Remarks about the C^∞-closing lemma for 3-dimensional Reeb flows

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Abstract We prove two refinements of the C^∞-closing lemma for 3-dimensional Reeb flows, which was proved by the author as an application of spectral invariants of *Embedded Contact Homology (ECH)*. Specifically, we prove the following two results: (i) for a C^∞-generic contact form on any closed 3-manifold, the union of periodic Reeb orbits representing ECH homology classes is dense; (ii) a certain real-analytic version of the C^∞-closing lemma for 3-dimensional Reeb flows. A few questions and conjectures related to these results are also discussed.

1. Introduction

1.1. Conventions

Throughout this paper, Y denotes a closed C^∞-manifold of dimension 3, and ξ denotes a *contact distribution* on Y. Namely, ξ is an oriented plane field on Y such that there exists $\lambda \in \Omega^1(Y)$ which satisfies the following conditions:

$$\ker \lambda = \xi, \qquad d\lambda|_\xi > 0. \tag{1}$$

Let $\Lambda(Y,\xi)$ denote the set of $\lambda \in \Omega^1(Y)$ satisfying (1). For any $\lambda \in \Lambda(Y,\xi)$, there exists a bijection $C^\infty(Y,\mathbb{R}_{>0}) \to \Lambda(Y,\xi)$; $f \mapsto f\lambda$. In this note, $\Lambda(Y,\xi)$ is equipped with the C^∞-topology, unless otherwise specified.

For each $\lambda \in \Lambda(Y,\xi)$, its *Reeb vector field* $R_\lambda \in \mathcal{X}(Y)$ is characterized by equations $i_{R_\lambda} d\lambda \equiv 0$ and $\lambda(R_\lambda) \equiv 1$. Let us consider the set of *periodic Reeb orbits* and the set of *periods* of these orbits:

$$\mathcal{P}(Y,\lambda) := \{\gamma : \mathbb{R}/T_\gamma\mathbb{Z} \to Y \mid T_\gamma \in \mathbb{R}_{>0}, \dot\gamma \equiv R_\lambda(\gamma)\},$$
$$\mathcal{T}(Y,\lambda) := \{T_\gamma \mid \gamma \in \mathcal{P}(Y,\lambda)\}.$$

Sometimes $\mathcal{P}(Y,\lambda)$, $\mathcal{T}(Y,\lambda)$ are abbreviated by $\mathcal{P}(\lambda)$, $\mathcal{T}(\lambda)$.

Let $(\varphi^t_{R_\lambda})_{t\in\mathbb{R}}$ denote the flow on Y generated by R_λ. Then, for every $\gamma \in \mathcal{P}(Y,\lambda)$, $\gamma(0)$ is a fixed point of $\varphi^{T_\gamma}_{R_\lambda}$; γ is considered nondegenerate if 1 is not an eigenvalue of $d\varphi^{T_\gamma}_{R_\lambda}|_{\xi_{\gamma(0)}}$. λ is considered nondegenerate if all elements of $\mathcal{P}(Y,\lambda)$ are nondegenerate. It is well known that a generic (with respect to the C^∞-topology) element of $\Lambda(Y,\xi)$ is nondegenerate.

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We also consider the set of embedded, unparameterized periodic orbits:

$$\mathcal{P}_{\mathrm{emb}}(Y,\lambda) := \bigl\{\gamma : \text{closed curve in } Y \bigm| \gamma \cong S^1, T_p\gamma = \mathbb{R}R_\lambda(p)\ (\forall p \in \gamma)\bigr\}.$$

Each $\gamma \in \mathcal{P}_{\mathrm{emb}}(Y,\lambda)$ is oriented so that R_λ is of positive direction, and we set $T_\gamma(\lambda) := \int_\gamma \lambda$. When there is no risk of confusion, $T_\gamma(\lambda)$ will be abbreviated as T_γ.

Recall that an $\mathbb{R}$-linear map

$$C : \Omega^1(Y) \to \mathbb{R}; \alpha \mapsto \langle C, \alpha\rangle$$

is called a 1-*dimensional current* on Y if $\lim_{k\to\infty} C(\alpha_k) = 0$ for any sequence $(\alpha_k)_k$ on $\Omega^1(Y)$ which converges to 0 in the C^∞-topology.

A *positive Reeb current* of (Y,λ) is a 1-dimensional current C on Y which is of the form

$$C = \sum_{1\le i\le k} a_i\gamma_i, \tag{2}$$

where k is a nonnegative integer, $a_1,\dots,a_k$ are positive real numbers, and $\gamma_1,\dots,\gamma_k$ are distinct elements of $\mathcal{P}_{\mathrm{emb}}(Y,\lambda)$. More explicitly, (2) means the following:

$$\langle C,\alpha\rangle = \sum_{1\le i\le k} a_i \int_{\gamma_i} \alpha \quad \bigl(\forall \alpha \in \Omega^1(Y)\bigr). \tag{3}$$

When $k = 0$, (2) means that $C = 0$ as a current.

Note that, for $C = \sum_{1\le i\le k} a_i\gamma_i$, the set $\{(a_i,\gamma_i)\}_{1\le i\le k}$ is uniquely determined from C. We define $\mathrm{supp}(C) \subset Y$ and $\mathrm{weight}(C) \in \mathbb{R}_{\ge 0}$ by

$$\mathrm{supp}(C) := \bigcup_{1\le i\le k} \gamma_i, \qquad \mathrm{weight}(C) := \sum_{1\le i\le k} a_i.$$

Note that, when $C = 0$, one has $\mathrm{supp}(0) = \emptyset$ and $\mathrm{weight}(0) = 0$.

Let $\mathcal{C}_+(Y,\lambda)$ denote the set of positive Reeb currents on (Y,λ). We also consider

$$\mathcal{C}_{\mathbb{N}}(Y,\lambda) := \Bigl\{\sum_{1\le i\le k} a_i\gamma_i \in \mathcal{C}_+(Y,\lambda) \Bigm| k \ge 0, a_1,\dots,a_k \in \mathbb{N}, \gamma_1,\dots,\gamma_k \in \mathcal{P}_{\mathrm{emb}}(Y,\lambda)\Bigr\},$$

where $\mathbb{N}$ denotes the set of positive integers.

1.2. Existence results of periodic Reeb orbits

One of the most fundamental questions on periodic Reeb orbits is the (generalized) Weinstein conjecture, which states that there exists at least one periodic Reeb orbit on any closed contact manifold. Viterbo in [25] proved this conjecture for contact-type hypersurfaces in symplectic vector spaces. In dimension 3, Hofer in [9] used holomorphic curves in symplectizations of contact 3-manifolds to prove the Weinstein conjecture for the three-sphere, orientable closed 3-manifolds with $\pi_2 \neq 0$, and closed 3-manifolds with overtwisted contact forms.

The Weinstein conjecture in dimension 3 was finally proved by Taubes in [23] using Seiberg–Witten Floer cohomology. Taubes's proof is strongly related to Hutchings's theory of embedded contact homology (ECH). Using ECH, Cristofaro-Gardiner and Hutchings in [4] proved the existence of at least two embedded periodic Reeb orbits on any closed contact 3-manifold. Cristofaro-Gardiner, Hutchings, and Pomerleano in [5] proved the following "two or infinity" theorem: if Y is a closed, connected 3-manifold, ξ is a contact distribution on Y such that the first Chern class $c_1(\xi)$ is torsion, and $\lambda \in \Lambda(Y,\xi)$ is nondegenerate, then $\sharp\mathcal{P}_{\mathrm{emb}}(Y,\lambda) \in \{2,\infty\}$. Quite recently, Colin, Dehornoy, and Rechtman in [2] claimed a proof of the "two or infinity" theorem for nondegenerate contact forms on all closed 3-manifolds.

In a different direction, the author in [15] used ECH to prove the following result.

THEOREM 1.1 ([15, Lemma 3.1, Theorem 1.1])

(i) *Let $\lambda \in \Lambda(Y,\xi)$, let U be a nonempty open set of Y, and let $\mathcal{N}$ be a neighborhood of 0 in $C^\infty(Y,\mathbb{R})$ with respect to the C^∞-topology. Then there exist $h \in \mathcal{N}$ and $\gamma \in \mathcal{P}_{\mathrm{emb}}((1+h)\lambda)$ such that $\gamma \cap U \neq \emptyset$.*

(ii) *There exists $\mathcal{R} \subset \Lambda(Y,\xi)$, which is residual with respect to the C^∞-topology, and $\bigcup_{\gamma\in\mathcal{P}_{\mathrm{emb}}(Y,\lambda)} \gamma$ is dense in Y for every $\lambda \in \mathcal{R}$.*

Part (i) is called the C^∞-closing lemma, and part (ii) is called the *C^∞-generic density theorem*. Part (ii) follows from part (i) by a simple (and perhaps well-known) argument. In the C^1-topology on the space of dynamical systems, the closing lemma was established by Pugh in [20] (for nonconservative systems) and by Pugh and Robinson in [21] (for conservative systems); however, such a result in the C^∞-topology was rather surprising. Indeed, "C^∞-closing lemma for general Hamiltonian systems" is not true even in dimension 4, since there is a celebrated counterexample by Herman in [8].

1.3. Plan of this paper

In Section 2, we recall quantitative aspects of ECH very briefly. The readers are strongly recommended to consult papers by Hutchings, for example, [11]. In Section 3, we discuss the asymptotic behavior of periodic Reeb orbits "representing" ECH classes. The main result of Section 3 is Theorem 3.6. A few related questions are also discussed. In Section 4 we prove Theorem 4.1, which is a certain real-analytic version of Theorem 1.1(i).

2. Preliminaries on embedded contact homology

For any closed 3-manifold Y, a contact distribution ξ on Y, and $\Gamma \in H_1(Y:\mathbb{Z})$, one can define a module $\mathrm{ECH}(Y,\xi,\Gamma)$ termed embedded contact homology (ECH). We also consider $\mathrm{ECH}(Y,\xi) := \bigoplus_{\Gamma\in H_1(Y:\mathbb{Z})} \mathrm{ECH}(Y,\xi,\Gamma)$. It is possible to define ECH over $\mathbb{Z}$; however, in this note we only consider ECH defined over $\mathbb{Z}/2$, since it will be sufficient for applications discussed in this paper.

Very roughly speaking, ECH is a homology group of a $\mathbb{Z}/2$-chain complex generated over finite subsets of periodic Reeb orbits, equipped with a boundary operator defined by counting rigid (up to the natural action of $\mathbb{R}$ on $Y \times \mathbb{R}$) and "mostly" embedded holomorphic curves in $Y \times \mathbb{R}$. One can also define a linear map U on ECH by counting rigid holomorphic curves passing through a generic fixed point in $Y \times \mathbb{R}$.

To proceed, we need the following notions. For any vector space V, a relative $\mathbb{Z}$-grading on V is a direct product decomposition $V \cong \bigoplus_{a \in A} V_a$ with a map $I : A \times A \to \mathbb{Z}$ satisfying the following properties:

- $I(a,b) + I(b,c) = I(a,c)$ for any $a,b,c \in A$,
- $I(a,b) = 0$ if and only if $a = b$.

$v \in V$ is considered homogeneous if there exists $a \in A$ such that $v \in V_a$. When v and w are nonzero homogeneous elements in V, one can define $a(v), a(w) \in A$ by $v \in V_{a(v)}$ and $w \in V_{a(w)}$, and we define $I(v,w) := I(a(v), a(w))$, which we call the *grading* of v relative to w.

If $c_1(\xi) + 2\,\mathrm{PD}(\Gamma) \in H^2(Y : \mathbb{Z})$ is torsion, where c_1 denotes the first Chern class and PD denotes the Poincaré dual, then $\mathrm{ECH}(Y, \xi, \Gamma)$ has a relative $\mathbb{Z}$-grading which we denote by I. The U-map on $\mathrm{ECH}(Y, \xi, \Gamma)$ decreases the grading by 2. It is known that if Y is connected and $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion, then there exists a sequence $(\sigma_k)_{k \geq 1}$ of nonzero homogeneous classes of $\mathrm{ECH}(Y, \xi, \Gamma)$ which satisfies the following condition:

($\star$) $U\sigma_{k+1} = \sigma_k$ for every $k \geq 1$.

Obviously, this condition implies the following condition:

($\star\star$) $I(\sigma_{k+1}, \sigma_k) = 2$ for every $k \geq 1$.

The existence of a sequence $(\sigma_k)_k$ satisfying the condition ($\star$) follows from results of Taubes in [24] and Kronheimer and Mrowka in [18] (see Corollary 2.2 of [4]).

For any $\sigma \in \mathrm{ECH}(Y, \xi) \setminus \{0\}$, Hutchings in [10] defined a function

$$c_\sigma : \Lambda(Y, \xi) \to \mathbb{R}_{\geq 0}$$

which we call *ECH spectral invariant*. We sometimes denote $c_\sigma(\lambda)$ as $c_\sigma(Y, \lambda)$.

To state some properties of ECH spectral invariants, let us first introduce the following notation: for any $S \subset \mathbb{R}$, let S_+ denote the set of finite sums of elements of S, that is,

$$S_+ := \{0\} \cup \{s_1 + \cdots + s_k \mid k \geq 1, s_1, \dots, s_k \in S\}.$$

Now let us state properties (i)–(v) of ECH spectral invariants. In (i)–(iv), σ is any nonzero element in $\mathrm{ECH}(Y, \xi)$, and λ is any element in $\Lambda(Y, \xi)$. The following hold:

(i) $c_\sigma(\lambda) \in \mathcal{T}(Y, \lambda)_+$;
(ii) $c_\sigma(a\lambda) = a c_\sigma(\lambda)$ for any $a \in \mathbb{R}_{>0}$;
(iii) $c_\sigma(f\lambda) \geq c_\sigma(\lambda)$ for any $f \in C^\infty(Y, \mathbb{R}_{\geq 1})$;
(iv) if $(f_j)_{j \geq 1}$ is a sequence in $C^\infty(Y, \mathbb{R}_{>0})$ which satisfies $\lim_{j \to \infty} \|f_j - 1\|_{C^0} = 0$, then $\lim_{j \to \infty} c_\sigma(f_j \lambda) = c_\sigma(Y, \lambda)$;

(v) if Y is connected and $(\sigma_k)_{k\geq 1}$ is a sequence of nonzero homogeneous elements in $\mathrm{ECH}(Y,\xi,\Gamma)$ (where $c_1(\xi)+2\,\mathrm{PD}(\Gamma)$ is torsion) which satisfies the condition $(\star\star)$, then

$$\lim_{k\to\infty}\frac{c_{\sigma_k}(\lambda)}{\sqrt{2k}}=\sqrt{\mathrm{vol}(Y,\lambda)}.$$

The properties (i)–(v) are, respectively, called *spectrality, conformality, monotonicity, C^0-continuity*, and *volume theorem*. Properties (i) and (ii) are easy consequences of the definition of ECH spectral invariants (see [15]). Property (iii) follows from the fact that ECH cobordism maps preserve action filtrations on ECH (see [14]). Property (iv) is an immediate consequence of (ii) and (iii). Property (v) is due to Cristofaro-Gardiner, Hutchings, and Ramos in [6]. Property (v) is perhaps the most surprising property and key to many applications to dynamics.

REMARK 2.1

By setting $e_k(\lambda):=c_{\sigma_k}(\lambda)-\sqrt{2k\,\mathrm{vol}(Y,\lambda)}$, the volume theorem is equivalent to $e_k=o(k^{1/2})$. This estimate has been improved in recent papers [22], [7], and [13] (in chronological order). In particular, [13] proves $e_k=O(k^{1/4})$ for contact-type hypersurfaces in $\mathbb{C}^2$; actually, this is a special case of Theorem 1.1 in [13], which applies to any compact domain with C^∞-boundary in $\mathbb{C}^2$. Moreover, [13] proposed a remarkable conjecture (Conjecture 1.5 of [13]), which states that for a generic nice star-shaped hypersurface in $\mathbb{C}^2$, the asymptotic behavior of $(e_k)_k$ recovers the Ruelle invariant of the Reeb flow on the hypersurface.

3. Asymptotic behavior of ECH representatives

3.1. Nice contact forms and ECH representatives

Let us first introduce the following definition.

DEFINITION 3.1

We call $\lambda\in\Lambda(Y,\xi)$ a *nice contact form* if it is nondegenerate and, for any distinct elements $\gamma_1,\dots,\gamma_k\in\mathcal{P}_{\mathrm{emb}}(Y,\lambda)$, their periods $T_{\gamma_1},\dots,T_{\gamma_k}$ are linearly independent over $\mathbb{Q}$. The set of nice contact forms in $\Lambda(Y,\xi)$ is denoted by $\Lambda_{\mathrm{nice}}(Y,\xi)$.

LEMMA 3.2

It holds that $\Lambda_{\mathrm{nice}}(Y,\xi)$ is residual in $\Lambda(Y,\xi)$ with respect to the C^∞-topology.

Proof

For each $N\in\mathbb{N}$, let $\Lambda_N(Y,\xi)$ denote the set of $\lambda\in\Lambda(Y,\xi)$ which satisfies the following conditions:

- If $\gamma\in\mathcal{P}(Y,\lambda)$ satisfies $T_\gamma\leq N$, then γ is nondegenerate.
- If $\gamma_1,\dots,\gamma_k$ are distinct elements in $\mathcal{P}_{\mathrm{emb}}(Y,\lambda)$ and $(m_1,\dots,m_k)\in\mathbb{Z}^k\setminus\{(0,\dots,0)\}$ such that $\sum_{j=1}^k|m_j|T_{\gamma_j}\leq N$, then $\sum_{j=1}^k m_jT_{\gamma_j}\neq 0$.

Then $\Lambda_{\mathrm{nice}}(Y,\xi)=\bigcap_{N=1}^{\infty}\Lambda_N(Y,\xi)$. Thus, to show that $\Lambda_{\mathrm{nice}}(Y,\xi)$ is residual in $\Lambda(Y,\xi)$, it is sufficient to show that $\Lambda_N(Y,\xi)$ is open and dense for every N.

The openness of $\Lambda_N(Y,\xi)$ follows from the following observation: let $(\lambda_k)_k$ be a sequence in $\Lambda(Y,\xi)$, and let $(\gamma_k)_k$ be a sequence of periodic Reeb orbits such that $\gamma_k\in\mathcal{P}(Y,\lambda_k)$ $(\forall k)$ and $\sup_k T_{\gamma_k}<\infty$. If there exists $\lambda_\infty\in\Lambda(Y,\xi)$ such that $\lim_{k\to\infty}\lambda_k=\lambda_\infty$ in the C^∞-topology, then there exist $\gamma_\infty\in\mathcal{P}(Y,\lambda_\infty)$ and a sequence of integers $k_1<k_2<\cdots<k_m<\cdots$ such that $\lim_{m\to\infty}T_{\gamma_{k_m}}=T_{\gamma_\infty}$. Moreover, if γ_{k_m} is degenerate for every sufficiently large m, then γ_∞ is also degenerate.

To show the denseness of $\Lambda_N(Y,\xi)$, let $\mathcal{U}$ be any nonempty open set in $\Lambda(Y,\xi)$; we would like to show that $\Lambda_N(Y,\xi)\cap\mathcal{U}\neq\emptyset$. Since generic elements in $\Lambda(Y,\xi)$ are nondegenerate, there exists $\lambda\in\mathcal{U}$ which is nondegenerate. Let

$$\{\gamma_1,\ldots,\gamma_k\}:=\bigl\{\gamma\in\mathcal{P}_{\mathrm{emb}}(Y,\lambda)\bigm| T_\gamma(\lambda)\le N\bigr\}.$$

Then there exists $\mathcal{U}'$ such that $\mathcal{U}'$ is an open neighborhood of λ in $\mathcal{U}$, and every $\lambda'\in\mathcal{U}'$ satisfies the following properties:

(a) every $\gamma\in\mathcal{P}(Y,\lambda')$ satisfying $T_\gamma\le N$ is nondegenerate,
(b) $\{\gamma\in\mathcal{P}_{\mathrm{emb}}(Y,\lambda')\mid T_\gamma(\lambda')\le N\}$ has at most k elements.

Let us take $(\rho_i,\varepsilon_i)_{i=1,\ldots,k}$ so that the following conditions hold.

(c) For each i, $\rho_i\in C^\infty(Y,\mathbb{R}_{\ge 0})$ and $\rho_i\equiv 1$ near γ_i. Moreover, $i\neq j\implies \operatorname{supp}\rho_i\cap\operatorname{supp}\rho_j=\emptyset$.

(d) $\varepsilon_1,\ldots,\varepsilon_k$ are small positive real numbers such that $e^{-(\varepsilon_1\rho_1+\cdots+\varepsilon_k\rho_k)}\lambda\in\mathcal{U}'$.

(e) $e^{-\varepsilon_1}T_{\gamma_1}(\lambda),\ldots,e^{-\varepsilon_k}T_{\gamma_k}(\lambda)$ are linearly independent over $\mathbb{Q}$.

Note that (d) and (e) can be simultaneously achieved, since (d) is an open condition and (e) is a generic condition. Now let $\lambda_{\mathrm{new}}:=e^{-(\varepsilon_1\rho_1+\cdots+\varepsilon_k\rho_k)}\lambda$. Then $\lambda_{\mathrm{new}}\in\mathcal{U}'$ by (d). Thus, every $\gamma\in\mathcal{P}(Y,\lambda_{\mathrm{new}})$ with $T_\gamma\le N$ is nondegenerate by (a). By (c), $\gamma_i\in\mathcal{P}_{\mathrm{emb}}(Y,\lambda_{\mathrm{new}})$ and $T_{\gamma_i}(\lambda_{\mathrm{new}})=e^{-\varepsilon_i}T_{\gamma_i}(\lambda)$ for every $i\in\{1,\ldots,k\}$. By (b),

$$\bigl\{\gamma\in\mathcal{P}_{\mathrm{emb}}(Y,\lambda_{\mathrm{new}})\bigm| T_\gamma(\lambda_{\mathrm{new}})\le N\bigr\}=\{\gamma_1,\ldots,\gamma_k\}.$$

By (e), $T_{\gamma_1}(\lambda_{\mathrm{new}}),\ldots,T_{\gamma_k}(\lambda_{\mathrm{new}})$ are linearly independent over $\mathbb{Q}$. Therefore, $\lambda_{\mathrm{new}}\in\Lambda_N(Y,\xi)$. Finally, $\Lambda_N(Y,\xi)\cap\mathcal{U}$ is not empty since it contains λ_{new}. □

For any $\lambda\in\Lambda_{\mathrm{nice}}(Y,\xi)$ and $\sigma\in\mathrm{ECH}(Y,\xi)\setminus\{0\}$, there exists a unique $C\in\mathcal{C}_{\mathbb{N}}(Y,\lambda)$ such that $\langle C,\lambda\rangle=c_\sigma(\lambda)$. Let us denote such C by $C_\sigma(\lambda)$, and say that C *represents* σ with respect to λ.

Let us assume that Y is connected, that $\Gamma\in H_1(Y:\mathbb{Z})$ such that $c_1(\xi)+2\,\mathrm{PD}(\Gamma)$ is torsion, and that $(\sigma_k)_k$ is a sequence of nonzero homogeneous elements of $\mathrm{ECH}(Y,\xi,\Gamma)$ which satisfies the condition $(\star\star)$. We are interested in asymptotic behavior of the ECH representatives $(C_{\sigma_k}(\lambda))_k$ for generic $\lambda\in\Lambda_{\mathrm{nice}}(Y,\xi)$. In the next section, we prove that $\bigcup_{k\ge 1}\operatorname{supp}(C_{\sigma_k}(\lambda))$ is dense in Y for generic λ

(Theorem 3.6). Let us speculate on how Theorem 3.6 can be improved. The most optimistic conjecture will be the following.

CONJECTURE 3.3

Let $(\sigma_k)_k$ *be a sequence of homogeneous elements of* $\mathrm{ECH}(Y,\xi)\setminus\{0\}$ *which satisfies the condition* $(\star\star)$*. Then there exists a residual set* $\mathcal{R}\subset\Lambda_{\mathrm{nice}}(Y,\xi)$ *such that any* $\lambda\in\mathcal{R}$ *satisfies*

$$\lim_{k\to\infty}\frac{C_{\sigma_k}(\lambda)}{\sqrt{2k}}=\frac{d\lambda}{\sqrt{\mathrm{vol}(Y,\lambda)}}, \tag{4}$$

where the limit is the weak convergence of currents, that is,

$$\lim_{k\to\infty}\frac{\langle C_{\sigma_k}(\lambda),\alpha\rangle}{\sqrt{2k}}=\frac{\int_Y\alpha\wedge d\lambda}{\sqrt{\mathrm{vol}(Y,\lambda)}}\quad\big(\forall\alpha\in\Omega^1(Y)\big).$$

REMARK 3.4

Since $\langle C_{\sigma_k}(\lambda),\lambda\rangle=c_{\sigma_k}(\lambda)$, if λ satisfies (4), then $\lim_{k\to\infty}\frac{c_{\sigma_k}(\lambda)}{\sqrt{2k}}=\sqrt{\mathrm{vol}(Y,\lambda)}$. Thus, Conjecture 3.3 recovers the volume theorem for generic $\lambda\in\Lambda(Y,\xi)$. Note that once one verifies the volume theorem for generic contact forms, then it is easy to deduce the volume theorem for all contact forms, assuming the conformality and monotonicity of ECH spectral invariants. We also note that, in [13] it is suggested that Conjecture 3.3 might (heuristically) imply Conjecture 1.5 of [13].

Currently, Conjecture 3.3 seems quite ambitious. So far the only partial result toward this conjecture is the following, which was inspired by a similar result for minimal hypersurfaces; Main Theorem in [19].

THEOREM 3.5 ([16, Theorem 1.1])

There exists a residual set $\mathcal{R}\subset\Lambda(Y,\xi)$ *such that for any* $\lambda\in\mathcal{R}$ *there exists a sequence* $(C_k)_{k\geq 1}$ *in* $\mathcal{C}_+(Y,\lambda)$ *which weakly converges to* $d\lambda$*.*

One can also formulate a toy model version of Conjecture 3.3 for generic *toric* domains and check that the toy model conjecture is true for strictly convex and concave toric domains from combinatorial formulas in [3], [1] and [12], combined with a version of isoperimetric inequality (see Section 3.4).

3.2. Density of ECH representatives for generic contact forms

The goal of this section is to prove the following theorem. The proof is a slight improvement of the argument in [15].

THEOREM 3.6

Let $(\sigma_k)_k$ *be a sequence in* $\mathrm{ECH}(Y,\xi)\setminus\{0\}$ *which satisfies the condition* $(\star\star)$*. Then there exists a residual set* $\mathcal{R}\subset\Lambda_{\mathrm{nice}}(Y,\xi)$ *such that* $\bigcup_{k=1}^\infty\mathrm{supp}(C_{\sigma_k}(\lambda))$ *is dense in* Y *for any* $\lambda\in\mathcal{R}$*.*

To prove Theorem 3.6, we first need the following lemma.

LEMMA 3.7
Let $(\lambda_i)_{i\ge 1}$ be a sequence in $\Lambda_{\mathrm{nice}}(Y,\xi)$, and let $\lambda_\infty \in \Lambda_{\mathrm{nice}}(Y,\xi)$ such that $\lim_{i\to\infty}\lambda_i = \lambda_\infty$ in the C^∞-topology. Then, for any $\sigma \in \mathrm{ECH}(Y,\xi)\setminus\{0\}$, there exist $m_1,\dots,m_k \in \mathbb{N}$, $I \in \mathbb{N}$, and distinct orbits $\gamma_{i,1},\dots,\gamma_{i,k} \in \mathcal{P}_{\mathrm{emb}}(Y,\lambda_i)$ $(i \in \{I,\dots,\infty\})$ such that the following properties hold:

- $C_\sigma(\lambda_i) = \sum_{j=1}^k m_j\gamma_{i,j}$ *for every* $i \in \{I,\dots,\infty\}$,
- $\lim_{i\to\infty}\gamma_{i,j} = \gamma_{\infty,j}$ *in the* C^∞*-topology for every* $j \in \{1,\dots,k\}$.

Proof
Let us denote

$$C_\sigma(\lambda_\infty) = \sum_{j=1}^k m_j\gamma_{\infty,j},$$

where $m_1,\dots,m_k \in \mathbb{N}$ and $\gamma_{\infty,1},\dots,\gamma_{\infty,k}$ are distinct elements of $\mathcal{P}_{\mathrm{emb}}(Y,\lambda_\infty)$. Since λ_∞ is nondegenerate, for sufficiently large i there exist $\gamma_{i,1},\dots,\gamma_{i,k} \in \mathcal{P}_{\mathrm{emb}}(Y,\lambda_i)$ such that $\lim_{i\to\infty}\gamma_{i,j} = \gamma_{\infty,j}$ for every $j \in \{1,\dots,k\}$. Now it is sufficient to prove

$$C_\sigma(\lambda_i) = \sum_{j=1}^k m_j\gamma_{i,j}$$

for every sufficiently large i. If this is not the case, then there exists an increasing sequence of positive integers $i_1 < i_2 < \cdots < i_p < \cdots$ such that for every positive integer p

$$C_\sigma(\lambda_{i_p}) = \sum_{j=1}^{k'} m'_j\gamma'_{p,j}$$

and the following conditions hold:

- $m'_1,\dots,m'_{k'}$ are positive integers and $\gamma'_{p,1},\dots,\gamma'_{p,k'}$ are distinct elements of $\mathcal{P}_{\mathrm{emb}}(Y,\lambda_{i_p})$;
- for each j, there exists $\gamma'_{\infty,j} \in \mathcal{P}_{\mathrm{emb}}(Y,\lambda_\infty)$ such that $\lim_{p\to\infty}\gamma'_{p,j} = \gamma'_{\infty,j}$;
- $\sum_{j=1}^{k'} m'_j\gamma'_{\infty,j} \ne \sum_{j=1}^k m_j\gamma_{\infty,j}$.

Since λ_∞ is nice, the last condition implies that $\sum_{j=1}^{k'} m'_j\int_{\gamma'_{\infty,j}}\lambda_\infty \ne c_\sigma(\lambda_\infty)$. On the other hand,

$$c_\sigma(\lambda_\infty) = \lim_{p\to\infty} c_\sigma(\lambda_{i_p}) = \lim_{p\to\infty}\sum_{j=1}^{k'} m'_j\int_{\gamma'_{p,j}}\lambda_{i_p} = \sum_{j=1}^{k'} m'_j\int_{\gamma'_{\infty,j}}\lambda_\infty,$$

which is a contradiction. □

Proof of Theorem 3.6
Let $(U_i)_{i\ge 1}$ be a basis of open sets of Y. For each i, let Λ_{U_i} be the set of

$\lambda \in \Lambda_{\mathrm{nice}}(Y,\xi)$ such that there exists $k \geq 1$ satisfying $\operatorname{supp}(C_{\sigma_k}(\lambda)) \cap U_i \neq \emptyset$. Lemma 3.7 shows that Λ_{U_i} is an open set of $\Lambda_{\mathrm{nice}}(Y,\xi)$. Once we prove that Λ_{U_i} is dense in $\Lambda_{\mathrm{nice}}(Y,\xi)$, then $\mathcal{R} := \bigcap_{i=1}^\infty \Lambda_{U_i}$ is residual, and if $\lambda \in \mathcal{R}$, then $\bigcup_{k=1}^\infty \operatorname{supp}(C_{\sigma_k}(\lambda))$ is dense in Y. Thus, it is sufficient to prove that Λ_{U_i} is dense, that is, any nonempty open set $\mathcal{U} \subset \Lambda_{\mathrm{nice}}(Y,\xi)$ intersects Λ_{U_i}. The proof consists of three steps.

Step 1. Take an open set $\mathcal{U}'$ of $\Lambda(Y,\xi)$ such that $\mathcal{U}' \cap \Lambda_{\mathrm{nice}}(Y,\xi) = \mathcal{U}$. Also take $\lambda \in \mathcal{U}'$ arbitrarily. We show that there exists $\lambda' \in \mathcal{U}'$ with the following two conditions:

(a) $\operatorname{supp}(\lambda' - \lambda) \subset U_i$,
(b) there exists $k \geq 1$ such that $c_{\sigma_k}(\lambda') \notin \mathcal{T}(Y,\lambda)_+$.

Indeed, take $h \in C^\infty(Y, \mathbb{R}_{\geq 0}) \setminus \{0\}$ with $\operatorname{supp} h \subset U_i$. Then there exists $c > 0$ such that $e^{th}\lambda \in \mathcal{U}'$ for any $t \in [0,c]$. Since $\operatorname{vol}(Y,\lambda) < \operatorname{vol}(Y, e^{ch}\lambda)$, there exists k such that $c_{\sigma_k}(\lambda) < c_{\sigma_k}(e^{ch}\lambda)$. Since $c_{\sigma_k}(e^{th}\lambda)$ is continuous on t and $\mathcal{T}(Y,\lambda)_+$ has zero Lebesgue measure (see [15]), there exists $t \in [0,c]$ with $c_{\sigma_k}(e^{th}\lambda) \notin \mathcal{T}(Y,\lambda)_+$. Then $\lambda' := e^{th}\lambda$ satisfies the conditions (a) and (b).

Step 2. Take λ' as in Step 1. We show that any $C \in \mathcal{C}_{\mathbb{N}}(Y,\lambda')$ with $\langle C, \lambda' \rangle = c_{\sigma_k}(\lambda')$ satisfies $\operatorname{supp}(C) \cap U_i \neq \emptyset$. Indeed, if $C \in \mathcal{C}_{\mathbb{N}}(Y,\lambda')$ satisfies $\operatorname{supp}(C) \cap U_i = \emptyset$, then (a) implies that $C \in \mathcal{C}_{\mathbb{N}}(Y,\lambda)$ and $\langle C,\lambda'\rangle = \langle C,\lambda\rangle$. Moreover, $C \in \mathcal{C}_{\mathbb{N}}(Y,\lambda)$ implies that $\langle C,\lambda\rangle \in \mathcal{T}(Y,\lambda)_+$. Therefore,

$$c_{\sigma_k}(\lambda') = \langle C,\lambda'\rangle = \langle C,\lambda\rangle \in \mathcal{T}(Y,\lambda)_+,$$

contradicting (b).

Step 3. Recall that $\mathcal{U} = \mathcal{U}' \cap \Lambda_{\mathrm{nice}}(Y,\xi)$. Since $\Lambda_{\mathrm{nice}}(Y,\xi)$ is residual (in particular, dense) in $\Lambda(Y,\xi)$, there exists a sequence $(\lambda_j)_{j\geq 1}$ in $\mathcal{U}$ which converges to λ' in the C^∞-topology. If $\operatorname{supp}(C_{\sigma_k}(\lambda_j)) \cap U_i = \emptyset$ for every $j \geq 1$, then by taking a limit of a certain subsequence of $(C_{\sigma_k}(\lambda_j))_{j\geq 1}$, we obtain $C \in \mathcal{C}_{\mathbb{N}}(Y,\lambda')$, which satisfies $\operatorname{supp}(C) \cap U_i = \emptyset$ and

$$\langle C,\lambda'\rangle = \lim_{j\to\infty} \big\langle C_{\sigma_k}(\lambda_j), \lambda_j \big\rangle = \lim_{j\to\infty} c_{\sigma_k}(\lambda_j) = c_{\sigma_k}(\lambda').$$

This contradicts Step 2. Thus, there exists j such that $\operatorname{supp}(C_{\sigma_k}(\lambda_j)) \cap U_i \neq \emptyset$, which implies that $\lambda_j \in \Lambda_{U_i}$; in particular, $\Lambda_{U_i} \cap \mathcal{U} \neq \emptyset$. □

3.3. Asymptotic behavior of weights

Let $(\sigma_k)_k$ be a sequence in $\mathrm{ECH}(Y,\xi) \setminus \{0\}$ which satisfies the condition $(\star\star)$. In this section, we briefly discuss asymptotic behavior of $\operatorname{weight}(C_{\sigma_k}(\lambda))$ as $k \to \infty$, where $\lambda \in \Lambda_{\mathrm{nice}}(Y,\xi)$.

It is easy to see that $\operatorname{weight}(C_{\sigma_k}(\lambda)) \leq \frac{c_{\sigma_k}(\lambda)}{\min_{\gamma\in\mathcal{P}(Y,\lambda)} T_\gamma}$ for every k. Indeed, setting $C_{\sigma_k}(\lambda) = \sum_{1\leq i\leq k} a_i\gamma_i$, where $a_1, \ldots, a_k$ are positive integers and $\gamma_1, \ldots, \gamma_k$ are distinct elements of $\mathcal{P}_{\mathrm{emb}}(Y,\lambda)$, we obtain

$$c_{\sigma_k}(\lambda) = \sum_i a_i T_{\gamma_i} \geq \Big(\sum_i a_i\Big) \cdot \min_i T_{\gamma_i} \geq \operatorname{weight}\big(C_{\sigma_k}(\lambda)\big) \cdot \min_{\gamma\in\mathcal{P}(Y,\lambda)} T_\gamma.$$

Since $c_{\sigma_k}(\lambda) = O(\sqrt{k})$, we obtain $\limsup_k \frac{\mathrm{weight}(C_{\sigma_k}(\lambda))}{\sqrt{k}} < \infty$. Now let us ask the following question.

QUESTION 3.8
Is it true that $\liminf_k \frac{\mathrm{weight}(C_{\sigma_k}(\lambda))}{\sqrt{k}} > 0$ for any $\lambda \in \Lambda_{\mathrm{nice}}(Y,\xi)$? Also, does $\lim_{k\to\infty} \frac{\mathrm{weight}(C_{\sigma_k}(\lambda))}{\sqrt{k}}$ exist for generic $\lambda \in \Lambda_{\mathrm{nice}}(Y,\xi)$?

3.4. Toy model questions for star-shaped toric domains

Following [16, Section 6], let us briefly discuss toy model versions of Conjecture 3.3 and Question 3.8, which are formulated for star-shaped toric domains.

We designate $\Omega \subset \mathbb{R}^2_{\geq 0}$ a star-shaped domain if there exists $\rho \in C^\infty([0,\pi/2], \mathbb{R}_{>0})$ such that

$$\Omega = \big\{(r\cos\theta, r\sin\theta) \bigm| 0 \leq \theta \leq \pi/2, 0 \leq r \leq \rho(\theta)\big\}.$$

We denote

$$\partial\Omega := \big\{\big(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta\big) \bigm| 0 \leq \theta \leq \pi/2\big\}$$

although this set is strictly smaller than the boundary of Ω in $\mathbb{R}^2$. Let us define $\mu : \mathbb{C}^2 \to \mathbb{R}^2_{\geq 0}$ by $\mu(z_1,z_2) := (\pi|z_1|^2, \pi|z_2|^2)$, and set

$$Y_\Omega := \mu^{-1}(\partial\Omega), \qquad \lambda_\Omega := \Big(\sum_{i=1}^{2} \frac{x_i\,dy_i - y_i\,dx_i}{2}\Big)\Big|_{Y_\Omega}, \qquad \xi_\Omega := \ker(\lambda_\Omega).$$

Here we set $z_i = x_i + \sqrt{-1}y_i$ for $i = 1,2$. It is known that $\mathrm{ECH}(Y_\Omega,\xi_\Omega) = \bigoplus_{k=1}^\infty (\mathbb{Z}/2)\sigma_k$, where $(\sigma_k)_{k\geq 1}$ satisfies $(\star)$, that is, $U\sigma_{k+1} = \sigma_k$ for every $k \geq 1$.

For any $p \in \partial\Omega$, let $\nu(p)$ denote the unit vector which is normal to $T_p(\partial\Omega)$ and satisfies $p \cdot \nu(p) > 0$. Let us set

$$\mathcal{R}(\partial\Omega) := \big\{\big(\rho(0),0\big), \big(0,\rho(\pi/2)\big)\big\} \cup \big\{p \in \partial\Omega \bigm| \nu(p) \in \mathbb{R}\cdot\mathbb{Q}^2\big\}.$$

For every $p \in \mathcal{R}(\partial\Omega)$, we define $n(p) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ as follows:

- If $p = (\rho(0),0)$, then $n(p) := (1,0)$.
- If $p = (0,\rho(\pi/2))$, then $n(p) := (0,1)$.
- Otherwise, $n(p)$ is a unique primitive element in $\mathbb{Z}^2$ such that there exists $a \in \mathbb{R}_{>0}$ with $n(p) = a\nu(p)$.

We define $w : \mathcal{R}(\partial\Omega) \to \mathbb{R}_{>0}$ by $w(p) := p \cdot n(p)$. We say that $\partial\Omega$ is *nice* if $w(p_1),\dots,w(p_k)$ are linearly independent over $\mathbb{Q}$ for any distinct elements $p_1,\dots,p_k$ of $\mathcal{R}(\partial\Omega)$. By Lemma 6.1 of [16], niceness is a C^∞-generic condition.

For any 1-dimensional current C on Y_Ω, we can define a distribution $\mathfrak{D}_C$ on $\partial\Omega$ by

$$\langle \mathfrak{D}_C, f\rangle := \big\langle C, (\mu|_{Y_\Omega})^* f \cdot \lambda_\Omega\big\rangle \quad \big(\forall f \in C^\infty(\partial\Omega)\big).$$

Let us state the following key lemma, which is a direct consequence of Lemma 6.2 of [16]. For any $p \in \partial\Omega$, let δ_p denote the Dirac measure at p, that is, $\langle \delta_p, f\rangle = f(p)$ for any $f \in C^\infty(\partial\Omega)$.

LEMMA 3.9

Suppose that $\partial\Omega$ is nice. Then, for any integer $k \geq 1$, there exists a unique set of pairs $\{(m_i, p_i)\}_{1\leq i\leq N}$ such that $m_1, \dots, m_N$ are positive integers, $p_1, \dots, p_N$ are distinct points of $\mathcal{R}(\partial\Omega)$, and

$$\mathfrak{D}_C = \sum_{1\leq i\leq N} m_i w(p_i)\delta_{p_i} \tag{5}$$

for any $C \in \mathcal{C}_{\mathbb{N}}(Y_\Omega, \lambda_\Omega)$ which represents σ_k with respect to λ_Ω.

For any integer $k \geq 1$ and any Ω such that $\partial\Omega$ is nice, let us take $\{(m_i, p_i)\}_{1\leq i\leq N}$ so that (5) is satisfied, and let us set

$$\mathfrak{D}_k(\Omega) := \sum_{1\leq i\leq N} m_i w(p_i)\delta_{p_i}, \qquad \mathrm{weight}_k(\Omega) := \sum_{1\leq i\leq N} m_i.$$

Also, let us define a distribution $\mathfrak{D}_{\mathrm{vol}}(\Omega)$ on $\partial\Omega$ by

$$\langle \mathfrak{D}_{\mathrm{vol}}(\Omega), f\rangle := \int_{Y_\Omega} (\mu|_{\lambda_\Omega})^* f \lambda_\Omega \wedge d\lambda_\Omega \quad \big(\forall f \in C^\infty(\partial\Omega)\big).$$

Then we can ask if

$$\lim_{k\to\infty} \frac{\mathfrak{D}_k(\Omega)}{\sqrt{2k}} = \frac{\mathfrak{D}_{\mathrm{vol}}(\Omega)}{\sqrt{\mathrm{vol}(Y_\Omega, \lambda_\Omega)}} \tag{6}$$

holds for generic Ω such that $\partial\Omega$ is nice; this is a toy model version of Conjecture 3.3. In [16] we checked that (6) holds true when $\partial\Omega$ is strictly convex or strictly concave, using combinatorial formulas in [3], [1], and [12], combined with a version of isoperimetric inequality.

One can also ask if $\lim_{k\to\infty} \frac{\mathrm{weight}_k(\Omega)}{\sqrt{k}}$ exists for generic Ω such that $\partial\Omega$ is nice. This is a toy model version of Question 3.8.

4. C^∞-Closing lemma by real-analytic perturbation

The goal of this section is to prove the following theorem, which is a real-analytic version of Theorem 1.1(i).

THEOREM 4.1

Let Y be a closed 3-manifold with a real-analytic structure, and let $C^\omega(Y, \mathbb{R})$ denote the space of real-analytic functions on Y, equipped with the C^∞-topology, that is, the topology induced from the C^∞-topology on $C^\infty(Y, \mathbb{R})$. Let U be a nonempty open set of Y, and let $\mathcal{N}$ be a neighborhood of 0 in $C^\omega(Y, \mathbb{R})$. Then there exist $h \in \mathcal{N}$ and $\gamma \in \mathcal{P}_{\mathrm{emb}}(Y, (1+h)\lambda)$ such that $\gamma \cap U \neq \emptyset$.

REMARK 4.2

On $C^\omega(Y, \mathbb{R})$, it is more natural to consider the so-called *direct limit topology* (see pp. 52–53 of [17]). However, it seems difficult to prove the version of Theorem 4.1 with respect to the direct limit topology. Also, it seems that the direct limit topology does not satisfy the Bairé property; thus, even if the closing lemma with the direct limit topology holds true, it does not directly imply the generic density theorem.

The proof of Theorem 1.1(i) in [15] uses a bump function supported on U, which cannot be real-analytic. The idea of the proof of Theorem 4.1 is to follow the argument in [15] "with estimates."

4.1. Proof of Theorem 4.1

We may assume that Y is connected. Also, by replacing $\mathcal{N}$ with a smaller neighborhood if necessary, we may assume that $\sup_{h\in\mathcal{N}} \|h\|_{C^2} < \infty$.

LEMMA 4.3
It holds that $\inf_{h\in\mathcal{N}}(\min \mathcal{T}((1+h)\lambda)) > 0$.

Proof
The assumption $\sup_{h\in\mathcal{N}} \|h\|_{C^2} < \infty$ implies that $\sup_{h\in\mathcal{N}} \|R_{(1+h)\lambda}\|_{C^1} < \infty$. On the other hand, there exists a constant $c > 0$ such that, for any C^1-vector field ζ on Y satisfying $\zeta(y) \neq 0$ $(\forall y \in Y)$, the minimal period of periodic orbits of ζ is bounded from below by $\frac{c}{\|\zeta\|_{C^1}}$. Then we can conclude

$$\inf_{h\in\mathcal{N}}\bigl(\min \mathcal{T}\bigl((1+h)\lambda\bigr)\bigr) \geq \frac{c}{\sup_{h\in\mathcal{N}} \|R_{(1+h)\lambda}\|_{C^1}} > 0. \qquad \square$$

We take a sequence $(\sigma_k)_{k\geq 1}$ in $\mathrm{ECH}(Y,\xi) \setminus \{0\}$ satisfying the condition $(\star)$, and fix it for the rest of the proof.

LEMMA 4.4

(i) *For any* $\lambda \in \Lambda(Y,\xi)$*, there exists* $C > 0$ *such that*

$$c_{\sigma_k}(Y,\lambda) \leq C\sqrt{k} \quad (\forall k \geq 1).$$

(ii) *For any nonempty open set* $V \subset Y$*,* $\lambda \in \Lambda(Y,\xi)$*, and* $\varepsilon > 0$*, there exists* $c > 0$ *which satisfies the following condition: for any* $h, h' \in C^\infty(Y, \mathbb{R}_{>0})$ *such that* $h > h'$*,* $h|_V \geq \varepsilon$*, and* $h'|_V \leq \varepsilon/2$*, there holds*

$$c_{\sigma_{k+1}}\bigl(Y,(1+h)\lambda\bigr) - c_{\sigma_k}\bigl(Y,(1+h')\lambda\bigr) \geq c \quad (\forall k \geq 1).$$

Proof
Part (i) is immediate from the volume property. To prove (ii), let us first set

$$B(c) := \bigl\{z \in \mathbb{C}^2 \bigm| \pi|z|^2 \leq c\bigr\} \quad (c > 0),$$

$$\lambda_{B(c)} := \Bigl(\sum_{i=1}^{2} \frac{x_i\,dy_i - y_i\,dx_i}{2}\Bigr)\Big|_{\partial B(c)},$$

$$\xi_{B(c)} := \ker(\lambda_{B(c)}).$$

Then $\mathrm{ECH}(\partial B(c), \xi_{B(c)}) = \bigoplus_{k=0}^{\infty} (\mathbb{Z}/2)\zeta_k$, where $(\zeta_k)_{k\geq 0}$ satisfies $U\zeta_{k+1} = \zeta_k$ for every $k \geq 0$. Then $c_{\zeta_1}(\partial B(c), \lambda_{B(c)}) = c$ by Corollary 1.3 of [10].

If $c > 0$ is sufficiently small, then there exists a symplectic embedding

$$\Big(B(c), \sum_{i=1}^{2} dx_i \wedge dy_i\Big) \to \big(\{(y,s) \mid y \in V, 1+\varepsilon/2 < s < 1+\varepsilon\}, d(s\lambda)\big).$$

If $h, h' \in C^\infty(Y, \mathbb{R}_{>0})$ satisfy $h > h'$, $h'|_V \leq \varepsilon/2$, and $h|_V \geq \varepsilon$, then

$$\{(y,s) \mid y \in V, 1+\varepsilon/2 < s < 1+\varepsilon\} \subset \{(y,s) \mid y \in Y, 1+h'(y) < s < 1+h(y)\}.$$

Then there exists a weakly exact symplectic cobordism (see Definition 2.2 of [10]) from $(Y,(1+h)\lambda)$ to $(Y,(1+h')\lambda) \sqcup (\partial B(c), \lambda_{B(c)})$, which implies a cobordism map

$$\Phi : \mathrm{ECH}(Y,\xi) \to \mathrm{ECH}(Y,\xi) \otimes \mathrm{ECH}\big(\partial B(c), \xi_{B(c)}\big).$$

Lemma 3.2 of [6] implies that

$$\Phi(\sigma_{k+1}) = \sum_{l=0}^{\infty} U^l \sigma_{k+1} \otimes \zeta_l.$$

Now we obtain

$$\begin{aligned} c_{\sigma_{k+1}}\big(Y,(1+h)\lambda\big) &\geq c_{\Phi(\sigma_{k+1})}\big(\big(Y,(1+h')\lambda\big) \sqcup \big(\partial B(c), \lambda_{B(c)}\big)\big) \\ &= \max_{\substack{l \geq 0 \\ U^l\sigma_{k+1} \neq 0}} c_{U^l\sigma_{k+1}}\big(Y,(1+h')\lambda\big) + c_{\zeta_l}\big(\partial B(c), \lambda_{B(c)}\big) \\ &\geq c_{\sigma_k}\big(Y,(1+h')\lambda\big) + c_{\zeta_1}\big(\partial B(c), \lambda_{B(c)}\big). \end{aligned}$$

The first inequality holds since the ECH cobordism map respects action filtration (see Section 3.1 (in particular, formula (52)) of [6]). The next equality follows from formula (5.6) of [10]. The last inequality is obtained by putting $l = 1$. Finally, we obtain

$$c_{\sigma_{k+1}}\big(Y,(1+h)\lambda\big) - c_{\sigma_k}\big(Y,(1+h')\lambda\big) \geq c_{\zeta_1}\big(\partial B(c), \lambda_{B(c)}\big) = c. \qquad \square$$

We take and fix a neighborhood $\mathcal{N}'$ of 0 in $C^\omega(Y,\mathbb{R})$ with the C^∞-topology such that

$$(7) \qquad s_1, s_2 \in [0,1], \qquad g_1, g_2 \in \mathcal{N}' \qquad \Longrightarrow \qquad s_1 g_1 + s_2 g_2 \in \mathcal{N}.$$

We also take and fix a nonempty open set $V \subset Y$ such that $\bar{V} \subset U$, where U is a nonempty open set in Y which appears in the statement of Theorem 4.1.

LEMMA 4.5

There exist $\varepsilon \in (0,1]$ and a sequence $(h'_j)_{j\geq 1}$ in $\mathcal{N}' \cap C^\omega(Y,[0,1])$ such that

- $\min h'_j|_{\bar{V}} \geq \varepsilon$ *for every j,*
- $h'_j|_{Y\setminus U}$ *converges to 0 in the C^∞-topology.*

Proof

By the definition of the C^∞-topology on $C^\omega(Y,\mathbb{R})$, there exists a neighborhood $\mathcal{N}''$ of 0 in $C^\infty(Y,\mathbb{R})$ such that $\mathcal{N}'' \cap C^\omega(Y,\mathbb{R}) = \mathcal{N}'$. Take $h' \in C^\infty(Y,[0,1/2]) \cap$

$\mathcal{N}''$ which is supported on U and $\min h'|_{\bar{V}} > 0$. Since $C^\omega(Y,\mathbb{R})$ is dense in $C^\infty(Y,\mathbb{R})$ with the C^∞-topology, there exists a sequence $(h'_j)_j$ in $C^\omega(Y,\mathbb{R})$ which converges to h' in the C^∞-topology, and $h_j \geq 0$ for every j. Then $h'_j \in \mathcal{N}' \cap C^\omega([0,1])$ for every sufficiently large j. Moreover, for any $\varepsilon \in (0, \min h'|_{\bar{V}})$, there holds $\min h'_j|_{\bar{V}} > \varepsilon$ for every sufficiently large j. □

Let us take $\varepsilon \in (0,1]$ and a sequence $(h'_j)_{j\geq 1}$ as in Lemma 4.5, and fix them for the rest of the proof.

By Lemma 4.4(ii), there exists $c_1 > 0$ such that

$$c_{\sigma_{k+1}}\big((1+h'_j+h'')\lambda\big) - c_{\sigma_k}\big((1+h'')\lambda\big) \geq c_1 \tag{8}$$

for any integer $j,k \geq 1$ and any $h'' \in C^\infty(Y,\mathbb{R}_{\geq 0})$ such that $h''|_{\bar{V}} \leq \varepsilon/2$. On the other hand, by Lemma 4.4(i), there exists $c_2 > 0$ such that $c_{\sigma_k}(\lambda) \leq c_2\sqrt{k}$ for every $k \geq 1$. Let us take an integer K so that $4c_2/\sqrt{K} \leq c_1/10$, and set $c_3 := 3c_2\sqrt{K}$.

LEMMA 4.6

There exists $h'' \in \mathcal{N}' \cap C^\omega(Y,[0,\varepsilon/2])$ such that every $\gamma \in \mathcal{P}((1+h'')\lambda)$ satisfying $T_\gamma \leq c_3$ is nondegenerate.

Proof

There exists a neighborhood $\mathcal{N}''$ of 0 in $C^\infty(Y,\mathbb{R})$ such that $\mathcal{N}'' \cap C^\omega(Y,\mathbb{R}) = \mathcal{N}'$. There exists $h''' \in \mathcal{N}'' \cap C^\infty(Y,(0,\varepsilon/2))$ such that all elements in $\mathcal{P}((1+h''')\lambda)$ are nondegenerate. Then take $h'' \in C^\omega(Y,\mathbb{R})$ which is sufficiently close to h''' in the C^∞-topology. □

We take and fix h'' as in Lemma 4.6. Then $\mathcal{T}((1+h'')\lambda) \cap [0,c_3]$ is discrete; thus, $\mathcal{T}((1+h'')\lambda)_+ \cap [0,c_3]$ is also discrete. Then there exists $\delta > 0$ such that each connected component of $B_\delta(\mathcal{T}((1+h'')\lambda)_+) \cap [0,c_3]$ is contained in an open interval of length 2δ. Here B_δ denotes the open δ-neighborhood. We may further assume that $\delta \leq c_1/10$.

For any contact form λ' on Y, we set

$$\mathcal{T}_U(\lambda') := \big\{T_\gamma \;\big|\; \gamma \in \mathcal{P}(\lambda'), \operatorname{Im}\gamma \cap U = \emptyset\big\}.$$

Before stating our next lemma, we need the following remark.

REMARK 4.7

For every integer $j \geq 1$ and $t \in [0,1]$, there holds $th'_j + h'' \in \mathcal{N}$ by $h'_j, h'' \in \mathcal{N}'$ and (7).

LEMMA 4.8

For every sufficiently large j,

$$\mathcal{T}_U\big((1+th'_j+h'')\lambda\big)_+ \cap [0,c_3] \subset B_\delta\big(\mathcal{T}\big((1+h'')\lambda\big)_+\big) \tag{9}$$

for every $t \in [0,1]$.

Proof
Suppose that this is not the case. Then there exist

- $(j_m)_{m\ge 1}$, a sequence of integers which goes to ∞ as $m\to\infty$;
- $(t_m)_{m\ge 1}$, a sequence in $[0,1]$; we set $\lambda_m := (1+t_m h'_{j_m}+h'')\lambda$;
- a sequence $(a_m)_{m\ge 1}$ such that $a_m \in (\mathcal{T}_U(\lambda_m)_+ \cap [0,c_3]) \setminus B_\delta(\mathcal{T}((1+h'')\lambda)_+)$.

For every $m\ge 1$ there exist $\gamma_{m,1},\dots,\gamma_{m,N_m}\in\mathcal{P}(\lambda_m)$ such that their images are disjoint from U and $a_m = T_{\gamma_{m,1}}+\cdots+T_{\gamma_{m,N_m}}$. Let $\tau := \inf_m(\min\mathcal{T}(\lambda_m))$. Then $\tau>0$ by Remark 4.7 and Lemma 4.3. Moreover,

$$\tau\cdot N_m \le T_{\gamma_{m,1}}+\cdots+T_{\gamma_{m,N_m}} = a_m \le c_3,$$

where the last inequality holds since we have taken a_m so that $a_m\in[0,c_3]$. Now we have $N_m\le c_3/\tau$ for every m, in particular, $\sup_m N_m<\infty$. Thus, we may assume that N_m does not depend on m and denote it by N. Since $h'_j|_{Y\setminus U}$ converges to 0 in the C^∞-topology as $j\to\infty$, $\lambda_m|_{Y\setminus U}$ converges to $(1+h'')\lambda|_{Y\setminus U}$ in the C^∞-topology as $m\to\infty$. Thus, for each $k\in\{1,\dots,N\}$, a certain subsequence of $(\gamma_{m,k})_m$ converges to an element of $\mathcal{P}((1+h'')\lambda)$ as $m\to\infty$. Therefore, a certain subsequence of $(a_m)_m$ converges to an element of $\mathcal{T}((1+h'')\lambda)_+$, contradicting the assumption that $a_m\notin B_\delta(\mathcal{T}((1+h'')\lambda)_+)$ for every m. □

Let us take sufficiently large j so that (9) holds for every $t\in[0,1]$, and set $\lambda_t := (1+th'_j+h'')\lambda$. By Remark 4.7, $th'_j+h''\in\mathcal{N}$ for every $t\in[0,1]$. Hence, to prove Theorem 4.1, it is sufficient to prove the following claim.

CLAIM 4.9
There exist $t\in[0,1]$ and $\gamma\in\mathcal{P}(\lambda_t)$ such that $\operatorname{Im}\gamma\cap U\ne\emptyset$.

If the claim does not hold, then $\mathcal{T}(\lambda_t)=\mathcal{T}_U(\lambda_t)$ for every $t\in[0,1]$. Then, by the spectrality, $c_{\sigma_l}(\lambda_t)\in\mathcal{T}_U(\lambda_t)_+$ for every integer l. On the other hand, for every $k\in\{1,\dots,K-1\}$ there holds

$$c_{\sigma_{k+1}}(\lambda_t) = c_{\sigma_{k+1}}\big((1+th'_j+h'')\lambda\big) \le 3c_{\sigma_{k+1}}(\lambda) \le 3c_2\sqrt{K} = c_3,$$

where the first inequality holds since $h'_j(y), h''(y)\in[0,1]$ for every $y\in Y$.

Now we obtain

$$c_{\sigma_{k+1}}(\lambda_t)\in\mathcal{T}_U(\lambda_t)_+\cap[0,c_3]\subset B_\delta\big(\mathcal{T}\big((1+h'')\lambda\big)_+\big)\cap[0,c_3].$$

Since the left-hand side is a continuous function of $t\in[0,1]$ and each connected component of the right-hand side is contained in an open interval of length 2δ, we obtain

$$c_{\sigma_{k+1}}\big((1+h'_j+h'')\lambda\big) - c_{\sigma_{k+1}}\big((1+h'')\lambda\big) = c_{\sigma_{k+1}}(\lambda_1) - c_{\sigma_{k+1}}(\lambda_0) \le 2\delta$$

for every $k\in\{1,\dots,K-1\}$.

On the other hand,

$$\min_{1\le k\le K-1} c_{\sigma_{k+1}}\big((1+h'')\lambda\big) - c_{\sigma_k}\big((1+h'')\lambda\big) \le c_{\sigma_K}\big((1+h'')\lambda\big)/(K-1)$$
$$\le 4c_2/\sqrt{K} \le c_1/10,$$

where the second inequality follows from $c_{\sigma_K}((1+h'')\lambda) \le 2c_2\sqrt{K}$ and $K-1 \ge K/2$. Finally, there exists $k \in \{1,\dots,K-1\}$ such that

$$c_{\sigma_{k+1}}\big((1+h'_j+h'')\lambda\big) - c_{\sigma_k}\big((1+h'')\lambda\big) \le 2\delta + c_1/10 \le 3c_1/10 < c_1,$$

which contradicts (8). This completes the proof of Theorem 4.1.

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