A critical point analysis of Landau–Ginzburg potentials with bulk in Gelfand–Cetlin systems

Yunhyung Cho, Yoosik Kim, and Yong-Geun Oh

In honor of Kenji Fukaya's 60th birthday

Abstract Using the bulk deformation of Floer cohomology by Schubert classes and non-Archimedean analysis of Fukaya–Oh–Ohta–Ono's bulk-deformed potential function, we prove that every complete flag manifold $\mathrm{Fl}(n)$ $(n \geq 3)$ with a monotone Kirillov–Kostant–Souriau (KKS) symplectic form carries a continuum of nondisplaceable Lagrangian tori which degenerates to a nontorus fiber in the Hausdorff limit. In particular, the Lagrangian S^3-fiber in $\mathrm{Fl}(3)$ is nondisplaceable, answering a question raised by Nohara and Ueda who computed its Floer cohomology to be vanishing.

1. Introduction

This article is a sequel to Part I of [6]. Here we focus on detecting nondisplaceable Lagrangian fibers of Gelfand–Cetlin (GC) systems.

Lagrangian Floer theory on GC systems was first studied by Nishinou, Nohara, and Ueda in [26]. They proved that the Lagrangian GC torus fiber at the center of the GC polytope is nondisplaceable. To show it, they calculated the potential function by constructing a toric degeneration from a GC system to a toric moment map and finding a weak bounding cochain such that the deformed Floer cohomology is nonvanishing. The GC system admits nontorus Lagrangian GC fibers at the lower-dimensional strata of the GC polytope, which makes Floer theory of the system more interesting and challenging. Using non-Abelian symmetry or discrete symmetry, particular fibers of limited cases of Grassmannians have been investigated in [8], [9], and [28].

Motivated by those works, we recently undertook a systematic study of topological/geometrical types of GC fibers and provided a complete classification thereof in terms of the combinatorial ladder diagram in [6]. In this paper, we discuss nondisplaceable GC fibers on complete flag manifolds equipped with monotone Kirillov–Kostant–Souriau symplectic forms (KKS forms for short).

Kyoto Journal of Mathematics, Vol. 61, No. 2 (2021), 259–304
DOI 10.1215/21562261-2021-0002,
Received November 12, 2019. Revised February 21, 2020. Accepted May 27, 2020.
First published online April 7, 2021.
2020 Mathematics Subject Classification: Primary 53D40; Secondary 53D12, 37J35, 14M15.

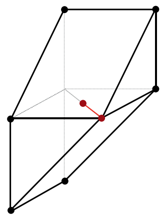

Figure 1. The positions of nondisplaceable GC Lagrangian fibers in $\mathrm{Fl}(3)$.

Let $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n)$ be a decreasing sequence of real numbers so that the coadjoint orbit $\mathcal{O}_\lambda$ is diffeomorphic to a complete flag manifold $\mathrm{Fl}(n)$. When the KKS form ω_λ is *monotone*, we shall show nondisplaceability of certain torus and nontorus fibers. First, we describe the putative positions of nondisplaceable fibers. By scaling ω_λ if necessary, we may assume that

$$\lambda = (\lambda_i := n - 2i + 1 \mid i = 1, \dots, n), \tag{1.1}$$

which is the case where $[\omega_\lambda] = c_1(T\mathcal{O}_\lambda)$. In this case, the GC polytope Δ_λ is a reflexive polytope so that it admits a unique lattice point in its interior (see [1, Corollary 2.2.4]). Each Lagrangian face—a face containing Lagrangian fibers in its relative interior—admits a unique point satisfying certain Bohr–Sommerfeld conditions, at which a monotone Lagrangian fiber is located (see [5]). A candidate is a line segment connecting the center of the polytope and the position of a monotone Lagrangian fiber (in the case of monotone complete flag manifolds).

To verify that a nontorus Lagrangian fiber is nondisplaceable, a family of Lagrangian tori whose Hausdorff limit is the nontorus fiber will be taken into account. Once we show that the tori are nondisplaceable, we then obtain nondisplaceability of the nontorus Lagrangian fiber.

We start from the simplest case, with the coadjoint orbit $\mathcal{O}_\lambda$ of a sequence $\lambda = (\lambda_1 > \lambda_2 > \lambda_3)$. In this case, Pabiniak investigated displaceable GC fibers.

THEOREM 1.1 ([29, Theorem 1.1])
For $\lambda = (\lambda_1 > \lambda_2 > \lambda_3)$, let $(\mathcal{O}_\lambda, \omega_\lambda)$ be as above.

(1) *If ω_λ is not monotone, then all the fibers but one over the center are displaceable.*

(2) *If ω_λ is monotone, then all the fibers but those over the line segment*

$$I := \left\{ (u_{1,1}, u_{1,2}, u_{2,1}) = (0, a - t, -a + t) \in \mathbb{R}^3 : 0 \le t \le a \right\} \tag{1.2}$$

are displaceable, where $2a = \lambda_1 - \lambda_2$. Observe that the line segment I is the red line in Figure 1.

Note that the line segment I connects the center $(0, a, -a)$ of the GC polytope Δ_λ and the position $(0,0,0)$ of the Lagrangian 3-sphere.

The following theorem asserts that every GC fiber in the family $\{\Phi_\lambda^{-1}(\mathbf{u}) \mid \mathbf{u} \in I\}$ is indeed nondisplaceable. Thus, together with Theorem 1.1, our result provides the complete classification of displaceable and nondisplaceable Lagrangian GC fibers when ω_λ is monotone.

THEOREM A (Theorem 2.9)

Let $\lambda = (\lambda_1 = 2 > \lambda_2 = 0 > \lambda_3 = -2)$, and consider the coadjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$. Then the fiber over a point $\mathbf{u} \in \Delta_\lambda$ is nondisplaceable if and only if $\mathbf{u} \in I$, where

$$I := \{(u_{1,1}, u_{1,2}, u_{2,1}) = (0, 1-t, -1+t) \in \mathbb{R}^3 | 0 \le t \le 1\}. \tag{1.3}$$

In particular, the Lagrangian 3-sphere $\Phi_\lambda^{-1}(0,0,0)$ is nondisplaceable.

REMARK 1.2

Nohara and Ueda [28] calculated a Floer cohomology of the Lagrangian 3-sphere $\Phi_\lambda^{-1}(0,0,0)$, which turns out to be zero over the Novikov field Λ; hence nondisplaceability of the fiber remained open. Theorem A resolves the question by showing that the fiber is nondisplaceable.

Next, we deal with a general case for an arbitrary positive integer $n \ge 4$ where λ is given as in (1.1). In this case, the GC polytope Δ_λ is a reflexive polytope whose center is

$$(u_{i,j} := j - i \mid i + j \le n) \in \Delta_\lambda \subset \mathbb{R}^{n(n-1)/2}.$$

Consider the face f_m of Δ_λ defined by

$$\{u_{i,j} = u_{i,j+1} \mid 1 \le i \le m, 1 \le j \le m-1\}$$
$$\cap \{u_{i+1,j} = u_{i,j} \mid 1 \le i \le m-1, 1 \le j \le m\}$$

for any integer m satisfying $2 \le m \le \lfloor \frac{n}{2} \rfloor$. Note that there are $(\lfloor \frac{n}{2} \rfloor - 1)$ such faces in Δ_λ.

The candidates for nondisplaceable Lagrangian fibers are the fibers over the line segment $I_m \subset \Delta_\lambda$ connecting the center of Δ_λ and the center of f_m for each $m \ge 2$. Explicitly, the line segment I_m is parameterized by $\{I_m(t) \in \Delta_\lambda \mid 0 \le t \le 1\}$, where

$$I_m(t) := \begin{cases} u_{i,j}(t) = (j-i) - (j-i)t & \text{if } \max(i,j) \le m, \\ u_{i,j}(t) = (j-i) & \text{if } \max(i,j) > m. \end{cases} \tag{1.4}$$

We denote by $L_m(t)$ the Lagrangian GC fiber over the point $I_m(t)$, that is, $L_m(t) := \Phi_\lambda^{-1}(I_m(t))$. The following is the main theorem of the present paper.

THEOREM B (Theorem 4.18)

Let $\lambda = (\lambda_i := n - 2i + 1 | i = 1, \ldots, n)$ be an n-tuple of real numbers for an arbitrary integer $n \ge 4$. Consider the coadjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$. Then each GC fiber $L_m(t)$ is nondisplaceable Lagrangian for every $2 \le m \le \lfloor \frac{n}{2} \rfloor$.

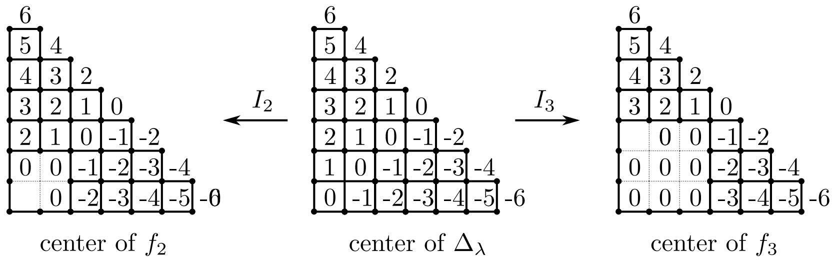

Figure 2. Positions of nondisplaceable Lagrangian GC fibers in Fl(7).

The main ingredient of the proof of the theorem is a careful non-Archimedean (or T-adic) analysis of the critical point equation of the bulk-deformed potential in the spirit of [14], [15], and [22], generalizing the analysis from the toric to the GC system.

EXAMPLE 1.3

A monotone complete flag manifold Fl(7) admits (at least) two line segments I_2 and I_3 in the GC polytope over which the fibers are nondisplaceable as depicted in Figure 2. Particularly, it has nondisplaceable Lagrangian fibers diffeomorphic to $\mathrm{U}(2) \times T^{17}$ and $\mathrm{U}(3) \times T^{12}$.

REMARK 1.4

The third author, along with Fukaya, Ohta, and Ono [14], found a continuum of nondisplaceable torus fibers on some compact toric manifolds including a non-monotone toric blowup of $\mathbb{C}P^2$ at two points (see also Woodward [32]). Using the degeneration models, those authors also produced a continuum of nondisplaceable Lagrangian tori on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the cubic surface, respectively, in [15] and [17]. Vianna [31] also showed a continuum of nondisplaceable tori in $(\mathbb{C}P^1)^{2n}$. Sun [30] found nondisplaceable Lagrangian tori near a chain of Lagrangian 2-spheres in a closed symplectic 4-manifold and del Pezzo surfaces.

2. Lagrangian Floer theory on Gelfand–Cetlin systems

In this section, after briefly recalling Lagrangian Floer theory and its deformation developed by the third author with Fukaya, Ohta, and Ono in a general context, we review the work of Nishinou, Nohara, and Ueda on the calculation of the potential function of a GC system. Then, using the combinatorial description of Schubert classes in complete flag manifolds by Kogan, we will compute the potential function deformed by a combination of Schubert classes of codimension 2 as a Laurent series. Finally, combining those ingredients, we give the proof of Theorem A.

2.1. Potential functions of Gelfand–Cetlin systems

We begin by setting up notation.

- $\Lambda := \{\sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty\}$,
- $\mathfrak{v}_T \colon \Lambda \backslash \{0\} \to \mathbb{R}, \mathfrak{v}_T(\sum_{j=1}^{\infty} a_j T^{\lambda_j}) := \inf_j \{\lambda_j \mid a_j \neq 0\}$,
- $\Lambda_0 := \mathfrak{v}_T^{-1}[0,\infty) \cup \{0\} = \{\sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \mid \lambda_i \geq 0\}$,
- $\Lambda_+ := \mathfrak{v}_T^{-1}(0,\infty) \cup \{0\} = \{\sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \mid \lambda_i > 0\}$,
- $\Lambda_U := \Lambda_0 \backslash \Lambda_+ = \{\sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda_0 \mid \mathfrak{v}_T(\sum_{i=1}^{\infty} a_i T^{\lambda_i}) = 0\}$.

For a relatively spin closed Lagrangian submanifold L in a closed symplectic manifold (X,ω), by the work of Fukaya [10], one can associate an A_∞-algebra $\{\mathfrak{m}_k\}_{k\geq 0}$ on the Λ_0-valued de Rham complex of L. Following the procedure in [11], the constructed A_∞-algebra can be converted into the canonical model on $H^\bullet(L;\Lambda_0)$.

A solution $b \in H^1(L;\Lambda_+)$ of the (weak) Maurer–Cartan equation

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) \equiv 0 \mod \mathrm{PD}[L]$$

is called a *(weak) bounding cochain*. The value of the *potential function* W at a bounding cochain b is assigned to be the multiple of the Poincaré dual $\mathrm{PD}[L]$ of L. Namely, we solve

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) = W(b) \cdot \mathrm{PD}[L], \quad \text{with } W(b) \in \Lambda_0.$$

Since $\mathrm{PD}[L]$ is the strict unit of the A_∞-algebra (on the de Rham model), the deformed map $\mathfrak{m}_1^b$ defined by

$$\mathfrak{m}_1^b(h) := \sum_{l,k} \mathfrak{m}_{l+k+1}(b^{\otimes l}, h, b^{\otimes k})$$

becomes a differential, and thus, its cohomology (deformed by b) over Λ_0 is defined. Let

- $HF((L,b);\Lambda_0) := \mathrm{Ker}(\mathfrak{m}_1^b)/\mathrm{Im}(\mathfrak{m}_1^b)$,
- $HF((L,b);\Lambda) := HF((L,b);\Lambda_0) \otimes_{\Lambda_0} \Lambda$.

The reader is referred to [12]–[14], and [16] for details.

Now, we restrict to the case of a Lagrangian GC torus fiber $\Phi_\lambda^{-1}(\mathbf{u})$ in a coadjoint orbit $\mathcal{O}_\lambda$. As a deformation of the action of the Borel subgroup, Kogan and Miller [24] realized the toric degeneration of a flag manifold constructed by Gonciulea and Lakshmibai [20]. Using the degeneration of a (partial) flag manifold to the GC toric variety, Nishinou, Nohara, and Ueda [26] constructed a degeneration of the GC system of $\mathcal{O}_\lambda$ to the moment map of the toric variety.

THEOREM 2.1 ([26, Theorem 1.2])

For any nonincreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, there exists a toric degeneration of the GC system Φ_λ on the coadjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ in the following sense.

(1) *There is a flat family* $f\colon \mathcal{X} \to I = [0,1]$ *of algebraic varieties and a 2-form* $\widetilde{\omega}$ *on* $\mathcal{X}$ *such that*

(a) $X_0 := f^{-1}(0)$ *is the toric variety associated with the GC polytope* Δ_λ *and* $\omega_0 := \widetilde{\omega}|_{X_0}$ *is a torus-invariant Kähler form.*

(b) $X_1 := f^{-1}(1)$ *is the coadjoint orbit* $\mathcal{O}_\lambda$ *and* $\omega_1 = \widetilde{\omega}|_{X_1}$ *is the KKS form* ω_λ.

(2) *There is a family* $\{\Phi_t \mid X_t \to \Delta_\lambda\}_{0\le t\le 1}$ *of completely integrable systems such that* Φ_0 *is the moment map for the torus action on* X_0 *and* Φ_1 *is the GC system.*

(3) *Let* $\Delta_\lambda^{\mathrm{sm}} := \Delta_\lambda \backslash \Phi_0(\mathrm{Sing}(X_0))$ *and* $X_t^{\mathrm{sm}} := \Phi_t^{-1}(\Delta_\lambda^{\mathrm{sm}})$, *where* $\mathrm{Sing}(X_0)$ *is the set of singular points of* X_0. *Then there exists a flow* ϕ_t *on* $\mathcal{X}$ *such that for each* $0 \le t \le s$, *the restricted flow* $\phi_t|_{X_s^{\mathrm{sm}}}\colon X_s^{\mathrm{sm}} \to X_{s-t}^{\mathrm{sm}}$ *respects the symplectic structures and the complete integrable systems:*

$$\begin{array}{ccc} (X_s^{\mathrm{sm}}, \omega_s) & \xrightarrow{\ \phi_t|_{X_s^{\mathrm{sm}}}\ } & (X_{s-t}^{\mathrm{sm}}, \omega_{s-t}) \\ & {\scriptstyle \Phi_s} \searrow \quad \swarrow {\scriptstyle \Phi_{s-t}} & \\ & \Delta_\lambda^{\mathrm{sm}} & \end{array}$$

Let $\phi_s'\colon X_s \to X_0$ be a (continuous) extension of the flow $\phi_s\colon X_s^{\mathrm{sm}} \to X_0^{\mathrm{sm}}$ in Theorem 2.1 (see [26, Section 8]). The extended map ϕ_s' transports Floer theory data from the toric moment map to a nearby integrable system. As the deformation in Theorem 2.1 is a family of Fano varieties and the GC toric variety admits a small resolution at the singular loci, any holomorphic disks bounded by L intersecting the loci collapsing to the singular loci of X_0 must have the Maslov index strictly greater than 2 so that such disks do *not* contribute to the potential function. Furthermore, because the Fredholm regularity is an open condition, the holomorphic disks of Maslov index 2 intersecting the toric divisor in the toric variety X_0 give rise to regular holomorphic disks at X_s for sufficiently small s. Set $X := X_s$, and let L be any Lagrangian torus fiber of $\Phi_s\colon X_s \to \Delta_\lambda$. Combining those with the results of Cho and Oh [4], Nishinou, Nohara, and Ueda proved the following.

(1) Each Lagrangian torus fiber L does not bound any nonconstant holomorphic disks whose classes are of Maslov index less than or equal to 0.

(2) Every class $\beta \in \pi_2(X, L)$ of Maslov index 2 is Fredholm regular.

(3) There is a one-to-one correspondence between the holomorphic disks of Maslov index 2 bounded by a Lagrangian GC torus fiber and the facets of a GC polytope.

(4) For each class β of Maslov index 2, the open Gromov–Witten invariant n_β, which counts the holomorphic disks passing through a generic point in L and representing β, is 1.

The above conditions imply that every 1-cochain in $H^1(L;\Lambda_0)$ is a weak bounding cochain; hence the potential function can be defined on $H^1(L;\Lambda_0)$. To extend deformation space from $H^1(L;\Lambda_+)$ to $H^1(L;\Lambda_0)$, one twists Floer theory by the holonomy of flat nonunitary line bundles as in [3]. Then the potential function is expressed as

$$W(L;b) = \sum_\beta n_\beta \cdot \exp(\partial\beta \cap b) T^{\omega(\beta)/2\pi}, \tag{2.1}$$

where the summation is taken over all homotopy classes in $\pi_2(X,L)$ of Maslov index 2.

Setting

$$\Gamma(n) = \{(i,j) \in \mathbb{N}^2 \mid 2 \le i+j \le n\}, \tag{2.2}$$

we fix the basis $\{\gamma_{i,j} \mid (i,j) \in \Gamma(n)\}$ for $H^1(L;\mathbb{Z})$ dual to the basis for $H_1(L,\mathbb{Z})$ consisting of the orbits generated by periodic Hamiltonians $\{u_{i,j} \mid (i,j) \in \Gamma(n)\}$ in [21]. A 1-cochain $b \in H^1(L;\Lambda_0)$ is expressed as the linear combination $\sum_{(i,j)\in\Gamma(n)} x_{i,j} \cdot \gamma_{i,j}$, and we take the exponential variables

$$y_{i,j} := e^{x_{i,j}}. \tag{2.3}$$

Then the potential function can be expressed as a Laurent polynomial $W(\mathbf{y})$ with respect to $\{y_{i,j} \mid (i,j) \in \Gamma(n)\}$. Setting $u_{i,n+1-i} := \lambda_i$, keep in mind that Δ_λ is defined by

$$\{(u_{i,j}) \in \mathbb{R}^{n(n-1)/2} \mid u_{i,j+1} - u_{i,j} \ge 0, u_{i,j} - u_{i+1,j} \ge 0 \text{ for } (i,j) \in \Gamma(n)\}.$$

THEOREM 2.2 ([26, Theorem 10.1])

Consider the Lagrangian torus L over a point $(u_{i,j} \mid (i,j) \in \Gamma(n))$, and set $u_{i,n+1-i} =: \lambda_i$. Then the potential function on L is given by

$$W(L;\mathbf{y}) = \sum_{(i,j)\in\Gamma(n)} \Big(\frac{y_{i,j+1}}{y_{i,j}} T^{u_{i,j+1}-u_{i,j}} + \frac{y_{i,j}}{y_{i+1,j}} T^{u_{i,j}-u_{i+1,j}} \Big). \tag{2.4}$$

For t with $0 \le t < 1$, we can arrange the potential function of the fibers $L_m(t)$ over $I_m(t)$ as

$$\begin{aligned} W(L_m(t);\mathbf{y}) &= \Big(\sum_{i=1}^{m}\sum_{j=1}^{m-1} \frac{y_{i,j+1}}{y_{i,j}} + \sum_{i=1}^{m-1}\sum_{j=1}^{m} \frac{y_{i,j}}{y_{i+1,j}} \Big) T^{1-t} \\ &\quad + \sum_{\max(i,j)\ge m+1} \Big(\frac{y_{i,j+1}}{y_{i,j}} + \frac{y_{i,j}}{y_{i+1,j}} \Big) T^1 \\ &\quad + \sum_{i=0}^{m-1} \Big(\frac{y_{m-i,m+1}}{y_{m-i,m}} + \frac{y_{m,m-i}}{y_{m+1,m-i}} \Big) T^{1+it}. \end{aligned} \tag{2.5}$$

For simplicity, we frequently omit $L_m(t)$ in $W(L_m(t);\mathbf{y})$ if $L_m(t)$ is clear from the context.

2.2. Bulk deformations by Schubert classes

We shall apply Lagrangian Floer theory deformed by ambient cycles of a symplectic manifold, as developed in [12] and [14]. We will exploit the cycles of the form

$$\mathfrak{b} := \sum_{j=1}^{B} \mathfrak{b}_j \cdot \mathscr{D}_j,$$

where each $\mathscr{D}_j$ is a cycle of degree 2 not intersecting L. The deformed potential function is denoted by $W^{\mathfrak{b}}$.

We first recall from [14] the formula for the potential function of a torus fiber L deformed by a combination of toric divisors $\mathscr{D}_j$, $\mathfrak{b} := \sum_{j=1}^{B} \mathfrak{b}_j \cdot \mathscr{D}_j$, in a compact toric manifold X.

THEOREM 2.3 ([14, Lemma 9.2, Equation (9.3)])
The bulk-deformed potential function, also called the potential function with bulk*, is*

$$W^{\mathfrak{b}}(L;b) = \sum_{\beta} n_\beta \cdot \exp\Big(\sum_{j=1}^{B} (\beta \cap \mathscr{D}_j)\mathfrak{b}_j\Big) \exp(\partial\beta \cap b) T^{\omega(\beta)/2\pi}, \tag{2.6}$$

where the summation is taken over all homotopy classes in $\pi_2(X,L)$ of Maslov index 2*.*

In the derivation of this in [14], the smoothness and the T^n-invariance of the relevant ambient cycles are used. Since Schubert classes that we will use are neither smooth nor T^n-invariant in general, we will provide details of the proof of this theorem for the current GC case modifying the arguments used in the proof of [14, Proposition 4.7] similarly as done in [26, Section 9] (see Section 5). The upshot is that we still have the same formula for the potential function with bulk in the current GC case (see (2.6)). Again by taking the system of exponential coordinates in (2.3), $W^{\mathfrak{b}}$ in (2.6) becomes a Laurent polynomial with respect to $\{y_{i,j} \mid (i,j) \in \Gamma(n)\}$.

THEOREM 2.4 ([14, Section 8])
If the bulk-deformed potential function $W^{\mathfrak{b}}(L;\mathbf{y})$ admits a critical point $\mathbf{y}$ whose components are in Λ_U, then L is nondisplaceable.

In his dissertation, Kogan [23] found an expression of a Schubert class in terms of a certain union of the inverse images of faces in the GC system of a complete flag manifold (see also Kogan and Miller [24]). Due to presence of nontorus fibers in [6], the inverse image of a *single* face might have boundary so that it does *not* form a cycle. What he proved is that a certain combination of faces can form a cycle because the boundaries are cancelled out.

We review the result in terms of ladder diagrams. A facet in a GC polytope is called *horizontal* (resp., *vertical*) if it is given by $u_{i,j} = u_{i+1,j}$ (resp., $u_{i,j+1} = u_{i,j}$). Let $P^{\text{hor}}_{i,i+1}$ (resp., $P^{\text{ver}}_{j+1,j}$) be the union of horizontal (resp., vertical) facets between the ith column and the $(i+1)$th column (resp., the $(j+1)$th row and the jth row). That is,

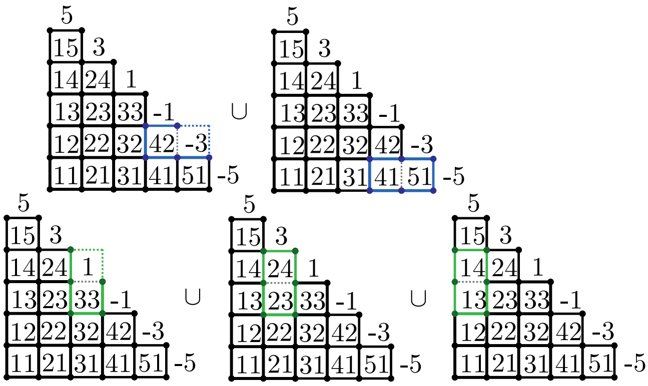

Figure 3. $P^{\text{hor}}_{4,5}$ and $P^{\text{ver}}_{4,3}$ in Fl(6).

$$P^{\text{hor}}_{i,i+1} := \bigcup_{s=1}^{n-i} \{u_{i,s} = u_{i+1,s}\}, \qquad P^{\text{ver}}_{j+1,j} := \bigcup_{r=1}^{n-j} \{u_{r,j+1} = u_{r,j}\}$$

for $1 \leq i, j \leq n-1$, where $\{u_{\bullet,\bullet} = u_{\bullet,\bullet}\}$ denotes the facet given by the equation inside. Let

$$\mathscr{D}^{\text{hor}}_{i,i+1} := \Phi_\lambda^{-1}(P^{\text{hor}}_{i,i+1}), \qquad \mathscr{D}^{\text{ver}}_{j+1,j} := \Phi_\lambda^{-1}(P^{\text{ver}}_{j+1,j}), \tag{2.7}$$

which are respectively called a *horizontal* and *vertical Schubert class* (of degree 2). (See Theorem 2.6 below.)

EXAMPLE 2.5

Consider the coadjoint orbit $\mathcal{O}_\lambda \simeq \text{Fl}(6)$, where $\lambda = (5, 3, 1, -1, -3, -5)$. $P^{\text{hor}}_{4,5}$ is the union of two horizontal facets $P^{\text{hor}}_{4,5} = \{u_{4,2} = -3\} \cup \{u_{4,1} = u_{5,1}\}$ as in Figure 3 and $P^{\text{ver}}_{4,3}$ is the union of three vertical facets $P^{\text{ver}}_{4,3} = \{1 = u_{3,3}\} \cup \{u_{2,4} = u_{2,3}\} \cup \{u_{1,4} = u_{1,3}\}$ as in Figure 3.

From the combinatorial process in [23] and [24], we observe that the Schubert varieties associated with the simple transpositions with a complex codimension 1 correspond to either unions of horizontal facets or unions of vertical facets. The opposite Schubert varieties correspond to the other (see [24, Remark 9]).

THEOREM 2.6 ([23, Theorem 2.3.1], [24, Theorem 8])

The inverse image $\mathscr{D}^{\text{hor}}_{\bullet,\bullet+1}$ (or $\mathscr{D}^{\text{ver}}_{\bullet+1,\bullet}$) represents a (or opposite) Schubert class of degree 2.

Now, we apply (2.6), which is a counterpart of (2.3), to calculate the bulk-deformed potential function. Using the one-to-one correspondence in property (3) of Section 2.1, let $\beta^{i,j}_{i+1,j}$(resp.,$\beta^{i,j+1}_{i,j}$) be the homotopy class in $\pi_2(\mathcal{O}_\lambda, L)$ represented by a holomorphic disk intersecting the facet $u_{i,j} = u_{i+1,j}$ (resp., $u_{i,j+1} = u_{i,j}$) once.

LEMMA 2.7

Let $\mathscr{D}$ be either a horizontal or a vertical Schubert class in X_s. Then we have

$$\beta^{i,j}_{i+1,j} \cap \mathscr{D} = \begin{cases} 1 & \text{if } \mathscr{D} = \mathscr{D}^{\text{hor}}_{i,i+1}, \\ 0 & \text{otherwise,} \end{cases} \qquad \beta^{i,j+1}_{i,j} \cap \mathscr{D} = \begin{cases} 1 & \text{if } \mathscr{D} = \mathscr{D}^{\text{ver}}_{j+1,j}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof

Let $\phi'_s\colon X_s \to X_0$ be a (continuous) extension of the flow $\phi_s\colon X_s^{\text{sm}} \to X_0^{\text{sm}}$ in Theorem 2.1 (see [26, Section 8]). Let $\varphi\colon (\mathbb{D}, \partial\mathbb{D}) \to (X_s, L_s)$ be a holomorphic disk in the class $\beta^{i,j}_{i+1,j}$ of Maslov index 2, for example. We then have a (topological) disk $\phi'_s \circ \varphi\colon (\mathbb{D}, \partial\mathbb{D}) \to (X_s, L_s) \to (X_0, L_0)$, representing $(\phi'_s)_* \beta^{i,j}_{i+1,j}$. Note that there exists a holomorphic disk φ_0 by [4] in the class $(\phi'_s)_* \beta^{i,j}_{i+1,j} = [\phi'_s \circ \varphi]$. Meanwhile, by our choice of $\mathscr{D}$, $\phi'_s(\mathscr{D})$ is the union of the components over either $P^{\text{hor}}_{i,i+1}$ or $P^{\text{ver}}_{j+1,j}$. Since the flow ϕ'_s gives rise to a symplectomorphism from X_s^{sm} to X_0^{sm} and the image of the disk φ is contained in X_s^{sm}, the (local) intersection number should be preserved through the flow ϕ'_s. To calculate the intersection number, we consider a small resolution $p\colon \widetilde{X}_0 \to X_0$. Because the intersection happens only outside the singular loci of X_0, we can lift the divisor and the disk φ_0 to $\widetilde{\mathscr{D}}_0$ and $\widetilde{\varphi}_0$ in $\widetilde{X}_0$ without any change of the intersection number. Then we have

$$\beta^{i,j}_{i+1,j} \cap \mathscr{D} = [\varphi] \cap \mathscr{D} = [\widetilde{\varphi}_0] \cap \widetilde{\mathscr{D}}_0,$$

which completes the proof. □

We take

$$\mathfrak{b} := \sum_i \mathfrak{b}^{\text{hor}}_{i,i+1} \cdot \phi'_{1-s}(\mathscr{D}^{\text{hor}}_{i,i+1}) + \sum_j \mathfrak{b}^{\text{ver}}_{j+1,j} \cdot \phi'_{1-s}(\mathscr{D}^{\text{ver}}_{j+1,j}), \tag{2.8}$$

where $\mathfrak{b}^{\text{hor}}_{i,i+1}, \mathfrak{b}^{\text{ver}}_{j+1,j} \in \Lambda_0$ and $\phi'_{1-s}\colon X_1 \to X_s$. By abuse of notation for simplicity, we denote $\phi'_{1-s}(\mathscr{D}^{\text{hor}}_{i,i+1})$ (resp., $\phi'_{1-s}(\mathscr{D}^{\text{ver}}_{j+1,j})$) by $\mathscr{D}^{\text{hor}}_{i,i+1}$ (resp., $\mathscr{D}^{\text{ver}}_{j+1,j}$). By the homotopy invariance of the A_∞-structures, we calculate the (bulk-deformed) Floer cohomology of $L_{m,s}(t)$ in X_s for s sufficiently close to 0. In particular, nondisplaceability of $L_m(t)$ can be achieved as long as the Floer cohomology of $L_{m,s}(t)$ is nonzero. Whenever turning on a bulk deformation, this process of passing to $L_{m,s}(t)$ and $\phi'_{1-s}(\mathscr{D})$ in X_s will be taken into consideration. Also, depending on the position t of a Lagrangian torus $L_m(\cdot)$, we need to consider different $\mathfrak{b}^{\text{hor}}_{i,i+1}$ and $\mathfrak{b}^{\text{ver}}_{j+1,j}$.

COROLLARY 2.8

Taking a bulk deformation parameter as in (2.8), the deformed potential function is expressed as

$$W^{\mathfrak{b}}(L;\mathbf{y}) = \sum_{(i,j)} \Big(\exp(\mathfrak{b}^{\mathrm{hor}}_{i,i+1}) \frac{y_{i,j}}{y_{i+1,j}} T^{u_{i,j}-u_{i+1,j}} + \exp(\mathfrak{b}^{\mathrm{ver}}_{j+1,j}) \frac{y_{i,j+1}}{y_{i,j}} T^{u_{i,j+1}-u_{i,j}} \Big). \tag{2.9}$$

Let

$$c^{\mathrm{hor}}_{i,i+1} := \exp(\mathfrak{b}^{\mathrm{hor}}_{i,i+1}), \qquad c^{\mathrm{ver}}_{j+1,j} := \exp(\mathfrak{b}^{\mathrm{ver}}_{j+1,j}).$$

By definition, they lie in Λ_U. With this notation, the logarithmic derivative of the bulk-deformed potential becomes

$$y_{i,j}\frac{\partial W^{\mathfrak{b}}}{\partial y_{i,j}}(\mathbf{y}) = -c^{\mathrm{ver}}_{j+1,j}\frac{y_{i,j+1}}{y_{i,j}} T^{u_{i,j+1}-u_{i,j}} - c^{\mathrm{hor}}_{i-1,i}\frac{y_{i-1,j}}{y_{i,j}} T^{u_{i-1,j}-u_{i,j}} + c^{\mathrm{hor}}_{i,i+1}\frac{y_{i,j}}{y_{i+1,j}} T^{u_{i,j}-u_{i+1,j}} + c^{\mathrm{ver}}_{j,j-1} \cdot \frac{y_{i,j}}{y_{i,j-1}} T^{u_{i,j}-u_{i,j-1}}. \tag{2.10}$$

2.3. Nondisplaceable Gelfand–Cetlin fibers in $\mathrm{Fl}(3)$

In this section, the case of $\mathrm{Fl}(3)$ will be discussed in detail. The following theorem will be proved as a warm-up towards the general theorem.

THEOREM 2.9 (Theorem A)

Let $\lambda = (\lambda_1 = 2 > \lambda_2 = 0 > \lambda_3 = -2)$. Consider the coadjoint orbit $\mathcal{O}_\lambda$, a complete flag manifold $\mathrm{Fl}(3)$ equipped with the monotone KKS form ω_λ. Then the GC fiber over a point $\mathbf{u} \in \Delta_\lambda$ is nondisplaceable if and only if $\mathbf{u} \in I$, where

$$I := \big\{ (u_{1,1}, u_{1,2}, u_{2,1}) = (0, 1-t, -1+t) \in \mathbb{R}^3 | 0 \le t \le 1 \big\}. \tag{2.11}$$

In particular, the Lagrangian 3-sphere $\Phi_\lambda^{-1}(0,0,0)$ is nondisplaceable.

To deform Floer theory, we employ a combination of the vertical and horizontal divisors in (2.7) any of which do *not* intersect the torus fibers. Let

$$\mathfrak{b} = \mathfrak{b}^{\mathrm{ver}}_{2,1} \cdot \mathscr{D}^{\mathrm{ver}}_{2,1} + \mathfrak{b}^{\mathrm{hor}}_{1,2} \cdot \mathscr{D}^{\mathrm{hor}}_{1,2} + \mathfrak{b}^{\mathrm{ver}}_{3,2} \cdot \mathscr{D}^{\mathrm{ver}}_{3,2} + \mathfrak{b}^{\mathrm{hor}}_{2,3} \cdot \mathscr{D}^{\mathrm{hor}}_{2,3}. \tag{2.12}$$

For the proof of Theorem 2.9, we need the following topological fact.

PROPOSITION 2.10

Let $\Phi \colon X \to \Delta \subset \mathbb{R}^d$ be a completely integrable system such that Φ is proper. If there exists a sequence $\{\mathbf{u}_i \mid i \in \mathbb{N}\}$ such that

1. *each $\Phi^{-1}(\mathbf{u}_i)$ is nondisplaceable,*
2. *the sequence $\mathbf{u}_i$ converges to some point $\mathbf{u}_\infty$ in Δ,*

then $\Phi^{-1}(\mathbf{u}_\infty)$ is also nondisplaceable.

Proof

For a contradiction, suppose that $\Phi^{-1}(\mathbf{u}_\infty)$ is displaceable. There are a Hamiltonian diffeomorphism ϕ and an open set U containing $\Phi^{-1}(\mathbf{u}_\infty)$ in X such that

$\phi(U)\cap U=\emptyset$. For each i, there exists a point $x_i\in\Phi^{-1}(\mathbf{u}_i)$ such that $x_i\notin U$ since $\Phi^{-1}(\mathbf{u}_i)$ is nondisplaceable. It implies that any subsequence of $\{x_i\}$ cannot converge to a point in U. On the other hand, passing to a subsequence, we may assume that x_i converges to x_∞ for some $x_\infty\in X$ since Φ is proper. By the continuity of Φ, we then have

$$\mathbf{u}_\infty=\lim_{i\to\infty}\mathbf{u}_i=\lim_{i\to\infty}\Phi(x_i)=\Phi(x_\infty).$$

It leads to a contradiction that $x_\infty\in\Phi^{-1}(\mathbf{u}_\infty)\subset U$. □

We now start the proof of Theorem 2.9.

Proof of Theorem 2.9

For any fixed t with $0\le t<1$, let $L(t)$ be the Lagrangian torus fiber over $(0,1-t,-1+t)\in I$ in (1.3). Let $L_s(t)$ be the fiber corresponding to $L(t)$ in X_s via a toric degeneration of completely integrable systems in Theorem 2.1. By taking $s>0$ sufficiently close to 0, the potential function of $L_s(t)$ can be arranged as

$$W(\mathbf{y})=\Big(\frac{y_{1,2}}{y_{1,1}}+\frac{y_{1,1}}{y_{2,1}}+y_{1,2}+\frac{1}{y_{2,1}}\Big)T^{1-t}+\Big(\frac{1}{y_{1,2}}+y_{2,1}\Big)T^{1+t}.$$

We use a combination of Schubert classes in (2.8) (or, equivalently, (2.12)) to deform the potential function. A strategy we take is to postpone determining bulk-deformation parameters. Namely, we start with a tentative parameter, determine solutions for $\mathbf{y}$ first, and then adjust the parameter to make the chosen $\mathbf{y}$ a critical point.

Take a *tentative* bulk parameter $\mathfrak{b}':=\mathfrak{b}^{\mathrm{ver}}_{2,1}\cdot\mathscr{D}^{\mathrm{ver}}_{2,1}$ such that $\exp(\mathfrak{b}^{\mathrm{ver}}_{2,1})=1+T^{2t}$, that is,

$$\mathfrak{b}^{\mathrm{ver}}_{2,1}=T^{2t}-\frac{1}{2}T^{4t}+\cdots\in\Lambda_+.$$

By Corollary 2.8, the potential function is deformed into

$$W^{\mathfrak{b}'}(\mathbf{y})=\Big(\frac{y_{1,2}}{y_{1,1}}+\frac{y_{1,1}}{y_{2,1}}+y_{1,2}+\frac{1}{y_{2,1}}\Big)T^{1-t}+\Big(\frac{y_{1,2}}{y_{1,1}}+\frac{1}{y_{2,1}}+\frac{1}{y_{1,2}}+y_{2,1}\Big)T^{1+t},$$

whose logarithmic derivatives are

$$\begin{cases} y_{1,1}\frac{\partial W^{\mathfrak{b}'}}{\partial y_{1,1}}(\mathbf{y})=(-\frac{y_{1,2}}{y_{1,1}}+\frac{y_{1,1}}{y_{2,1}})T^{1-t}+(-\frac{y_{1,2}}{y_{1,1}})T^{1+t},\\ y_{1,2}\frac{\partial W^{\mathfrak{b}'}}{\partial y_{1,2}}(\mathbf{y})=(\frac{y_{1,2}}{y_{1,1}}+y_{1,2})T^{1-t}+(\frac{y_{1,2}}{y_{1,1}}-\frac{1}{y_{1,2}})T^{1+t},\\ y_{2,1}\frac{\partial W^{\mathfrak{b}'}}{\partial y_{2,1}}(\mathbf{y})=(-\frac{y_{1,1}}{y_{2,1}}-\frac{1}{y_{2,1}})T^{1-t}+(-\frac{1}{y_{2,1}}+y_{2,1})T^{1+t}. \end{cases}$$

We set $y_{1,2}=1, y_{2,1}=1$ and take $y_{1,1}$ as the solution of $(y_{1,1})^2=1+T^{2t}$ satisfying $y_{1,1}\equiv-1 \mod T^{>0}$. It is easy to see that $y_{1,1}\frac{\partial W^{\mathfrak{b}'}}{\partial y_{1,1}}(\mathbf{y})=0$. Note that $y_{1,1}$ is of the form $y_{1,1}\equiv-1-\frac{1}{2}T^{2t} \mod T^{>2t}$.

We now adjust a bulk deformation parameter from $\mathfrak{b}'$ to $\mathfrak{b}$ in order for the chosen $(y_{1,1},y_{1,2},y_{2,1})$ to be a critical point of $W^{\mathfrak{b}}$. Let

$$\mathfrak{b}:=\mathfrak{b}'+\mathfrak{b}^{\mathrm{ver}}_{3,2}\cdot\mathscr{D}^{\mathrm{ver}}_{3,2}+\mathfrak{b}^{\mathrm{hor}}_{2,3}\cdot\mathscr{D}^{\mathrm{hor}}_{2,3}.$$

Since $\mathscr{D}^{\mathrm{ver}}_{3,2}$ and $\mathscr{D}^{\mathrm{hor}}_{2,3}$ do not intersect the disks of Maslov index 2 emanating from $\mathscr{D}^{\mathrm{ver}}_{2,1}$ and $\mathscr{D}^{\mathrm{hor}}_{1,2}$ in $\pi_2(\mathcal{O}_\lambda, \Phi_\lambda^{-1}(t))$, we have

$$y_{1,1}\frac{\partial W^{\mathfrak{b}}}{\partial y_{1,1}}(\mathbf{y}) = y_{1,1}\frac{\partial W^{\mathfrak{b}'}}{\partial y_{1,1}}(\mathbf{y}).$$

Plugging in the chosen $y_{i,j}$'s, we have

$$\begin{cases} y_{1,2}\frac{\partial W^{\mathfrak{b}}}{\partial y_{1,2}}(\mathbf{y}) = (-\frac{1}{2} - \exp(\mathfrak{b}^{\mathrm{ver}}_{3,2}))T^{1+t} + h^{(1,2)} \cdot T^{1+t}, \\ y_{2,1}\frac{\partial W^{\mathfrak{b}}}{\partial y_{2,1}}(\mathbf{y}) = (-\frac{1}{2} + \exp(\mathfrak{b}^{\mathrm{hor}}_{2,3}))T^{1+t} + h^{(2,1)} \cdot T^{1+t} \end{cases}$$

for some constant $h^{(1,2)}, h^{(2,1)} \in \Lambda_+$. By choosing $\mathfrak{b}^{\mathrm{ver}}_{3,2}, \mathfrak{b}^{\mathrm{hor}}_{2,3} \in \Lambda_0$ such that $\exp(\mathfrak{b}^{\mathrm{ver}}_{3,2}) = -\frac{1}{2} + h^{(1,2)}$ and $\exp(\mathfrak{b}^{\mathrm{hor}}_{2,3}) = \frac{1}{2} - h^{(2,1)}$, we can make $W^{\mathfrak{b}}(\mathbf{y})$ admit a critical point. By Theorem 2.4, $L_s(t)$ has a nonvanishing (bulk-)deformed Floer cohomology. By the Hamiltonian invariance of A_∞-structures, so does $L(t)$ and therefore, it is nondisplaceable.

In sum, each torus fiber over the line segment in (1.3) is nondisplaceable. Furthermore, Proposition 2.10 yields nondisplaceability of the Lagrangian 3-sphere. □

3. Decompositions of the gradient of potential function

In this section, in order to prove Theorem B, we introduce the *split leading term equation* (SLT-equation for short) of the potential function in (2.5), which is the analogue of the *leading term equation* in [13] and [14]. We discuss the relation between its solvability and the nontriviality of Floer cohomology under a certain choice of bulk deformation.

3.1. Outline of Sections 3 and 4

Thanks to Theorem 2.4, the proof of Theorem B boils down to finding a bulk deformation parameter $\mathfrak{b}$ such that the bulk-deformed potential function $W^{\mathfrak{b}}$ admits a critical point. Sections 3 and 4 will be devoted to discussing how to determine them.

Before giving the outline, we start by explaining why this process is nontrivial by pointing out the differences from the toric case. In the toric case, the (generalized) leading term equation was introduced to detect nondisplaceable toric fibers effectively in [14, Section 11]. Roughly speaking, it consists of the initial terms of the gradient of a (bulk-deformed) potential function with respect to a suitable choice of exponential variables. It is proved therein that there always exists a bulk parameter $\mathfrak{b}$ so that the complex solution becomes a critical point of $W^{\mathfrak{b}}$ if the leading term equation admits a solution whose components are nonzero. Indeed, the locations where the leading term equation is solvable are characterized by the intersection of certain tropicalizations in [22]. The key features for proving the above statements are as follows. First, there is a one-to-one correspondence between the *honest* holomorphic disks bounded by a torus fiber of Maslov index 2 and the facets of the moment polytope. Second, the preimage of each facet represents a cycle of degree 2. Thus, all terms corresponding to the facets can be independently controlled.

In the GC system, the inverse image of a single facet is *not* a cycle in general because of the appearance of nontorus fibers at the boundary. We need to take into account a particular union of facets to represent a cycle of degree 2. The terms of W *cannot* be independently controlled if taking such a cycle for a bulk deformation. Nonetheless, for the family of Lagrangian tori $L_m(t)$ over (1.4) in $\mathcal{O}_\lambda \simeq \mathrm{Fl}(n)$ with the monotone symplectic form ω_λ, we shall show the existence of a bulk deformation parameter $\mathfrak{b}$ and a critical point associated to $\mathfrak{b}$.

In Section 3, we introduce the *split leading term equation* or *SLT-equation* (see Definition 3.2). We then demonstrate how to determine a bulk deformation parameter and promote a solution of the SLT-equation to a critical point of the bulk-deformed potential function once a solution of the SLT-equation is given. In Section 4, we show that the SLT-equation always admits a solution. One might think the situation looks somewhat similar to that of Bernstein [2] and Kushnirenko [25] which, however, treats the *generic* system of Laurent polynomial equations while the system in our situation does not have much freedom. For a given fixed system of multivariable equations, finding a solution is not simple at all even with the aid of a computer. Yet, guided by the ladder diagrams decorated by the exponential variables and using the freedom in the choice of *seeds* (see Section 4.1) which provides some freedom of changing the coefficients of the equations, we are able to show the existence of a solution for the system of our current interest.

A brief description of our procedure of solving the critical point equation is now in order. Let m be any fixed integer with $2 \le m \le \lfloor n/2 \rfloor$, and let $B(m)$ be the subdiagram consisting of $(m \times m)$ lower left unit boxes in the ladder diagram $\Gamma(n)$ of $\mathrm{Fl}(n)$. The diagrams $\Gamma(n)$ and $B(m)$ are often regarded as collections of double indices as follows:

$$\begin{aligned} \Gamma(n) &= \left\{(i,j) \in \mathbb{N}^2 \mid 2 \le i+j \le n\right\}, \\ B(m) &= \left\{(i,j) \in \mathbb{N}^2 \mid 1 \le i,j \le m\right\}. \end{aligned} \tag{3.1}$$

Recalling (2.5), the potential function of $L_m(t)$ is rearranged into several groups by energy filtration. The valuation of $\partial_{(i,j)} W(\mathbf{y})$ for $(i,j) \in B(m)$ is $(1-t)$ and that for $(i,j) \in \Gamma(n) \backslash B(m)$ is 1, where

$$\partial_{(i,j)} W(\mathbf{y}) := y_{i,j} \frac{\partial}{\partial y_{i,j}} W(\mathbf{y})$$

is the logarithmic derivative of W.

Decomposing the gradient of the potential function deformed by $\mathfrak{b}$ in (2.8) into two pieces by cutting along the boundary of $B(m)$, we shall determine a critical point and a bulk parameter using the following steps.

(1) Find a solution $(y^{\mathbb{C}}_{i,j} \in \mathbb{C}^*)$ of the system of equations $\partial_{(i,j)} W^{\mathfrak{b}}(\mathbf{y}) \equiv 0 \mod T^{>1}$ in $(\Gamma(n) \backslash B(m)) \cup \{(m,m)\}$ and equations relating the variables adjacent to $B(m)$ in Section 4. Also, in this step, the complex part $\mathfrak{b}^{\mathbb{C}}$ of a bulk parameter $\mathfrak{b}$ will be determined.

(2) Find a solution $(y^{\mathbb{C}}_{i,j} \in \mathbb{C}^*)$ of $\partial_{(i,j)} W^{\mathfrak{b}}(\mathbf{y}) \equiv 0 \mod T^{>1-t}$ in $B(m)$ in Section 3.3.

(3) Find a solution $(y_{i,j} \in \Lambda_U)$ of $\partial_{(i,j)} W^{\mathfrak{b}}(\mathbf{y}) = 0$ in $B(m)$ such that $y_{i,j} \equiv y^{\mathbb{C}}_{i,j} \mod T^{>0}$ in Section 3.4.

(4) Find a solution $(y_{i,j} \in \Lambda_U)$ of $\partial_{(i,j)} W^{\mathfrak{b}}(\mathbf{y}) = 0$ in $\Gamma(n) \backslash B(m)$ and determine a bulk parameter $\mathfrak{b}$ such that $y_{i,j} \equiv y^{\mathbb{C}}_{i,j}$ and $\mathfrak{b} \equiv \mathfrak{b}^{\mathbb{C}} \mod T^{>0}$ in Section 3.5.

The *split leading term equation* (see Definition 3.2) arises in step (1).

EXAMPLE 3.1

In the coadjoint orbit $\mathcal{O}_\lambda$, where $\lambda = (5, 3, 1, -1, -3, -5)$, the potential function of $L_2(t)$ is

$$W(L_2(t); \mathbf{y}) = \Big(\frac{y_{1,2}}{y_{1,1}} + \frac{y_{1,1}}{y_{2,1}} + \frac{y_{1,2}}{y_{2,2}} + \frac{y_{2,2}}{y_{2,1}} \Big) T^{1-t} + \Big(\frac{y_{1,4}}{y_{1,3}} + \frac{y_{1,3}}{y_{2,3}} + \cdots \Big) T^1 + \Big(\frac{y_{1,3}}{y_{1,2}} + \frac{y_{2,1}}{y_{3,1}} \Big) T^{1+t}.$$

In this example, the valuation of partial derivatives of W jumps along the red line in Figure 4.

Turning on bulk deformation, it follows from (2.10) that any solution complex number $y_{i,j} \in \mathbb{C}^*$ of the SLT-equation has to satisfy the system

$$\begin{cases} -c^{\mathrm{ver}}_{2,1} \cdot \frac{y_{1,2}}{y_{1,1}} + c^{\mathrm{hor}}_{1,2} \cdot \frac{y_{1,1}}{y_{2,1}} = 0, & c^{\mathrm{ver}}_{2,1} \cdot \frac{y_{1,2}}{y_{1,1}} + c^{\mathrm{hor}}_{1,2} \cdot \frac{y_{1,2}}{y_{2,2}} = 0, \\ -c^{\mathrm{hor}}_{1,2} \cdot \frac{y_{1,1}}{y_{2,1}} - c^{\mathrm{ver}}_{2,1} \cdot \frac{y_{2,2}}{y_{2,1}} = 0, & -c^{\mathrm{hor}}_{1,2} \cdot \frac{y_{1,2}}{y_{2,2}} + c^{\mathrm{ver}}_{2,1} \cdot \frac{y_{2,2}}{y_{2,1}} = 0 \end{cases} \tag{3.2}$$

and

$$\begin{cases} -c^{\mathrm{ver}}_{4-i+1,4-i} \frac{y_{i,4-i+1}}{y_{i,4-i}} + c^{\mathrm{hor}}_{i,i+1} \frac{y_{i,4-i}}{y_{i+1,4-i}} = 0 \\ \qquad \text{for } i = 1, \ldots, 3, \\ -c^{\mathrm{ver}}_{5-i+1,5-i} \frac{y_{i,5-i+1}}{y_{i,5-i}} - c^{\mathrm{hor}}_{i-1,i} \frac{y_{i-1,5-i}}{y_{i,5-i}} \\ \qquad + c^{\mathrm{ver}}_{5-i,5-i-1} \frac{y_{i,5-i}}{y_{i,5-i-1}} + c^{\mathrm{hor}}_{i,i+1} \cdot \frac{y_{i,5-i}}{y_{i+1,5-i}} = 0 \\ \qquad \text{for } i = 1, \ldots, 4, \\ -c^{\mathrm{ver}}_{6-i+1,6-i} \frac{1}{y_{i,6-i}} - c^{\mathrm{hor}}_{i-1,i} \frac{y_{i-1,6-i}}{y_{i,6-i}} + c^{\mathrm{ver}}_{6-i,6-i-1} \frac{y_{i,6-i}}{y_{i,6-i-1}} \\ \qquad + c^{\mathrm{hor}}_{i,i+1} y_{i,6-i} = 0 \\ \qquad \text{for } i = 1, \ldots, 5. \end{cases} \tag{3.3}$$

Equations (3.2) and (3.3) come from the leading parts of the partial derivatives inside $B(2)$ and of the partial derivatives inside $\Gamma(6) \backslash B(2) \cup \{(2,2)\}$, respectively. Solving (3.2) corresponds to step (2) and solving (3.3) corresponds to step (1). In the rest of this section, assuming that the SLT-equation is solvable, we explain how to complete the remaining steps.

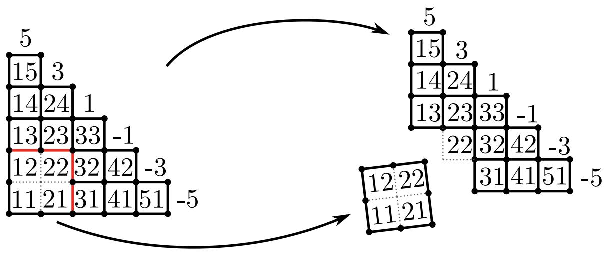

Figure 4. Decomposition of the gradient of the potential function in Fl(6).

3.2. Split leading term equations

Let $\lambda = (\lambda_i := n - 2i + 1 \mid i = 1, \dots, n)$ be an n-tuple of real numbers for an arbitrary integer $n \geq 4$. Consider the coadjoint orbit $\mathcal{O}_\lambda$, a complete flag manifold $\mathrm{Fl}(n)$ equipped with the monotone form ω_λ. Fix the one-parameter family of Lagrangian GC tori $L_m(t)$ over $I_m(t)$ for $0 \leq t < 1$ in $\mathcal{O}_\lambda$ as in (1.4).

We define the split leading term equation arising from the potential function of $L_m(t)$. Let

$$k := \lceil n/2 \rceil,$$

that is, $n = 2k - 1$ or $2k$. We denote

$$f^{\mathfrak{b}}_{(i,j)}(\mathbf{y}) := \partial_{(i,j)} W^{\mathfrak{b}}(\mathbf{y}) \cdot T^{-\nu} \quad \text{where } \nu := \mathfrak{v}_T\big(\partial_{(i,j)} W^{\mathfrak{b}}(\mathbf{y})\big). \tag{3.4}$$

We also denote $f_{(i,j)} := f^{\mathfrak{b}}_{(i,j)}$ when $\mathfrak{b} = 0$. For our purpose, it will suffice to take a bulk deformation parameter of the type

$$\mathfrak{b} := \sum_{i \geq k} \mathfrak{b}^{\mathrm{hor}}_{i,i+1} \cdot \mathscr{D}^{\mathrm{hor}}_{i,i+1} + \sum_{j \geq k} \mathfrak{b}^{\mathrm{ver}}_{j+1,j} \cdot \mathscr{D}^{\mathrm{ver}}_{j+1,j}. \tag{3.5}$$

Note that horizontal (resp., vertical) facets in (3.5) are supported outside of the maximal diagonal (of $(k \times k)$-size) embedded in $\Gamma(n)$. Therefore, we have

$$\begin{cases} c^{\mathrm{hor}}_{i,i+1} := \exp(\mathfrak{b}^{\mathrm{hor}}_{i,i+1} = 0) = 1 & \text{for } i < k, \\ c^{\mathrm{ver}}_{j+1,j} := \exp(\mathfrak{b}^{\mathrm{ver}}_{j+1,j} = 0) = 1 & \text{for } j < k. \end{cases}$$

DEFINITION 3.2

Let $2 \leq m \leq k$ be given. The *split leading term equation* (abbreviated as *SLT-equation*) *of* $\Gamma(n)$ *associated with* $B(m)$ is the system of $\mathbb{C}$-*valued* equations given by

$$\begin{cases} \ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0 & \text{for all } (i,j) \in \Gamma(n), \\ \ell^{m}_{(l)}(\mathbf{y}) = 0 & \text{for all } l \text{ with } 1 \leq l \leq m, \end{cases} \tag{3.6}$$

where we define

$$\begin{aligned} \ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) := & -c^{\mathrm{ver}}_{j+1,j} \cdot \frac{y_{i,j+1}}{y_{i,j}} - c^{\mathrm{hor}}_{i-1,i} \cdot \frac{y_{i-1,j}}{y_{i,j}} \\ & + c^{\mathrm{hor}}_{i,i+1} \cdot \frac{y_{i,j}}{y_{i+1,j}} + c^{\mathrm{ver}}_{j,j-1} \cdot \frac{y_{i,j}}{y_{i,j-1}}, \end{aligned} \tag{3.7}$$

$$\ell^m_{(l)}(\mathbf{y}) := (-1)^{m+1-l} \cdot \frac{y_{l,m+1}}{y_{m,m}} + \frac{y_{m,m}}{y_{m+1,l}}. \tag{3.8}$$

Here we split the system into two, one on $B(m)$ and the other outside $B(m)$, with the following constraints:

(1) For $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$ with $(i,j) \in B(m)$, we set

$$\begin{cases} y_{r,m+1} = y_{0,r} := 0 & \text{for } r \leq m, \\ y_{m+1,s} = y_{s,0} := \infty & \text{for } s \leq m. \end{cases} \tag{3.9}$$

(2) For $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$ with $(i,j) \in \Gamma(n) \backslash B(m)$, we set

$$\begin{cases} y_{r,m} := \infty \quad \text{for } r < m, & y_{m,s} := 0 \quad \text{for } s < m, \\ y_{\bullet,0} := \infty, \qquad y_{0,\bullet} := 0, & y_{s,n+1-s} := 1 \quad \text{for } 1 \leq s \leq n. \end{cases} \tag{3.10}$$

We explain the implication of the above constraints in writing the SLT-equation of $\Gamma(n)$ associated with $B(m)$. Cutting the boxes $B(m) \backslash \{(m,m)\}$ off from the diagram $\Gamma(n)$, we have decomposed blocks as in Figure 4.

For an index $(i,j) \in B(m)$, ignoring the terms containing the variables not in $B(m)$, $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y})$ consists of the initial terms of the logarithmic derivative of $W^{\mathfrak{b}}$. Also, for an index $(i,j) \in \Gamma(n) \backslash B(m)$, ignoring the terms containing the variables within $B(m) \backslash \{(m,m)\}$, $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y})$ consists of the initial terms of the logarithmic derivative of $W^{\mathfrak{b}}$ whose formula is in (2.10). In addition to them, we have the equation $\ell^m_{(l)}(\mathbf{y}) = 0$ relating two variables $y_{l,m+1}$ and $y_{m+1,l}$, which are adjacent to $B(m)$ and are in the $(l+m)$th diagonal. It will be explained in (3.24) why the latter equation occurs.

Now introduce the following linear orders on $\Gamma(n)$.

DEFINITION 3.3

We say that $(i,j) \prec_{\mathrm{hor}} (i',j')$ on $\Gamma(n)$ if one of the following alternatives holds:

(1) $i + j < i' + j'$,
(2) $i + j = i' + j'$ and $i < i'$.

We also similarly define $\prec_{\mathrm{ver}}$ by replacing $i < i'$ by $j < j'$ in (2) above.

The existence of a solution for the SLT-equation (3.6) will guarantee that the assumption of the following lemma holds. We will repeatedly employ it in order to promote a solution in $\mathbb{C}^*$ to that in Λ_U for the critical point equation of the (bulk-deformed) potential function.

LEMMA 3.4

Suppose that we are given $y_{i-1,j} \in \Lambda_U \cup \{0\}$, $y_{i,j-1} \in \Lambda_U \cup \{\infty\}$, and $y_{i,j}, y_{i,j+1} \in \Lambda_U$. If there is a nonzero *complex solution $y^{\mathbb{C}}_{i+1,j}$ of $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0 \mod T^{>0}$ for*

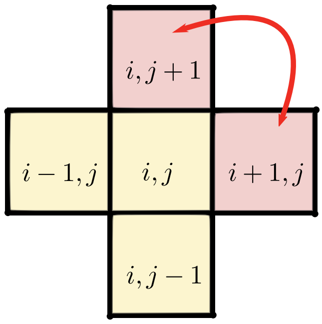

Figure 5. Graphical description of Lemma 3.4.

some $c^{\text{ver}}_{j+1,j}, c^{\text{hor}}_{i-1,i}, c^{\text{hor}}_{i,i+1}$*, and* $c^{\text{ver}}_{j,j-1}$ *in* Λ_U*, then* $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$ *has a unique solution* $y_{i+1,j} \in \Lambda_U$ *such that* $y_{i+1,j} \equiv y^{\mathbb{C}}_{i+1,j} \mod T^{>0}$*. The same holds by changing the roles of* $y_{i+1,j}$ *and* $y_{i,j+1}$*. (See Figure 5.)*

Proof
We observe that $(i+1,j)$ is maximal with respect to the order $\prec_{\text{hor}}$ among the four points $(i-1,j), (i,j-1), (i,j+1), (i+1,j)$. Then the proof immediately follows from the equation $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$. It shows that $y_{i+1,j}$ can be expressed as a rational function in the other variables by the formula (3.7). □

The following lemma is also an immediate consequence of the valuation analysis of the equation $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$.

LEMMA 3.5
If in addition to the assumption of Lemma 3.4, we assume that

(1) $c^{\text{ver}}_{j+1,j}, c^{\text{hor}}_{i-1,i}, c^{\text{hor}}_{i,i+1}$*, and* $c^{\text{ver}}_{j,j-1}$ *are nonzero complex numbers, and*
(2) $\mathfrak{v}_T(y_{i,j} - y^{\mathbb{C}}_{i,j}), \mathfrak{v}_T(y_{i-1,j} - y^{\mathbb{C}}_{i-1,j})$*, and* $\mathfrak{v}_T(y_{i,j-1} - y^{\mathbb{C}}_{i,j-1}) > \lambda$*,*

then $\mathfrak{v}_T(y_{i+1,j} - y^{\mathbb{C}}_{i+1,j}) = \lambda$ *if and only if* $\mathfrak{v}_T(y_{i,j+1} - y^{\mathbb{C}}_{i,j+1}) = \lambda$*.*

Now we ready to state the main theorem of this section.

THEOREM 3.6
If there exist nonzero complex numbers $c^{\text{hor},\mathbb{C}}_{i,i+1}$ *and* $c^{\text{ver},\mathbb{C}}_{j+1,j}$ $(i,j \geq k)$ *for which the SLT-equation (3.6) admits a solution*

$$\mathbf{y}^{\mathbb{C}} := \left(y^{\mathbb{C}}_{i,j} \in \mathbb{C}^* \mid (i,j) \in \Gamma(n) \backslash B(m) \cup \{(m,m)\}\right),$$

then there exists a bulk deformation $\mathfrak{b}$ *(depending on* m *and* t*) of the form (3.5) such that*

(1) *the bulk-deformed potential* $W^{\mathfrak{b}}(\mathbf{y})$ *has a critical point* $(y_{i,j} \in \Lambda_U \mid (i,j) \in \Gamma(n))$ *satisfying*

$$y^{\mathbb{C}}_{i,j} \equiv y_{i,j} \mod T^{>0} \textit{for } (i,j) \in \Gamma(n)\backslash B(m) \cup \{(m,m)\},$$

$$(2)\ \exp(\mathfrak{b}^{\text{hor}}_{i,i+1}) \equiv c^{\text{hor},\mathbb{C}}_{i,i+1}, \exp(\mathfrak{b}^{\text{ver}}_{j+1,j}) \equiv c^{\text{ver},\mathbb{C}}_{j+1,j} \mod T^{>0}.$$

This main theorem will be proved in later subsections.

3.3. Solving the SLT-equation with $\mathfrak{b} = 0$ within $B(m)$

The goal of the section is to find a "symmetric" complex solution of

$$(3.11)\quad \ell^{0,m}_{(i,j)}(\mathbf{y}) = -\frac{y_{i,j+1}}{y_{i,j}} - \frac{y_{i-1,j}}{y_{i,j}} + \frac{y_{i,j}}{y_{i+1,j}} + \frac{y_{i,j}}{y_{i,j-1}} = 0 \quad \text{for all } (i,j) \in B(m).$$

Notice that with the choice we made in (3.5), $\mathfrak{b}^{\text{hor}}_{i,i+1} = 0$ (resp., $\mathfrak{b}^{\text{ver}}_{j+1,j} = 0$) for $i < k$ (resp., $j < k$). This in turn implies that $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y})$ given in (3.7) actually coincides with $\ell^{0,m}_{(i,j)}(\mathbf{y})$ for $(i,j) \in B(m)$.

LEMMA 3.7

Let $\mathbf{y}^{\mathbb{C}} := (y^{\mathbb{C}}_{i,j} \in \mathbb{C}^* \mid (i,j) \in B(m))$ be a solution of (3.11). Then

$$\widetilde{y^{\mathbb{C}}_{i,j}} := c \cdot y^{\mathbb{C}}_{i,j} \quad \textit{for } (i,j) \in B(m)$$

is also a solution of (3.11) for any nonzero complex number c.

Proof

The lemma immediately follows from $\ell^{0,m}_{(i,j)}(\mathbf{y}) = \ell^{0,m}_{(i,j)}(c \cdot \mathbf{y})$. □

LEMMA 3.8

There exists a solution $\mathbf{y}^{\mathbb{C}} := (y^{\mathbb{C}}_{i,j} \in \mathbb{C}^* \mid i + j \leq m+1)$ of the system of equations

$$(3.12)\qquad \ell^{0,m}_{(i,j)}(\mathbf{y}) = 0 \quad \textit{for } i + j \leq m$$

such that it is symmetric in a way that

$$(3.13)\qquad y^{\mathbb{C}}_{i,j} = (y^{\mathbb{C}}_{j,i})^{-1}.$$

Proof

It is straightforward to check that

$$(3.14)\qquad y^{\mathbb{C}}_{i,j} := \begin{cases} 1 & \text{for } i = j, \\ \prod_{r=0}^{j-i-1}(2i+2r) & \text{for } i < j, \\ \prod_{r=0}^{i-j-1}(2j+2r)^{-1} & \text{for } i > j \end{cases}$$

is a symmetric solution for (3.12). □

We are ready to prove the existence of a symmetric solution in the sense of (3.13).

PROPOSITION 3.9

There exists a symmetric solution $\mathbf{y}^{\mathbb{C}} := (y^{\mathbb{C}}_{i,j} \in \mathbb{C}^* \mid (i,j) \in B(m))$ of (3.11) satisfying $y^{\mathbb{C}}_{i,i} = \pm 1$ for all $1 \leq i \leq m$.

Proof
We start with a solution $y^{\mathbb{C}}_{i,j} \in \mathbb{C}^*$ for $i+j \le m+1$ given in Lemma 3.8. By the choice, it satisfies $y^{\mathbb{C}}_{i,i} = 1$ for all i with $2i \le m$. For the remaining indices (i,j) with $i+j > m+1$, we put

$$y^{\mathbb{C}}_{i,j} := (-1)^{i+j-m-1} y^{\mathbb{C}}_{m+1-j,m+1-i}. \tag{3.15}$$

We now show that this choice gives rise to a solution for (3.11).

For $(i,j) \in B(m)$ with $i+j \ge m+2$, it is straightforward to check that

$$\ell^{0,m}_{(i,j)}(\mathbf{y}^{\mathbb{C}}) = -\ell^{0,m}_{(m+1-j,m+1-i)}(\mathbf{y}^{\mathbb{C}}) = 0$$

from the choice made in Lemma 3.8 for $i+j \le m+1$. For the indices (i,j) with $i+j = m+1$, the definition (3.15) implies that $y^{\mathbb{C}}_{i,j+1} = -y^{\mathbb{C}}_{i-1,j}$ and hence,

$$\begin{aligned}\ell^{0,m}_{(i,j)}(\mathbf{y}^{\mathbb{C}}) &= -\frac{y^{\mathbb{C}}_{i,j+1}}{y^{\mathbb{C}}_{i,j}} - \frac{y^{\mathbb{C}}_{i-1,j}}{y^{\mathbb{C}}_{i,j}} + \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i+1,j}} + \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i,j-1}} \\ &= \frac{y^{\mathbb{C}}_{i-1,j}}{y^{\mathbb{C}}_{i,j}} - \frac{y^{\mathbb{C}}_{i-1,j}}{y^{\mathbb{C}}_{i,j}} + \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i+1,j}} - \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i+1,j}} = 0.\end{aligned}$$

The symmetry (3.13) for (i,j) with $i+j \le m+1$ and the definition (3.15) also immediately imply the symmetry of the solution and that $y_{i,i} = \pm 1$ for all $1 \le i \le m$. □

COROLLARY 3.10
For any $c \in \mathbb{C}^*$, there is a solution $\mathbf{y}^{\mathbb{C}} := (y^{\mathbb{C}}_{i,j} \in \mathbb{C}^* \mid (i,j) \in B(m))$ of (3.11) such that

(1) $y^{\mathbb{C}}_{i,j} \cdot y^{\mathbb{C}}_{j,i} = c^2$,
(2) $y^{\mathbb{C}}_{m,m} = c$,
(3) $y^{\mathbb{C}}_{i,i}$ *is either c or $-c$ for $1 \le i < m$.*

Proof
The component $y^{\mathbb{C}}_{m,m}$ of a solution from Proposition 3.9 is either 1 or -1. By multiplying the solution found therein by $\pm c$ and using Lemma 3.7, we produce another solution that satisfies (1), (2), and (3). □

3.4. Determination of $\mathbf{y}$ inside $B(m)$

HYPOTHESIS 3.11
We assume that the SLT-equation of $\Gamma(n)$ associated with $B(m)$ has a solution for some nonzero complex numbers $c^{\mathrm{hor},\mathbb{C}}_{i,i+1}$ and $c^{\mathrm{ver},\mathbb{C}}_{j+1,j}$ for $i,j \ge k = \lceil n/2 \rceil$.

Consider the variables

$$y^{\mathbb{C}}_{m,m} \qquad \text{and} \qquad y^{\mathbb{C}}_{i,m+1} \quad \text{with } 1 \le i \le m, \tag{3.16}$$

where $y^{\mathbb{C}}_{i,j} \in \mathbb{C}^*$ is the (i,j)th component of a solution of the SLT-equation. In order to emphasize that (3.16) has been predetermined, we denote the values by

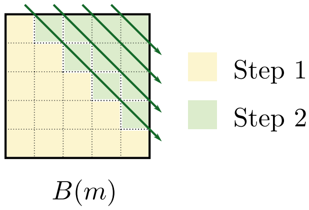

Figure 6. Pictorial outline of Section 3.4.

$$d_{i,j} := y^{\mathbb{C}}_{i,j}, \quad (i,j) = (m,m) \text{ or } (i, m+1) \text{ with } 1 \leq i \leq m. \tag{3.17}$$

Setting it as the initial part for a solution and using Lemma 3.4, we promote it to a solution of (3.7) over Λ_U *in the increasing order of* $(i,j) \in B(m)$ *with respect to the order* $\prec_{\text{hor}}$. For a pictorial outline of this section, see Figure 6.

Step 1: $(i,j) \in B(m)$ with $i + j \leq m+1$. We begin by taking $y_{i,j} := y^{\mathbb{C}}_{i,j} \in \mathbb{C}^* \subset \Lambda_U$, where $(y^{\mathbb{C}}_{i,j} \mid (i,j) \in B(m))$ is a solution satisfying $y^{\mathbb{C}}_{m,m} = d_{m,m}$ from Corollary 3.10 for all (i,j) with $i + j \leq m+1$.

Step 2: $(i,j) \in B(m)$ with $m + 2 \leq i + j \leq 2m$. We first compute

$$\partial_{(j,m)} W(\mathbf{y}) = \Big(-\frac{y_{j-1,m}}{y_{j,m}} + \frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}}\Big) T^{1-t} + \Big(-\frac{y_{j,m+1}}{y_{j,m}}\Big) T^{1+(m-j)t}$$

from which we have

$$\nu = \mathfrak{v}_T\big(\partial_{(j,m)} W(\mathbf{y})\big) = 1 - t.$$

Then by the definition of $f_{(j,m)}$ in (3.4) with $\mathfrak{b} = 0$, we have

$$f_{(j,m)}(\mathbf{y}) = \Big(-\frac{y_{j-1,m}}{y_{j,m}} + \frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}}\Big) + \Big(-\frac{y_{j,m+1}}{y_{j,m}}\Big) T^{(m-j+1)t}.$$

For an index j with $1 \leq j < m$, we decompose $f_{(j,m)}(\mathbf{y})$:

$$f_{(j,m)}(\mathbf{y}) = h^{(1)}_{(j,m)}(\mathbf{y}) + h^{(2)}_{(j,m)}(\mathbf{y}) T^{(m-j+1)t},$$

where

$$\begin{cases} h^{(1)}_{(j,m)}(\mathbf{y}) := \big(-\frac{y_{j-1,m}}{y_{j,m}} + \frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}}\big) \\ \qquad\qquad + \big(-a_j \frac{y_{j,m}}{y_{j+1,m}} - a_j \frac{y_{j,m}}{y_{j,m-1}}\big) T^{(m-j+1)t}, \\ h^{(2)}_{(j,m)}(\mathbf{y}) := -\frac{y_{j,m+1}}{y_{j,m}} + a_j \frac{y_{j,m}}{y_{j+1,m}} + a_j \frac{y_{j,m}}{y_{j,m-1}}. \end{cases}$$

Here we recall that $y_{0,m} \equiv 0$ from (3.9). This term appears only for the case $j = 1$.

We will solve $f_{(j,m)}(\mathbf{y}) = 0$ inductively in the increasing order of $\prec_{\text{hor}}$. For this purpose, we utilize the following sufficient conditional equation.

LEMMA 3.12

A solution of the system

$$(3.18)\qquad \begin{cases} h^{(1)}_{(j,m)}(\mathbf{y}) = 0, \\ h^{(2)}_{(j,m)}(\mathbf{y}) - a_j \cdot f_{(j,m)}(\mathbf{y}) \\ \qquad = -\frac{y_{j,m+1}}{y_{j,m}} + a_j(\frac{y_{j-1,m}}{y_{j,m}} + \frac{y_{j,m+1}}{y_{j,m}} T^{(m-j+1)t}) = 0 \end{cases}$$

is also a solution of $f_{(j,m)}(\mathbf{y}) = 0$.

Proof
The proof is omitted. □

As the first step of the induction, suppose that we are given a solution

$$\big(y_{r,s} \in \Lambda_U \mid (r,s) \in B(m) \cap \Gamma(m+j)\big)$$

of $f_{(r,s)}(\mathbf{y}) = 0$ for all $(r,s) \in B(m) \cap \Gamma(m+j-1)$ such that for any $(r,s) \in B(m) \cap \Gamma(m+j)$,

$$(3.19)\qquad \mathfrak{v}_T(y_{r,s} - y^{\mathbb{C}}_{r,s}) \geq (m-j+2)t.$$

By Corollary 3.10, we have

$$y^{\mathbb{C}}_{r,s} \neq 0 \quad \text{for all } (r,s) \in B(m).$$

Therefore, we have $y^{\mathbb{C}}_{j-1,m}/y^{\mathbb{C}}_{j,m} \neq 0$ for $j \geq 2$ in particular. Furthermore, thanks to the second equation, (3.18), which is equivalent to the equation

$$\begin{cases} h^{(2)}_{(j,m)}(\mathbf{y}) - a_j \cdot f_{(j,m)}(\mathbf{y}) = 0, \\ y_{j,m+1} = d_{j,m+1} \end{cases}$$

uniquely determines an element $a_j \in \Lambda_U$. Then the equation $h^{(1)}_{(j,m)}(\mathbf{y}) = 0$ gives rise to

$$\begin{aligned} &\frac{y^{\mathbb{C}}_{j,m}}{y_{j+1,m}}(1 - a_j T^{(m-j+1)t}) \\ &\qquad \equiv \Big(\frac{y^{\mathbb{C}}_{j-1,m}}{y^{\mathbb{C}}_{j,m}} - \frac{y^{\mathbb{C}}_{j,m}}{y^{\mathbb{C}}_{j,m-1}}\Big) + a_j \frac{y^{\mathbb{C}}_{j,m}}{y^{\mathbb{C}}_{j,m-1}} T^{(m-j+1)t} \mod T^{>(m-j+1)t}. \end{aligned}$$

Recall that the $y^{\mathbb{C}}_{r,s}$'s arising from Corollary 3.10 are nonzero and satisfy

$$(3.20)\qquad \ell^{0,m}_{(j,m)}(\mathbf{y}^{\mathbb{C}}) = -\frac{y^{\mathbb{C}}_{j-1,m}}{y^{\mathbb{C}}_{j,m}} + \frac{y^{\mathbb{C}}_{j,m}}{y^{\mathbb{C}}_{j+1,m}} + \frac{y^{\mathbb{C}}_{j,m}}{y^{\mathbb{C}}_{j,m-1}} = 0.$$

Therefore, $y_{j+1,m} \in \Lambda_U$ is uniquely determined by induction with respect to $y^{\mathbb{C}}_{i,\ell}$'s with $i + \ell \leq j + m$ and satisfies

$$\mathfrak{v}_T(y_{j+1,m} - y^{\mathbb{C}}_{j+1,m}) = (m-j+1)t.$$

Next, as the second step of the induction along each diagonal, suppose that

$$\big\{(y_{r,s} \in \Lambda_U) \mid (r,s) \in B(m), (r,s) \prec_{\text{hor}} (i+j, m-i+1)\big\} \qquad \text{and}$$

$$y_{i+j,m-i+1} \in \Lambda_U$$

are given, and satisfy

$$\mathfrak{v}_T(y_{r,s} - y^{\mathbb{C}}_{r,s}) = (m - j + 1)t$$

for (r,s) with $r+s=m+j+1$. We will then determine $y_{i+j+1,m-i}$ therefrom. By Lemmas 3.4 and 3.5, the equation $f_{(r,s-1)}(\mathbf{y})=0$ uniquely determines $y_{r+1,s-1} \in \Lambda_U$ so that

$$\mathfrak{v}_T(y_{r+1,s-1} - y^{\mathbb{C}}_{r+1,s-1}) = (m - j + 1)t. \tag{3.21}$$

In order to find $y_{m+1,j}$, we convert $f_{(m,j)}(\mathbf{y})=0$ into $h^{(2)}_{(m,j)}(\mathbf{y})=0$ by inserting the previously determined $y_{i,j}$'s. For $0 \le i+j \le m-1$, thanks to (3.21), we may set

$$y_{r+1,s-1} \equiv y^{\mathbb{C}}_{r+1,s-1} + A_{r-j} \cdot T^{(m-j+1)t} \mod T^{>(m-j+1)t}, \tag{3.22}$$

where $A_{r-j} \in \mathbb{C}^*$ for all $r \le i+j$.

LEMMA 3.13

A recurrence relation for the A_i's to satisfy is given by

$$\begin{cases} A_0 = -a_j \cdot \frac{(y^{\mathbb{C}}_{j+1,m})^2}{(y^{\mathbb{C}}_{j,m})^2} \cdot y^{\mathbb{C}}_{j-1,m}, \\ A_i = -\frac{(y^{\mathbb{C}}_{i+j+1,m-i})^2}{(y^{\mathbb{C}}_{i+j,m-i})^2} A_{i-1}. \end{cases}$$

Proof

This follows from a straightforward valuation analysis on $\partial_{(i+j,m-i)} W(\mathbf{y}) = 0$ with the insertion of (3.22) into $\mathbf{y}$ for $(r,s) = (i+j+1, m-i)$. □

Because of (3.19) and (3.22), we have

$$\begin{cases} -\frac{y_{m,j+1}}{y_{m,j}} \equiv -\frac{y^{\mathbb{C}}_{m,j+1}}{y^{\mathbb{C}}_{m,j}} - \frac{A_{m-j-1}}{y^{\mathbb{C}}_{m,j}} T^{(m-j+1)t} & \mod T^{>(m-j+1)t}, \\ -\frac{y_{m-1,j}}{y_{m,j}} + \frac{y_{m,j}}{y_{m,j-1}} \equiv -\frac{y^{\mathbb{C}}_{m-1,j}}{y^{\mathbb{C}}_{m,j}} + \frac{y^{\mathbb{C}}_{m,j}}{y^{\mathbb{C}}_{m,j-1}} & \mod T^{>(m-j+1)t}, \end{cases}$$

which yields

$$\begin{aligned} f_{(m,j)}(\mathbf{y}) &= \Big(-\frac{y_{m,j+1}}{y_{m,j}} - \frac{y_{m-1,j}}{y_{m,j}} + \frac{y_{m,j}}{y_{m,j-1}}\Big) + \frac{y_{m,j}}{y_{m+1,j}} T^{(m-j+1)t} \\ &\equiv \Big(-\frac{y^{\mathbb{C}}_{m,j+1}}{y^{\mathbb{C}}_{m,j}} - \frac{y^{\mathbb{C}}_{m-1,j}}{y^{\mathbb{C}}_{m,j}} + \frac{y^{\mathbb{C}}_{m,j}}{y^{\mathbb{C}}_{m,j-1}}\Big) \\ &\quad + \Big(-\frac{A_{m-j-1}}{y^{\mathbb{C}}_{m,j}} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}}\Big) T^{(m-j+1)t} \\ &\equiv \Big(-\frac{A_{m-j-1}}{y^{\mathbb{C}}_{m,j}} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}}\Big) T^{(m-j+1)t} \mod T^{>(m-j+1)t}. \end{aligned}$$

Here the last equality follows from our choice of complex solutions in Proposition 3.9 and Corollary 3.10. By applying Lemma 3.13 iteratively, we further see that

$$
\begin{aligned}
&- \frac{A_{m-j-1}}{y^{\mathbb{C}}_{m,j}} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}} \\
&\quad \equiv (-1)^{m-j+1} a_j \frac{y^{\mathbb{C}}_{j-1,m}}{y^{\mathbb{C}}_{m,j}} \Big(\prod_{i=0}^{m-j-1} \frac{(y^{\mathbb{C}}_{i+j+1,m-i})^2}{(y^{\mathbb{C}}_{i+j,m-i})^2} \Big) \\
&\quad + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}} \mod T^{>0}.
\end{aligned}
$$

By the symmetry of complex solutions within $B(m)$ in Corollary 3.10 ($c = d_{m,m}$), we have

$$
\begin{aligned}
-\frac{A_{m-j-1}}{y^{\mathbb{C}}_{m,j}} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}} &\equiv (-1)^{m-j+1} a_j \frac{y^{\mathbb{C}}_{j-1,m}}{y^{\mathbb{C}}_{m,j}} \frac{(y^{\mathbb{C}}_{m,j})^2}{(d_{m,m})^2} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}} \\
&= (-1)^{m-j+1} a_j \frac{y^{\mathbb{C}}_{m,j}}{y^{\mathbb{C}}_{m,j-1}} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}},
\end{aligned}
$$

where the last equality is from $y^{\mathbb{C}}_{j-1,m} \cdot y^{\mathbb{C}}_{m,j-1} = (d_{m,m})^2$.

Combining the above discussions, we have derived

$$
y^{\mathbb{C}}_{m,j} \Big((-1)^{m-j+1} a_j \frac{1}{y^{\mathbb{C}}_{m,j-1}} + \frac{1}{y_{m+1,j}} \Big) + k^{(2)}_{(m,j)}(a_1, \ldots, a_j) = 0 \tag{3.23}
$$

for some $k^{(2)}_{(m,j)}(a_1, \ldots, a_j) \in \Lambda_+$. We then derive from (3.23), Corollary 3.10, and (3.18)

$$
\begin{aligned}
y_{m+1,j} &\equiv (-1)^{m-j} \frac{y_{m,j-1}}{a_j} \equiv (-1)^{m-j} \frac{(d_{m,m})^2}{a_j \cdot y_{j-1,m}} \\
&\equiv (-1)^{m-j} \frac{(d_{m,m})^2}{y_{j,m+1}} \mod T^{>0},
\end{aligned} \tag{3.24}
$$

which explains why the equation $\ell^m_{(j)}(\mathbf{y}) = 0$ in the system (3.6) appears. In other words, (3.8) provides a sufficient condition to solve $y_{m+1,j}$ over Λ_U in (3.23).

Finally, we convert $f_{(m,m)}(\mathbf{y}) = 0$ into $h^{(2)}_{(m,m)}(\mathbf{y}) = 0$ as follows. Inserting $y_{m-1,m}, y_{m,m}, y_{m,m-1}$ and $y_{m,m+1} = d_{m,m+1}$ into $f_{(m,m)}(\mathbf{y}) = 0$, we derive

$$
h^{(2)}_{(m,m)}(\mathbf{y}) = \Big(-\frac{y_{m,m+1}}{y^{\mathbb{C}}_{m,m}} + \frac{y^{\mathbb{C}}_{m,m}}{y_{m+1,m}} \Big) + k^{(2)}_{(m,m)}(\mathbf{a}) = 0 \tag{3.25}
$$

for some $k^{(2)}_{(m,m)}(\mathbf{a}) \in \Lambda_+$. We obtain

$$
y_{m+1,m} \equiv \frac{(y_{m,m})^2}{y_{m,m+1}} \mod T^{>0}
$$

and determine $y_{m+1,m}$ in Λ_U.

We summarize the above discussion into the following.

PROPOSITION 3.14

For any tuple $(d_{m,m}, d_{1,m+1}, \ldots, d_{m,m+1})$ *of nonzero complex numbers, we can* *find $y_{i,j} \in \Lambda_U$ for all $(i,j) \in B(m)$ and $(i, m+1), (m+1, i)$ with $1 \leq i \leq m$ so that they satisfy*

(1) $y_{m,m} \equiv d_{m,m} \mod T^{>0}$,
(2) $y_{i,m+1} = d_{i,m+1}$ *for each* $i = 1, \ldots, m$,
(3) $f_{(i,j)}(\mathbf{y}) = 0$ *for* $(i,j) \in B(m)$,
(4) $(-1)^{m+1-l} \frac{y_{l,m+1}}{y_{m,m}} + \frac{y_{m,m}}{y_{m+1,l}} \equiv 0 \mod T^{>0}$.

3.5. Solving for $(\mathfrak{b}; \mathbf{y})$ outside $B(m)$

In this section, we discuss how to determine a bulk deformation parameter $\mathfrak{b}$ in (3.5) from $c^{\mathrm{ver},\mathbb{C}}_{i+1,i}$'s and $c^{\mathrm{hor},\mathbb{C}}_{j,j+1}$'s and extend it to a solution over Λ_U from $y^{\mathbb{C}}_{i,j}$'s under the following hypothesis.

HYPOTHESIS 3.15

Suppose that the hypothesis of Theorem 3.6 holds—that is, suppose that we are given a complex solution $(y^{\mathbb{C}}_{i,j} \in \mathbb{C}^* \mid (i,j) \in \Gamma(n) \backslash B(m) \cup \{(m,m)\})$ for (3.6) together with nonzero complex numbers $c^{\mathrm{ver},\mathbb{C}}_{i+1,i}$ and $c^{\mathrm{hor},\mathbb{C}}_{j,j+1}$.

In the setup of (3.10), to apply Lemma 3.4, we rewrite the equation $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$ into

$$\begin{cases} c^{\mathrm{hor}}_{i,i+1} \cdot \frac{1}{y_{i+1,j}} = h^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) & \text{if } i \geq j, \\ c^{\mathrm{ver}}_{j+1,j} \cdot y_{i,j+1} = h^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) & \text{if } i < j, \end{cases} \tag{3.26}$$

where

$$h^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) := \begin{cases} -c^{\mathrm{ver}}_{j,j-1} \cdot \frac{1}{y_{i,j-1}} + \frac{1}{(y_{i,j})^2}(c^{\mathrm{ver}}_{j+1,j} \cdot y_{i,j+1} + c^{\mathrm{hor}}_{i-1,i} \cdot y_{i-1,j}) \\ \qquad \text{if } i \geq j, \\ -c^{\mathrm{hor}}_{i-1,i} \cdot y_{i-1,j} + (y_{i,j})^2 (c^{\mathrm{hor}}_{i,i+1} \cdot \frac{1}{y_{i+1,j}} + c^{\mathrm{ver}}_{j,j-1} \cdot \frac{1}{y_{i,j-1}}) \\ \qquad \text{if } i < j. \end{cases} \tag{3.27}$$

As one can see, both sides of (3.26) depend on bulk parameters.

Now, we introduce a new coordinate system $(z_{i,j} \mid (i,j) \in \Gamma(n) \backslash B(m) \cup \{(m,m)\})$ with respect to which the system of equations $h^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$ does *not* depend on the choice of a bulk deformation parameter $\mathfrak{b}$ for those (i,j) satisfying $i + j < n$. We set

$$\begin{cases} z_{i+1,\bullet} := (\prod_{r=k}^{i} c^{\mathrm{hor}}_{r,r+1})^{-1} y_{i+1,\bullet} & \text{if } i \geq k, \\ z_{\bullet,j+1} := (\prod_{r=k}^{j} c^{\mathrm{ver}}_{r+1,r}) y_{\bullet,j+1} & \text{if } j \geq k, \\ z_{i,j} := y_{i,j} & \text{otherwise}, \end{cases} \tag{3.28}$$

where $k = \lceil n/2 \rceil$. Setting the product over the empty set to be 1 such as in $\prod_{r=k}^{k-1} c^{\mathrm{ver}}_{r+1,r} = 1$, the same inductive isolation procedure applied to $h^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$ under this coordinate system leads us to the following Laurent polynomials:

$$
(3.29)\qquad k^{\mathfrak{b},m}_{(i,j)}(\mathbf{z}) := \begin{cases} -\frac{1}{z_{i,j-1}} + \frac{1}{(z_{i,j})^2}(z_{i,j+1}+z_{i-1,j}) \\ \qquad (=\frac{1}{z_{i+1,j}}) \quad \text{if } i \geq j, i+j<n, \\ -z_{i-1,j} + (z_{i,j})^2(\frac{1}{z_{i+1,j}}+\frac{1}{z_{i,j-1}}) \\ \qquad (=z_{i,j+1}) \quad \text{if } i<j, i+j<n, \\ -\frac{1}{z_{i,j-1}} + \frac{1}{(z_{i,j})^2}((\prod_{r=k}^{i-1} c^{\text{hor}}_{r,r+1})^{-1} + z_{i-1,j}) \\ \qquad (=\prod_{r=k}^{i} c^{\text{hor}}_{r,r+1}) \quad \text{if } i \geq j, i+j=n, \\ -z_{i-1,j} + (z_{i,j})^2((\prod_{r=k}^{j-1} c^{\text{ver}}_{r+1,r})^{-1} + \frac{1}{z_{i,j-1}}) \\ \qquad (=\prod_{r=k}^{j} c^{\text{ver}}_{r+1,r}) \quad \text{if } i<j, i+j=n. \end{cases}
$$

Here the variables inside the parentheses are the isolated variables to be determined by $k^{\mathfrak{b},m}_{(i,j)}(\mathbf{z})$, respectively. Observe the following:

- For any (i,j) with $i+j<n$, (3.29) does *not* involve any bulk parameters.
- A solution $(y_{i,j} \in \Lambda_U)$ of (3.26) exists if and only if a solution $(z_{i,j} \in \Lambda_U)$ exists.

Assume that $m < \lceil n/2 \rceil =: k$. We will separately deal with the case where $m = \lceil n/2 \rceil$ later on. When $n = 2k$ (resp., $n = 2k-1$), we take

$$
(3.30)\qquad \begin{aligned} \mathcal{I}_{\text{seed}} &:= \{(m,m)\} \cup \{(s,m+1) \in \Gamma(n) \mid 1 \leq s \leq m\} \\ &\cup \{(m+1,m+1),(m+1,m+2),\ldots,(m+r,m+r), \\ &(m+r,m+r+1),\ldots,(k,k)\} \\ \big(\text{resp., } \mathcal{I}_{\text{seed}} &:= \{(m,m)\} \cup \{(s,m+1) \in \Gamma(n) \mid 1 \leq s \leq m\} \\ &\cup \{(m+1,m+1),(m+1,m+2),\ldots,(m+r,m+r), \\ &(m+r,m+r+1),\ldots,(k-1,k)\}\big). \end{aligned}
$$

Also, let

$$
(3.31)\qquad \mathcal{I}_{\text{initial}} := \mathcal{I}_{\text{seed}} \backslash \{(m,m)\}.
$$

Proof of Theorem 3.6
We start by taking $z_{i,j} = y_{i,j} := y^{\mathbb{C}}_{i,j}$ for $(i,j) \in \mathcal{I}_{\text{initial}}$ by (3.28) and fixing a complex solution from Corollary 3.10 such that $c = y^{\mathbb{C}}_{m,m}$ within $B(m)$. When extending a complex solution to that of a solution over Λ_U, we shall have $z_{i,j} = y_{i,j} \in \mathbb{C}^*$ for any $(i,j) \in \mathcal{I}_{\text{initial}}$, while $y_{m,m} = c + \cdots \in \Lambda_U$.

REMARK 3.16
We will define a *seed* in Definition 4.1 to generate a candidate solution for the SLT-equation. The equation (3.30) is the collection of indices where the corresponding variables will be chosen as the initial step.

By the argument in Section 3.4, the chosen element $z_{1,m+1} \in \mathbb{C}^*$ determines $z_{i+1,m-i+1} = y_{i+1,m-i+1} \in \Lambda_U$ for $1 \leq i \leq m$. In particular, we have $z_{m+1,1} =$

$y_{m+1,1} \in \Lambda_U$. We then proceed to the next diagonal given by $i+j=m+3$. Moreover, we find that $z_{i,j} = y_{i,j}$ in Λ_U for (i,j) with $i \le m+1$ and $j \le m+1$ in the diagonal satisfying (3.24). Applying Lemma 3.4 to (3.29), we determine the other variables $z_{i,j} \in \Lambda_U$ in the diagonal $i+j=m+3$. Namely, starting from $z_{2,m+1} \in \mathbb{C}^*$ and $z_{m+1,2} \in \Lambda_U$ determined in Section 3.4, we solve $z_{1,m+2}(\prec_{\mathrm{hor}} z_{2,m+1})$ and $z_{m+2,1}(\prec_{\mathrm{ver}} z_{m+1,2})$. Notice that Lemma 3.4 is applicable because of the existence of a complex solution of the SLT-equation. Proceeding inductively, we obtain a solution $(z_{i,j} \in \Lambda_U \mid (i,j) \in \Gamma(n))$ for the system (3.29) of equations over $(i,j) \in \Gamma(n-1)$. Finally, we solve $c^{\mathrm{hor}}_{r,r+1}$ and $c^{\mathrm{ver}}_{r+1,r}$ to make the equations (3.29) over (i,j) with $i+j=n$ hold. Thus, Theorem 3.6 is now verified when $m < \lceil n/2 \rceil$.

In this case when $n = 2k$ and $m = k$, we shall take $c = 1$ for $y^{\mathbb{C}}_{m,m}$. Corollary 3.10 will give us the initial parts $y^{\mathbb{C}}_{i,j}$ of $y_{i,j}$'s for $(i,j) \in B(m)$. We then follow Section 3.4 to extend to $z_{i,j} = y_{i,j}$'s over Λ_U in $B(m)$. If one uses both $\mathfrak{b}^{\mathrm{ver}}_{m+1,m}$ and $\mathfrak{b}^{\mathrm{hor}}_{m,m+1}$ to deform $f_{(m,m)} = 0$, then we have two extra variables $c^{\mathrm{ver}}_{m+1,m}$ and $c^{\mathrm{hor}}_{m,m+1}$ in $f^{\mathfrak{b}}_{(m,m)} = 0$. For our convenience, we choose $\mathfrak{b}^{\mathrm{hor}}_{m,m+1} = 0$. Now, we need to take $\mathfrak{b}^{\mathrm{ver}}_{m+1,m}$ so that $f^{\mathfrak{b}}_{(m,m)} = 0$. It yields that $1 = c^{\mathrm{ver},\mathbb{C}}_{m+1,m} = \exp(\mathfrak{b}^{\mathrm{ver}}_{m+1,m}) \mod T^{>0}$. After fixing $\mathfrak{b}^{\mathrm{ver}}_{m+1,m} \in \Lambda_+$, we obtain $z_{\bullet,m+1} = y_{\bullet,m+1}$ by solving $h^{\mathfrak{b},(2)}_{(\bullet,m)} = 0$, where

$$h^{\mathfrak{b},(2)}_{(1,m)}(\mathbf{y}) := -\exp(\mathfrak{b}^{\mathrm{ver}}_{m+1,m}) \frac{y_{1,m+1}}{y_{1,m}} + a_1 \frac{y_{1,m}}{y_{1,m-1}} = 0,$$

$$h^{\mathfrak{b},(2)}_{(j,m)}(\mathbf{y}) := -\exp(\mathfrak{b}^{\mathrm{ver}}_{m+1,m}) \frac{y_{j,m+1}}{y_{j,m}} + a_j \left(\frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}} \right) = 0 \quad \text{for } j \ge 2.$$

The remaining steps are similar. □

4. Solvability of the split leading term equation

This section aims to verify the assumption for Theorem 3.6 when the SLT-equation (3.6) comes from the line segment $I_m \subset \Delta_\lambda$ in (1.4). To find its solution, we introduce the notion of *seeds* generating a candidate for the solution and prove that there exists a "good" choice of seeds such that the candidate is indeed a solution.

4.1. Seeds

Recall the definitions $\Gamma(n)$ and $B(m)$ from (3.1). Consider a nested sequence of fields of rational functions

$$\mathbb{C}(\mathbf{y}_{(2)}) \subset \mathbb{C}(\mathbf{y}_{(3)}) \subset \cdots \subset \mathbb{C}(\mathbf{y}_{(n)}), \tag{4.1}$$

where $\mathbb{C}(\mathbf{y}_{(n)})$ is filtered by the variables in the diagonals of $\Gamma(n)$ and

$$\begin{aligned} \mathbb{C}(\mathbf{y}_{(r)}) &:= \mathbb{C}\big(\{(i,j) \in \Gamma(n) \backslash \big(B(m) \backslash \{y_{m,m}\}\big) \mid (i,j) \prec_{\mathrm{hor}} (1,r)\}\big) \\ &= \mathbb{C}\Big(\bigcup_{s=2}^{r} \{y_{1,s-1}, \dots, y_{s-1,1}\} \backslash \big(B(m) \backslash \{y_{m,m}\}\big)\Big). \end{aligned}$$

Such a filtration will be called the *diagonal filtration associated to* $\Gamma(n)$ and $B(m)$. Notice that when $(i,j) \in \Gamma(n)\backslash B(m)$ the Laurent polynomials appearing in the left-hand side of (3.6) are actually of the following types:

$$\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) \in \mathbb{C}(\mathbf{y}_{(i+j+1)}), \qquad \ell^{m}_{(l)}(\mathbf{y}) \in \mathbb{C}\big(\mathbf{y}_{(l+m+1)} \cup \{y_{m,m}\}\big).$$

We begin with the definition of a seed.

DEFINITION 4.1

A *seed* of $\Gamma(n)$ *associated with* $B(m)$ consists of the two data $(\mathcal{I}, \mathbf{d}_{\mathcal{I}})$:

- an $(n-m)$-tuple $\mathcal{I}$ of double indices

$$\mathcal{I} = \big\{(m,m),(i_1,j_1),\dots,(i_{n-m-1},j_{n-m-1})\big\} \subset \big\{(m,m)\big\} \cup \big(\Gamma(n)\backslash B(m)\big)$$

such that one index (i_k,j_k) is selected from each diagonal not intersecting $B(m)$;

- an $(n-m)$-tuple $\mathbf{d}_{\mathcal{I}}$ of elements in Λ_U indexed by the elements of $\mathcal{I}$.

We are particularly interested in the seeds $(\mathcal{I},\mathbf{d})$ of the form

- $\mathcal{I} := \mathcal{I}_{\text{seed}}$ in (3.30),
- $\mathbf{d}_{\mathcal{I}}$ is a tuple of *nonzero real* numbers.

Let $\mathbf{y}_{\mathcal{I}}$ denote the components of $\mathbf{y}$ associated with the set $\mathcal{I}$ of indices. Namely,

$$\mathbf{y}_{\mathcal{I}} := (y_{m,m}, y_{i_1,j_1}, \dots, y_{i_{n-m-1},j_{n-m-1}}). \tag{4.2}$$

Then, as the initial step, we take $\mathbf{y}_{\mathcal{I}} := \mathbf{d}_{\mathcal{I}}$. So, the double indices designate the places in which the components of $\mathbf{d}_{\mathcal{I}}$ are plugged. Since $\mathcal{I}$ is always taken to be $\mathcal{I}_{\text{seed}}$, $\mathcal{I}$ will be often omitted from now on. We instead set $d_{i,j}$ to denote the component of $\mathbf{d}_{\mathcal{I}}$ corresponding to (i,j).

We shall determine a solution of the SLT-equation recursively from a seed according to the order $\prec_{\text{hor}}$ (or $\prec_{\text{ver}}$) and the diagonal filtration (4.1). By the SLT-equation and Lemma 3.4 over the field of complex numbers, $\mathbf{y}_{(r)}$, $y_{m,m}$, and one datum of a seed (4.2) in the diagonal $i+j=r+1$ in $\mathbf{y}_{(r+1)}\backslash\mathbf{y}_{(r)}$ generate the other variables in the diagonal with a suitable choice of complex numbers

$$\mathbf{c} := (c^{\text{hor}}_{k,k+1},\dots,c^{\text{hor}}_{n-1,n},c^{\text{ver}}_{k+1,k},\dots,c^{\text{ver}}_{n,n-1}).$$

Namely, we isolate and express one undetermined variable in terms of the already determined variables for the remaining $y_{i,j}$'s and $\mathbf{c}$ inductively over the order $\prec_{\text{hor}}$ (or $\prec_{\text{ver}}$) by rewriting the SLT-equation. However, the undetermined variable might be *zero* or *undefined* which should be avoided. A *good* choice of seed, what we call a *generic* seed, will do for this purpose.

Now we will give the precise definition of a "generic seed." In the setup of (3.10), we rewrite $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$ into (3.27). We first note that (3.27) involves only the variables $y_{(r,s)}$ such that $(r,s) \prec_{\text{hor}} (i+1,j)$ (resp., $(r,s) \prec_{\text{ver}} (i,j+1)$) for $i \geq j$ (resp., for $i < j$). Therefore, we can determine $y_{(i+1,j)}$ (resp., $y_{(i,j+1)}$) if $i \geq j$ (resp., if $i<j$), *as long as the value of* $h^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) \neq 0$ *or is defined, that is, provided that*

$$\mathbf{y} \notin \operatorname{Zero}(h^{\mathfrak{b},m}_{(i,j)}) \cup \operatorname{Pole}(h^{\mathfrak{b},m}_{(i,j)}). \tag{4.3}$$

DEFINITION 4.2

A seed $(\mathcal{I}, \mathbf{d}_{\mathcal{I}})$ is called *generic* if the seed $\mathbf{y}_{\mathcal{I}} = \mathbf{d}_{\mathcal{I}}$ generates inductive solutions $y_{(r,s)}$ for $(r,s) \prec_{\text{hor}} (i+1,j)$ (resp., $(r,s) \prec_{\text{ver}} (i,j+1)$) such that

$$h^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) \neq 0 \mod T^{>0} \tag{4.4}$$

for all (i,j) with $i \geq j$ (resp., for all (i,j) with $i < j$).

EXAMPLE 4.3

A straightforward calculation asserts that the tuples

(1) $\mathcal{I} = \mathcal{I}_{\text{seed}} = ((2,2),(1,3),(2,3),(3,3),(3,4))$,
(2) $\mathbf{d}_{\mathcal{I}} = (-1,1,1,-1,1)$

form a generic seed of $\Gamma(7)$ to $B(2)$. The tuples

(1) $\mathcal{I} = \mathcal{I}_{\text{seed}} = ((2,2),(1,3),(2,3),(3,3),(3,4))$,
(2) $\mathbf{d}_{\mathcal{I}} = (-1,1,1,1,1)$

form a seed of $\Gamma(7)$ to $B(2)$, but not a generic seed because $h^{\mathfrak{b},m=2}_{(1,5)}(\mathbf{y}) = 0$.

The main proposition of this section is the following existence of a generic seed, whose proof will occupy the rest of this section.

PROPOSITION 4.4

For each $m \in \mathbb{Z}$, where $2 \leq m \leq k = \lceil n/2 \rceil$, a generic seed of $\Gamma(n)$ associated to $B(m)$ exists.

As a corollary, we assert the solvability of the SLT-equation.

COROLLARY 4.5

The SLT-equation of $\Gamma(n)$ associated with $B(m)$ has a solution each component of which is a nonzero complex number.

Proof

Assume that a seed has the property (4.4), and that the remaining $y_{i,j}$'s and a sequence $\mathbf{c}$ are (uniquely) determined in $\mathbb{C}^*$ by exactly the same process as in Section 3.5. □

4.2. Pregeneric elements

We again exploit the coordinate system $\{z_{i,j} \mid (i,j) \in \Gamma(n) \backslash B(m) \cup \{(m,m)\}\}$ in (3.28). Recall the system of rational functions in (3.29).

We have the following lemma.

LEMMA 4.6
We have $k^{\mathfrak{b},m}_{(i,j)}(\mathbf{z})\neq 0$ *if and only if* $h^{\mathfrak{b},m}_{(i,j)}(\mathbf{y})\neq 0$ *for* $(i,j)\in\Gamma(n)\backslash B(m)$.

Proof
Under the coordinate change (3.28), (3.27) is converted into (3.29). □

When $m<k=\lceil n/2\rceil$, since $\mathcal{I}\subset B(k)$ and $\mathbf{y}_{\mathcal{I}}=\mathbf{z}_{\mathcal{I}}$ by (3.28), we may insert $\mathbf{d}_{\mathcal{I}}$ into $\mathbf{z}_{\mathcal{I}}$ as the starting point. For the case $m=k$, we have taken $c^{\mathrm{hor},\mathbb{C}}_{m,m+1}=1$ and $c^{\mathrm{ver},\mathbb{C}}_{m+1,m}=1$ and hence $\mathbf{z}_{\mathcal{I}}=\mathbf{y}_{\mathcal{I}}=\mathbf{d}_{\mathcal{I}}$ as well. As in (4.1), consider the diagonal filtration of the field of rational functions (with respect to z)

$$\mathbb{C}(\mathbf{z}_{(2)})\subset\mathbb{C}(\mathbf{z}_{(3)})\subset\cdots\subset\mathbb{C}(\mathbf{z}_{(n)}),$$

where

$$\mathbf{z}_{(r)}:=\bigcup_{s=2}^{r}\{z_{1,s-1},\dots,z_{s-1,1}\}\backslash\big(\mathbf{z}_{B(m)}\backslash\{z_{m,m}\}\big)\qquad\text{and}$$
$$\mathbf{z}_{B(m)}=\big\{z_{i,j}\mid(i,j)\in B(m)\big\}.$$

By choosing one component in a diagonal generically, we can make the first two equations of (3.29) nonzero because of the following lemma. For a given fixed $(r,s)\in B(m)\setminus\{(m,m)\}$, we set $X(0):=z_{r,s}$ and denote by $X(i)$ the rational function obtained by solving $z_{r-i,s+i}$ in terms of $X(0)$ with the coefficients $\mathbb{C}(\mathbf{z}_{(r+s-1)})$. If the above-mentioned inductive determination does not meet an obstruction, its corresponding value at $\mathbf{y}$ is supposed to determine the value of $z_{r-i,s+i}$. We only show the case for $i>0$ since the case where $i<0$ is similar.

PROPOSITION 4.7
Let us take $X(0):=z_{(r,s)}$. *The rational function* $X(i)$ *is a nonconstant fractional linear map of* $X(0)$ *in coefficients* $\mathbb{C}(\mathbf{z}_{(r+s-1)})$.

Proof
By (3.29), we derive a recurrence relation for $X(i)$'s of the form

$$X(i)=[i]+\frac{[i,i-1]}{X(i-1)},\tag{4.5}$$

where

$$[i]:=-z_{r-i-1,s+i-1}+\frac{(z_{r-i,s+i-1})^2}{z_{r-i,s+i-2}}\qquad\text{and}\qquad[i,i-1]:=(z_{r-i,s+i-1})^2.\tag{4.6}$$

We note that $[i],[i,i-1]\in\mathbb{C}(\mathbf{z}_{(r+s-1)})$, while $X(i)\in\mathbb{C}(\mathbf{z}_{(r+s)})$ for all i.

Composing (4.5) several times, $X(i)$ is expressed as a continued fraction in terms of $X(0)=z_{r,s}$. If we set the initial condition $A(0)=1,B(0)=0$, then we can express $X(i)$ as

$$X(i)=\frac{A(i)\cdot X(0)+B(i)}{A(i-1)\cdot X(0)+B(i-1)}\tag{4.7}$$

for the elements $A(i),B(i)\in\mathbb{C}(\mathbf{z}_{(r+s-1)})$. Thus, we have $X(i)\in\mathbb{C}(\mathbf{z}_{(r+s-1)})(z_{r,s})$.

To show that every $X(i)$ is *nonconstant* with respect to $X(0) = z_{(r,s)}$, we investigate properties of $A(i)$'s and $B(i)$'s. By the induction, we obtain the following lemma.

LEMMA 4.8

Consider $\{i, i-1, \dots, 1, 0\}$. *Let* $\mathcal{P}_{A(i)}$ *be the partitions of* $\{i, i-1, \dots, 1\}$ *into one single or two consecutive numbers, and let* $\mathcal{P}_{B(i)}$ *be the partitions of* $\{i, i-1, \dots, 1, 0\}$ *into one single or two consecutive numbers containing the subset* $[1, 0]$. *Then*

$$A(i) = \sum_{I \in \mathcal{P}_{A(i)}} [I] \qquad \textit{and} \qquad B(i) = \sum_{I \in \mathcal{P}_{B(i)}} [I].$$

For instance, $A(3)$ and $B(3)$ are expressed as

$$A(3) = [3][2][1] + [3][2,1] + [3,2][1],$$
$$B(3) = [3][2][1,0] + [3,2][1,0].$$

COROLLARY 4.9

We have

$$A(i) = [i] \cdot A(i-1) + [i, i-1] \cdot A(i-2),$$
$$B(i) = [i] \cdot B(i-1) + [i, i-1] \cdot B(i-2).$$

Note that $X(0)$ and $X(1)$ are nonconstant Laurent polynomials with respect to $X(0) = z_{r,s}$. Suppose to the contrary that $X(i)$ is a constant function with the value C and that all $X(j)$'s for all $j < i$ are nonconstant rational functions of $X(0) (= z_{r,s})$.

Suppose that $X(i) \equiv C$. By recalling that $A(i), B(i) \in \mathbb{C}(\mathbf{z}_{(r+s-1)})$ and substituting the values $z_{r,s} = 0, 1$ into (4.7), respectively, we obtain

$$C \cdot A(i-1) = A(i) = [i] \cdot A(i-1) + [i, i-1] \cdot A(i-2),$$
$$C \cdot B(i-1) = B(i) = [i] \cdot B(i-1) + [i, i-1] \cdot B(i-2).$$

Then $(C - [i]) \cdot A(i-1) = [i, i-1] \cdot A(i-2)$.

We claim that $C - [i]$ is a nonzero rational function contained in $\mathbb{C}(\mathbf{z}_{(r+s-1)})$. Otherwise, we would have $A(i-2) = B(i-2) = 0$ because $[i, i-1] = (z_{r-i,s+i-1})^2 \neq 0$. It would yield that $X(i-2) \equiv 0$, contradicting the assumption that $X(i-2)$ is not constant with respect to $z_{r,s}$. We then have

$$A(i-1) = C' \cdot A(i-2),$$
$$B(i-1) = C' \cdot B(i-2),$$

where $C' = [i, i-1]/(C - [i]) \in \mathbb{C}(\mathbf{z}_{(r+s-1)})$. Then we deduce that $X(i-1) = C'$ which is constant with respect to $z_{r,s}$, a contradiction to the standing hypothesis. This finishes the proof of the proposition. $\square$

COROLLARY 4.10

Suppose that the set $\mathbf{z}_{(r+s-1)}$ *is determined. There exists a nonzero real number* $d_{r,s}$ *such that if we set* $z_{r,s} = d_{r,s}$*, then we achieve*

$$k^{\mathfrak{b},m}_{(i,j)}(\mathbf{z}) \neq 0 \tag{4.8}$$

for all (i,j)*'s satisfying* $i+j = r+s-1$.

Proof

Since each $X(i)$ is a nonconstant fractional linear map of $z_{r,s}$, there are only finitely many $z_{r,s}$'s making $z_{r-i,s+i}$ zero or not defined. Avoid these values when choosing $d_{r,s}$. □

DEFINITION 4.11

Suppose that the set $\mathbf{z}_{(r+s-1)}$ is given. For an index $(r,s) \in \Gamma(n)\backslash B(m)$, an element $d_{r,s}$ in a seed is said to be *pregeneric with respect to* $\mathbf{z}_{(r+s-1)}$ if (4.8) holds for any $(i,j) \in \Gamma(r+s-1)\backslash(B(m) \cup \Gamma(r+s-2))$.

For a later purpose, we prove the following property of pregeneric elements. Choose $m < k$, and fix $d_{m,m} \in \mathbb{C}^*$. By making a sequence $\{d_{1,m+1}, d_{2,m+1}, \dots, d_{s,m+1}\}$ of pregeneric seeds for $s < m$, one generates a solution $\mathbf{z}_{(m+s+1)}$ of the corresponding SLT-equation (3.6) with respect to the variables $\mathbf{z}$ in (3.28) (whenever any equation in (3.6) makes sense). Notice that $\mathbf{z}_{(m+s+1)}$ depends on $d_{m,m}$ because $z_{i,m+1}$ and $d_{m,m}$ determine $z_{m+1,i}$. Now, we make another choice $d'_{m,m} \in \mathbb{C}^*$ and denote by $\mathbf{z}'_{(m+s+1)}$ another solution of the SLT-equation generated by the same sequence $\{d_{1,m+1}, \dots, d_{s,m+1}\}$. The following lemma tells us that the pregenericity of $d_{s+1,m+1}$ does not depend on $d_{m,m}$ (but it may depend on $\{d_{1,m+1}, \dots, d_{s,m+1}\}$).

LEMMA 4.12

Suppose that $d_{s+1,m+1}$ *is pregeneric with respect to* $\mathbf{z}_{(m+s+1)}$*, Then* $d_{s+1,m+1}$ *is also pregeneric with respect to* $\mathbf{z}'_{(m+s+1)}$.

Proof

If $m < k$, then we see that $y_{i,j} = z_{i,j}$ for $(i,j) \in B(m)$ by (3.28). For any $(i+1,j) \notin B(m)$ with $i+1+j = s+1+m+1$, we claim that

$$z_{j,i+1} = (-1)^{i+j} \cdot \frac{(d_{m,m})^2}{z_{i+1,j}} \tag{4.9}$$

by induction over the order $\prec_{\text{ver}}$. Recall from (3.8) that

$$d_{m+1,s+1} =: z_{m+1,s+1} = (-1)^{m+s-1} \cdot \frac{(d_{m,m})^2}{z_{s+1,m+1}},$$

which provides the initial step for the induction. Next suppose that (4.9) holds for all $(r,s) \prec_{\text{ver}} (j,i+1)$. Then we observe that all four subindices appearing in the equation

$$(4.10) \qquad -\frac{z_{i,j+1}}{z_{i,j}} - \frac{z_{i-1,j}}{z_{i,j}} + \frac{z_{i,j}}{z_{i+1,j}} + \frac{z_{i,j}}{z_{i,j-1}} = 0$$

are smaller than $(j, i+1)$. By the induction hypothesis, we substitute (4.9) into (4.10), and we derive the equation

$$0 = \frac{z_{j,i}}{z_{j+1,i}} + \frac{z_{j,i}}{z_{j,i-1}} + (-1)^{i+j-1}\frac{(d_{m,m})^2}{z_{i+1,j}\cdot z_{j,i}} - \frac{z_{j-1,i}}{z_{j,i}}.$$

Then by a back-substitution of (4.10) thereinto, we obtain

$$\frac{z_{j,i+1}}{z_{j,i}} + (-1)^{i+j-1}\frac{(d_{m,m})^2}{z_{i+1,j}z_{j,i}} = 0,$$

which is equivalent to

$$z_{j,i+1} = (-1)^{i+j}\cdot\frac{(d_{m,m})^2}{z_{i+1,j}}.$$

Thus, (4.9) is established.

It follows from (4.9) that even if we choose different $d'_{m,m}$, we still have

(1) $z_{j,i+1} = z'_{j,i+1}$,
(2) $z_{i+1,j}$ and $z'_{i+1,j}$ differ by a *nonzero* constant multiple

for any pair $(j, i+1) \notin B(m)$ with $i+j = s+m+1$ and $i \geq j$ as long as we keep the sequence $\{d_{1,m+1}, \ldots, d_{s,m+1}\}$ the same. Hence, Lemma 4.12 follows. □

4.3. Generic seeds

Applying Corollary 4.10, we make the first two expressions in (3.29) nonzero by generically choosing the value of one entry of each diagonal. To make the last two terms of (3.29) nonzero at the same time, we need to carefully adjust this choice we have made. We recall that $k := \lceil n/2 \rceil$. Depending on the parity of n, either $n = 2k-1$ or $n = 2k$. We consider the two cases separately.

Case 1: $n = 2k-1$. We need several lemmas.

LEMMA 4.13

Assume that $\mathbf{z}_{(n-2)}$ is given. Suppose that either $d_{k-1,k-1} = -1$ is pregeneric or $k-1 = m$. Then one can retake a real number $d_{k-1,k-1}$ (sufficiently close to -1, but not equal to -1) and take a nonzero real number $d_{k-1,k}$ such that if $z_{k-1,k-1} = d_{k-1,k-1}$ and $z_{k-1,k} = d_{k-1,k}$, then

$$(4.11) \qquad k^{\mathfrak{b},m}_{(i,j)}(\mathbf{z}) \neq 0 \mod T^{>0}$$

for all (i,j) with $i+j = n-1$ and $i+j = n$.

Note that the $k^{\mathfrak{b},m}_{(i,j)}(\mathbf{z})$'s for (i,j) with $i+j = n$ provide the last two expressions of (3.29).

Proof
Assuming that $d_{k-1,k-1}=-1$ is pregeneric, by definition, every $z_{i,n-1-i}$ is defined and becomes nonzero if we set $z_{k-1,k-1}=d_{k-1,k-1}=-1$. By Corollary 4.10, we can choose and fix a pregeneric element $d_{k-1,k}$ for $z_{k-1,k}$ so that the entries $z_{i,n-i}$ are also determined.

We emphasize that $d_{k-1,k-1}=-1$ is *never* a component of a generic seed because of the following reason. It is straightforward to see that the equations $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y})=0$ in terms of $\mathbf{z}$ for (i,j)'s with $i+j\geq n$ and $i<j$ read

$$(4.12)\quad \begin{cases} \prod_{r=k}^{k} c^{\text{ver}}_{r+1,r} = -z_{k-2,k} + (z_{k-1,k})^2(1+\frac{1}{z_{k-1,k-1}}), \\ \prod_{r=k}^{k+1} c^{\text{ver}}_{r+1,r} = -z_{k-3,k+1} + (z_{k-2,k+1})^2((\prod_{r=k}^{k} c^{\text{ver}}_{r+1,r})^{-1} + \frac{1}{z_{k-2,k}}), \\ \cdots \\ \prod_{r=k}^{n-2} c^{\text{ver}}_{r+1,r} = -z_{1,n-2} + (z_{2,n-2})^2((\prod_{r=k}^{n-3} c^{\text{ver}}_{r+1,r})^{-1} + \frac{1}{z_{2,n-3}}), \\ \prod_{r=k}^{n-1} c^{\text{ver}}_{r+1,r} = (z_{1,n-1})^2((\prod_{r=k}^{n-2} c^{\text{ver}}_{r+1,r})^{-1} + \frac{1}{z_{1,n-2}}). \end{cases}$$

If one chooses $z_{k-1,k-1}=d_{k-1,k-1}=-1$, then we obtain

$$\prod_{r=k}^{j} c^{\text{ver}}_{r+1,r} = -z_{n-j-1,j}$$

from (4.12) for $j=k,k+1,\ldots,n-2$ and

$$(4.13)\quad k^{\mathfrak{b},m}_{(1,n-1)}(\mathbf{z}) = \prod_{r=k}^{n-1} c^{\text{ver}}_{r+1,r} = 0.$$

Thus, the seed $\mathbf{d}_{\mathcal{I}}$ is *not* generic. Nevertheless, we claim that there exists a choice of $d_{k-1,k-1}$ not equal to -1 but close to -1 so that (4.11) is satisfied.

Note that the fixed $d_{k,k-1}$ remains pregeneric even if we perturb the value $z_{k-1,k-1}$ from -1 with a sufficiently small amount. This is because the expression $k^{\mathfrak{b},m}_{(i,j)}(\mathbf{z})$ for each index (i,j) with $i+j=n-1$ is a continuous function with respect to $z_{k-1,k-1}$ at -1 after inserting $d_{k,k-1}$ into $z_{k,k-1}$. Also, by Proposition 4.7, there exists a dense set of pregeneric elements for $d_{k-1,k-1}$. Therefore, (4.11) is satisfied for $i+j=n-1$.

Also, we observe that as $z_{k-1,k-1}\to -1$, because of (3.29) and (4.12), $k^{\mathfrak{b},m}_{(n-j,j)}(\mathbf{z})\to -z_{n-j-1,j}$ when $j\geq k$. Because $-z_{n-j-1,j}\neq 0$ for j with $k\leq j<n-1$, we still have $k^{\mathfrak{b},m}_{(n-j,j)}(\mathbf{z})\neq 0$ for j with $k\leq j<n-1$ if $d_{k-1,k-1}$ is sufficiently close to -1. Finally, we claim that $k^{\mathfrak{b},m}_{(1,n-1)}(\mathbf{z})\neq 0$ if $z_{k-1,k-1}\neq -1$ so that the issue in (4.13) is solved. From (4.12) and $z_{k-1,k-1}\neq -1$, it follows that

$$c^{\text{ver}}_{k+1,k} \neq -z_{k-2,k}.$$

Proceeding inductively, we deduce that $k^{\mathfrak{b},m}_{(1,n-1)}(\mathbf{z})\neq 0$ from (4.12). The discussion on the part where $j<k$ is omitted because the argument is symmetrical. Consequently, we may choose a generic $d_{k-1,k-1}$ sufficiently close to -1 so that (4.11) holds for all (i,j) with $i+j=n-1$ and $i+j=n$.

It remains to take care of the case where $k-1=m$. We note that $(k-1,k-1)=(m,m)$ is contained in the box $B(m)$ so that $d_{k-1,k-1}$ can be freely chosen by Corollary 3.10. Thus, the exact same argument can be applied as above. □

By applying a similar argument, we can prove the following lemma.

LEMMA 4.14
For any fixed $i \in \mathbb{Z}$ with $i \geq m$, suppose that either $d_{i,i} = \pm 1$ is pregeneric. Then one can retake a real number $d_{i,i}$ (sufficiently close to ± 1, but not equal to ± 1) and take a nonzero real number $d_{i,i+1}$ so that $d_{i+1,i+1} = \mp 1$ becomes pregeneric.

We are now ready to start the proof of Proposition 4.4 for the case where $n = 2k-1$ and $m < k := \lceil n/2 \rceil$.

Proof of Proposition 4.4
We start with a tentative choice of $d_{m,m} = \pm 1$. Choosing pregeneric elements from $d_{1,m+1} := z_{1,m+1}$ to $d_{m-1,m+1} := z_{m-1,m+1}$, we find $\mathbf{z}_{(2m)}$ so that (4.11) is satisfied for each index (i,j) with $i+j \leq 2m-1$. Due to Lemma 4.14, we may select $d_{m,m}$ sufficiently close to ± 1 and $d_{m,m+1}$ so that $d_{m+1,m+1} = \mp 1$ becomes pregeneric. Because of Lemma 4.12, note that $d_{1,m+1}, \ldots, d_{m,m+1}$ remain pregeneric even if we choose another $d_{m,m}$. Moreover, applying Lemma 4.14 repeatedly, we assert that $d_{k-1,k-1} = -1$ is also pregeneric by suitably choosing $d_{\bullet,\bullet}$. Hence, we have (4.11) for all indices (i,j) with $i+j \leq n-2$. Finally, Lemma 4.13 says that there is $d_{k-1,k-1}$ and $d_{k,k-1}$ such that (4.11) holds for $i+j = n-1, n$. Thus, we have just found a generic seed. □

Case 2: $n = 2k$ and $m < k$. We can modify the proofs of Lemmas 4.13 and 4.14 to prove the following.

LEMMA 4.15
Assume that $\mathbf{z}_{(n-2)}$ is given. Suppose that $d_{k-1,k} = -1$ is pregeneric. Then one can retake a real number $d_{k-1,k}$ (sufficiently close to -1, but not equal to -1) and take a nonzero real number $d_{k,k}$ such that if $z_{k-1,k} = d_{k-1,k}$ and $z_{k,k} = d_{k,k}$, then

$$k^{\mathfrak{b},m}_{(i,j)}(\mathbf{z}) \neq 0 \mod T^{>0}$$

for all (i,j) with $i+j = n-1$ and $i+j = n$.

Suppose that $d_{i-1,i} = \pm 1$ is pregeneric for $i \geq m+1$. There is a real number $d_{i-1,i}$ (sufficiently close to ± 1) and a nonzero real number $d_{i,i}$ so that $d_{i,i+1} = \mp 1$ becomes pregeneric.

Also, we need the following lemma, which serves as the starting point for obtaining the desired $d_{\bullet,\bullet}$'s.

LEMMA 4.16
We have that $d_{m,m+1} = \pm 1$ can be pregeneric.

Proof
As in Lemma 3.8, one sees that

$$\widetilde{z}_{i,m+j} := \begin{cases} 1 & \text{for } i = j, \\ \prod_{r=0}^{j-i-1}(2i+2r) & \text{for } i < j, \\ \prod_{r=0}^{i-j-1}(2j+2r)^{-1} & \text{for } i > j \end{cases} \tag{4.14}$$

is a solution of $\ell^{\mathfrak{b},m}_{(i,j)}(\mathbf{y}) = 0$ under the coordinate change (3.28) for $m+i+j < n$. Also, by Lemma 3.7, so is

$$z_{i,m+j} := a \cdot \widetilde{z}_{i,m+j} \tag{4.15}$$

for any nonzero complex number a. Selecting

$$a := \prod_{r=0}^{m-2}(2+2r),$$

$d_{m,m+1} = z_{m,m+1}$ becomes 1. Because of Lemma 4.12, no matter what we choose, any nonzero complex number $d_{m,m}$, $d_{m,m+1}$ is pregeneric (with respect to the previously determined $\mathbf{z}_{(2m)}$). □

Proof of Proposition 4.4 (continued)
Combining Lemmas 4.15 and 4.16, we conclude that Proposition 4.4 holds for the case where $n = 2k$ and $m < k$. □

Case 3: $n = 2k$ and $m = k$. In this case, we have taken $d_{m,m} = 1$. Because of Lemma 3.8, note that

$$\begin{aligned} \widetilde{z}_{i,m+j} &:= \begin{cases} 1 & \text{for } i = j, \\ \prod_{r=0}^{j-i-1}(2i+2r) & \text{for } i < j, \\ \prod_{r=0}^{i-j-1}(2j+2r)^{-1} & \text{for } i > j, \end{cases} \\ \widetilde{z}_{m+i,j} &:= \begin{cases} (-1)^{m+i+j-1} & \text{for } i = j, \\ (-1)^{m+i+j-1}\prod_{r=0}^{i-j-1}(2j+2r)^{-1} & \text{for } i > j, \\ (-1)^{m+i+j-1}\prod_{r=0}^{j-i-1}(2i+2r) & \text{for } i < j, \end{cases} \end{aligned} \tag{4.16}$$

respectively, form a solution of $\ell^{\mathfrak{b},m}_{(i,m+j)}(\mathbf{y}) = 0$ and $\ell^{\mathfrak{b},m}_{(m+i,j)}(\mathbf{y}) = 0$ for $m+i+j < n$. Also, our choice makes $\ell^{m}_{(l)}(\mathbf{y}) = 0$ in (3.8) because $c^{\text{ver}}_{m+1,m} = 1$ and $c^{\text{hor}}_{m,m+1} = 1$.

Furthermore,

$$z_{i,m+j} := a \cdot \widetilde{z}_{i,m+j}, \qquad z_{m+i,j} := a^{-1} \cdot \widetilde{z}_{m+i,j} \tag{4.17}$$

are also solutions for any nonzero complex number a. Thus, we have a one-parameter family of solutions. Then the expressions $k^{\mathfrak{b},m}_{(i,m+j)}(\mathbf{z})$ and $k^{\mathfrak{b},m}_{(m+i,j)}(\mathbf{z})$ with $m+i+j = n$ can be considered as a function with respect to a.

Because of Lemma 4.15, it suffices to deal with the starting point.

LEMMA 4.17

There exists a choice of the number a in (4.17) such that

$$k^{\mathfrak{b},m}_{(m-i,m+i)}(\mathbf{z}) \neq 0 \qquad \text{and} \qquad k^{\mathfrak{b},m}_{(m+i,m-i)}(\mathbf{z}) \neq 0 \mod T^{>0} \tag{4.18}$$

in (3.29).

Proof

We claim that $k^{\mathfrak{b},m}_{(m-i,m+i)}(\mathbf{z})/z_{m-i,m+i}$ is a nonconstant rational function with respect to a. We observe that

$$\frac{k^{\mathfrak{b},m}_{(m-1,m+1)}(\mathbf{z})}{z_{m-1,m+1}} = -\frac{z_{m-2,m+1}}{z_{m-1,m+1}} + z_{m-1,m+1} = -(2m-4) + a \cdot \Big(\prod_{r=0}^{m-3} (2+2r)^{-1}\Big)$$

is a nonconstant linear function with respect to a. By the induction hypothesis, assume that

$$\frac{k^{\mathfrak{b},m}_{(m-i,m+i)}(\mathbf{z})}{z_{m-i,m+i}} := \frac{P_i(a)}{Q_i(a)}$$

is a nonconstant rational function with respect to a. Then we see

$$\begin{aligned}
&\frac{k^{\mathfrak{b},m}_{(m-i-1,m+i+1)}(\mathbf{z})}{z_{m-i-1,m+i+1}} \\
&\quad = \Big(-\frac{z_{m-i-2,m+i+1}}{z_{m-i-1,m+i+1}} + \frac{z_{m-i-1,m+i+1}}{z_{m-i-1,m+i}}\Big) + \frac{z_{m-i-1,m+i+1}}{k^{\mathfrak{b},m}_{(m-i,m+i)}(\mathbf{z})} \\
&\quad = \Big(-\frac{\widetilde{z}_{m-i-2,m+i+1}}{\widetilde{z}_{m-i-1,m+i+1}} + \frac{\widetilde{z}_{m-i-1,m+i+1}}{\widetilde{z}_{m-i-1,m+i}}\Big) + \frac{\widetilde{z}_{m-i-1,m+i+1}}{\widetilde{z}_{m-i,m+i}} \cdot \frac{Q_i(a)}{P_i(a)},
\end{aligned}$$

which is also a nonconstant rational function. Similarly, one can see that $k^{\mathfrak{b},m}_{(m+i,m-i)}(\mathbf{z})$ is also a nonconstant rational function for $i \geq 1$. Thus, (4.18) is established if choosing a generically. □

We are ready to prove Proposition 4.4 for the case where $n = 2k$ and $m = k := \lceil n/2 \rceil$.

Proof of Proposition 4.4 (continued)

By Lemma 4.17, we choose $d_{i,m+1} := a \cdot \widetilde{z}_{i,m+1}$ from (4.17) as a generic seed. This completes the proof. □

4.4. Proof of Theorem B

Finally, we are ready to prove the following main theorem.

THEOREM 4.18 (Theorem B)

Let $\lambda = \{\lambda_i := n - 2i + 1 \mid i = 1, \dots, n\}$ be an n-tuple of real numbers for an arbitrary integer $n \geq 4$. Consider the coadjoint orbit $\mathcal{O}_\lambda$, a complete flag manifold

$\mathrm{Fl}(n)$ equipped with the monotone KKS form ω_λ. Then each GC fiber $L_m(t)$ is nondisplaceable Lagrangian for every $2 \le m \le \lfloor \frac{n}{2} \rfloor$.

In particular, there exists a family of nondisplaceable nontorus Lagrangian fibers

$$\left\{ L_m(1) \mid 2 \le m \le \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{Z} \right\},$$

where $L_m(1)$ is diffeomorphic to $\mathrm{U}(m) \times T^{\frac{n(n-1)}{2} - m^2}$.

Proof

By Corollary 4.5, the SLT-equation (3.6) has a desired solution for some nonzero complex numbers $c_{i+1,i}^{\mathrm{ver},\mathbb{C}}$ and $c_{j,j+1}^{\mathrm{hor},\mathbb{C}}$ $(i, j \ge k)$. Theorem 3.6 ensures that for each Lagrangian torus $L_m(t)$ $(0 \le t < 1)$, there exists a suitable bulk deformation parameter $\mathfrak{b}$ of the form (2.8) so that $W^{\mathfrak{b}}$ admits a critical point. By Theorem 2.4, each GC torus fiber $L_m(t)$ for $0 \le t < 1$ is nondisplaceable. Furthermore, Lemma 2.10 implies that $L_m(1)$ is nondisplaceable. □

5. Calculation of potential function deformed by Schubert classes

Since the main steps of the derivation of (2.6) are the same as that of the proof of [14, Proposition 4.7] given in Section 7 therein, we will only explain modifications we need to make to apply them to the current GC case. Also, for the purpose of proving the counterpart of Theorem 2.3 in the present paper, the fact that a Fano manifold X_ϵ has a toric degeneration and that we only have to consider codimension 2 cycles also helps us simplify the study of holomorphic disks contributing to the potential functions. We closely follow [26, Section 9].

Let L be a Lagrangian submanifold in a symplectic manifold X. Let $\mathcal{M}_{k+1;\ell}(X, L; \beta)$ denote the moduli space of stable maps in the class $\beta \in \pi_2(X, L)$ from a bordered Riemann surface Σ of genus 0 with $(k+1)$ marked points $\{z_s\}_{s=0}^k$ on the boundary $\partial\Sigma$ respecting the counterclockwise orientation and ℓ marked points $\{z_r^+\}_{r=1}^\ell$ at the interior of Σ. It naturally comes with two types of evaluation maps, at the ith boundary marked point

$$\begin{aligned} &\mathrm{ev}_i \colon \mathcal{M}_{k+1;\ell}(X, L; \beta) \to L; \\ &\mathrm{ev}_i\big(\big[\varphi \colon \Sigma \to X, \{z_s\}_{s=0}^{k+1}, \{z_r^+\}_{r=1}^\ell\big]\big) = \varphi(z_i) \end{aligned} \tag{5.1}$$

and at the jth interior marked point

$$\begin{aligned} &\mathrm{ev}_j^{\mathrm{int}} \colon \mathcal{M}_{k+1;\ell}(X, L; \beta) \to X; \\ &\mathrm{ev}_j^{\mathrm{int}}\big(\big[\varphi \colon \Sigma \to X, \{z_s\}_{s=0}^{k+1}, \{z_r^+\}_{r=1}^\ell\big]\big) = \varphi(z_j^+). \end{aligned} \tag{5.2}$$

Set $\mathcal{M}_{k+1}(X, L; \beta) := \mathcal{M}_{k+1;\ell=0}(X, L; \beta)$, the moduli space without interior marked points, and let

$$\mathbf{ev}_+ := (\mathrm{ev}_1, \dots, \mathrm{ev}_k).$$

Recall that an A_∞-structure with the operators

$$\mathfrak{m}_k = \sum_{\beta} \mathfrak{m}_{k,\beta} \cdot T^{\omega(\beta)/2\pi}, \quad \mathfrak{m}_{k,\beta}(b_1, \ldots, b_k) := (\mathrm{ev}_0)!(\mathbf{ev}_+)^*(\pi_1^* b_1 \otimes \cdots \otimes \pi_k^* b_k)$$

on the de Rham complex $\Omega(L)$ is defined via a smooth correspondence

$$\begin{array}{ccc} & \mathcal{M}_{k+1}(X, L; \beta) & \\ {}^{\mathbf{ev}_+}\swarrow & & \searrow^{\mathrm{ev}_0} \\ L^k & & L \end{array} \tag{5.3}$$

where $\pi_i \colon L^k \to L$ denotes the projection to the ith copy of L. For a general symplectic manifold, one should choose a system of compatible Kuranishi structures and continuous family perturbations (CF-perturbations) on $\mathcal{M}_{k+1,l}(X, L; \beta)$'s in order to apply the above smooth correspondence (see [10], [18], [19] for details on construction). For a Lagrangian toric fiber L in a $2n$-dimensional toric manifold, by constructing a system of compatible T^n-equivariant Kuranishi structures and multisections, the smooth correspondence can be applied without adding an auxiliary space for perturbing multisections to make it submersive (see [14, Section 12]). This is because ev_0 is submersive by the T^n-equivariance.

We now recall the computation of the potential function of a torus fiber $L_\varepsilon \subset X_\varepsilon$ in [26]. There, the authors were able to exploit the presence of toric degeneration of X_ϵ to X_0 in their computation, the explanation of which is now in order. For the study of holomorphic disks in X_0, which is not smooth, they used the following notion in Nishinou and Siebert [27].

DEFINITION 5.1 ([27, Definition 4.1])

A holomorphic curve in a toric variety X is called *torically transverse* if it is disjoint from all toric strata of codimension greater than 1. A stable map $\varphi \colon \Sigma \to X$ is *torically transverse* if $\varphi(\Sigma) \subset X$ is torically transverse and $\varphi^{-1}(\mathrm{Int}\, X) \subset \Sigma$ is dense. Here, $\mathrm{Int}\, X$ is the complement of the toric divisors in X.

We denote by $S_0 := \mathrm{Sing}(X_0)$ the singular locus of X_0. Using the classification result in [4] of holomorphic disks attached to a Lagrangian toric fiber in a smooth toric manifold and the property of the small resolution, Nishinou, Nohara, and Ueda [26] proved the following.

LEMMA 5.2 ([26, Proposition 9.5, Lemma 9.15])

Any holomorphic disk $\varphi \colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (X_0, L_0)$ can be deformed into a holomorphic disk with the same boundary condition that is torically transverse. Furthermore the moduli space $\mathcal{M}_1(X_0, L_0; \beta)$ is empty if the Maslov index of β is less than 2.

LEMMA 5.3 ([26, Lemma 9.9])

There is a small neighborhood W_0 of the singular locus $S_0 \subset X_0$ such that no holomorphic disks of Maslov index 2 intersect W_0.

Now let $\phi'_\epsilon\colon X_\epsilon \to X_0$ be a (continuous) extension of the flow $\phi_\epsilon\colon X_\epsilon^{\mathrm{sm}} \to X_0^{\mathrm{sm}}$ given in Theorem 2.1 (see [26, Section 8]). The following is the key proposition which relates the above-mentioned holomorphic disks in (X_0, L_0) to those of (X_ϵ, L_ϵ).

PROPOSITION 5.4 ([26, Proposition 9.16])
For any $\beta \in \pi_2(X_0, L_0)$ of Maslov index 2, there are positive real numbers $0 < \varepsilon \le 1$ and a diffeomorphism

$$\psi\colon \mathcal{M}_1(X_0, L_0; \beta) \to \mathcal{M}_1(X_\varepsilon, L_\varepsilon; \beta)$$

such that the diagram

$$\begin{array}{ccc} H_*(\mathcal{M}_1(X_0, L_0; \beta)) & \xrightarrow{(\mathrm{ev}_0)_*} & H_*(L_0) \\ {\scriptstyle \psi_*}\downarrow & & \downarrow{\scriptstyle (\phi_\varepsilon)_*^{-1}} \\ H_*(\mathcal{M}_1(X_\varepsilon, L_\varepsilon; \beta)) & \xrightarrow{(\mathrm{ev}_0)_*} & H_*(L_\varepsilon) \end{array}$$

is commutative.

LEMMA 5.5 ([26, Lemma 9.22])
Let $W_\varepsilon := (\phi'_\varepsilon)^{-1}(W_0)$. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$, any holomorphic curve bounded by L_ε in a class of Maslov index 2 does not intersect W_ε.

We now combine the diffeomorphisms $\psi\colon \mathcal{M}_1(X_0, L_0; \beta) \to \mathcal{M}_1(X_\varepsilon, L_\varepsilon; \beta)$ and $\phi'_\varepsilon\colon X_0 \to X_\varepsilon$ to define an isomorphism between the following two correspondences:

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1}(X_0, L_0; \beta) & & \\ & \swarrow^{\mathbf{ev}_+} & & \searrow^{\mathrm{ev}_0} & \\ L_0^k & & & & L_0 \end{array} \qquad \text{and}$$

(5.4)

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1}(X_\varepsilon, L_\varepsilon; \beta) & & \\ & \swarrow^{\mathbf{ev}_+} & & \searrow^{\mathrm{ev}_0} & \\ L_\varepsilon^k & & & & L_\varepsilon \end{array}$$

Although they did not explicitly mention a choice of compatible systems of Kuranishi structures or perturbations, Nishinou, Nohara, and Ueda [26] essentially constructed an A_∞-structure on $L_\varepsilon \subset X_\epsilon$ and computed its potential function in the same way as on a Fano toric manifold (see [4], [13]) using Proposition 5.4 and Lemma 5.5. Thus, they were able to take advantage of properties of T^N-equivariant perturbations in a toric manifold, an open submanifold of a toric variety X_0 where $N = \dim_{\mathbb{C}} \mathrm{Fl}(n)$. We denote the corresponding compatible system of multisections by $\mathfrak{s} = \mathfrak{s}_{k+1,\beta}$ (see [13], [14] for the meaning of this notation).

Next we need to involve bulk deformations for our purpose of constructing a continuum of nondisplaceable Lagrangian tori in X, whose construction is now in order. Denote by $\mathscr{A}^2_{GS}(\mathbb{Z})$ the free abelian group generated by the horizontal and vertical Schubert classes of codimension 2:

$$\{\mathscr{D}^{\mathrm{hor}}_{i,i+1} \mid 1 \le i \le n-1\} \cup \{\mathscr{D}^{\mathrm{ver}}_{j+1,j} \mid 1 \le j \le n-1\}. \tag{5.5}$$

Since $L \cap \mathscr{D}^{\mathrm{hor}}_{i,i+1} = \emptyset = L \cap \mathscr{D}^{\mathrm{ver}}_{j+1,j}$ for any i, j, the cap product of $\beta \in \pi_2(X, L)$ with any element thereof is well defined. Putting $\mathscr{A}^2_{GS}(\Lambda_0) := \mathscr{A}^2_{GS}(\mathbb{Z}) \otimes \Lambda_0$, any element $\mathfrak{b} \in \mathscr{A}^2_{GS}(\Lambda_0)$ can be expressed as

$$\mathfrak{b} = \sum_{i=1}^{n-1} \mathfrak{b}^{\mathrm{hor}}_{i,i+1} \mathscr{D}^{\mathrm{hor}}_{i,i+1} + \sum_{j=1}^{n-1} \mathfrak{b}^{\mathrm{ver}}_{j+1,j} \mathscr{D}^{\mathrm{ver}}_{j+1,j}, \tag{5.6}$$

where $\mathfrak{b}^{\mathrm{hor}}_{i,i+1}, \mathfrak{b}^{\mathrm{ver}}_{j+1,j} \in \Lambda_0$. We formally denote

$$\beta \cap \mathfrak{b} = \sum_{i=1}^{n-1} \mathfrak{b}^{\mathrm{hor}}_{i,i+1} (\beta \cap \mathscr{D}^{\mathrm{hor}}_{i,i+1}) + \sum_{j=1}^{n-1} \mathfrak{b}^{\mathrm{ver}}_{j+1,j} (\beta \cap \mathscr{D}^{\mathrm{ver}}_{j+1,j}). \tag{5.7}$$

For simplicity, let us fix an enumeration $\{\mathscr{D}_j \mid j = 1, \dots, B\}$ of the elements in (5.5), where $B = 2(n-2)$, and set $\mathfrak{b}_j$ to be the coefficient corresponding to $\mathscr{D}_j$ in (5.6).

The following is the statement of the counterpart of Theorem 2.3.

THEOREM 5.6

Let $\mathfrak{b} \in \mathscr{A}^2_{GS}(\Lambda_0)$, and let L_ε be a torus Lagrangian fiber in X_ε. Then the bulk-deformed potential function is written as

$$W^{\mathfrak{b}}(L_\varepsilon; b) = \sum_\beta n_\beta \cdot \exp(\beta \cap \mathfrak{b}) \cdot \exp(\partial\beta \cap b) T^{\omega(\beta)/2\pi}, \tag{5.8}$$

where the summation is taken over all homotopy classes in $\pi_2(X_\varepsilon, L_\varepsilon)$ of Maslov index 2.

The remaining part of this section is reserved for the proof of Theorem 5.6.

For a Lagrangian submanifold L of X, denote by

$$\mathrm{ev}^{\mathrm{int}}_i \colon \mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L, \beta) \to X$$

the evaluation map at the ith interior marked point for $i = 1, \dots, \ell$. We put $\underline{B} = \{1, \dots, B\}$ and denote the set of all maps $\mathbf{p} \colon \{1, \dots, \ell\} \to \underline{B}$ by $\mathrm{Map}(\ell, \underline{B})$. We write $|\mathbf{p}| = \ell$ if $\mathbf{p} \in \mathrm{Map}(\ell, \underline{B})$. We define a fiber product

$$\mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L, \beta; \mathbf{p}) = \mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L, \beta)_{(\mathrm{ev}^{\mathrm{int}}_1, \dots, \mathrm{ev}^{\mathrm{int}}_\ell)} \times_{X^\ell} \prod_{i=1}^{\ell} \mathscr{D}_{\mathbf{p}(i)} \tag{5.9}$$

and consider the evaluation maps

$$\mathrm{ev}_i : \mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L,\beta;\mathbf{p}) \to L \text{ by } \mathrm{ev}_i\big((\Sigma,\varphi,\{z_i^+\},\{z_i\})\big) = \varphi(z_i).$$

Note that the image of a Schubert horizontal or vertical class is the (union of) components of the toric divisor via the map ϕ'_ε so that $\mathscr{D}_j$ can be regarded as the (union of) components of toric divisors corresponding to Schubert horizontal or vertical classes. Also, by Proposition 5.4 and Lemma 5.5, the holomorphic disks of Maslov index 2 intersect at the smooth locus of cycles. For a toric fiber L_0 in X_0, there is a system $\mathfrak{s} = \{\mathfrak{s}_{k+1,\beta;\mathbf{p}}\}$ of T^N-equivariant multisections on the moduli spaces $\mathcal{M}_{k+1,l}(X_0, L_0;\beta;\mathbf{p})$ for all classes β with $\mu(\beta) = 2$ in [14, Lemma 6.5], where $N = \dim_{\mathbb{C}} \mathrm{Fl}(n)$.

As in the correspondence in (5.4), applying a smooth correspondence into

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1,l}(X_\varepsilon, L_\varepsilon;\beta;\mathbf{p}) & & \\ & \swarrow^{\mathbf{ev}_+} & & \searrow^{\mathrm{ev}_0} & \\ L^k & & & & L \end{array} \tag{5.10}$$

we define

$$\mathfrak{q}_{k,\ell;\beta}(\mathbf{p}; b^{\otimes k}) := (\mathrm{ev}_0)!\big(\mathbf{ev}_+^*(\pi_1^* b \otimes \cdots \otimes \pi_k^* b)\big).$$

Since X_ε is Fano, Lemma 5.2 and $L \cap \mathscr{D}^{\mathrm{hor}}_{i,i+1} = \emptyset = L \cap \mathscr{D}^{\mathrm{ver}}_{j+1,j}$ yield that $\mathcal{M}_{k+1;\ell}(X_\varepsilon, L_\varepsilon,\beta;\mathbf{p})$ is empty if one of the following is satisfied:

$$\begin{aligned} &(1) \quad \mu(\beta) < 0, \qquad (2) \quad \mu(\beta) = 0 \quad \text{and} \quad \beta \neq 0, \\ &(3) \quad \beta = 0 \quad \text{and} \quad l > 0. \end{aligned} \tag{5.11}$$

Because of the compatibility of the forgetful map forgetting the boundary marked points (see [10, Section 5]),

$$\mathfrak{q}_{k,\ell;\beta}(\mathbf{p}; b^{\otimes k}) = \frac{1}{k!}(\partial\beta \cap b)^k \cdot \mathfrak{q}_{0,\ell;\beta}(\mathbf{p};1). \tag{5.12}$$

Since any moduli spaces satisfying one of the conditions in (5.11) are empty, $\mathfrak{q}_{0,\ell;\beta}(\mathbf{p};1)$ represents a cycle, yielding that L_ε is weakly unobstructed with respect to $\mathfrak{b}$ in (5.6). Passing to the canonical model (see [12], [11]), we obtain

$$\mathfrak{q}_{0,\ell;\beta}(\mathbf{p};1) = n_\beta(\mathbf{p}) \cdot \mathrm{PD}[L] \tag{5.13}$$

for some $n_\beta(\mathbf{p}) \in \mathbb{Q}$. As a consequence, we obtain that every 1-cochain is a weak bounding cochain with respect to $\mathfrak{b}$. In particular, the potential function with bulk is defined on $H^1(L_\varepsilon;\Lambda_+)$.

Under our situation, $n_\beta(\mathbf{p})$ is well defined. Especially when $\dim \mathscr{D}_\bullet = 2n-2$, $\mu(\beta) = 2$, we recall that this is precisely the situation where the divisor axiom of the Gromov–Witten theory applies (see [7, p. 193], [14, Lemma 9.2]). In particular, we can calculate $n_\beta(\mathbf{p})$ in the homology level and therefore

$$n_\beta(\mathbf{p}) = n_\beta \cdot \prod_{i=1}^{|\mathbf{p}|} (\beta \cap \mathscr{D}_{\mathbf{p}(i)}). \tag{5.14}$$

Following [14, Section 7] and using (5.12), (5.13), (5.14), and (5.7), we obtain that

$$\begin{aligned}
\sum_{k=0}^{\infty}\mathfrak{m}_k^{\mathfrak{b}}(b^{\otimes k}) &:= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{\beta;\mu(\beta)=2}\frac{1}{\ell!}\mathfrak{q}_{k,\ell;\beta}(\mathfrak{b}^{\otimes \ell};b^{\otimes k})T^{\omega(\beta)/2\pi} \\
&= \sum_{\ell=0}^{\infty}\sum_{\mathbf{p};|\mathrm{p}|=\ell}\sum_{\beta;\mu(\beta)=2}\exp(\partial\beta\cap b)\cdot\frac{1}{\ell!}\mathfrak{b}^{\mathbf{p}}\mathfrak{q}_{0,\ell;\beta}(\mathbf{p};1)T^{\omega(\beta)/2\pi} \\
&= \sum_{\beta;\mu(\beta)=2}\Big(\sum_{\ell=0}^{\infty}\sum_{\mathbf{p};|\mathrm{p}|=\ell}\exp(\partial\beta\cap b)\cdot\frac{1}{\ell!}\mathfrak{b}^{\mathbf{p}}n_\beta(\mathbf{p})\Big)T^{\omega(\beta)/2\pi}\cdot PD[L] \\
&= \sum_{\beta;\mu(\beta)=2} n_\beta\cdot\Big(\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{\mathbf{p};|\mathrm{p}|=\ell}\prod_{i=1}^{|\mathbf{p}|}\mathfrak{b}_{\mathbf{p}(i)}(\beta\cap\mathscr{D}_{\mathbf{p}(i)})\Big) \\
&\qquad\times\exp(\partial\beta\cap b)\cdot T^{\omega(\beta)/2\pi}\cdot PD[L] \\
&= \sum_{\beta;\mu(\beta)=2} n_\beta\cdot\exp(\beta\cap\mathfrak{b})\cdot\exp(\partial\beta\cap b)\cdot T^{\omega(\beta)/2\pi}\cdot PD[L],
\end{aligned}$$

where $\mathfrak{b}^{\mathbf{p}} = \prod_{i=1}^{\ell}\mathfrak{b}_{\mathbf{p}(i)}$. Finally, incorporating this with the deformation of nonunitary flat line bundles by Cho [3], we extend the domain of the bulk-deformed potential to $H^1(L_\varepsilon;\Lambda_0)$. This completes the proof of Theorem 5.6.

Acknowledgments. The authors would like to thank for the referees for valuable comments.

Cho's work was partially supported by National Research Foundation of Korea (NRF) grant NRF-2020R1C1C1A01010972 funded by the Korean Ministry of Science, ICT and Future Planning (MSIP). Kim's work was supported by the Simons Collaboration grant on "Homological mirror symmetry and applications." Oh's work was supported by Institute for Basic Science (IBS) project IBS-R003-D1.

References

[1] V. V. Batyrev, I. Ciocan-Fontanine, B. Kim, and D. van Straten, *Mirror symmetry and toric degenerations of partial flag manifolds*, Acta Math. **184** (2000), no. 1, 1–39. MR 1756568. DOI 10.1007/BF02392780.

[2] D. N. Bernstein, *The number of roots of a system of equations* (in Russian), Funktsional. Anal. i Prilozhen **9** (1975), no. 3, 1–4; English translation in Functional Anal. Appl. **9** (1975), no. 3, 183–185 MR 0435072.

[3] C.-H. Cho, *Non-displaceable Lagrangian submanifolds and Floer cohomology with non-unitary line bundle*, J. Geom. Phys. **58** (2008), no. 11, 1465–1476. MR 2463805. DOI 10.1016/j.geomphys.2008.06.003.

[4] C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), no. 4, 773–814. MR 2282365. DOI 10.4310/AJM.2006.v10.n4.a10.

[5] Y. Cho and Y. Kim, *Monotone Lagrangians in flag varieties*, Int. Math. Res. Not. IMRN, published electronically 1 November 2019. DOI 10.1093/imrn/rnz227.

[6] Y. Cho, Y. Kim, and Y.-G. Oh, *Lagrangian fibers of Gelfand–Cetlin systems*, Adv. Math. **372** (2020), art. ID 107304. MR 4126721. DOI 10.1016/j.aim.2020.107304.

[7] D. A. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, Math. Surveys Monogr. **68**, Amer. Math. Soc., Providence, 1999. MR 1677117. DOI 10.1090/surv/068.

[8] J. D. Evans and Y. Lekili, *Floer cohomology of the Chiang Lagrangian*, Selecta Math. (N.S.) **21** (2015), no. 4, 1361–1404. MR 3397452. DOI 10.1007/s00029-014-0171-9.

[9] ———, *Generating the Fukaya categories of Hamiltonian G-manifolds*, J. Amer. Math. Soc. **32** (2019), no. 1, 119–162. MR 3868001. DOI 10.1090/jams/909.

[10] K. Fukaya, *Cyclic symmetry and adic convergence in Lagrangian Floer theory*, Kyoto J. Math. **50** (2010), no. 3, 521–590. MR 2723862. DOI 10.1215/0023608X-2010-004.

[11] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, "Canonical models of filtered A_∞-algebras and Morse complexes" in *New Perspectives and Challenges in Symplectic Field Theory*, CRM Proc. Lecture Notes **49**, Amer. Math. Soc., Providence, 2009, 201–227. MR 2555938. DOI 10.1090/crmp/049/07.

[12] ———, *Lagrangian Intersection Floer Theory: Anomaly and Obstruction, Part I*, AMS/IP Stud. Adv. Math. **46.1**, Amer. Math. Soc., Providence, 2009. MR 2553465. *Part II*, MR 2548482.

[13] ———, *Lagrangian Floer theory on compact toric manifolds, I*, Duke Math. J. **151** (2010), no. 1, 23–174. MR 2573826. DOI 10.1215/00127094-2009-062.

[14] ———, *Lagrangian Floer theory on compact toric manifolds, II: Bulk deformations*, Selecta Math. (N.S.) **17** (2011), no. 3, 609–711. MR 2827178. DOI 10.1007/s00029-011-0057-z.

[15] ———, *Toric degeneration and nondisplaceable Lagrangian tori in* $S^2 \times S^2$, Int. Math. Res. Not. IMRN **2012**, no. 13, 2942–2993. MR 2946229. DOI 10.1093/imrn/rnr128.

[16] ———, *Lagrangian Floer Theory and Mirror Symmetry on Compact Toric Manifolds*, Astérisque **376**, Soc. Math. France, Paris, 2016. MR 3460884.

[17] ______, *Spectral invariants with bulk, quasi-morphisms and Lagrangian Floer theory*, Mem. Amer. Math. Soc. **260** (2019), no. 1254. MR 3986938. DOI 10.1090/memo/1254.

[18] ______, *Kuranishi structure, pseudo-holomorphic curve, and virtual fundamental chain, I*, preprint, arXiv:1503.07631v1 [math.SG].

[19] ______, *Kuranishi structure, pseudo-holomorphic curve, and virtual fundamental chain, II*, preprint, arXiv:1704.01848v1 [math.SG].

[20] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transform. Groups **1** (1996), no. 3, 215–248. MR 1417711. DOI 10.1007/BF02549207.

[21] V. Guillemin and S. Sternberg, *The Gel'fand–Cetlin system and quantization of the complex flag manifolds*, J. Funct. Anal. **52** (1983), no. 1, 106–128. MR 0705993. DOI 10.1016/0022-1236(83)90092-7.

[22] Y. Kim, J. Lee, and F. Sanda, *Detecting non-displaceable toric fibers on compact toric manifolds via tropicalizations*, Internat. J. Math. **30** (2019), no. 1, art. ID 1950003. MR 3916270. DOI 10.1142/S0129167X19500034.

[23] M. Kogan, *Schubert geometry of flag varieties and Gelfand–Cetlin theory*, Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA, 2000. MR 2716977.

[24] M. Kogan and E. Miller, *Toric degeneration of Schubert varieties and Gelfand–Tsetlin polytopes*, Adv. Math. **193** (2005), no. 1, 1–17. MR 2132758. DOI 10.1016/j.aim.2004.03.017.

[25] A. G. Kushnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), no. 1, 1–31. MR 0419433. DOI 10.1007/BF01389769.

[26] T. Nishinou, Y. Nohara, and K. Ueda, *Toric degenerations of Gelfand–Cetlin systems and potential functions*, Adv. Math. **224** (2010), no. 2, 648–706. MR 2609019. DOI 10.1016/j.aim.2009.12.012.

[27] T. Nishinou and B. Siebert, *Toric degenerations of toric varieties and tropical curves*, Duke Math. J. **135** (2006), no. 1, 1–51. MR 2259922. DOI 10.1215/S0012-7094-06-13511-1.

[28] Y. Nohara and K. Ueda, *Floer cohomologies of non-torus fibers of the Gelfand–Cetlin system*, J. Symplectic Geom. **14** (2016), no. 4, 1251–1293. MR 3601889. DOI 10.4310/JSG.2016.v14.n4.a9.

[29] M. Pabiniak, *Displacing (Lagrangian) submanifolds in the manifolds of full flags*, Adv. Geom. **15** (2015), no. 1, 101–108. MR 3300713. DOI 10.1515/advgeom-2014-0025.

[30] Y. Sun, *A_n-type surface singularity and nondisplaceable Lagrangian tori*, Internat. J. Math. **31** (2020), no. 3, art. ID 2050020. MR 4083244. DOI 10.1142/S0129167X20500202.

[31] R. Vianna, *Continuum families of non-displaceable Lagrangian tori in* $(\mathbb{C}P^1)^{2m}$, J. Symplectic Geom. **16** (2018), no. 3, 857–883. MR 3882175. DOI 10.4310/JSG.2018.v16.n3.a8.

[32] C. T. Woodward, *Gauged Floer theory of toric moment fibers*, Geom. Funct. Anal. **21** (2011), no. 3, 680–749. MR 2810861. DOI 10.1007/s00039-011-0119-6.

Cho: Department of Mathematics Education, Sungkyunkwan University, Seoul, Republic of Korea; yunhyung@skku.edu

Kim: Department of Mathematics, Brandeis University, Waltham, Massachusetts, USA; Center of Mathematical Sciences and Applications, Harvard University, Cambridge, Massachusetts, USA; yoosik@brandeis.edu; yoosik@cmsa.fas.harvard.edu

Oh: Center for Geometry and Physics, Institute for Basic Science, Pohang, Republic of Korea; Department of Mathematics, Pohang University of Science and Technology, Pohang, Republic of Korea; yongoh1@postech.ac.kr