeed, according to Lemma 3.3 any trajectory entering U_i spends time at least 1 there, and hence, in time at most T it can cross only finitely many domains U_j. Note that $L_Y\widehat{\theta} = \iota(Y)\, d\widehat{\theta} + d(\iota(Y)\widehat{\theta}) = 0$ and $L_Y\mu = \iota(Y)\eta + d(\mu(Y)) = \widehat{\theta} + dH'$, where $H' := \mu(Y)$. Hence, by defining $H'_t := \int_0^t (H' \circ Y^s)\, ds$, we compute $(Y^t)^*\widehat{\theta} = \widehat{\theta}$ and $(Y^t)^*\mu = \mu + t\widehat{\theta} + dH'_t$. We can now define the required isotopy by the formula $f_t := f_0 \circ Y^t$. Then

$$
\begin{aligned}
f_1^*\mu &= (Y^1)^* f_0^*\mu = (Y^1)^*(\mu - \theta) = (Y^1)^*(\mu - \widehat{\theta} - dG) \\
&= \mu + \widehat{\theta} + dH_1' - \widehat{\theta} - d(G \circ Y^1) = \mu + d(H_1' - G \circ Y^1). \qquad \square
\end{aligned}
$$

Hence, if in Theorem 3.1 we assume that $\omega = d\lambda$, $\eta = d\mu$, and $(X, d\mu)$ is symplectically convex, then we can arrange the constructed symplectic embeddings in (1) and (3) to be exact and the symplectic isotopies in (2) and (4) to be Hamiltonian.

4. Liouville homotopy versus symplectomorphism

The following notion of Liouville homotopy, which formalizes the concept of a smooth family of convex symplectic structures on a given manifold, was introduced in [1]. A smooth family μ_s, $s \in [0,1]$, of Liouville forms on a manifold X is called a *simple Liouville homotopy* if there exists a smooth family of exhaustions $X = \bigcup_{k=1}^{\infty} X_s^k$ by compact domains $X_s^k \subset X$ with smooth boundaries along which the corresponding Liouville field Z_s is outward pointing. A *Liouville homotopy* is a composition of finitely many simple Liouville homotopies. It was shown in [1, Proposition 11.8] that given a Liouville homotopy μ_s there exists an isotopy $\varphi_s \colon X \to X$, $s \in [0,1]$, starting from $\varphi_0 = \mathrm{id}_X$ such that $\varphi_s^*\mu_s = \mu_0 + dH_s$, and in particular, the forms $\widehat{\mu}_s := \varphi_s^*\mu_s$ are Liouville for the same symplectic structure $\omega = d\mu_0$. The following proposition shows that the converse is also true.

PROPOSITION 4.1

Let (X,ω) and (X',ω') be two symplectomorphic convex symplectic manifolds. Then there exist a symplectomorphism $\varphi\colon (X,\omega) \to (X',\omega')$ and a Liouville homotopy μ_s connecting $\mu_0 = \mu$ and $\mu_1 = \varphi^*\mu'$.

Proof

According to Proposition 3.2 the symplectomorphism φ can be chosen to satisfy $\varphi^*\mu' = \mu + dH$ for a smooth function H on X. Choose the exhaustions $X = \bigcup_{j=1}^{\infty} X_j^0$ and $X = \bigcup_{j=1}^{\infty} X_j^1$ defining convex structures for forms $\mu_0 := \mu$ and $\mu_1 := \mu + dH$, respectively. We can arrange that

$$X_j^1 \subset \operatorname{Int} X_j^0 \subset X_j^0 \subset \operatorname{Int} X_{j+1}^1$$

for all $j \geq 1$. Let $\widetilde{H}$ be a smooth function which is equal to 0 on $\mathcal{O}p(\bigcup_{j=1}^{\infty} \partial X_j^0)$ and equal to H on $\mathcal{O}p(\bigcup_{j=1}^{\infty} \partial X_j^1)$. Then the required Liouville homotopy can be defined as the composition of two simple Liouville homotopies:

$$
\mu_s := \begin{cases} \mu_0 + 2s\, d\widetilde{H}, & s \in [0,1/2], \\ \mu_0 + d\widetilde{H} + (2s-1)\, d(H - \widetilde{H}), & s \in [1/2,1], \end{cases}
$$

with the constant exhaustions $X = \bigcup_{j=1}^{\infty} X_j^0$ and $X = \bigcup_{j=1}^{\infty} X_j^1$, respectively. $\square$

The notion of *Weinstein homotopy* can be defined in a similar way. However, it is unknown whether two Weinstein structures on the same symplectic manifold are homotopic.

5. Proof of Theorem 1.1

If $n = 2$, then $\dim V = 2$. Any 2-dimensional convex symplectic manifold is Weinstein and the theorem is trivially true. Hence, we assume that $n \geq 3$.

Choose a Liouville form μ_V on the convex symplectic manifold V, and denote by μ the corresponding stabilized Liouville form $\mu_V + \frac{1}{2}(x\,dy - y\,dx)$ on $X = V \times \mathbb{R}^2$. Denote $\eta := d\mu$. By the assumption, X is of Morse type at most n. Take an exhausting Morse function $\phi\colon X \to \mathbb{R}$ without critical points of index greater than n. Applying Theorem 2.2(1) to the pair (η, ϕ), we obtain a flexible Weinstein structure $\mathfrak{W} = (\omega = d\lambda, \phi)$ on X such that symplectic forms η and ω are homotopic as nondegenerate 2-forms. For the sake of convenience, the ambient space of $\mathfrak{W}$ is denoted by W instead of X. Thus, there exists a pair (φ, Φ_s), where $\varphi\colon X \to W$ is the identity and $\Phi_s\colon TX \to TW$, $s \in [0,1]$, is a homotopy of bundle isomorphisms covering φ starting at $\Phi_0 = d\varphi$ and ending at a symplectic isomorphism $\Phi_1 = \Phi\colon (TX, \eta) \to (TW, \omega)$.

The goal of this section is to construct an exact symplectomorphism $F\colon (X, \mu) \to (W, \lambda)$. This will be given by the telescope construction, the so-called *Mazur trick* (see [11]), following the scheme of the proof in [4, Proposition 2.2.A]. Take exhaustions $X_1 \subset X_2 \subset \cdots \subset X$, $\bigcup_{i=1}^{\infty} X_i = X$, by Liouville subdomains of X and $W_1 \subset W_2 \subset \cdots \subset W$, $\bigcup_{i=1}^{\infty} W_i = W$, by Weinstein subdomains of W.

Set $\mu_i = \mu|_{X_i}$ and $\lambda_i = \lambda|_{W_i}$ for $i \geq 1$. The construction is split into several steps.

LEMMA 5.1
For each $i \geq 1$ there exists an exact symplectic embedding $f_i\colon (X_i, \mu_i) \to (W, \lambda)$ which is formally isotopic to $(\varphi|_{X_i}, \Phi|_{TX_i})$.

Proof
Since V is an open symplectic manifold of dimension $2n - 2 < \dim W$, we can apply Theorem 3.1(1) to $(\varphi|_V, \Phi_s|_{TV})$ and obtain an exact symplectic embedding $V = V \times \{0\} \to W$. Moreover, it extends to an open neighborhood U of V, and hence, we get an exact symplectic embedding $h_U\colon (U, \mu|_U) \to (W, \lambda)$ which is formally isotopic to $(\varphi|_U, \Phi|_{TU})$.

We have $\mathrm{Core}(X_i, \mu_i) \subset V \subset U$, and therefore, there exists $t_i > 0$ such that $Z_\mu^{-t_i}(X_i) \subset U$, where Z_μ^t stands for the flow generated by the Liouville vector field of μ. Set $h_i := h_U|_{Z_\mu^{-t_i}(X_i)}$. Using the flow Z_λ^t of the Liouville field Z_λ we construct an exact symplectic embedding $f_i\colon (X_i, \mu_i) \to (W, \lambda)$ by the formula

$$f_i = Z_\lambda^{t_i} \circ h_i \circ Z_\mu^{-t_i}.$$

Indeed,

$$\begin{aligned} f_i^* \lambda &= (Z_\mu^{-t_i})^* \circ h_i^* \circ (Z_\lambda^{t_i})^*(\lambda) = (Z_\mu^{-t_i})^* \circ h_i^*(e^{t_i}\lambda) = e^{t_i}(Z_\mu^{-t_i})^*(\mu_i + dH) \\ &= e^{t_i}\big(e^{-t_i}\mu_i + d(H \circ Z_\mu^{-t_i})\big) = \mu_i + d\big(e^{t_i}(H \circ Z_\mu^{-t_i})\big). \end{aligned}$$

By the construction, f_i is formally isotopic to $(\varphi|_{X_i}, \Phi|_{TX_i})$. □

The next lemma is a special case of Theorem 2.3(1).

LEMMA 5.2

There exists an exact symplectic embedding $g_i\colon (W_i,\lambda_i)\to(X,\mu)$ *which is formally isotopic to* $(\varphi^{-1}|_{W_i},\Phi^{-1}|_{TW_i})$.

There exist subfamilies $\{X_{i_k}\}$ and $\{W_{j_k}\}$ such that $f_{i_k}(X_{i_k})\subset W_{j_k}$, $\varphi(X_{i_k})\subset W_{j_k}$, $g_{j_k}(W_{j_k})\subset X_{i_{k+1}}$, and $\varphi^{-1}(W_{j_k})\subset X_{i_{k+1}}$. After renumbering, we have the following diagram:

$$\begin{array}{ccccccccc}
(X_1,\mu_1) & \xhookrightarrow{\iota_{X_1}} & (X_2,\mu_2) & \xhookrightarrow{\iota_{X_2}} & (X_3,\mu_3) & \xhookrightarrow{\iota_{X_3}} & \cdots\cdots\cdots & \hookrightarrow & (X,\mu) \\
\downarrow f_1 & \nearrow g_1 & \downarrow f_2 & \nearrow g_2 & \downarrow f_3 & \nearrow g_3 & \downarrow & & \\
(W_1,\lambda_1) & \xhookrightarrow[\iota_{W_1}]{} & (W_2,\lambda_2) & \xhookrightarrow[\iota_{W_2}]{} & (W_3,\lambda_3) & \xhookrightarrow[\iota_{W_3}]{} & \cdots\cdots\cdots & \hookrightarrow & (W,\lambda)
\end{array}$$

where ι_{X_k} and ι_{W_k} are the inclusions.

LEMMA 5.3

The compositions $g_k\circ f_k\colon X_k\to X_{k+1}$ *and* $f_{k+1}\circ g_k\colon W_k\to W_{k+1}$ *are Hamiltonian isotopic to the inclusions* $\iota_{X_k}\colon X_k\to X_{k+1}$ *and* $\iota_{W_k}\colon W_k\to W_{k+1}$, *respectively.*

Proof

By Lemmas 5.1 and 5.2, $g_k\circ f_k$ is formally isotopic to $\iota_{X_k}=\varphi^{-1}|_{W_k}\circ\varphi|_{X_k}$. Applying Theorem 3.1(2) to this formal isotopy restricted on $V_k=V\cap X_k$, we get a Hamiltonian isotopy $h_k^s\colon V_k\to X_{k+1}$, $s\in[0,1]$, such that $h_k^0=\iota_{X_k}|_{V_k}$ and $h_k^1=g_k\circ f_k|_{V_k}$. Arguing as in the proof of Lemma 5.1, we can define a Hamiltonian isotopy $\psi_k^s\colon X_k\to X_{k+1}$ between $\psi_k^0=\iota_{X_k}$ and $\psi_k^1=g_k\circ f_k$ by the formula

$$\psi_k^s := Z_\mu^{t_k}\circ\widetilde{h}_k^s\circ Z_\mu^{-t_k}.$$

Here $\widetilde{h}_k^s$ is an extension of h_k^s to an open neighborhood U_k of V_k as in the proof of Lemma 5.1 and t_k is a sufficiently large number so that $Z_\mu^{-t_k}(X_k)\subset U_k$. Similarly, we use Theorem 2.3(2) to construct a Hamiltonian isotopy connecting $f_{k+1}\circ g_k$ and ι_{W_k}. □

Proof of Theorem 1.1

We construct an exact symplectomorphism from X to W by induction over k. First, set $F_1:=f_1\colon(X_1,\mu_1)\to(W_1,\lambda_1)$. Lemma 5.3 constructs for any $k\ge 1$ a Hamiltonian isotopy connecting the inclusion $\iota_{X_k}\colon X_k\to X_{k+1}$ with the composition $g_k\circ f_k$. Hence, cutting off this isotopy outside X_k we get a Hamiltonian isotopy $G_k^s\colon(X,\mu)\to(X,\mu)$, $s\in[0,1]$, such that $G_k^0=\mathrm{id}_X$, $G_k^1|_{X_k}=g_k\circ f_k$, and $\mathrm{supp}(G_k^s)\subset X_{k+1}$. Similarly, Lemma 5.3 allows us to construct a Hamiltonian isotopy $H_k^s\colon(W,\lambda)\to(W,\lambda)$, $s\in[0,1]$, such that $H_k^0=\mathrm{id}_W$, $H_k^1|_{W_k}=f_{k+1}\circ g_k$, and $\operatorname{supp}(H_k^s) \subset W_{k+1}$. Set $G_k := G_k^1$ and $H_k := H_k^1$. For $k \geq 2$, we define the exact symplectic embedding

$$F_k := (H_{k-1} \circ \cdots \circ H_1)^{-1} \circ f_k \circ (G_{k-1} \circ \cdots \circ G_1)|_{X_k} : (X_k, \mu_k) \to (W_k, \lambda_k).$$

Since $\operatorname{supp}(G_j) \subset X_{j+1}$ and $H_{k-1}^{-1} \circ f_k \circ G_{k-1}|_{X_{k-1}} = f_{k-1}$, the restriction $F_k|_{X_{k-1}}$ is equal to F_{k-1}, $k \geq 2$. Hence, we can define an exact symplectic embedding $F\colon X \to W$ by setting $F := F_k$ on X_k for $k \geq 1$. Let us show that F is surjective. Note that $W_{k-1} \subset F_k(X_k)$ for $k \geq 2$. Indeed, we have

$$\begin{aligned} H_{k-1} \circ H_{k-2} \circ \cdots \circ H_1(W_{k-1}) &= H_{k-1}(W_{k-1}) \\ &= f_k\big(g_{k-1}(W_{k-1})\big) \\ &\subset f_k(X_k) \\ &= f_k\big(G_{k-1} \circ \cdots \circ G_1(X_k)\big), \end{aligned}$$

and thus, $W_{k-1} \subset F_k(X_k)$. Hence,

$$W = \bigcup_{k \geq 1} W_k = \bigcup_{k \geq 2} F_k(X_k) = F\Big(\bigcup_{k \geq 2} X_k\Big) = F(X).$$

Therefore, $F\colon (X, \mu) \to (W, \lambda)$ is an exact symplectomorphism. ☐

6. Proof of Theorem 1.3

The proof follows the same scheme as the proof of Theorem 1.1. As in that proof let W be a flexible Weinstein manifold which is formally symplectomorphic to X, let $\{(X_i, \mu_i) \mid i \geq 1\}$ and $\{(W_i, \lambda_i) \mid i \geq 1\}$ be exhaustions by Liouville and Weinstein subdomains of X and W, respectively, and let (φ, Φ_t) be the formal symplectic embeddings of (X, μ) into (W, λ). We begin by proving Lemmas 5.1, 5.2, and 5.3 in the current context.

Step 1: Construction of exact symplectic embeddings $f_i\colon (X_i, \mu_i) \to (W, \lambda)$, $i \geq 1$, in the formal isotopy class of $(\varphi|_{X_i}, \Phi|_{TX_i})$. Denote $S_i := \bigcap_t Z_\mu^{-t}(X_i)$. The attractor S_i is compact, we have $S_i \subset \operatorname{Core}(X, \mu)$, and hence, there exists an integer N such that $S_i \subset \bigcup_{j \leq N} C_j$.

Since the first subset C_1 is compact, there exists a finite open cover $\{U_{p_1}, \ldots, U_{p_{k_1}}\}$ of C_1 such that each intersection $U_{p_j} \cap C_1$ admits a symplectic extension Σ_{p_j} of positive codimension. Choose also a cover $\{U'_{p_1}, \ldots, U'_{p_{k_1}}\}$ such that $U'_{p_j} \Subset U_{p_j}$, $j = 1, \ldots, k_1$. Applying Theorem 3.1(1) to $(\varphi|_{\Sigma_{p_1}}, \Phi_t|_{\Sigma_{p_1}})$, we obtain an exact symplectic embedding $f_1^{(1)}\colon \Sigma_{p_1} \to W$, and moreover, we can modify Φ_t so that $\Phi_1|_{T\Sigma_{p_1}} = df_1^{(1)}$. The symplectic neighborhood theorem then allows us to extend $f_1^{(1)}$ to a neighborhood $\Omega_1 \supset \Sigma_{p_1}$ in U_{p_1}. We will keep the notation $f_1^{(1)}$ for this extension. Denote $A_1 := \overline{C_1 \cap U'_{p_1}} \cap U_{p_2}$. Note that every point of A_1 has an access to infinity in Σ_{p_2}. Hence, applying Theorem 3.1(3) we find a symplectic embedding $f_2^{(1)}\colon \Sigma_{p_2} \to W$ which coincides with $f_1^{(1)}$ on $\mathcal{O}p\, A_1 \subset \Sigma_{p_2}$. We further modify Φ_t so that $\Phi_1|_{T\Sigma_{p_2}} = df_2^{(1)}$ and then use the symplectic neighborhood theorem to extend the exact symplectic embedding

$f_2^{(1)}$ to a neighborhood $\Omega_2 \supset \Sigma_{p_2}$ in U_{p_2} so that the extended embedding $f_2^{(1)}$ coincides with $f_1^{(1)}$ on $\mathcal{O}p\, A_1$ in U_{p_2}. Continuing this process we construct an exact symplectic embedding $f^{(1)}\colon \mathcal{O}p\, C_1 \to W$. Choosing a sufficiently small neighborhood $U_1 \supset C_1$, where $f^{(1)}$ is defined, we find a finite open cover $\{U^2_{p_1}, \dots, U^2_{p_{k_2}}\}$ of the compact set $C_2 \setminus U_1$ such that each intersection $U^2_{p_j} \cap C_2$ admits a symplectic extension $\Sigma^2_{p_j}$ of positive codimension. Repeating the above process inductively over elements of the cover $U^2_{p_j}$ of $C_2 \setminus U_1$ and then continuing a similar process for $C_3, \dots, C_N$ we construct an exact symplectic embedding $h_i\colon \mathcal{O}p(C_1 \cup \dots \cup C_N) \to W$. By our construction $Z_\mu^{-T}(X_i)$ for a sufficiently large T is contained in a neighborhood of $\bigcup_{j \le N} C_j$ where h_i is defined. Hence, the formula $f_i := Z_\lambda^T \circ h_i \circ Z_\mu^{-T}$ defines the required exact symplectic embedding $f_i\colon X_i \to W$.

Step 2: Construction of exact symplectic embeddings $g_i\colon (W_i, \lambda_i) \to (X, \mu)$, $i \ge 1$, in the formal isotopy class of $(\varphi^{-1}|_{W_i}, \Phi^{-1}|_{TW_i})$. This is a corollary of Theorem 2.3(1), as in the case of Theorem 1.1.

Step 3: Proof that the compositions $g_k \circ f_k\colon X_k \to X_{k+1}$ and $f_{k+1} \circ g_k\colon W_k \to W_{k+1}$ are Hamiltonian isotopic to the inclusions $\iota_{X_k}\colon X_k \to X_{k+1}$ and $\iota_{W_k}\colon W_k \to W_{k+1}$, respectively (after readjusting the indices as in the proof of Theorem 1.1). Steps 1 and 2 imply that $g_k \circ f_k$ is formally isotopic to $\iota_{X_k} = \varphi^{-1}|_{W_k} \circ \varphi|_{X_k}$. To construct a genuine Hamiltonian isotopy connecting $g_k \circ f_k$ and ι_{X_k} we repeat the proof in Step 1, but using instead Theorems 3.1(2) and 3.1(4). The existence of a Hamiltonian isotopy connecting $f_{k+1} \circ g_k$ and ι_{W_k} is even more straightforward using Theorem 2.3(2).

Step 4. With the analogues of Lemmas 5.1, 5.2, and 5.3 established, we construct the required exact symplectomorphism $f\colon X \to W$ using the telescope construction exactly as in the proof of Theorem 1.1. □

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