Smooth Kuranishi structure of the space of Morse trajectories

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Abstract Our aim here is to explain a new technique for the construction of a smooth Kuranishi structure, which we used in our previous article about the construction of symplectic field theory. We explain this technique in the case of a Morse chain complex.

1. Introduction

In [5], we used various new techniques for the construction of symplectic field theory, one of which was about the smoothness of Kuranishi structures. There we constructed the Kuranishi structures of the spaces of pseudoholomorphic curves. A Kuranishi structure consists of a Kuranishi neighborhood of each point (counterpart of local chart) and the embeddings between them (counterpart of coordinate change), and we need smooth Kuranishi neighborhoods with smooth embeddings. If we restrict to each stratum consisting of the curves of the same diffeomorphism type, then it is easy to construct its natural smooth structure. This is because it can be locally represented as the zero set of a Fredholm map between a pair of Banach spaces. (For example, if this Fredholm map is transverse to zero, then the zero set is a smooth manifold.) However, we cannot use the same pair of Banach spaces for the curves of different diffeomorphism types. Hence we need to define the smooth structure artificially and prove the smoothness of the embeddings.

Originally, the construction of smooth Kuranishi structures was treated by Fukaya, Oh, Ohta, and Ono briefly in [1], and later in more detail in [2]. The key to the proof of smoothness is the following easy fact: if a continuous function f on $\mathbb{R}$ is continuously differentiable on the complement of a point and the differential has a limit at this point, then f is continuously differentiable on the whole of $\mathbb{R}$ (see also Lemma 6.6). Hence for the proof of the smoothness of the embeddings, it is enough to obtain appropriate estimates of the differentials on a neighborhood of the boundary of each stratum. They estimated the differentials by estimating approximate solutions appearing in Newton's method. In [5], we also use the same argument, but we use another way to estimate the differentials

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on neighborhoods of the strata boundaries. As we noted, we can represent each stratum as the zero set of a Fredholm map between Banach spaces if we fix a family of diffeomorphisms of the domain curves. We estimate the differentials of this Fredholm map, and we use these estimates to obtain the estimates of the differentials of the family of solutions. Although this technique is simple, [5] may be difficult for readers unfamiliar with pseudoholomorphic curves or Kuranishi theory. The aim of this article is to explain this technique in the case of a Morse chain complex. Although some arguments and calculations are almost the same as those of [5], we repeat them here for the reader's convenience.

Let f be a Morse function on a closed manifold M, and let g be an arbitrary Riemannian metric of M. We denote the set of critical points and the set of critical values of f by $P_f \subset M$ and $\operatorname{CriV} f \subset \mathbb{R}$, respectively. Define $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. For each $m \geq 0$, let $\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m$ be the space obtained by identifying $+\infty \in \overline{\mathbb{R}}_i$ $(= \overline{\mathbb{R}})$ with $-\infty \in \overline{\mathbb{R}}_{i+1}$ for all $1 \leq i \leq m-1$. We say that a map $\theta : \overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m \to \overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m$ is a translation map if $\theta(\overline{\mathbb{R}}_i) = \overline{\mathbb{R}}_i$ and $\theta|_{\overline{\mathbb{R}}_i}(t) = t + c_i$ for some $c_i \in \mathbb{R}$. Let $\gamma : \overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m \to M$ be a continuous map. We call a pair $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, \gamma)$ a *stable trajectory* (without marked points) if

- $\gamma|_{\mathbb{R}_i} : \mathbb{R}_i \to M$ satisfies the equation
$$\frac{d\gamma}{dt} + \nabla f(\gamma) = 0$$
for each i, and
- each $\gamma|_{\mathbb{R}_i}$ is not a constant map.

Two stable trajectories $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, \gamma)$ and $(\overline{\mathbb{R}}_1 \sqcup \cdots \sqcup \overline{\mathbb{R}}_{m'}, \gamma')$ are the same if $m = m'$ and there exists a translation map θ such that $\gamma = \gamma' \circ \theta$. Let $q_\pm \in P_f$ be critical points of f. We say that a stable trajectory $p = (\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, \gamma)$ is from q_- to q_+ if γ takes q_- and q_+ at $-\infty_1 \in \overline{\mathbb{R}}_1$ and $+\infty_m \in \overline{\mathbb{R}}_m$, respectively. More generally, we define its history $P_p = (q_0 = q_-, q_1, \ldots, q_m = q_+)$ by $q_i = \gamma(+\infty_i)$. (For $i = 0$, we read $+\infty_0$ as $-\infty_1$.)

Let $\overline{\mathcal{M}}_0(q_-, q_+)$ be the space of stable trajectories from q_- to q_+. We define its topology and prove its topological properties in Section 3. For any sequence $P_0 = (q_0 = q_-, q_1, \ldots, q_m = q_+)$ such that $f(q_0) > f(q_1) > \cdots > f(q_m)$, let $\overline{\mathcal{M}}_0(q_-, q_+)^{P_0} \subset \overline{\mathcal{M}}_0(q_-, q_+)$ be the subspace of stable trajectories p whose history contains P_0 as a subsequence. Then the decomposition of the stable trajectories into pieces defines a homeomorphism

$$\overline{\mathcal{M}}_0(q_-, q_+)^{P_0} \cong \overline{\mathcal{M}}_0(q_-, q_1) \times \overline{\mathcal{M}}_0(q_1, q_2) \times \cdots \times \overline{\mathcal{M}}_0(q_{m-1}, q_+).$$

More generally, for any decreasing sequence $\widehat{P}_0 \supset P_0$,

$$\overline{\mathcal{M}}_0(q_-, q_+)^{\widehat{P}_0} \cong \overline{\mathcal{M}}_0(q_-, q_1)^{\widehat{P}_1} \times \overline{\mathcal{M}}_0(q_1, q_2)^{\widehat{P}_2} \times \cdots \times \overline{\mathcal{M}}_0(q_{m-1}, q_+)^{\widehat{P}_m},$$

where $\widehat{P}_i$ is a sequence defined by $\widehat{P}_i = \{q \in \widehat{P}_0; f(q_{i-1}) \geq f(q) \geq f(q_i)\}$. In this article, we construct global Kuranishi neighborhoods of $\overline{\mathcal{M}}_0(q_-, q_+)$ compatible with these relations.

DEFINITION 1.1

For a compact Hausdorff space X, a *global Kuranishi neighborhood* (V, E, s, ψ) of X (with trivial automorphism group) is a 4-tuple of a manifold V with boundary and corners, a finite vector space E, a smooth function $s : V \to E$, and a homeomorphism $\psi : s^{-1}(0) \cong X$. We say that (V, E, s, ψ) is of class C^N if V is a smooth manifold of class C^N and s is of class C^N.

In the above definition, a manifold V with boundary and corners is a Hausdorff space with a smooth structure defined by a covering of open subsets $\{\mathcal{U}_\alpha\}$ and homeomorphisms $\varphi_\alpha : \mathcal{U}_\alpha \to [0,\infty)^n \subset \mathbb{R}^n$ onto open subsets of $[0,\infty)^n$ (for some $n \in \mathbb{N}$) whose coordinate changes $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ are smooth. In the present article, $\partial^k V \subset V$ is the subset of V defined by $\partial^k V = \bigcup \varphi_\alpha^{-1}(\{(x_i) \in [0,\infty)^n ; \#\{i; x_i = 0\} \geq k\})$.

We prove this proposition.

PROPOSITION 1.2

For any $N \geq 1$, each $\overline{\mathcal{M}}_0(q_-, q_+)$ has a global Kuranishi neighborhood $(V_{q_-,q_+}, E_{q_-,q_+}, s_{q_-,q_+}, \psi_{q_-,q_+})$ of class C^N. Furthermore, the following hold.

- *For each point $x \in V_{q_-,q_+}$, its history P_x is defined, and $P_x = P_{\psi_{q_-,q_+}(x)}$ for each $x \in s_{q_-,q_+}^{-1}(0)$. For each decreasing sequence $P_0 = (q_-, q_1, \ldots, q_+)$, $V_{q_-,q_+}^{P_0} = \{x \in V_{q_-,q_+} ; P_x \supset P_0\}$ is a codimension $(\#P_0 - 2)$ submanifold of V_{q_-,q_+} contained in $\partial^{\#P_0 - 2} V_{q_-,q_+}$.*
- *For each decreasing sequence $P_0 = (q_0 = q_-, q_1, \ldots, q_m = q_+)$, E_{q_-,q_+} is the direct sum of $E_{q_i,q_{i+1}}$, and there exists an open embedding*

$$\phi_{P_0} : V_{q_-,q_+}^{P_0} \to V_{q_-,q_1} \times V_{q_1,q_2} \times \cdots \times V_{q_{m-1},q_+}$$

(of class C^N) such that $(s_{q_-,q_1} \oplus s_{q_1,q_2} \oplus \cdots \oplus s_{q_{m-1},q_+}) \circ \phi_{P_0} = s_{q_-,q_+}$ and $(\psi_{q_-,q_1} \times \psi_{q_1,q_2} \times \cdots \times \psi_{q_{m-1},q_+}) \circ \phi_{P_0} = \psi_{q_-,q_+}$ on $V_{q_-,q_+}^{P_0} \cap s_{q_-,q_+}^{-1}(0)$.
- *For any decreasing sequence $\widehat{P}_0 \supset P_0$, the image of $V_{q_-,q_+}^{\widehat{P}_0}$ by ϕ_{P_0} is contained in $V_{q_-,q_1}^{\widehat{P}_1} \times V_{q_1,q_2}^{\widehat{P}_2} \times \cdots \times V_{q_{m-1},q_+}^{\widehat{P}_m}$, where $\widehat{P}_i = \{q \in \widehat{P}_0 ; f(q_{i-1}) \geq f(q) \geq f(q_i)\}$. Furthermore, $\phi_{\widehat{P}_0} = (\phi_{\widehat{P}_1} \times \phi_{\widehat{P}_2} \times \cdots \times \phi_{\widehat{P}_m}) \circ \phi_{P_0}$ on $V_{q_-,q_+}^{\widehat{P}_0}$.*

REMARK 1.3

Using the above proposition, we can construct a Morse chain complex of M as follows. By induction on $f(q_-) - f(q_+)$, we can perturb the functions s_{q_-,q_+} to (single-valued) functions $s_{q_-,q_+}^\epsilon = s_{q_-,q_+} + \epsilon_{q_-,q_+} : V_{q_-,q_+} \to E_{q_-,q_+}$ transverse to $0 \subset E_{q_-,q_+}$ which satisfy the following condition: for each decreasing sequence $P_0 = (q_0 = q_-, q_1, \ldots, q_m = q_+)$, $(\epsilon_{q_-,q_1} \oplus \epsilon_{q_1,q_2} \oplus \cdots \oplus \epsilon_{q_{m-1},q_+}) \circ \phi_{P_0} = \epsilon_{q_-,q_+}$ on $V_{q_-,q_+}^{P_0}$. We assume that ϵ_{q_-,q_+} are sufficiently small, which ensures that the zero sets of $s_{q_-,q_+} + \epsilon_{q_-,q_+}$ are contained in some compact neighborhoods of $s_{q_-,q_+}^{-1}(0) \subset V_{q_-,q_+}$. Then we can define the Morse chain complex by using the numbers $\#(s_{q_-,q_+} + \epsilon_{q_-,q_+})^{-1}(0) \in \mathbb{Z}_2$ (or $\mathbb{Z}$) instead of the numbers of Morse trajectories $\#\overline{\mathcal{M}}_0(q_-, q_+) = \#s_{q_-,q_+}^{-1}(0) \in \mathbb{Z}_2$ (or $\mathbb{Z}$) in the case of Morse–Smale type. (For the counting over $\mathbb{Z}$, we need to define the orientations of the Kuranishi neighborhoods; see [1], [3], or [5] for the orientation of Kuranishi structures.) More precisely, the chain complex (C_*, ∂_*) is defined by $C_* = \bigoplus_{q \in P_f} \mathbb{Z}_2 q$ and $\partial q_- = \sum_{q_+ \in P_f, \dim \overline{\mathcal{M}}_0(q_-, q_+) = 0} \# \overline{\mathcal{M}}_0(q_-, q_+) q_+$. It satisfies $\partial \circ \partial = 0$, and furthermore we can prove that its homology is isomorphic to the usual homology of M by further arguments.

REMARK 1.4

Actually, the above proposition holds for $C^N = C^\infty$. However, for the generalizations such as the construction of global Kuranishi neighborhoods of the space of marked stable trajectories (or Kuranishi spaces for symplectic field theory), we can only construct a compatible family of C^N-structures for arbitrary fixed $N > 0$ (see Section 10).

REMARK 1.5

In the case of a general moduli space such as the space of stable maps or the space of holomorphic buildings, we cannot construct a global Kuranishi neighborhood. In these cases, we cover the moduli space by Kuranishi neighborhoods the dimension of whose base space V and the rank of whose vector space E vary, and we relate them by embeddings. This is called *Kuranishi structure*.

We mention that in [6], Wehrheim also constructed a smooth structure of the space of Morse trajectories in the case of Morse–Smale type. This is the case where we do not need Kuranishi structure and $\overline{\mathcal{M}}_0(q_-, q_+)$ itself becomes a smooth manifold. (Namely, $E_{q_-, q_+} = 0$ in this case.) We also mention that Hofer, Wysocki, and Zehnder [4] also treated the smoothness problem as sc-smoothness in their polyfold theory.

Our work here is organized as follows. In Section 2 and 3, we define the space of the Morse trajectories and that of their domain curves. In Section 4, we explain the construction of the Kuranishi neighborhoods of the space of the trajectories, and in Section 5, we prove the linearized gluing lemma used in the construction. In Section 6, we explain the meaning of the smoothness of Kuranishi neighborhoods and we give its proof. In Section 7, we prove the smoothness of the coordinate change. In Section 8, we study the structure of the boundary and the corners of the space. Finally, in Section 9, patching the Kuranishi neighborhoods, we construct the global Kuranishi neighborhood of the space of the trajectories.

2. Space of marked stable lines

In this section, we explain the space of the domains of Morse trajectories. Let $t = (t_1, t_2, \dots, t_\mu)$ be a sequence of points in $\mathbb{R}_1 \sqcup \cdots \sqcup \mathbb{R}_m$. We call a pair $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t)$ a (μ-marked) *semistable line*. We do not assume that the marked points t_j are different points. We call m the *length* of $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t)$. We consider two semistable lines $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t)$ and $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_{m'}, t')$ identical if $m = m'$ and there exists a translation map θ such that $\theta(t_j) = t'_j$ for all $1 \le j \le \mu$.

Similarly, we define the automorphism group $\mathrm{Aut}(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t)$ of a semistable line $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t)$ by the group of translation maps which preserve the marked points. We consider $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t)$ to be stable if its automorphism group is trivial. This is equivalent to the condition that every $\mathbb{R}_i$ contains at least one marked point.

Let $\overline{\mathcal{M}}_{\mu}^{\mathrm{SL}}$ be the space of μ-marked stable lines. It has the structure of a manifold with corners, as follows. (We will use a stronger differential structure for the construction of smooth Kuranishi structures in Section 6.) For a stable line $p = (\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t^0 = (t_j^0)) \in \overline{\mathcal{M}}_{\mu}^{\mathrm{SL}}$, let $\mathcal{U}_{t^0}$ be a smooth slice in a small neighborhood of $(t_j^0) \in \prod_{j=1}^{\mu}(\mathbb{R}_1 \sqcup \cdots \sqcup \mathbb{R}_m)$ with respect to the translation action (the diagonal action of $\mathrm{Aut}(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m)$, that is). We always assume that $\mathcal{U}_{t^0}$ contains t^0. Fix constants $T_i^- < T_i^+$ such that the union of $(T_i^-, T_i^+) \subset \overline{\mathbb{R}}_i$ contains all marked points. Define $\mathring{I}_i = [T_i^-, T_i^+]$. We regard $[-\infty, T_i^+]$ and $[T_i^+, +\infty]$ in $\overline{\mathbb{R}}_i$ as $[-\infty, 0]$ and $[0, \infty]$ by translations, respectively, and we regard $\overline{\mathbb{R}}_i$ as $\overline{\mathbb{R}}_i = [-\infty, 0]_i \cup \mathring{I}_i \cup [0, +\infty]_i$.

For a sequence $\rho = (\rho_1, \ldots, \rho_{m-1}) \in [0,1)^{m-1}$, we replace $[0, +\infty]_i \cup [-\infty, 0]_{i+1}$ with a shorter interval $[0, -\frac{1}{2}\log\rho_i] \cup [\frac{1}{2}\log\rho_i, 0]$ for each i, and we define a semistable line ℓ_ρ by

$$\begin{aligned} \ell_\rho = {}& [-\infty, 0]_1 \cup \mathring{I}_1 \cup \Big[0, -\frac{1}{2}\log\rho_1\Big]_1 \\ & \cup \bigcup_i \Big(\Big[\frac{1}{2}\log\rho_{i-1}, 0\Big]_i \cup \mathring{I}_i \cup \Big[0, -\frac{1}{2}\log\rho_i\Big]_i \Big) \\ & \cup \Big[\frac{1}{2}\log\rho_{m-1}, 0\Big]_m \cup \mathring{I}_m \cup [0, +\infty]_m. \end{aligned} \tag{1}$$

We define a coordinate $[0,1)^{m-1} \times \mathcal{U}_{t^0} \to \overline{\mathcal{M}}_{\mu}^{\mathrm{SL}}$ of a neighborhood of each p by $(\rho, t) \mapsto (\ell_\rho, t)$. It is easy to check that these coordinates define the topology and the manifold structure of $\overline{\mathcal{M}}_{\mu}^{\mathrm{SL}}$. It is also easy to check that $\overline{\mathcal{M}}_{\mu}^{\mathrm{SL}}$ is compact.

REMARK 2.1

For a marked semistable curve $(\Sigma, z = (z_j))$, we usually assume that the marked points z_j are disjoint, and if two marked points converge to the same point, then the limit curve contains a new irreducible component on which there are marked points corresponding to these marked points. This works well because holomorphic structure has scale invariance. (For example, constant multiplications in $\mathbb{C}$ preserve the complex structure.) In contrast, for a semistable line, we need length and direction instead of holomorphic structure, and length does not have such an invariance. Hence we do not assume that marked points are disjoint for a semistable line.

3. Space of stable trajectories

In this section, we define the space of marked stable trajectories, and we prove its topological properties. Let $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t)$ be a semistable line, and let

$\gamma : \overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m \to M$ be a continuous map. We say that $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t, \gamma)$ is a (marked) stable trajectory if it satisfies the following conditions:

- $\gamma|_{\mathbb{R}_i} : \mathbb{R}_i \to M$ satisfies the equation

$$\frac{d\gamma}{dt} + \nabla f(\gamma) = 0$$

for each i, and

- if $\gamma|_{\mathbb{R}_i}$ is a constant map, then $\mathbb{R}_i$ contains at least one marked point.

Two stable trajectories $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t, \gamma)$ and $(\overline{\mathbb{R}}_1 \sqcup \cdots \sqcup \overline{\mathbb{R}}_{m'}, t', \gamma')$ are the same if $m = m'$ and there exists a translation map θ such that $\theta(t_j) = t'_j$ and $\gamma = \gamma' \circ \theta$.

For two critical points $q_-, q_+ \in P_f$, we say that a stable trajectory $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t, \gamma)$ is from q_- to q_+ if γ takes q_- and q_+ at $-\infty \in \overline{\mathbb{R}}_1$ and $+\infty \in \overline{\mathbb{R}}_m$, respectively. We denote the space of μ-marked stable trajectories from q_- to q_+ by $\overline{\mathcal{M}}_\mu(q_-, q_+)$.

We use the following well-known estimate for the definition of the topology and the Kuranishi neighborhoods of $\overline{\mathcal{M}}_\mu(q_-, q_+)$.

LEMMA 3.1

Let $\delta_0^+ > 0$ be the minimal positive eigenvalue of the Hessian of a Morse function $f : M \to \mathbb{R}$ at its critical point $q \in M$. Then for any $0 < \delta < \delta_0^+$ and any trajectory $\gamma : [0, \infty) \to M$ convergent to q, there exists some constant $C > 0$ such that $\mathrm{dist}(\gamma(t), q) \le Ce^{-\delta t}$ *for $t \in [0, \infty)$.*

We define the topology of $\overline{\mathcal{M}}_\mu(q_-, q_+)$ as follows. For a stable trajectory $p_0 = (\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_{m_0}, t^0, \gamma_0)$, we fix constants $T_i^- < T_i^+$ such that the union of the interiors of $\mathring{I}_i = [T_i^-, T_i^+] \subset \overline{\mathbb{R}}_i$ contains all marked points t^0. As in the previous section, we regard each $\overline{\mathbb{R}}_i$ as $\overline{\mathbb{R}}_i = [-\infty, 0]_i \cup \mathring{I}_i \cup [0, +\infty]_i$, and we define ℓ_ρ by (1) for each $\rho = (\rho_i) \in [0,1)^{m_0 - 1}$. Let $U_{t^0} \subset \prod^\mu (\coprod_{i=1}^{m_0} \mathring{I}_i)$ be a neighborhood of t^0. (This is not a slice.) Then for an open neighborhood U of $(0, t^0) \in [0,1)^{m_0-1} \times U_{t^0}$ and a positive number $\epsilon > 0$, we define a subset $\mathcal{W}_{p_0}(U, \epsilon) \subset \overline{\mathcal{M}}_\mu(q_-, q_+)$ as follows. A stable trajectory $p = (\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t, \gamma) \in \overline{\mathcal{M}}_\mu(q_-, q_+)$ is contained in $\mathcal{W}_{p_0}(U, \epsilon)$ if there exists a point $(\rho, t) \in U$ which corresponds to the domain semistable line of p, and the L^∞-distance of the maps $\gamma : \ell_\rho \to M$ and γ_0 on $I_i^\rho = [\frac{1}{2}\log \rho_{i-1}, 0]_i \cup \mathring{I}_i \cup [0, -\frac{1}{2}\log \rho_i]_i$ is less than ϵ for each $1 \le i \le m_0$, where the L^∞-distance $\mathrm{dist}_{L^\infty}(\gamma|_{I_i^\rho}, \gamma_0|_{I_i^\rho})$ is defined by the L^∞-norm of the function $\mathrm{dist}(\gamma(t), \gamma_0(t))$ on I_i^ρ. We define the neighborhoods of a point $p_0 \in \overline{\mathcal{M}}_\mu(q_-, q_+)$ by the sets which contain some $\mathcal{W}_{p_0}(U, \epsilon)$.

First we check that this definition of neighborhood is independent of the choice of $\mathring{I}_i$. Let $[\tilde{T}_i^-, \tilde{T}_i^+]$ be another choice of intervals. We may assume that $[T_i^-, T_i^+] \cap [\tilde{T}_i^-, \tilde{T}_i^+] \ne \emptyset$. (It is enough to compare $[T_i^-, T_i^+]$ and $[\min(T_i^-, \tilde{T}_i^-), \max(T_i^+, \tilde{T}_i^+)]$.) By translation, we may assume that $0 \in [T_i^-, T_i^+] \cap [\tilde{T}_i^-, \tilde{T}_i^+]$. For a sufficiently small $\rho = (\rho_i)_{1 \le m_0 - 1}$, we define $\tilde{\rho} = (\tilde{\rho}_i)_{1 \le i \le m_0 - 1}$ by

$$T_i^+ + (-\log \rho_i) - T_{i+1}^- = \tilde{T}_i^+ + (-\log \tilde{\rho}_i) - \tilde{T}_{i+1}^-.$$

Then $J_i^\rho = [0, T_i^+]_i \cup [0, -\frac{1}{2}\log \rho_i]_i \cup [\frac{1}{2}\log \rho_i, 0]_{i+1} \cup [T_{i+1}^-, 0]$ coincides with $\tilde{J}_i^{\tilde{\rho}} = [0, \tilde{T}_i^+]_i \cup [0, -\frac{1}{2}\log \tilde{\rho}_i]_i \cup [\frac{1}{2}\log \tilde{\rho}_i, 0]_{i+1} \cup [\tilde{T}_{i+1}^-, 0]$ as intervals. Since the L^∞-distance of $\gamma_0|_{J_i^\rho}$ and $\gamma_0|_{\tilde{J}_i^{\tilde{\rho}}}$ converges to 0 as ρ_i converges to 0, the definition of the neighborhood does not depend on the choice of the intervals $\mathring{I}_i$.

Next we show that the topology of $\overline{\mathcal{M}}_\mu(q_-, q_+)$ is well defined by these neighborhood systems. It is enough to prove that, for any $p_1 \in W_{p_0}(U, \epsilon)$, there exist some U_1 and $\epsilon_1 > 0$ such that $W_{p_1}(U_1, \epsilon_1) \subset W_{p_0}(U, \epsilon)$. Since $p_1 = (\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_{m_1}, t^1, \gamma_1)$ is contained in $W_{p_0}(U, \epsilon)$, there exists some $(\rho^1, t^1) \in U$ such that the domain of p_1 coincides with (ℓ_{ρ^1}, t^1), and $\mathrm{dist}_{L^\infty}(\gamma_1|_{I_i^{\rho^1}}, \gamma_0|_{I_i^{\rho^1}}) < \epsilon'$ for some $0 < \epsilon' < \epsilon$. Define $\epsilon_1 = \epsilon - \epsilon' > 0$.

Let $i = i_1, \ldots, i_{m_1 - 1}$ be the indices such that $\rho_i^1 = 0$. For the definition of neighborhoods of p_1, we may use the intervals $\mathring{I}_j^1 = [T_j^{1,-}, T_j^{1,+}]$ $(1 \le j \le m_1)$ defined by the minimal interval which contains all intervals $\mathring{I}_i$ in the same $\overline{\mathbb{R}}$ in ℓ_{ρ_1}. Namely,

$$\mathring{I}_j^1 = \mathring{I}_{i_{j-1}+1} \cup \Big[0, -\frac{1}{2}\log \rho^1_{i_{j-1}+1}\Big]_{i_{j-1}+1} \cup \cdots \cup \Big[\frac{1}{2}\log \rho^1_{i_j - 1}, 0\Big]_{i_j} \cup \mathring{I}_{i_j},$$

where we read i_0 and i_{m_1} as $i_0 = 0$ and $i_{m_1} = m_0$, respectively. Let U_1 be a sufficiently small open neighborhood of (ℓ_{ρ_1}, t^1). For any $p = (\ell, t, \gamma) \in W_{p_1}(U_1, \epsilon_1)$, there exist some $\rho_{i_1}, \ldots, \rho_{i_{m_1-1}} \ge 0$ such that $\ell = \ell_{(\rho_i)}$, where ρ_i for $i \notin \{i_j\}$ are defined by $\rho_i = \rho_i^1$, $((\rho_i), t) \in U_1$, and the L^∞-distance of γ and γ_1 on

$$(I_j^1)^\rho = \Big[\frac{1}{2}\log \rho_{i_{j-1}+1}, 0\Big]_{i_{j-1}+1} \cup \mathring{I}_j^1 \cup \Big[0, -\frac{1}{2}\log \rho_{i_j}\Big]_{i_j}$$

is less than ϵ_1. Since each $I_i^\rho = [\frac{1}{2}\log \rho_{i-1}, 0]_i \cup \mathring{I}_i \cup [0, -\frac{1}{2}\log \rho_i]_i$ is contained in some $(I_j^1)^\rho$, the L^∞-distance of γ and γ_0 on I_i^ρ is $< \epsilon_1 + \epsilon' = \epsilon$. Hence p is contained in $W_{p_0}(U, \epsilon)$.

We prove the topological properties of $\overline{\mathcal{M}}_\mu(q_-, q_+)$.

LEMMA 3.2

We have that $\overline{\mathcal{M}}_\mu(q_-, q_+)$ is compact-Hausdorff and second-countable.

Proof

Let $\mathcal{A} = \{\alpha\} \subset \mathbb{R} \setminus \mathrm{CriV}\, f$ be a finite set which separates any two different critical values of f. Namely, for any critical values $c_0, c_1 \in \mathrm{CriV}\, f$ such that $c_0 < c_1$, we assume that $\mathcal{A}$ contains some α such that $c_0 < \alpha < c_1$. For each stable trajectory $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t, \gamma) \in \overline{\mathcal{M}}_\mu(q_-, q_+)$, we define additional marked points $t_\alpha^+ = (f \circ \gamma)^{-1}(\alpha)$ for $\alpha \in \mathcal{A}_{q_-, q_+} = \{\alpha \in \mathcal{A}; f(q_-) < \alpha < f(q_+)\}$. Then for any $\epsilon > 0$, there exists some $T > 0$ such that for any stable trajectory $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t, \gamma)$, the values of γ on the complement of the T-neighborhood of $\{t_\alpha^+\}$ are contained in the ϵ-neighborhood of the critical points of f.

First we check the second countability. For each $m \ge 1$ and intervals $\mathring{I}_i \subset \mathbb{R}_i$ $(1 \le i \le m)$, let $\mathcal{U}(m, (\mathring{I}_i)) \subset \overline{\mathcal{M}}_\mu(q_-, q_+)$ be the set of stable trajectories $p =$

$(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t, \gamma) \in \overline{\mathcal{M}}_\mu(q_-, q_+)$ such that $t \cup \{t^+_\alpha\} \subset \coprod_i \mathring{I}_i$ (after an appropriate translation of $\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m$). If we choose appropriate countable pairs $(m, (\mathring{I}_i))$, then we can cover $\overline{\mathcal{M}}_\mu(q_-, q_+)$ by $\{\mathcal{U}(m, (\mathring{I}_i))\}$. Hence it is enough to construct a countable family of open subsets of $\overline{\mathcal{M}}_\mu(q_-, q_+)$ for each $\mathcal{U}(m, (\mathring{I}_i))$ which contains a basis of a neighborhood system for every point of $\mathcal{U}(m, (\mathring{I}_i))$. For any $T > 0$, let $\mathring{I}_i^T$ be the T-neighborhood of $\mathring{I}_i$. Then the restrictions of the maps to $\mathring{I}_i^T$ are equicontinuous for the stable curves contained in $\mathcal{U}(m, (\mathring{I}_i))$. (More precisely, they are equicontinuous for all stable curves with arbitrary translation of $\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m$ which satisfy the condition $t \cup \{t^+_\alpha\} \subset \coprod_i \mathring{I}_i$.) Furthermore, for any $\epsilon > 0$, if $T > 0$ is sufficiently large, then the images of the maps on the complement of $\bigcup_i \mathring{I}_i^T$ are contained in the ϵ-neighborhood of the critical points of f. Therefore, we can construct a required countable family of open subsets.

Next we prove the compactness. It is enough to show that any sequence $(\ell^j, t^j, \gamma^j) \in \overline{\mathcal{M}}_\mu(q_-, q_+)$ has a convergent subsequence. For each j, define $t^{j,+} = (t^{j,+}_\alpha)_{\alpha \in \mathcal{A}_{q_-,q_+}}$ by $t^{j,+}_\alpha = (f \circ \gamma^j)^{-1}(\alpha)$. Since $\overline{\mathcal{M}}^{\mathrm{SL}}_{\mu+\mu^+}$ is compact, we may assume that $(\ell^j, t^j \cup t^{j,+})$ converges to a stable line $(\ell, t \cup t^+)$. Let $\mathring{I}_i \subset \mathbb{R}_i$ be intervals whose interiors cover $t \cup t^+$. For each $\rho = (\rho_i)$, we define ℓ_ρ by (1). Then there exists a sequence $\rho^j = (\rho^j_i) \to 0$ and isomorphisms $\ell^j = \ell_{\rho^j}$ such that under these isomorphisms, t^j and $t^{j,+}$ converge to t and t^+, respectively. For any $\epsilon > 0$, there exists some $T > 0$ such that the values of γ^j on the complement of the T-neighborhood of the union of $\mathring{I}_i$ are contained in the ϵ-neighborhood of the critical points of f. Hence passing to a subsequence, we may assume that each $\gamma^j|_{[\frac{1}{2}\log \rho_{i-1}, 0]_i \cup \mathring{I}_i \cup [0, -\frac{1}{2}\log \rho_i]_i}$ uniformly converges to some trajectory $\gamma_i : \overline{\mathbb{R}}_i \to M$, and that their union defines a stable trajectory (ℓ, t, γ). By the definition of the topology, (ℓ^j, t^j, γ^j) converges to (ℓ, t, γ).

Finally, we prove the Hausdorffness. It is enough to show that if a sequence (ℓ^j, t^j, γ^j) converges to two points (ℓ, t, γ) and $(\tilde{\ell}, \tilde{t}, \tilde{\gamma})$, then these two points coincide. Define $t^+ = (t^+_\alpha)_{\alpha \in \mathcal{A}_{q_-,q_+}}$ and $\tilde{t}^+ = (t^+_\alpha)_{\alpha \in \mathcal{A}_{q_-,q_+}}$ by $t^+_\alpha = (f \circ \gamma)^{-1}(\alpha)$ and $\tilde{t}^+_\alpha = (f \circ \tilde{\gamma})^{-1}(\alpha)$, respectively. Let $\mathring{I}_i \subset \mathbb{R}_i$ and $\mathring{I}'_i \subset \mathbb{R}_i$ be intervals whose interiors cover $t \cup t^+$ and $\tilde{t} \cup \tilde{t}^+$, respectively. Then there exist sequences $\rho^j \to 0$, $\tilde{\rho}^j \to 0$ and isomorphisms $\ell^j \cong \ell_{\rho^j} \cong \tilde{\ell}_{\tilde{\rho}^j}$ such that under these isomorphisms, t^j converges to t and $\tilde{t}$, respectively, $\mathrm{dist}_{L^\infty}(\gamma^j|_{I_i^{\rho^j}}, \gamma|_{I_i^{\rho^j}}) \to 0$, and $\mathrm{dist}_{L^\infty}(\gamma^j|_{(I'_i)^{\tilde{\rho}^j}}, \tilde{\gamma}|_{(I'_i)^{\tilde{\rho}^j}}) \to 0$, where $I_i^\rho = [\frac{1}{2}\log \rho_{i-1}, 0]_i \cup \mathring{I}_i \cup [0, -\frac{1}{2}\log \rho_i]_i$ and $(I'_i)^{\tilde{\rho}} = [\frac{1}{2}\log \tilde{\rho}_{i-1}, 0]_i \cup \mathring{I}'_i \cup [0, -\frac{1}{2}\log \tilde{\rho}_i]_i$. In particular, under the isomorphisms $\ell^j \cong \ell_{\rho^j}$ and $\ell^j \cong \tilde{\ell}_{\tilde{\rho}^j}$, $t^{j,+}$ converges to t^+ and $\tilde{t}^+$, respectively. In particular, $(\ell^j, t^j \cup t^{j,+})$ converges to $(\ell, t \cup t^+)$ and $(\tilde{\ell}, \tilde{t} \cup \tilde{t}^+)$ in $\overline{\mathcal{M}}_\mu(q_-, q_+)$, which implies that $(\ell, t \cup t^+) = (\tilde{\ell}, \tilde{t} \cup \tilde{t}^+)$. Hence we may assume that $\mathring{I}_i = \mathring{I}'_i$, $t = \tilde{t}$, $t^+ = \tilde{t}^+$, and $\rho^j = \tilde{\rho}^j$. Then $\mathrm{dist}_{L^\infty}(\gamma^j|_{I_i^{\rho^j}}, \gamma|_{I_i^{\rho^j}}) \to 0$ and $\mathrm{dist}_{L^\infty}(\gamma^j|_{(I'_i)^{\tilde{\rho}^j}}, \tilde{\gamma}|_{(I'_i)^{\tilde{\rho}^j}}) \to 0$ imply that $\gamma = \tilde{\gamma}$. □

4. Construction of Kuranishi neighborhoods

In this section, we explain the construction of Kuranishi neighborhoods of the points in $\overline{\mathcal{M}}_0(q_-, q_+)$. (The case of the space of marked stable trajectories is similar; see Section 10.) First we explain the Sobolev spaces we use. For a semistable line $\ell = \overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m$, we fix constants $T_i^- < T_i^+$ and we regard $\overline{\mathbb{R}}_i$ as $\overline{\mathbb{R}}_i = [-\infty, 0]_i \cup \mathring{I}_i \cup [0, +\infty]_i$, where $\mathring{I}_i = [T_i^-, T_i^+]$. Then for nonnegative constants $\delta = (\delta_i)_{i=0,1,\dots,m}$ and $1 < p < \infty$, we define the L^p_δ-norm of a function ξ on ℓ by

$$\|\xi\|^p_{L^p_\delta} = \sum_{1 \le i \le m} \Big(\int_{(-\infty,0]_i} |e^{-\delta_{i-1} t} \xi|^p \, dt + \int_{\mathring{I}_i} |\xi|^p \, dt + \int_{[0,\infty)_i} |e^{\delta_i t} \xi|^p \, dt \Big).$$

The Sobolev space $W^{1,p}_\delta(\ell)$ is the space of continuous functions ξ on ℓ whose $W^{1,p}_\delta(\ell)$-norm

$$\|\xi\|^p_{W^{1,p}_\delta} = \sum_{1 \le i \le m} \Big(\int_{(-\infty,0]_i} \big(|e^{-\delta_{i-1} t} \xi|^p + |e^{-\delta_{i-1} t} \nabla_t \xi|^p \big) \, dt + \int_{\mathring{I}_i} \big(|\xi|^p + |\nabla_t \xi|^p \big) \, dt + \int_{[0,\infty)_i} \big(|e^{\delta_i t} \xi|^p + |e^{\delta_i t} \nabla_t \xi|^p \big) \, dt \Big)$$

is finite.

For a family of deformations of ℓ, we need to choose an appropriate family of norms to obtain uniform estimates. As in Section 2, for each $\rho = (\rho_i)_{1 \le i \le m-1} \in [0,1)^{m-1}$, we define ℓ_ρ by (1). Then we define the norm of $L^p_\delta(\ell_\rho)$ by

$$\|\xi\|^p_{L^p_\delta(\ell_\rho)} = \sum_{1 \le i \le m} \Big(\int_{[\frac{1}{2} \log \rho_{i-1}, 0]_i} |e^{-\delta_{i-1} t} \xi|^p \, dt + \int_{\mathring{I}_i} |\xi|^p \, dt + \int_{[0, -\frac{1}{2} \log \rho_i]_i} |e^{\delta_i t} \xi|^p \, dt \Big),$$

where we read $\frac{1}{2} \log \rho_0$ and $-\frac{1}{2} \log \rho_m$ as $-\infty$ and $+\infty$, respectively. We also define the norm of $W^{1,p}_\delta(\ell_\rho)$ similarly.

We sometimes use the following notation: for nonnegative functions A and B, $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \le CB$.

Fix finite numbers $\mathcal{A} = \{\alpha\} \subset \mathbb{R} \setminus \operatorname{CriV} f$ which separate any two different critical values of f as in the proof of Lemma 3.2. For each stable trajectory $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, \gamma)$ from q_- to q_+, we define additional marked points $t^+ = (t^+_\alpha)_{\alpha \in \mathcal{A}_{q_-,q_+}}$ by $t^+_\alpha = (f \circ \gamma)^{-1}(\alpha)$, where the index set is $\mathcal{A}_{q_-,q_+} = \{\alpha \in \mathcal{A}; f(q_-) < \alpha < f(q_+)\}$. Then $(\overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t^+)$ is a stable line.

For each $\alpha \in \mathcal{A}$, we construct a finite vector space E^0_α and a linear map $\lambda_\alpha : E^0_\alpha \to C^\infty(\mathbb{R} \times M; TM)$ whose support is contained in $(-C_\alpha, C_\alpha) \times M \subset \mathbb{R} \times M$ for some $C_\alpha > 0$. For each pair (q_-, q_+) of critical points, we define a vector space $E^0_{q_-,q_+}$ by $E^0_{q_-,q_+} = \bigoplus_{\alpha \in \mathcal{A}_{q_-,q_+}} E^0_\alpha$.

For a stable trajectory $(\ell = \overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, \gamma) \in \overline{\mathcal{M}}_0(q_-, q_+)$, let $\delta_{0,i} > 0$ be the minimum of the absolute values of the eigenvalues of the Hessian at the critical point $q_i = \gamma(+\infty_i)$ for each $0 \le i \le m$. Let $\delta = (\delta_i)_{0 \le i \le m}$ be a sequence of positive constants such that $0 \le \delta_i < \delta_{0,i}$. We define a linear map

$$W^{1,p}_\delta(\ell, \gamma^* TM) \oplus E^0_{q_-,q_+} \to L^p_\delta(\ell, \gamma^* TM) \tag{2}$$

by

$$\big(\xi,(h_\alpha)\big)\mapsto\Big(D_p\xi(t)+\sum_{\alpha\in\mathcal{A}_{q_-,q_+}}\lambda_\alpha(h_\alpha)\big(o_{t^+_\alpha}(t),\gamma(t)\big)\Big),$$

where D_p is the linearization of the equation of trajectories, and $o_{t^+_\alpha}:[t^+_\alpha-C_\alpha,t^+_\alpha+C_\alpha]\to[-C_\alpha,C_\alpha]$ is the map defined by translation.

We assume that (2) is surjective for all stable trajectories $(\ell=\overline{\mathbb{R}}_1\cup\cdots\cup\overline{\mathbb{R}}_m,\gamma)\in\bigcup_{q_-,q_+\in P_f}\overline{\mathcal{M}}_0(q_-,q_+)$. The linearized gluing lemma (Lemma 5.1) and the compactness of $\bigcup_{q_-,q_+\in P_f}\overline{\mathcal{M}}_0(q_-,q_+)$ imply that, if we choose appropriate vector spaces E^0_α and linear maps λ_α, then (2) is surjective for all stable trajectories.

Using these finite vector spaces, we construct a Kuranishi neighborhood of each stable trajectory $p_0=(\ell=\overline{\mathbb{R}}_1\cup\cdots\cup\overline{\mathbb{R}}_m,\gamma_0)\in\overline{\mathcal{M}}_0(q_-,q_+)$ as follows. Define $t^{0,+}=(t^{0,+}_\alpha)_{\alpha\in\mathcal{A}_{q_-,q_+}}$ by $t^{0,+}_\alpha=(f\circ\gamma_0)^{-1}(\alpha)$. Fix constants $T^-_i\ll T^+_i$ such that the union of $(T^-_i,T^+_i)\subset\overline{\mathbb{R}}_i$ contains $[t^{0,+}_\alpha-C_\alpha,t^{0,+}_\alpha+C_\alpha]$ for all $\alpha\in\mathcal{A}_{q_-,q_+}$. Define $\mathring{I}_i=[T^-_i,T^+_i]$; we regard each $\overline{\mathbb{R}}_i$ as $\overline{\mathbb{R}}_i=[-\infty,0]_i\cup\mathring{I}_i\cup[0,+\infty]_i$. We may assume that $\gamma_0([0,+\infty]_i\cup[-\infty,0]_{i+1})\subset M$ is contained in a chart U_i of M for each $0\le i\le m$. Recall that a coordinate $[0,1)^{m-1}\times\mathcal{U}_{t^{0,+}}\to\overline{\mathcal{M}}^{\mathrm{SL}}_{\#\mathcal{A}_{q_-,q_+}}$ of a neighborhood of $(\overline{\mathbb{R}}_1\cup\cdots\cup\overline{\mathbb{R}}_m,t^{0,+})$ is defined by $(\rho,t^+)\mapsto(\ell_\rho,t^+)$, where ℓ_ρ is defined by (1). We may assume that $\mathcal{U}_{t^{0,+}}$ is sufficiently small so that $[t^+_\alpha-C_\alpha,t^+_\alpha+C_\alpha]$ is contained in some $\mathring{I}_i$ for any $\alpha\in\mathcal{A}_{q_-,q_+}$ and $t^+\in\mathcal{U}_{t^{0,+}}$. We fix a constant $0<\kappa_i<\delta_{0,i}$ for each i.

First we construct a piecewise smooth map $\gamma_\rho:\ell_\rho\to M$ for each $\rho\in[0,1)^{m-1}$ as an approximate solution. We also construct a map $\Phi_\rho:\gamma^*_\rho TM\to M$ to represent maps near the approximate solution as a section of $\gamma^*_\rho TM$.

We define γ_ρ as follows. On each $\mathring{I}_i\subset\ell_\rho$, we define $\gamma_\rho|_{\mathring{I}_i}=\gamma_0|_{\mathring{I}_i}$. On each $[0,-\frac{1}{2}\log\rho_i]_i\subset\ell_\rho$, we define

$$\gamma_\rho|_{[0,-\frac{1}{2}\log\rho_i]_i}(t)=\gamma_0|_{[0,+\infty]_i}\Big(-\frac{1}{\kappa_i}\log\Big(\frac{e^{-\kappa_i t}-\rho_i^{\kappa_i/2}}{1-\rho_i^{\kappa_i/2}}\Big)\Big).$$

Similarly, we define

$$\gamma_\rho|_{[\frac{1}{2}\log\rho_i,0]_{i+1}}(t)=\gamma_0|_{[-\infty,0]_{i+1}}\Big(\frac{1}{\kappa_i}\log\Big(\frac{e^{\kappa_i t}-\rho_i^{\kappa_i/2}}{1-\rho_i^{\kappa_i/2}}\Big)\Big)$$

on each $[\frac{1}{2}\log\rho_i,0]_{i+1}\subset\ell_\rho$.

The map $\Phi_\rho:\gamma^*_\rho TM\to M$ is defined as follows. Since $\gamma_0([0,+\infty]_i\cup[-\infty,0]_{i+1})\subset M$ is contained in a chart U_i for each i, we can trivialize $\gamma^*_\rho TM|_{[0,-\frac{1}{2}\log\rho_i]_i\cup[\frac{1}{2}\log\rho_i,0]_{i+1}}$ by the coordinate of this chart. Then on $[0,-\frac{1}{2}\log\rho_i]_i\cup[\frac{1}{2}\log\rho_i,0]_{i+1}$, we define $\Phi_\rho:([0,-\frac{1}{2}\log\rho_i]_i\cup[\frac{1}{2}\log\rho_i,0]_{i+1})\times\mathbb{R}^m\to M$ by $\Phi_\rho(t,\xi)=\gamma_\rho(t)+\xi$ in the coordinate of U_i. Identifying $([0,-\frac{1}{2}\log\rho_i]_i\cup[\frac{1}{2}\log\rho_i,0]_{i+1})\times\mathbb{R}^m$ with $\gamma^*_\rho TM|_{[0,-\frac{1}{2}\log\rho_i]_i\cup[\frac{1}{2}\log\rho_i,0]_{i+1}}$ by the differential of Φ_ρ at $([0,-\frac{1}{2}\log\rho_i]_i\cup[\frac{1}{2}\log\rho_i,0]_{i+1})\times\{0\}$, we regard Φ_ρ as a map from $\gamma^*_\rho TM|_{[0,-\frac{1}{2}\log\rho_i]_i\cup[\frac{1}{2}\log\rho_i,0]_{i+1}}$ to M.

Next on each $\mathring{I}_i$, let $\Phi_0 : \gamma_0^* TM|_{\mathring{I}_i} \to M$ be a smooth map whose differential at the zero section is the identity map and whose restriction to the joints $\partial([0, -\frac{1}{2}\log\rho_i]_i \cup [\frac{1}{2}\log\rho_i, 0]_{i+1}) = \{0\}_i \sqcup \{0\}_{i+1}$ coincides with the above Φ_ρ. We define Φ_ρ on each $\gamma_\rho^* TM|_{\mathring{I}_i} = \gamma_0^* TM|_{\mathring{I}_i}$ by $\Phi_\rho|_{\gamma_0^* TM|_{\mathring{I}_i}} = \Phi_0|_{\gamma_0^* TM|_{\mathring{I}_i}}$. These define a family of piecewise smooth maps $\Phi_\rho : \gamma_\rho^* TM \to M$.

Define a family of Fredholm maps

$$F^{(\rho,t^+)} : W^{1,p}_\delta(\ell_\rho, \gamma_\rho^* TM) \oplus E^0_{q_-,q_+} \to L^p_\delta(\ell_\rho, \gamma_\rho^* TM)$$

by

$$\begin{aligned} F^{(\rho,t^+)}(\xi, h)(t) &= \frac{d}{dt}\big(\Phi_\rho\big(\xi(t)\big)\big) + \nabla f\big(\Phi_\rho\big(\xi(t)\big)\big) \\ &\quad + \sum_{\alpha \in \mathcal{A}_{q_-,q_+}} \lambda_\alpha(h_\alpha)\big(o_{t^+_\alpha}(t), \Phi_\rho\big(\xi(t)\big)\big), \end{aligned}$$

where $h = (h_\alpha) \in E^0_{q_-,q_+} = \bigoplus_{\alpha \in \mathcal{A}_{q_-,q_+}} E^0_\alpha$. We note that

$$F^{(\rho,t^+)}(\xi, h) = \frac{d}{dt}(\gamma_\rho + \xi) + \nabla f(\gamma_\rho + \xi)$$

on $[0, -\frac{1}{2}\log\rho_i]_i$ and $[\frac{1}{2}\log\rho_i, 0]_{i+1}$. We also define Fredholm maps

$$\begin{aligned} F^{(\rho,t^+)+} : W^{1,p}_\delta(\ell_\rho, \gamma_\rho^* TM) &\oplus E^0_{q_-,q_+} \\ &\to L^p_\delta(\ell_\rho, \gamma_\rho^* TM) \oplus \operatorname{Ker} DF^{(0,t^{0,+})}_{(0,0)} \end{aligned}$$

by

$$\begin{aligned} &F^{(\rho,t^+)+}(\xi, h) \\ &\quad = \Big(F^{(\rho,t^+)}(\xi, h), \sum_j \Big(\langle \xi, \xi_j\rangle_{L^2(\bigcup_i \mathring{I}_i)} + \sum_\alpha \langle h_\alpha, h_{j,\alpha}\rangle_{E^0_\alpha}\Big) \cdot x_j\Big), \end{aligned}$$

where $\{x_j = (\xi_j, (h_{j,\alpha}))\}_j$ is an orthonormal basis of $\operatorname{Ker} DF^{(0,t^{0,+})}_{(0,0)}$ with respect to the inner product given by

$$\big\langle \big(\xi, (h_\alpha)\big), \big(\xi', (h'_\alpha)\big)\big\rangle = \langle \xi, \xi'\rangle_{L^2(\bigcup_i \mathring{I}_i)} + \sum_\alpha \langle h_\alpha, h'_\alpha\rangle_{E^0_\alpha}$$

for some inner products $\langle , \rangle_{E^0_\alpha}$ of E^0_α. (This is a nondegenerate inner product if $T_i^- \ll 0$ and $T_i^+ \gg 0$.)

To apply the inverse function theorem to each $F^{(\rho,t^+)+}$, first we prove some estimates.

LEMMA 4.1

For any $0 \le \delta_i < \delta'_i < \delta_{0,i}$ and $1 < p < \infty$, there exists a constant $C > 0$ such that

$$\big\|F^{(\rho,t^+)}(0,0)|_{[0,-\frac{1}{2}\log\rho_i]_i \cup [\frac{1}{2}\log\rho_i, 0]_{i+1}}\big\|_{L^p_{\delta_i}} \le C\rho_i^{\min(\kappa_i, \delta'_i - \delta_i)/2}(-\log\rho_i)^{1/p}.$$

Proof
Since $\kappa_i < \delta_{0,i}$, we may assume that $\delta_i' > \kappa_i$. Since γ_0 is a solution, $F^{(\rho,t^+)}(0, 0)|_{[0,-\frac{1}{2}\log\rho_i]_i}$ is equal to

$$\frac{d\gamma_\rho}{dt} + \nabla f(\gamma_\rho) = \frac{\rho_i^{\kappa_i/2}}{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}} \cdot \frac{d\gamma_0}{dt}\Big(-\frac{1}{\kappa_i}\log\Big(\frac{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}}{1-\rho_i^{\kappa_i/2}}\Big)\Big).$$

The inequality $|\frac{d\gamma_0}{dt}| \lesssim e^{-\delta_i' t}$ (Lemma 3.1) implies that

$$\Big|\frac{d\gamma_0}{dt}\Big(-\frac{1}{\kappa_i}\log\Big(\frac{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}}{1-\rho_i^{\kappa_i/2}}\Big)\Big)\Big| \lesssim \Big(\frac{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}}{1-\rho_i^{\kappa_i/2}}\Big)^{\delta_i'/\kappa_i}.$$

Hence

$$\begin{aligned}
&\big\|F^{(\rho,t^+)}(0,0)|_{[0,-\frac{1}{2}\log\rho_i]_i}\big\|^p_{L^p_{\delta_i}} \\
&\lesssim \int_0^{-\frac{1}{2}\log\rho_i}\Big(\frac{\rho_i^{\kappa_i/2}}{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}} \cdot \Big(\frac{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}}{1-\rho_i^{\kappa_i/2}}\Big)^{\delta_i'/\kappa_i} e^{\delta_i t}\Big)^p\, dt \\
&= \frac{\rho_i^{p\kappa_i/2}}{(1-\rho_i^{\kappa_i/2})^{p\delta_i'/\kappa_i}} \cdot \int_0^{-\frac{1}{2}\log\rho_i}\big((e^{-\kappa_i t} - \rho_i^{\kappa_i/2})^{\frac{\delta_i'}{\kappa_i}-1} e^{\delta_i t}\big)^p\, dt.
\end{aligned}$$

Since $\frac{\delta_i'}{\kappa_i} - 1 > 0$,

$$\begin{aligned}
\big((e^{-\kappa_i t} - \rho_i^{\kappa_i/2})^{\frac{\delta_i'}{\kappa_i}-1} e^{\delta_i t}\big)^p &\le e^{-p(\delta_i'-\delta_i-\kappa_i)t} \\
&\le \rho_i^{p\min(0,\delta_i'-\delta_i-\kappa_i)/2}
\end{aligned}$$

for $t \in [0, -\frac{1}{2}\log\rho_i]$ in the above inequality. Therefore,

$$\big\|F^{(\rho,t^+)}(0,0)|_{[0,-\frac{1}{2}\log\rho_i]_i}\big\|^p_{L^p_{\delta_i}} \lesssim \rho_i^{p\min(\kappa_i,\delta_i'-\delta_i)/2}(-\log\rho_i).$$

The estimate of the $L^p_{\delta_i}$-norm of $F^{(\rho,t^+)}(0,0)|_{[\frac{1}{2}\log\rho_i,0]_{i+1}}$ is similar. □

Since $F^{(\rho,t^+)}(0,0)|_{\mathring{I}_i} = 0$, the above lemma implies that $\|F^{(\rho,t^+)}(0,0)\|_{L^p_\delta} \to 0$ as $\rho \to 0$.

Next we prove that $DF^{(\rho,t^+)+}_{(\xi,h)}$ is invertible and the norm of its inverse is uniformly bounded for all $(\rho,t^+) \in [0,1)^{m-1} \times \mathcal{U}_{t^{0,+}}$ sufficiently close to $(0,t^{0,+})$ and sufficiently small $(\xi,h) \in W^{1,p}_\delta(\ell,\gamma_\rho^*TM) \oplus E^0_{q_-,q_+}$. The case of $(\xi,h) = (0,0)$ follows from the linearized gluing lemma (Lemma 5.1), and the general case is due to the following lemma.

LEMMA 4.2
For any $0 \le \delta_i < \delta_{0,i}$, there exists a constant $C > 0$ such that, for any $(\rho,t^+) \in [0,1)^{m-1} \times \mathcal{U}_{t^{0,+}}$ sufficiently close to $(0,t^{0,+})$ and for any sufficiently small $(\xi,h) \in W^{1,p}_\delta(\ell,\gamma_\rho^*TM) \oplus E^0_{q_-,q_+}$,

$$\begin{aligned}&\big\|DF^{(\rho,t^+)}_{(\xi,h)}(\hat\xi,\hat h)-DF^{(\rho,t^+)}_{(0,0)}(\hat\xi,\hat h)\big\|_{L^p_\delta}\\&\qquad\le C\big(\|\xi\|_\infty\big(\|\hat\xi\|_{W^{1,p}_\delta}+|\hat h|\big)+\big(\|\xi\|_{W^{1,p}_\delta}+|h|\big)\|\hat\xi\|_\infty\big).\end{aligned}$$

In particular,

$$\big\|DF^{(\rho,t^+)}_{(\xi,h)}(\hat\xi,\hat h)-DF^{(\rho,t^+)}_{(0,0)}(\hat\xi,\hat h)\big\|_{L^p_\delta}\lesssim\big(\|\xi\|_{W^{1,p}_\delta}+|h|\big)\big(\|\hat\xi\|_{W^{1,p}_\delta}+|\hat h|\big).$$

We can easily prove the above lemma by direct calculation.

By the inverse function theorem, Lemma 4.1 and the uniform invertibility of $DF^{(\rho,t^+)+}_{(\xi,h)}$ imply that there exist some $\epsilon>0$, $C>0$, and a neighborhood $X\subset[0,1)^{m-1}\times\mathcal{U}_{t^{0,+}}$ of $(0,t^{0,+})$ such that, for each $(\rho,t^+)\in X$, there exists a smooth map

$$\phi^{\rho,t^+}:\operatorname{Ker}DF^{(0,t^{0,+})}_{(0,0)}\supset B_\epsilon(0)\to B_C(0)\subset W^{1,p}_\delta(\ell_\rho,\gamma_\rho^*TM)\oplus E^0_{q_-,q_+}$$

such that, for any $(\xi,h)\in B_C(0)$ and $x\in B_\epsilon(0)$, $F^{(\rho,t^+)+}(\xi,h)=(0,x)$ if and only if $(\xi,h)=\phi^{\rho,t^+}(x)$.

We define a manifold $V=X\times B_\epsilon(0)$ and a function $s:V\to E_{q_-,q_+}=\bigoplus_{\alpha\in\mathcal{A}_{q_-,q_+}}\mathbb{R}\oplus E^0_{q_-,q_+}$ by

$$s(\rho,t^+,x)=\big(f\big(\Phi_\rho\big(\xi_{(\rho,t^+,x)}(t^+_\alpha)\big)\big)-\alpha,h_{(\rho,t^+,x)}\big),$$

where $\xi_{(\rho,t^+,x)}$ and $h_{(\rho,t^+,x)}$ are defined by $\phi^{\rho,t^+}(x)=(\xi_{(\rho,t^+,x)},h_{(\rho,t^+,x)})$. Since the zero set of this function consists of Morse trajectories, we can define a map $\psi:s^{-1}(0)\to\overline{\mathcal{M}}_0(q_-,q_+)$ by $\psi(\rho,t^+,x)=(\ell_\rho,\Phi_\rho(\xi_{(\rho,t^+,x)}))$.

LEMMA 4.3

The map $\psi:s^{-1}(0)\to\overline{\mathcal{M}}_0(q_-,q_+)$ is a homeomorphism onto a neighborhood of $p_0\in\overline{\mathcal{M}}_0(q_-,q_+)$.

Proof

The continuity is easy to check. Next we prove the injectivity. Since t^+_α is defined by $\Phi_\rho(\xi_{(\rho,t^+,x)})(t^+_\alpha)\in\{x\in M;f(x)=\alpha\}$, we have that (ρ,t^+) is determined by the image of ψ. The injectivity of each restriction of ψ to the fiber at fixed point $(\rho,t^+)\in X$ is an inverse function theorem consequence. Hence ψ is injective.

Finally, we prove that the image of ψ contains a neighborhood of p_0. Let $p^j=(\ell^j,\gamma^j)\in\overline{\mathcal{M}}_0(q_-,q_+)$ be a sequence convergent to p_0. We prove that the image of ψ contains p^j for all large j. By the definition of topology, there exists a sequence $\rho^j=(\rho^j_i)\to0$ and the isomorphisms $\ell^j\cong\ell_{\rho^j}$ such that $\operatorname{dist}_{L^\infty}(\gamma^j|_{\mathring{I}^{\rho^j}_i},\gamma|_{\mathring{I}^{\rho^j}_i})\to0$. Then $t^{j,+}_\alpha=(f\circ\gamma^j)^{-1}(\alpha)\in\coprod_i\mathring{I}_i$ $(\subset\ell_{\rho^j})$ converges to $t^+_\alpha\in\coprod_i\mathring{I}_i$ for all $\alpha\in\mathcal{A}_{q_-,q_+}$. Define sections $\xi^j\in W^{1,p}_\delta(\ell_{\rho^j},\gamma^*_{\rho^j}TM)$ by $\gamma^j=\Phi_{\rho^j}(\xi^j)$. Then $\|\xi^j\|_{L^\infty}$ converges to 0. We prove that the sequence of their $W^{1,p}_\delta$-norms $\|\xi^j\|_{W^{1,p}_\delta}$ also converges to 0. This implies that for a sufficiently large j, $(\xi^j,0)$ is a solution of $F^{(\rho^j,t^{j,+})}$ contained in $B_C(0)\subset W^{1,p}_\delta(\ell_{\rho^j},\gamma^*_{\rho^j}TM)\oplus E^0_{q_-,q_+}$, and there exists some $x^j \in B_\epsilon(0)$ such that $\phi^{\rho^j,t^{j,+}}(x^j) = (\xi^j, 0)$. Hence p^j is contained in the image of ψ.

Now we estimate the $W^{1,p}_\delta$-norm of ξ^j. Consider the following equation:

$$
\begin{aligned}
F^{(\rho^j,t^{j,+})+}(\xi^j,0) &= F^{(\rho^j,t^{j,+})+}(0,0) + DF^{(\rho^j,t^{j,+})+}_{(0,0)}(\xi^j,0) \\
&\quad + \int_0^1 (DF^{(\rho^j,t^{j,+})+}_{(s\xi^j,0)} - DF^{(\rho^j,t^{j,+})+}_{(0,0)})(\xi^j,0)\,ds.
\end{aligned} \tag{3}
$$

The L^p_δ-part of $F^{(\rho^j,t^{j,+})+}(\xi^j,0)$ is 0 and its $E^0_{q_-,q_+}$-part is defined by the integration of ξ^j on intervals of finite length. Hence $\|F^{(\rho^j,t^{j,+})+}(\xi^j,0)\|_{L^p_\delta \oplus E^0_{q_-,q_+}} \lesssim \|\xi\|_{L^\infty}$. Lemma 4.1 implies that

$$\left\|F^{(\rho^j,t^{j,+})+}(0,0)\right\|_{L^p_\delta \oplus E^0_{q_-,q_+}} \to 0$$

as $j \to \infty$. Lemma 4.2 implies that

$$(DF^{(\rho^j,t^{j,+})+}_{(s\xi^j,0)} - DF^{(\rho^j,t^{j,+})+}_{(0,0)})(\xi^j,0) \le C\|\xi^j\|_{L^\infty}\|\xi^j\|_{W^{1,p}_\delta}$$

for some $C > 0$. Furthermore, the uniform invertibility of $DF^{(\rho^j,t^{j,+})+}_{(0,0)}$ implies that

$$\left\|DF^{(\rho^j,t^{j,+})+}_{(0,0)}(\xi^j,0)\right\|_{L^p_\delta \oplus E^0_{q_-,q_+}} \ge \epsilon\|\xi^j\|_{W^{1,p}_\delta}$$

for some constant $\epsilon > 0$. Hence if $C\|\xi^j\|_{L^\infty} < \epsilon$, then we can estimate $\|\xi^j\|_{W^{1,p}_\delta}$ by (3), and we see that $\|\xi^j\|_{W^{1,p}_\delta} \to 0$ as $j \to 0$. Hence p^j is contained in the image of ψ for all large j.

Since we may shrink V to its relatively compact subset, ψ is a homeomorphism onto its image. $\square$

We call a 4-tuple $(V, E_{q_-,q_+}, s, \psi)$ a *Kuranishi neighborhood* of a point $p_0 \in \overline{\mathcal{M}}_0(q_-,q_+)$.

5. Linearized gluing lemma

In this section, we prove the linearized gluing lemma (Lemma 5.1) used in the preceding section. Let $\ell = [-\infty,0]_1 \cup \mathring{I}_1 \cup [0,+\infty]_1 \cup \cdots \cup [-\infty,0]_m \cup \mathring{I}_m \cup [0,+\infty]_m$ be a semistable line, and let $S : \ell \to \mathrm{gl}(l,\mathbb{R})$ be a continuous matrix-valued function whose values at the infinity points $+\infty_i$ $(0 \le i \le m)$ are symmetric matrices. We assume that these symmetric matrices are invertible, and we let $\delta_{0,i} > 0$ be the minimum of the absolute values of the eigenvalues of each $S(+\infty_i)$. Let $\delta = (\delta_i)_{0\le i\le m}$ be a sequence of nonnegative constants such that $0 \le \delta_i < \delta_{0,i}$. Then for any $1 < p < \infty$, $D = \frac{d}{dt} + S(t) : W^{1,p}_\delta(\ell,\mathbb{R}^l) \to L^p_\delta(\ell,\mathbb{R}^l)$ is a Fredholm operator. Let $\lambda : E \to L^p_\delta(\ell,\mathbb{R}^l)$ be a linear map from a finite vector space E which makes $D \oplus \lambda : W^{1,p}_\delta(\ell,\mathbb{R}^l) \oplus E \to L^p_\delta(\ell,\mathbb{R}^l)$ surjective. Define a linear map $(D\oplus\lambda)^+ : W^{1,p}_\delta(\ell,\mathbb{R}^l) \oplus E \to L^p_\delta(\ell,\mathbb{R}^l) \oplus \mathrm{Ker}(D\oplus\lambda)$ by

$$(D\oplus\lambda)^+(\xi,h)=\Big(D\xi+\lambda(h),\sum_j\big(\langle\xi,\xi_j\rangle_{L^2(\bigcup_i\mathring{I}_i)}+\langle h,h_j\rangle_E\big)x_j\Big),$$

where $\{x_j=(\xi_j,h_j)\}$ is an orthonormal basis of $\mathrm{Ker}(D\oplus\lambda)$ with respect to the inner product

$$\big\langle(\xi,h),(\xi',h')\big\rangle=\langle\xi,\xi'\rangle_{L^2(\bigcup_i\mathring{I}_i)}+\langle h,h'\rangle_E$$

for some inner product $\langle,\rangle_E$ of E.

For each $\rho=(\rho_i)\in[0,1)^{m-1}$, we regard the semistable line ℓ_ρ defined by (1) as a line obtained by patching subsets of ℓ, and we regard S as a discontinuous function on ℓ_ρ. Namely, $S:\ell_\rho\to\mathrm{gl}(m,\mathbb{R})$ is discontinuous (or not well defined) at $\pm\frac{1}{2}\log\rho_i$. We prove the following estimate of a family of linear operators $D=\frac{d}{dt}+S:W^{1,p}_\delta(\ell_\rho,\mathbb{R}^l)\to L^p_\delta(\ell_\rho,\mathbb{R}^l)$.

LEMMA 5.1 (Linearized gluing lemma)
There exist constants $\epsilon>0$ and $C>0$ such that for any $\rho=(\rho_i)\in[0,\epsilon)^{m-1}$,

$$\|\xi\|_{W^{1,p}_\delta(\ell_\rho)}+|h|_E\le C\Big(\big\|D\xi+\lambda(h)\big\|_{L^p_\delta(\ell_\rho)}+\sum_j\big|\langle\xi,\xi_j\rangle_{L^2(\bigcup_i\mathring{I}_i)}+\langle h,h_j\rangle_E\big|\Big).\tag{4}$$

Proof
We may assume that $S=S(+\infty_i)$ on a neighborhood of $+\infty$. This is because if we change S, then the operator norm of the difference of D is estimated by the L^∞-norm of the difference of S. We assume that $S=S(+\infty_i)$ on $[-\frac{1}{6}\log\rho_i,\infty]_i\cup[-\infty,\frac{1}{6}\log\rho_i]_{i+1}$. Since $D\oplus\lambda:W^{1,p}_\delta(\ell,\mathbb{R}^l)\oplus E\to L^p_\delta(\ell,\mathbb{R}^l)$ is surjective, there exists some $C>0$ such that

$$\|\tilde\xi\|_{W^{1,p}_\delta(\ell)}+|h|_E\le C\Big(\big\|D\tilde\xi+\lambda(h)\big\|_{L^p_\delta(\ell)}+\sum_j\big|\langle\tilde\xi,\xi_j\rangle_{L^2(\bigcup_i\mathring{I}_i)}+\langle h,h_j\rangle_E\big|\Big)\tag{5}$$

for $(\tilde\xi,h)\in W^{1,p}_\delta(\ell,\mathbb{R}^l)\oplus E$.

For each $\xi\in W^{1,p}_\delta(\ell_\rho,\mathbb{R}^l)$, we construct a function $\tilde\xi\in W^{1,p}_\delta(\ell,\mathbb{R}^l)$ on ℓ as follows, and we apply (5) to $\tilde\xi$. Then we will obtain a required estimate. We define $\tilde\xi$ on each $\mathring{I}_i$ by $\tilde\xi|_{\mathring{I}_i}=\xi|_{\mathring{I}_i}$. Let $\{\phi^{(i)}_j\}_j$ be a basis of $\mathbb{R}^l$ consisting of eigenvectors of $S(+\infty_i)$ for each i, and let $\lambda^{(i)}_j$ be the eigenvalue of $\phi^{(i)}_j$. Let

$$\begin{aligned}\xi\Big(-\frac{1}{2}\log\rho_i\Big)&=\sum a^{(i)}_j e^{-\lambda^{(i)}_j(-\frac{1}{2}\log\rho_i)}\phi^{(i)}_j\\&=\sum b^{(i)}_j e^{-\lambda^{(i)}_j\cdot\frac{1}{2}\log\rho_i}\phi^{(i)}_j\end{aligned}$$

be the expansions by the eigenvectors for each $-\frac{1}{2}\log\rho_i\in[0,-\frac{1}{2}\log\rho_i]_i$. We note that $b^{(i)}_j=e^{-\lambda^{(i)}_j(-\log\rho_i)}a^{(i)}_j$. We define $\tilde\xi$ on $[0,+\infty]_i\cup[-\infty,0]_{i+1}$ by

$$\tilde\xi|_{[0,+\infty]_i}(t)=\begin{cases}\xi(t)-\chi_{-\frac{1}{2}\log\rho_i}(t)\sum_{\lambda^{(i)}_j<0}a^{(i)}_j e^{-\lambda^{(i)}_j t}\phi^{(i)}_j, & 0\le t\le-\frac{1}{2}\log\rho_i,\\ \sum_{\lambda^{(i)}_j>0}a^{(i)}_j e^{-\lambda^{(i)}_j t}\phi^{(i)}_j, & -\frac{1}{2}\log\rho_i\le t\le\infty,\end{cases}$$

$$\tilde{\xi}|_{[-\infty,0]_{i+1}}(t)=\begin{cases}\xi(t)-\chi_{-\frac{1}{2}\log\rho_i}(-t)\sum_{\lambda_j^{(i)}>0}b_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}, & \frac{1}{2}\log\rho_i\le t\le 0,\\ \sum_{\lambda_j^{(i)}<0}b_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}, & -\infty\le t\le\frac{1}{2}\log\rho_i,\end{cases}$$

where $\chi:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ is a smooth function such that $\chi|_{[0,1/3]}=0$ and $\chi|_{[2/3,\infty)}=1$, and $\chi_r:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ is defined by $\chi_r(t)=\chi(t/r)$ for each $r>0$.

We can easily check the following inequalities, where $C>0$ is some constant and $0<\epsilon\le 1$ is arbitrary ($C>0$ does not depend on ϵ):

$$\begin{aligned}
C\|\tilde{\xi}\|_{W^{1,p}_\delta(\ell)} &\ge \epsilon\|\tilde{\xi}\|_{W^{1,p}_\delta(\ell_\rho\setminus\coprod_i([-\frac{1}{2}\log\rho_i,\infty)_i\cup(-\infty,\frac{1}{2}\log\rho_i]_{i+1}))}\\
&\quad+\sum_i\Big\|\sum_{\lambda_j^{(i)}>0}a_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{W^{1,p}_{\delta_i}([-\frac{1}{2}\log\rho_i,\infty)_i)}\\
&\quad+\sum_i\Big\|\sum_{\lambda_j^{(i)}<0}b_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{W^{1,p}_{\delta_i}((-\infty,\frac{1}{2}\log\rho_i]_{i+1})}\\
&\ge \epsilon\Big(\|\xi\|_{W^{1,p}_\delta(\ell_\rho)}-C\sum_i\Big\|\sum_{\lambda_j^{(i)}<0}a_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{W^{1,p}_{\delta_i}([0,-\frac{1}{2}\log\rho_i]_i)}\\
&\qquad-C\sum_i\Big\|\sum_{\lambda_j^{(i)}>0}b_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{W^{1,p}_{\delta_i}([\frac{1}{2}\log\rho_i,0]_{i+1})}\Big)\\
&\quad+\sum_i\Big\|\sum_{\lambda_j^{(i)}>0}a_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{W^{1,p}_{\delta_i}([-\frac{1}{2}\log\rho_i,\infty)_i)}\\
&\quad+\sum_i\Big\|\sum_{\lambda_j^{(i)}<0}b_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{W^{1,p}_{\delta_i}((-\infty,\frac{1}{2}\log\rho_i]_{i+1})},
\end{aligned}\tag{6}$$

$$\begin{aligned}
&\big\|D\tilde{\xi}+\lambda(h)\big\|_{L^p_\delta(\ell)}\\
&\le C\Big(\big\|D\xi+\lambda(h)\big\|_{L^p_\delta(\ell_\rho)}\\
&\quad+\sum_i(-\log\rho_i)^{-1}\Big\|\sum_{\lambda_j^{(i)}<0}a_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{L^p_{\delta_i}([0,-\frac{1}{2}\log\rho_i]_i)}\\
&\quad+\sum_i(-\log\rho_i)^{-1}\Big\|\sum_{\lambda_j^{(i)}>0}b_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{L^p_{\delta_i}([\frac{1}{2}\log\rho_i,0]_{i+1})}\Big),
\end{aligned}\tag{7}$$

$$\begin{aligned}
&\Big\|\sum_{\lambda_j^{(i)}<0}a_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{W^{1,p}_{\delta_i}([0,-\frac{1}{2}\log\rho_i]_i)}\\
&\quad\le\Big\|\sum_{\lambda_j^{(i)}<0}b_j^{(i)}e^{-\lambda_j^{(i)}t}\phi_j^{(i)}\Big\|_{W^{1,p}_{\delta_i}((-\infty,\frac{1}{2}\log\rho_i]_{i+1})},
\end{aligned}\tag{8}$$

$$
\begin{aligned}
&\Big\| \sum_{\lambda_j^{(i)}>0} b_j^{(i)} e^{-\lambda_j^{(i)} t} \phi_j^{(i)} \Big\|_{W^{1,p}_{\delta_i}([\frac{1}{2}\log\rho_i, 0]_{i+1})} \\
&\qquad \le \Big\| \sum_{\lambda_j^{(i)}>0} a_j^{(i)} e^{-\lambda_j^{(i)} t} \phi_j^{(i)} \Big\|_{W^{1,p}_{\delta_i}([-\frac{1}{2}\log\rho_i, \infty)_i)}.
\end{aligned} \tag{9}
$$

Apply equation (5) to $\tilde{\xi}$, and use (6) and (7). Then we obtain

$$
\begin{aligned}
&\epsilon\|\xi\|_{W^{1,p}_{\delta}(\ell_\rho)} + |h|_E + \sum_i \Big\| \sum_{\lambda_j^{(i)}>0} a_j^{(i)} e^{-\lambda_j^{(i)} t} \phi_j^{(i)} \Big\|_{W^{1,p}_{\delta_i}([-\frac{1}{2}\log\rho_i, \infty)_i)} \\
&\quad + \sum_i \Big\| \sum_{\lambda_j^{(i)}<0} b_j^{(i)} e^{-\lambda_j^{(i)} t} \phi_j^{(i)} \Big\|_{W^{1,p}_{\delta_i}((-\infty, \frac{1}{2}\log\rho_i]_{i+1})} \\
&\le C\Big(\big\|D\xi + \lambda(h)\big\|_{L^p_\delta(\ell_\rho)} + \sum_j \big|\langle \tilde{\xi}, \xi_j\rangle_{L^2(\bigcup_i \mathring{I}_i)} + \langle h, h_j\rangle_E\big| \\
&\quad + \sum_i \big(\epsilon + (-\log\rho_i)^{-1}\big) \Big\| \sum_{\lambda_j^{(i)}<0} a_j^{(i)} e^{-\lambda_j^{(i)} t} \phi_j^{(i)} \Big\|_{L^p_{\delta_i}([0, -\frac{1}{2}\log\rho_i]_i)} \\
&\quad + \sum_i \big(\epsilon + (-\log\rho_i)^{-1}\big) \Big\| \sum_{\lambda_j^{(i)}>0} b_j^{(i)} e^{-\lambda_j^{(i)} t} \phi_j^{(i)} \Big\|_{L^p_{\delta_i}([\frac{1}{2}\log\rho_i, 0]_{i+1})} \Big),
\end{aligned}
$$

where $0 < \epsilon \le 1$ is arbitrary and $C > 0$ is some constant independent of ϵ. Inequalities (8) and (9) and this inequality imply that, if $\rho = (\rho_i)$ is sufficiently small and we choose sufficiently small ϵ, then we obtain (4). □

6. Smoothness of the Kuranishi neighborhoods

In this section, we prove the smoothness of Kuranishi neighborhoods. More precisely, we prove that if we change the differential structure of $\overline{\mathcal{M}}_\mu^{\mathrm{SL}}$ to a stronger one, then for the product differential structure of $V = X \times B_\epsilon(0)$, the map

$$
\begin{aligned}
&V \hookrightarrow X \times C^l\Big(\coprod_i \mathring{I}_i, M\Big) \times E^0_{q_-,q_+}, \\
&(\rho, t^+, x) \mapsto \big((\rho, t^+), \Phi_\rho(\xi_{(\rho,t^+,x)})|_{\coprod_i \mathring{I}_i}, h_{(\rho,t^+,x)}\big)
\end{aligned} \tag{10}
$$

is a smooth embedding for any $l \ge 1$. (The differential structure of $X \subset \overline{\mathcal{M}}_\mu^{\mathrm{SL}}$ on the right-hand side is also the stronger one.)

The stronger differential structure of $\overline{\mathcal{M}}_\mu^{\mathrm{SL}}$ is defined as follows. Let

$$
\begin{aligned}
&[0,1)^{m-1} \times \mathcal{U}_{t^0} \to \overline{\mathcal{M}}_\mu^{\mathrm{SL}}, \\
&(\rho, t^+) \mapsto (\ell_\rho, t^+)
\end{aligned}
$$

be the coordinate defined in Section 2. We change the variables ρ_i with $\hat{\rho}_i$ defined by $\rho_i = \hat{\rho}_i^\beta$ for some fixed large number $\beta \ge 1$. It is easy to check that the new coordinates $((\hat{\rho}_i), t^+)$ define a stronger differential structure of $\overline{\mathcal{M}}_\mu^{\mathrm{SL}}$. In the following argument, we prove that for any $N \geq 1$, (10) is of class C^N if we choose sufficiently large $\beta \geq 1$. Note that once we prove that (10) is of class N, then we can easily see that (10) is an embedding. This is because x is determined by the equation

$$x = \sum_j \Big(\langle \xi, \xi_j \rangle_{L^2(\bigcup_i \mathring{I}_i)} + \sum_\alpha \langle h_\alpha, h_{j,\alpha} \rangle_{E^0_\alpha} \Big) \cdot x_j$$

for $((\rho, t^+), \Phi_\rho(\xi)|_{\bigcup_i \mathring{I}_i}, h) \in X \times C^l(\bigcup_i \mathring{I}_i, M) \times E^0_{q_-,q_+}$, where $\{x_j = (\xi_j, (h_{j,\alpha}))\}_j$ is an orthonormal basis of $\operatorname{Ker} DF^{(0,t^{0,+})}_{(0,0)}$. We also note that once we prove that (10) is a smooth embedding, then for any intervals $I_i^{\circ\circ} \subset \mathring{I}_i$,

$$V \hookrightarrow X \times C^l\Big(\coprod_i I_i^{\circ\circ}, M \Big) \times E^0_{q_-,q_+}$$

is also a smooth embedding. This is because if some differential vanishes on $\coprod_i I_i^{\circ\circ}$, then it also vanishes on $\coprod_i \mathring{I}_i$ since it is a solution of a homogeneous linear differential equation.

Let $X = \coprod_\Pi X_\Pi$ be the decomposition defined by

$$X_\Pi = \big\{ (\rho, t^+) \in X; \rho_i \neq 0 \text{ if and only if } i \in \Pi \big\}$$

for $\Pi \subset \{1, \dots, m-1\}$. We prove the differentiability of ϕ on each $X_\Pi \times B_\epsilon(0)$ and investigate the behavior of its differentials near the boundary.

Fix one point $(\rho, t^+) \in X_\Pi$. For the other points $(\tilde{\rho}, \tilde{t}^+) \in X_\Pi$ in the same stratum, we rewrite the operators $F^{(\tilde{\rho}, \tilde{t}^+)}$ as operators on ℓ_ρ by the identification of ℓ_ρ and $\ell_{\tilde{\rho}}$ given by the following piecewise smooth map Ψ. On each $\mathring{I}_i \subset \ell_\rho$, we define Ψ by the identity map. On $[0, -\frac{1}{2}\log \rho_i]_i$, we define $\Psi(t) = \tilde{t} \in [0, -\frac{1}{2}\log \tilde{\rho}_i]_i$ by

$$\frac{e^{-\kappa_i \tilde{t}} - \tilde{\rho}_i^{\kappa_i/2}}{1 - \tilde{\rho}_i^{\kappa_i/2}} = \frac{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}}{1 - \rho_i^{\kappa_i/2}}.$$

Similarly, we define $\Psi|_{[\frac{1}{2}\log \rho_i, 0]_{i+1}}(t) = \tilde{t} \in [\frac{1}{2}\log \tilde{\rho}_i, 0]_{i+1}$ by

$$\frac{e^{\kappa_i \tilde{t}} - \tilde{\rho}_i^{\kappa_i/2}}{1 - \tilde{\rho}_i^{\kappa_i/2}} = \frac{e^{\kappa_i t} - \rho_i^{\kappa_i/2}}{1 - \rho_i^{\kappa_i/2}}.$$

Then under this identification, each $F^{(\tilde{\rho}, \tilde{t}^+)} : W^{1,p}_\delta(\ell_\rho, \gamma_\rho^* TM) \oplus E^0_{q_-,q_+} \to L^p_\delta(\ell_\rho, \gamma_\rho^* TM)$ is written as

$$F^{(\tilde{\rho}, \tilde{t}^+)}(\xi, h) = \Big(1 + \Big(\frac{1 - \rho_i^{\kappa_i/2}}{1 - \tilde{\rho}_i^{\kappa_i/2}} \tilde{\rho}_i^{\kappa_i/2} - \rho_i^{\kappa_i/2} \Big) e^{\kappa_i t} \Big) \frac{d}{dt}(\gamma_\rho + \xi) + \nabla f(\gamma_\rho + \xi)$$

on $[0, -\frac{1}{2}\log \rho_i]_i$. Define $N_i = [0, -\frac{1}{2}\log \rho_i]_i \cup [\frac{1}{2}\log \rho_i, 0]_{i+1}$ for each i.

Although $(\tilde{t}^+_\alpha)_{\alpha \in \mathcal{A}_{q_-,q_+}}$ are not independent variables of $\mathcal{U}_{t^{0,+}}$, we can also define $F^{(\tilde{\rho}, \tilde{t}^+)}$ for $(\tilde{t}^+_\alpha)_{\alpha \in \mathcal{A}_{q_-,q_+}}$ in a neighborhood of $\mathcal{U}_{t^{0,+}} \subset \prod_{\alpha \in \mathcal{A}_{q_-,q_+}} (\mathbb{R}_1 \sqcup \cdots \sqcup \mathbb{R}_{m_0})$ by the same equation and regard them as independent variables.

We note that $\partial^k_{\tilde{\rho}_i} F^{(\tilde{\rho},\tilde{t}^+)}$ vanishes on the complement of N_i for $k > 0$. Similarly, $\partial^l_{\tilde{t}^+_\alpha} F^{(\tilde{\rho},\tilde{t}^+)}$ vanishes on the complement of $\mathring{I}_i$ for $l > 0$, where we assume that $[t^+_\alpha - C_\alpha, t^+_\alpha + C_\alpha]$ is contained in $\mathring{I}_i$.

LEMMA 6.1

(i) *Let $\delta = (\delta_i)_{0\le i\le m}$ and $\delta' = (\delta'_i)_{1\le i\le m-1}$ be nonnegative sequences such that $0 \le \delta_i \le \delta'_i < \kappa_i$ for $1 \le i \le m-1$ and $0 \le \delta_i < \delta_{0,i}$ for $i = 0, m$. Then for any $1 < p < \infty$, $i \in \Pi$, and $k > 0$, there exist constants $C > 0$ and $c_0 > 0$ such that the following hold for $\|\xi\|_{W^{1,p}_{\delta'}(\ell_\rho)} \le c_0$:*

$$\big\|(\partial^k_{\tilde{\rho}_i} F^{(\rho,t^+)})(\xi,h)\big\|_{L^p_\delta(\ell_\rho)} \le C\rho_i^{(\delta'_i-\delta_i)/2-k},$$

$$\big\|(D\partial^k_{\tilde{\rho}_i} F^{(\rho,t^+)})_{(\xi,h)}(\hat{\xi},\hat{h})\big\|_{L^p_\delta(\ell_\rho)} \le C\rho_i^{(\delta'_i-\delta_i)/2-k}\|\hat{\xi}\|_{\widetilde{W}^{1,p}_{\delta'_i}(N_i)},$$

$$D^n\partial^k_{\tilde{\rho}_i} F^{(\rho,t^+)} \equiv 0 \quad (n \ge 2).$$

(ii) *For any $1 < p < \infty$, $\alpha \in \mathcal{A}_{q_-,q_+}$, $l > 0$, and $n \ge 0$, there exist constants $C > 0$ and $c_0 > 0$ such that the following holds for (ξ, h) such that $\|\xi\|_{W^{1,p}(\bigcup_i \mathring{I}_i)} + |h_\alpha|_{E^0_\alpha} \le c_0$:*

$$\begin{aligned}&\big\|D^n\partial^l_{\tilde{t}^+_\alpha} F^{(\rho,t^+)}_{(\xi,h)}(\hat{\xi}^{(n)},\hat{h}^{(n)})\cdots(\hat{\xi}^{(1)},\hat{h}^{(1)})\big\|_{W^{1,p}(\bigcup_i \mathring{I}_i)}\\ &\quad\le C\prod_{j=1}^n\big(\|\hat{\xi}^{(j)}\|_{W^{1,p}(\bigcup_i \mathring{I}_i)} + |\hat{h}^{(j)}_\alpha|_{E^0_\alpha}\big).\end{aligned}$$

Proof

The proof is in two parts.

(i) We use a change of variable $\mathring{\rho}_i = (\tilde{\rho}_i)^{\kappa_i/2}$. Then on $[0, -\frac{1}{2}\log\rho_i]_i$,

$$\partial^k_{\mathring{\rho}_i} F^{(\tilde{\rho},\tilde{t}^+)}(\xi,h)|_{(\tilde{\rho},\tilde{t}^+)=(\rho,t^+)} = k!(1-\rho_i^{\kappa_i/2})^{-k}e^{\kappa_i t}\frac{d}{dt}(\gamma_\rho + \xi).$$

Since $|\frac{d\gamma_0}{dt}|_{[0,\infty)_i}(t)| \lesssim e^{-\delta'_{0,i}t}$ for any $\kappa_i < \delta'_{0,i} < \delta_{0,i}$,

$$\frac{d\gamma_\rho}{dt}\Big|_{[0,-\frac{1}{2}\log\rho_i]_i}(t) = \frac{e^{-\kappa_i t}}{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}}\frac{d\gamma_0}{dt}\Big(-\frac{1}{\kappa_i}\log\Big(\frac{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}}{1-\rho_i^{\kappa_i/2}}\Big)\Big)$$

satisfies

$$\begin{aligned}\Big|\frac{d\gamma_\rho}{dt}\Big|e^{\delta'_i t} &\lesssim \frac{e^{-\kappa_i t}}{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}}\cdot\Big(\frac{e^{-\kappa_i t} - \rho_i^{\kappa_i/2}}{1-\rho_i^{\kappa_i/2}}\Big)^{\delta'_{0,i}/\kappa_i}e^{\delta'_i t}\\ &= e^{-(\kappa_i-\delta'_i)t}\cdot\frac{(e^{-\kappa_i t} - \rho_i^{\kappa_i/2})^{\delta'_{0,i}/\kappa_i - 1}}{(1-\rho_i^{\kappa_i/2})^{\delta'_{0,i}/\kappa_i}}\\ &\lesssim e^{-(\kappa_i-\delta'_i)t}.\end{aligned}$$

Hence $\|\frac{d\gamma_\rho}{dt}\|_{L^p_{\delta'_i}([0,-\frac{1}{2}\log\rho_i]_i)} \lesssim 1$. The assumption $\|\frac{d\xi}{dt}\|_{L^p_{\delta'_i}([0,-\frac{1}{2}\log\rho_i]_i)} \lesssim 1$ and

$$e^{\kappa_i t}\Big|\frac{d}{dt}(\gamma_\rho+\xi)\Big|e^{\delta_i t}\le \rho_i^{(\delta'_i-\delta_i-\kappa_i)/2}\cdot\Big|\frac{d}{dt}(\gamma_\rho+\xi)\Big|e^{\delta'_i t}$$

for $t\in[0,-\frac{1}{2}\log\rho_i]_i$ imply that

$$\big\|\partial^k_{\mathring{\rho}_i}F^{(\tilde\rho,\tilde t^+)}(\xi,h)|_{(\tilde\rho,\tilde t^+)=(\rho,t^+)}\big\|_{L^p_\delta(\ell_\rho)}\lesssim\rho_i^{(\delta'_i-\delta_i)/2-\kappa_i/2}.$$

Similarly,

$$\big\|D\partial^k_{\mathring{\rho}_i}F^{(\tilde\rho,\tilde t^+)}_{(\xi,h)}(\hat\xi,\hat h)|_{(\tilde\rho,\tilde t^+)=(\rho,t^+)}\big\|_{L^p_\delta(\ell_\rho)}\lesssim\rho_i^{(\delta'_i-\delta_i)/2-\kappa_i/2}\|\hat\xi\|_{W^{1,p}_{\delta'_i}(N_i)}$$

and

$$D^2\partial^k_{\mathring{\rho}_i}F^{(\tilde\rho,\tilde t^+)}_{(\xi,h)}\equiv 0.$$

The claim follows from these inequalities because $\partial_{\tilde\rho_i}=\frac{\kappa_i}{2}(\tilde\rho_i)^{\kappa_i/2-1}\partial_{\mathring{\rho}_i}$.

(ii) The estimates follow from

$$\partial^l_{\tilde t^+_\alpha}F^{\rho,t^+}(\xi,h)(t)=(-1)^l\big(\partial^l_t\lambda_\alpha(h_\alpha)\big)\big(o_{t^+_\alpha}(t),\Phi\big(\xi(t)\big)\big).\qquad\square$$

Choose appropriate variables, and let $(t^+_{\alpha_d})_d$ be a coordinate of $\mathcal{U}_{t^{0,+}}$. Then Lemma 6.1 immediately implies the following. (In Lemma 6.1, we regard $(\tilde t^+_\alpha)$ as independent variables, but in the following corollary, we restrict $F^{\tilde\rho,\tilde t^+}$ to $\tilde t^+\in\mathcal{U}_{t^{0,+}}$.)

COROLLARY 6.2

(i) *Let $\delta=(\delta_i)_{0\le i\le m}$ and $\delta'=(\delta'_i)_{1\le i\le m-1}$ be nonnegative sequences such that $0\le\delta_i\le\delta'_i<\kappa_i$ for $1\le i\le m-1$ and $0\le\delta_i<\delta_{0,i}$ for $i=0,m$. Then for any $1<p<\infty$, $i\in\Pi$, and $k>0$, there exist constants $C>0$ and $c_0>0$ such that the following hold for $\|\xi\|_{W^{1,p}_{\delta'}(\ell_\rho)}\le c_0$:*

$$\big\|(\partial^k_{\tilde\rho_i}F^{(\rho,t^+)})(\xi,h)\big\|_{L^p_\delta(\ell_\rho)}\le C\rho_i^{(\delta'_i-\delta_i)/2-k},$$

$$\big\|(D\partial^k_{\tilde\rho_i}F^{(\rho,t^+)})_{(\xi,h)}(\hat\xi,\hat h)\big\|_{L^p_\delta(\ell_\rho)}\le C\rho_i^{(\delta'_i-\delta_i)/2-k}\|\hat\xi\|_{W^{1,p}_{\delta'_i}(N_i)},$$

$$D^n\partial^k_{\tilde\rho_i}F^{(\rho,t^+)}\equiv 0\quad(n\ge 2).$$

(ii) *For any $1<p<\infty$, $n\ge 0$, and multi-index $(l_{\alpha_d})\ne 0$, there exist constants $C>0$ and $c_0>0$ such that the following holds for (ξ,h) such that $\|\xi\|_{W^{1,p}(\bigcup_i\mathring{I}_i)}+|h|_{E^0_{q_-,q_+}}\le c_0$:*

$$\big\|D^n\partial^{(l_{\alpha_d})}_{(\tilde t^+_{\alpha_d})}F^{(\rho,t^+)}_{(\xi,h)}(\hat\xi^{(n)},\hat h^{(n)})\cdots(\hat\xi^{(1)},\hat h^{(1)})\big\|_{W^{1,p}(\bigcup_i\mathring{I}_i)}$$
$$\le C\prod_{j=1}^n\big(\|\hat\xi^{(j)}\|_{W^{1,p}(\bigcup_i\mathring{I}_i)}+|\hat h^{(j)}|_{E^0_{q_-,q_+}}\big).$$

Let $U\subset X_\Pi$ be a neighborhood of (ρ,t^+) in X_Π, and regard the family of smooth maps

$$\phi^{\tilde{\rho},\tilde{t}^+} : \operatorname{Ker} DF_{(0,0)}^{(0,t^{0,+})} \supset B_\epsilon(0) \to W_\delta^{1,p}(\ell_\rho, \gamma_\rho^* TM) \times E_{q_-,q_+}^0$$

as a map

$$\phi : U \times B_\epsilon(0) \to W_\delta^{1,p}(\ell_\rho, \gamma_\rho^* TM) \times E_{q_-,q_+}^0. \tag{11}$$

We estimate the derivative of ϕ at (ρ, t^+, x).

PROPOSITION 6.3

Let $\delta = (\delta_i)_{0\le i\le m}$ and $\delta' = (\delta'_i)_{1\le i\le m-1}$ be nonnegative sequences such that $0 \le \delta_i \le \delta'_i < \kappa_i$ for $1 \le i \le m-1$ and $0 \le \delta_i < \delta_{0,i}$ for $i = 0, m$. Then for any $1 < p < \infty$ and multi-index $((k_i)_{i\in\Pi}, (l_{\alpha_d})_d, l_x)$, there exists some constant $C > 0$ such that

$$\left\|\partial_{(\rho_i)}^{(k_i)} \partial_{(t_{\alpha_d}^+)}^{(l_{\alpha_d})} \partial_x^{l_x} \phi(\rho, t_\alpha^+, x)\right\|_{W_\delta^{1,p}(\ell_\rho, \gamma_\rho^* TM)\oplus E_{q_-,q_+}^0} \le C \prod_{\substack{i \\ k_i>0}} \rho_i^{(\delta'_i - \delta_i)/2 - k_i}.$$

Proof

We prove the estimate by induction on $|(k_i, l_{\alpha_d}, l_x)| = \sum_i k_i + \sum_d l_{\alpha_d} + |l_x|$. The case $(k_i, l_{\alpha_d}, l_x) = (0,0,0)$ is obvious. Assuming the estimates for $|(k_i, l_{\alpha_d}, l_x)| < N$, we prove the estimates for $|(k_i, l_{\alpha_d}, l_x)| = N$. Differentiating the equation $F^{(\tilde{\rho},\tilde{t}^+)+}(\phi(\tilde{\rho}, \tilde{t}^+, x)) = (0, x)$ by $\partial_{(\rho_i)}^{(k_i)} \partial_{(t_{\alpha_d}^+)}^{(l_{\alpha_d})} \partial_x^l$, we obtain an equation in the following form:

$$\begin{aligned}
&(DF^{(\rho,t^+)+})_{\phi(\rho,t^+,x)} \partial_{(\rho_i)}^{(k_i)} \partial_{(t_{\alpha_d}^+)}^{(l_{\alpha_d})} \partial_x^l \phi(\rho, t^+, x) \\
&\qquad + \sum_{i_0} \sum_{\star_1} (D^n \partial_{\rho_{i_0}}^{k_{i_0}} F^{(\rho,t^+)+})_{\phi(\rho,t^+,x)} (\hat{\xi}^{(n)}, \hat{h}^{(n)}) \cdots (\hat{\xi}^{(1)}, \hat{h}^{(1)}) \\
&\qquad + \sum_{\star_2} (D^n \partial_{(\rho_i)}^{(k'_i)} \partial_{(t_{\alpha_d}^+)}^{(l'_{\alpha_d})} F^{(\rho,t^+)+})_{\phi(\rho,t^+,x)} (\hat{\xi}^{(n)}, \hat{h}^{(n)}) \cdots (\hat{\xi}^{(1)}, \hat{h}^{(1)}) \\
&\quad = 0,
\end{aligned}$$

where each $(\hat{\xi}^{(i)}, \hat{h}^{(i)})$ is some derivative of ϕ, and the sum of the indices of the differentials in each term is equal to (k_i, l_{α_d}, l_x); in the sum $\star_1$, $k_{i_0} > 0$; and in the sum $\star_2$, $k'_i < k_i$ if $k_i > 0$. (As we noted, the term is 0 if more than one of k'_i are nonzero.)

First we estimate the norm of each term in the sum $\star_1$. For each $1 \le i_0 \le m-1$, define a sequence of nonnegative constants $(\delta''_i)_{0\le i\le m}$ by $\delta''_{i_0} = \delta'_{i_0}$ and $\delta''_i = \delta_i$ for $i \ne i_0$. Corollary 6.2(i) and the assumption of the induction imply that the L_δ^p-norm of each term in the sum $\star_1$ is bounded by

$$\begin{aligned}
&\left\|(D^n \partial_{\rho_{i_0}}^{k_{i_0}} F^{(\rho,t^+)+})_{\phi(\rho,t^+,x)} (\hat{\xi}^{(n)}, \hat{h}^{(n)}) \cdots (\hat{\xi}^{(1)}, \hat{h}^{(1)})\right\|_{L_\delta^p(\ell_\rho)} \\
&\quad \lesssim \rho_{i_0}^{(\delta''_{i_0} - \delta_{i_0})/2 - k_{i_0}} \prod_j \|\hat{\xi}^{(j)}\|_{W_{\delta''}^{1,p}(\ell_\rho)} \\
&\quad \lesssim \prod_{\substack{i \\ k_i>0}} \rho_i^{(\delta'_i - \delta_i)/2 - k_i}.
\end{aligned} \tag{12}$$

Next we estimate the norm of each term in the sum $\star_2$. Note that if $k_i > 0$, then the differential ∂_{ρ_i} appears in some $(\xi^{(i)}, h^{(i)})$. Hence Corollary 6.2(i) for $\delta' = \delta$, (ii) and the assumption of the induction imply that

$$\begin{aligned}
&\big\|(D^n \partial_{(\rho_i)}^{(k'_i)} \partial_{(t^+_{\alpha_d})}^{(l'_{\alpha_d})} F^{(\rho,t^+)+})_{\phi(\rho,t^+,x)}(\hat{\xi}^{(n)}, \hat{h}^{(n)}) \cdots (\hat{\xi}^{(1)}, \hat{h}^{(1)})\big\|_{L^p_\delta(\ell_\rho)} \\
&\lesssim \prod_i \rho_i^{-k'_i} \cdot \prod_{j=1}^n \big(\|\hat{\xi}^{(j)}\|_{W^{1,p}_\delta(\ell_\rho)} + |\hat{h}^{(j)}|_{E^0_{q_-,q_+}}\big) \qquad (13)\\
&\lesssim \prod_{\substack{i\\ k_i>0}} \rho_i^{(\delta'_i-\delta_i)/2-k_i}.
\end{aligned}$$

Equations (12) and (13) imply that

$$\big\|(DF^{(\rho,t^+)+})_{\phi(\rho,t^+,x)} \partial_{(\rho_i)}^{(k_i)} \partial_{(t^+_{\alpha_d})}^{(l_{\alpha_d})} \partial_x^l \phi(\rho,t^+,x)\big\|_{L^p_\delta(\ell_\rho)} \lesssim \prod_{\substack{i\\ k_i>0}} \rho_i^{(\delta'_i-\delta_i)/2-k_i}.$$

Since the norm of $(DF^{(\rho,t^+)+})^{-1}_{\phi(\rho,t^+,x)}$ is uniformly bounded, this implies the claim. □

Next we regard the family of smooth maps

$$\phi^{\rho,t^+} : \operatorname{Ker} DF^{(0,t^{0,+})}_{(0,0)} \supset B_\epsilon(0) \to W^{1,p}_\delta(\ell_\rho, \gamma_\rho^* TM) \times E^0_{q_-,q_+}$$

as a map

$$\begin{aligned}
&\phi : X \times B_\epsilon(0) \to W^{1,p}\Big(\coprod_i \mathring{I}_i, \gamma_0^* TM|_{\coprod_i \mathring{I}_i}\Big) \times E^0_{q_-,q_+}, \qquad (14)\\
&(\rho, t^+, x) \mapsto \big(\Phi_\rho(\xi_{(\rho,t^+,x)})|_{\coprod_i \mathring{I}_i}, h_{(\rho,t^+,x)}\big).
\end{aligned}$$

For a neighborhood $U \subset X_\Pi$ of each point $(\rho, t^+) \in X_\Pi$, the restriction of (14) to $U \times B_\epsilon(0)$ is the composition of (11) and the projection $W^{1,p}_\delta(\ell_\rho, \gamma_\rho^* TM) \times E^0_{q_-,q_+} \to W^{1,p}(\coprod_i \mathring{I}_i, \gamma_0^* TM|_{\coprod_i \mathring{I}_i}) \times E^0_{q_-,q_+}$. Furthermore, the norm of this projection is uniform with respect to (ρ, t^+). Therefore, Proposition 6.3 implies that

$$\big\|\partial_{(\rho_i)}^{(k_i)} \partial_{(t^+_{\alpha_d})}^{(l_{\alpha_d})} \partial_x^{l_x} \phi(\rho, t^+_\alpha, x)\big\|_{W^{1,p}(\coprod_i \mathring{I}_i, \gamma_0^* TM|_{\coprod_i \mathring{I}_i}) \oplus E^0_{q_-,q_+}} \le C \prod_{\substack{i\\ k_i>0}} \rho_i^{(\delta'_i-\delta_i)/2-k_i}.$$

The same estimate holds for any Sobolev norm $W^{k,p}$ or C^l-norm instead of $W^{1,p}$ if we change the constant $C > 0$ because we can estimate the higher derivatives of $\xi_{(\rho,t^+,x)}|_{\coprod_i \mathring{I}_i}$ by the differential equation

$$\begin{aligned}
&\frac{d}{dt}\big(\Phi_\rho\big(\xi_{(\rho,t^+,x)}(t)\big)\big) + \nabla f\big(\Phi_\rho\big(\xi_{(\rho,t^+,x)}(t)\big)\big)\\
&\quad + \sum_{\alpha \in \mathcal{A}_{q_-,q_+}} \lambda_\alpha\big((h_{(\rho,t^+,x)})_\alpha\big)\big(o_{t^+_\alpha}(t), \Phi_\rho\big(\xi_{(\rho,t^+,x)}(t)\big)\big) = 0.
\end{aligned}$$

Since the above estimates hold for arbitrary $0 \le \delta_i \le \delta'_i < \kappa_i$, the following corollary holds true.

COROLLARY 6.4

For any $l \geq 1$, $0 < \delta'_i < \kappa_i$, $\Pi \subset \{1, \dots, m-1\}$ *and any multi-index* $((k_i)_{i\in\Pi}, (l_{\alpha_d})_d, l_x)$*, there exists some constant* $C > 0$ *such that*

$$\left\|\partial_{(\rho_i)}^{(k_i)}\partial_{(t^+_{\alpha_d})}^{(l_{\alpha_d})}\partial_x^{l_x}\phi(\rho, t^+_\alpha, x)\right\|_{C^l(\coprod_i \mathring{I}_i, \gamma_0^* TM|_{\coprod_i \mathring{I}_i})\oplus E^0_{q_-,q_+}} \leq C \prod_{\substack{i\\ k_i>0}} \rho_i^{\delta'_i/2-k_i} \tag{15}$$

for all $(\rho, t^+, x) \in X_\Pi \times B_\epsilon(0)$ *sufficiently close to* $(0, t^{0,+}, 0)$.

Recall that we define a strong differential structure of X by using a large constant $\beta > 0$, and we define the differential structure of V by the product differential structure.

COROLLARY 6.5

For any $N \geq 1$,

$$\phi : V \to C^l\Big(\coprod_i \mathring{I}_i, M\Big) \times E^0_{q_-,q_+},$$

$$(\rho, t^+, x) \mapsto \big(\Phi_\rho(\xi_{(\rho,t^+,x)})|_{\coprod_i \mathring{I}_i}, h_{(\rho,t^+,x)}\big)$$

is of class C^N *if* $\beta > 0$ *is sufficiently large.*

Proof

If we change the coordinate ρ_i to $\hat\rho_i$ by $\rho_i = \hat\rho_i^\beta$, then (15) in the previous corollary is equivalent to

$$\left\|\partial_{(\hat\rho_i)}^{(k_i)}\partial_{(t^+_{\alpha_d})}^{(l_{\alpha_d})}\partial_x^{l_x}\phi(\rho, t^+_\alpha, x)\right\|_{C^l(\coprod_i \mathring{I}_i, \gamma_0^* TM|_{\coprod_i \mathring{I}_i})\oplus E^0_{q_-,q_+}} \leq C \prod_{\substack{i\\ k_i>0}} \hat\rho_i^{\beta\delta'_i/2-k_i}.$$

If $\beta > 0$ is sufficiently large, then $\beta\delta'_i/2 - N > 0$. Hence the claim follows from the following easy lemma. □

LEMMA 6.6

Let $f : \mathbb{R}^m \to \mathbb{R}$ *be a continuous map whose restriction to* $\mathbb{R}^m_\Pi = \{(t_i) \in \mathbb{R}^m; t_i \neq 0$ *if and only if* $i \in \Pi\}$ *is smooth for each* $\Pi \in \{1, \dots, m\}$*. We assume that there exist constants* $A > N$ *and* $C > 0$ *such that*

$$\left|\partial_{t_1}^{k_1} \cdots \partial_{t_m}^{k_m} f(t)\right| \leq C \prod_{k_i>0} t_i^{A-k_i}$$

on $\mathbb{R}^m_\Pi$ *for any* $\Pi \subset \{1, \dots, m\}$ *and multi-index* $(k_i)_{i\in\Pi}$*. Then* $f : \mathbb{R}^m \to \mathbb{R}$ *is of class* C^N *on the whole of* $\mathbb{R}^m$.

Proof

For any multi-index $(k_i)_{1\leq i\leq m}$, $\partial_{t_1}^{k_1} \cdots \partial_{t_m}^{k_m} f$ is defined on $\bigcup_{\Pi\supset\{i;k_i\neq 0\}} \mathbb{R}^m_\Pi$. (We do not need to assume that $\partial_{t_1}^{k_1} \cdots \partial_{t_m}^{k_m} f$ is continuous on $\bigcup_{\Pi\supset\{i;k_i\neq 0\}} \mathbb{R}^m_\Pi$. It is enough to assume that it is continuous on each $\mathbb{R}^m_\Pi$. Its continuity on $\bigcup_{\Pi\supset\{i;k_i\neq 0\}} \mathbb{R}^m_\Pi$ follows from the argument below.) We define $\partial_{t_1}^{k_1} \cdots \partial_{t_m}^{k_m} f$ on the complement of $\bigcup_{\Pi\supset\{i;k_i\neq 0\}}\mathbb{R}^m_\Pi$ by 0 for any multi-index $(k_i)_{1\le i\le m}$ such that $\sum_i k_i\le N$. It is enough to show that these functions satisfy

$$\partial^{(k_i)}_{(t_i)}f(t')-\partial^{(k_i)}_{(t_i)}f(t)=\sum_j(t'_j-t_j)\int_0^1\partial_{t_j}\partial^{(k_i)}_{(t_i)}f\big(t+(t'-t)s\big)\,ds \tag{16}$$

for any $t,t'\in\mathbb{R}^m$ and multi-index $(k_i)_{1\le i\le m}$ such that $\sum_i k_i\le N$. (Note that the integrands on the right-hand side are integrable by the assumption.) If we choose an appropriate $0<a<1$, then $t''=t+a(t'-t)$ satisfies the following condition: $t''_i\neq 0$ if $t_i\neq 0$ or $t'_i\neq 0$ for each i. It is enough to prove the above equality for the pair of t and t'' (and the pair of t' and t''). Hence we may assume that $t'_i\neq 0$ if $t_i\neq 0$. Namely, if $t'\in\mathbb{R}^m_{\Pi'}$ and $t\in\mathbb{R}^m_\Pi$, then $\Pi'\supset\Pi$.

First we consider the case where $(k_i)_i=0$. In this case, the smoothness of f on $\mathbb{R}^m_{\Pi'}$ implies that for any small $\epsilon>0$,

$$f(t')-f\big(t+\epsilon(t'-t)\big)=\sum_j(t'_j-t_j)\int_\epsilon^1\partial_{t_j}f\big(t+(t'-t)s\big)\,ds.$$

Hence the continuity of f implies (16).

Next we consider the case where $k_i=0$ for all $i\notin\Pi$. We can prove this case by differentiating (16) for $(k_i)_i=0$ by the differential $\partial^{(k_i)}_{(t_i)}$.

Next we consider the case where $k_i=0$ for all $i\notin\Pi'$ and $k_i\neq 0$ for some $i\in\Pi$. In this case, the smoothness of f on $\mathbb{R}^m_{\Pi'}$ implies that

$$\partial^{(k_i)}_{(t_i)}f(t')-\partial^{(k_i)}_{(t_i)}f\big(t+\epsilon(t'-t)\big)=\sum_j(t'_j-t_j)\int_\epsilon^1\partial_{t_j}\partial^{(k_i)}_{(t_i)}f\big(t+(t'-t)s\big)\,ds$$

for small $\epsilon>0$. Since $\partial^{(k_i)}_{(t_i)}f(t+\epsilon(t'-t)))$ converges to 0 as ϵ converges to 0, (16) holds.

Finally, we consider the case where $k_i\neq 0$ for some $i\in\Pi'$. In this case, (16) holds because both sides are 0.

Hence f is of class C^N on the whole of $\mathbb{R}^m$. □

7. Smoothness of coordinate changes

Let $(V_1,E_{q_-,q_+},s_1,\psi_1)$ and $(V_2,E_{q_-,q_+},s_2,\psi_2)$ be Kuranishi neighborhoods of $p_1=(\ell^1,\gamma^1_1)$ and $p_2=(\ell^2,\gamma^2_2)$ in $\overline{\mathcal{M}}_0(q_-,q_+)$, respectively. Assume that $\psi_1(s_1^{-1}(0))\cap\psi_2(s_2^{-1}(0))\neq\emptyset$. In this section, for each point $p_0\in\psi_1(s_1^{-1}(0))\cap\psi_2(s_2^{-1}(0))$, we construct an open neighborhood $V_1^0\subset V_1$ of $\psi_1^{-1}(p_0)\in V_1$ and a natural open embedding $\phi:V_1^0\to V_2$ such that $s_2\circ\phi=s_1$ on V_1^0 and $\psi_2\circ\phi=\psi_1$ on $s_1^{-1}(0)\cap V_1^0$.

We may regard V_1 and V_2 as $V_1\subset X_1\times C^{l_1}(\coprod_i I_i^{\circ\circ,1},M)\times E^0_{q_-,q_+}$ and $V_2\subset X_2\times C^{l_2}(\coprod_i I_i^{\circ\circ,2},M)\times E^0_{q_-,q_+}$ for arbitrary intervals $I_i^{\circ\circ,1}\subset\mathring{I}_i^1\subset\ell^1$ and $I_i^{\circ\circ,2}\subset\mathring{I}_i^2\subset\ell^2$. We may assume that $l_1\gg l_2$. We denote the points of V_1 and V_2 as $(\rho^1,t^{1,+},\gamma^1,h^1)$ and $(\rho^2,t^{2,+},\gamma^2,h^2)$, respectively, where $(\rho^1,t^{1,+})\in X_1$, $\gamma^1\in C^{l_1}(\coprod_i I_i^{\circ\circ,1},M)$, $h^1\in E^0_{q_-,q_+}$ and so on.

Let $(\rho_0^1, t_0^{1,+}, \gamma_0^1, h_0^1) \in V_1$ and $(\rho_0^2, t_0^{2,+}, \gamma_0^2, h_0^2) \in V_2$ be the points corresponding to p_0. (Namely, they are $\psi_1^{-1}(p_0)$ and $\psi_2^{-1}(p_0)$, respectively.) Let $V_1^0 \subset V_1$ be a neighborhood of $(\rho_0^1, t_0^{1,+}, \gamma_0^1, h_0^1) \in V_1$. We construct a map $\phi : (\rho^1, t^{1,+}, \gamma^1, h^1) \mapsto (\rho^2, t^{2,+}, \gamma^2, h^2)$ from V_1^0 to V_2 as follows.

First we define $h^2 = h^1$.

Since X_1 and X_2 are the coordinates of $\overline{\mathcal{M}}^{\mathrm{SL}}_{\#\mathcal{A}_{q_-,q_+}}$, the coordinate change on $\overline{\mathcal{M}}^{\mathrm{SL}}_{\#\mathcal{A}_{q_-,q_+}}$ defines a diffeomorphism $(\rho^1, t^{1,+}) \mapsto (\rho^2, t^{2,+})$ from a neighborhood of $(\rho_0^1, t_0^{1,+})$ to a neighborhood of $(\rho_0^2, t_0^{2,+})$. For each point $(\rho^1, t^{1,+})$ in a neighborhood of $(\rho_0^1, t_0^{1,+})$, let $\Xi_{(\rho^1, t^{1,+})} : \ell^1_{\rho^1} \to \ell^2_{\rho^2}$ be the translation map which maps $t_\alpha^{1,+}$ to $t_\alpha^{2,+}$ for all $\alpha \in \mathcal{A}_{q_-,q_+}$.

First we assume that $\Xi_{(\rho^1,t^{1,+})}(\coprod_i I_i^{\circ\circ,1}) \supset \coprod_i I_i^{\circ\circ,2}$. Then we define $\gamma^2 \in C^{l_2}(\coprod_i I_i^{\circ\circ,2}, M)$ by $\gamma^2 = \gamma^1 \circ (\Xi_{(\rho^1,t^{1,+})})^{-1}$.

It is clear that the map $\phi : (\rho^1, t^{1,+}, \gamma^1, h^1) \mapsto (\rho^2, t^{2,+}, \gamma^2, h^2)$ constructed above is smooth. Since we can construct its inverse by the same way, it is a diffeomorphism. (For the construction of the inverse, we may use different intervals $I_i^{\circ\circ,l} \subset \mathring{I}_i^l$ $(l = 1, 2)$ such that $\Xi_{(\rho^1,t^{1,+})}(\coprod_i I_i^{\circ\circ,1}) \subset \coprod_i I_i^{\circ\circ,2}$.)

So far we have assumed that $\Xi_{(\rho^1,t^{1,+})}(\coprod_i I_i^{\circ\circ,1}) \supset \coprod_i I_i^{\circ\circ,2}$. Since $I_i^{\circ\circ,l} \subset \mathring{I}_i^l$ $(l = 1, 2)$ are arbitrary nonempty intervals, if $\mathring{I}_i^2$ are sufficiently large, then we may assume this condition. First, using this argument, we can prove that the smooth structure of the Kuranishi neighborhood of a point does not depend on the choice of the intervals $\mathring{I}_i$ used for the construction. Then we can take arbitrary intervals $I_i^{\circ\circ,l} \subset \mathbb{R}_i$ in the above argument. Hence the coordinate change $\phi : V_1^0 \to V_2$ is always smooth. It is clear that ϕ satisfies $s_2 \circ \phi = s_1$ on V_1^0 and $\psi_2 \circ \phi = \psi_1$ on $s_1^{-1}(0) \cap V_1^0$.

8. Structure of the boundary and corners

Let $p_0 = (\ell^0, \gamma^0) \in \overline{\mathcal{M}}_0(q_0, q_2)$ be a stable trajectory obtained by concatenating two stable trajectories $p_1 = (\ell^1, \gamma^1) \in \overline{\mathcal{M}}_0(q_0, q_1)$ and $p_2 = (\ell^2, \gamma^2) \in \overline{\mathcal{M}}_0(q_1, q_2)$. Let $(V_0, E_{q_0,q_2}, s_0, \psi_0)$, $(V_1, E_{q_0,q_1}, s_1, \psi_1)$, and $(V_2, E_{q_1,q_2}, s_2, \psi_2)$ be the Kuranishi neighborhoods of p_0, p_1, and p_2, respectively. We regard V_1 and V_2 as $V_1 \subset X_1 \times C^{l_1}(\coprod_i \mathring{I}_i^1, M) \times E^0_{q_0,q_1}$ and $V_2 \subset X_2 \times C^{l_2}(\coprod_i \mathring{I}_i^2, M) \times E^0_{q_1,q_2}$, respectively. Since we may use the intervals $\mathring{I}_i^1$ and $\mathring{I}_i^2$ for the construction of the Kuranishi neighborhood of p_0, we may regard V_0 as $V_0 \subset X_0 \times C^{l_1}(\coprod_i \mathring{I}_i^1 \sqcup \coprod_i \mathring{I}_i^2, M) \times E^0_{q_0,q_2}$.

Note that $X_1 \times X_2 \subset X_0$ is a part of the boundary of $\overline{\mathcal{M}}^{\mathrm{SL}}_{\#\mathcal{A}_{q_0,q_2}}$, and that $E^0_{q_0,q_2} = E^0_{q_0,q_1} \oplus E^0_{q_1,q_2}$. Hence we can construct the natural embedding $V_1 \times V_2 \hookrightarrow V_0$ by $((\rho^1, t^{1,+}, \gamma^1, h^1), (\rho^2, t^{2,+}, \gamma^2, h^2)) \mapsto ((\rho^1, t^{1,+}, \rho^2, t^{2,+}), (\gamma^1 \sqcup \gamma^2), (h^1, h^2))$. It is clear that its image is a part of the boundary of V_0. More precisely, let $V_0^{(q_0,q_1,q_2)} \subset V_0$ be the submanifold defined by $\rho_{m_1} = 0$, where m_1 is the length of p_1. Then we can construct a natural open embedding $V_0^{(q_0,q_1,q_2)} \hookrightarrow V_1 \times V_2$ if we shrink V_0 to a neighborhood of $\psi_{p_0}^{-1}(p_0)$. More generally, let $P_0 = (q_0 = q_-, q_1, \dots, q_k = q_+)$ be a decreasing sequence of critical points, and assume that a stable trajectory $p_0 \in \overline{\mathcal{M}}_0(q_-, q_+)$ is the concatenation of stable trajectories $p_i \in \overline{\mathcal{M}}_0(q_{i-1}, q_i)$ $(1 \le i \le k)$. Then we can construct a natural open embedding $V_0^{P_0} \hookrightarrow V_1 \times \cdots \times V_k$, where $(V_i, E_{q_{i-1},q_i}, s_i, \psi_i)$ are the Kuranishi neighborhoods of p_i, and $V_0^{P_0} \subset V_0$ is the submanifold defined by $\rho_i = 0$ for all $i \in \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_{k-1}\}$. (Here m_i is the length of p_i.)

9. Construction of global Kuranishi neighborhoods

For each point $p \in \overline{\mathcal{M}}_0(q_-, q_+)$, let $(V_p, E_{q_-,q_+}, s_p, \psi_p)$ be the Kuranishi neighborhood of p. The argument of Section 7 implies that for two points $p_1, p_2 \in \overline{\mathcal{M}}_0(q_-, q_+)$ such that $\psi_{p_1}(s_{p_1}^{-1}(0)) \cap \psi_{p_2}(s_{p_2}^{-1}(0)) \ne \emptyset$, there exists an open neighborhood $V_{p_2,p_1} \subset V_{p_1}$ of $\psi_{p_1}^{-1}(\psi_{p_2}(s_{p_2}^{-1}(0)))$ and an open embedding $\phi_{p_2,p_1} : V_{p_2,p_1} \to V_{p_2}$ such that $s_{p_2} \circ \phi_{p_2,p_1} = s_{p_1}$ on V_{p_2,p_1} and $\psi_{p_2} \circ \phi_{p_2,p_1} = \psi_{p_1}$ on $s_{p_1}^{-1}(0) \cap V_{p_2,p_1}$. Furthermore, for any triple $p_1, p_2, p_3 \in \overline{\mathcal{M}}_0(q_-, q_+)$ such that $\psi_{p_1}(s_{p_1}^{-1}(0)) \cap \psi_{p_2}(s_{p_2}^{-1}(0)) \cap \psi_{p_3}(s_{p_3}^{-1}(0)) \ne \emptyset$, $\phi_{p_3,p_2} \circ \phi_{p_2,p_1} = \phi_{p_3,p_1}$ on (at least) some neighborhood $V_{p_3,p_2,p_1} \subset V_{p_1}$ of $\psi_{p_1}^{-1}(\psi_{p_2}(s_{p_2}^{-1}(0)) \cap \psi_{p_3}(s_{p_3}^{-1}(0)))$. Shrinking V_{p_2,p_1} if necessary, we assume that $\phi_{p_2,p_1}(V_{p_2,p_1}) = V_{p_1,p_2}$ and $\phi_{p_1,p_2} = \phi_{p_2,p_1}^{-1}$. Similarly, we assume that $V_{p_2,p_3,p_1} = V_{p_3,p_2,p_1}$ and $\phi_{p_2,p_1}(V_{p_3,p_2,p_1}) = V_{p_3,p_1,p_2}$. In this section, we construct a global Kuranishi neighborhood $(V_{q_-,q_+}, E_{q_-,q_+}, s_{q_-,q_+}, \psi_{q_-,q_+})$ by patching the Kuranishi neighborhoods $(V_p, E_{q_-,q_+}, s_p, \psi_p)$.

Let $\overline{\mathcal{M}}_0(q_-, q_+) = \bigcup_j K_j$ be a finite cover by compact subsets such that for each j, there exists some point $p_j \in \overline{\mathcal{M}}_0(q_-, q_+)$ such that $K_j \subset \psi_{p_j}(s_{p_j}^{-1}(0))$. Let $\mathring{V}_{p_j} \Subset V_{p_j}$ be open neighborhoods of $\psi_{p_j}^{-1}(K_j)$ which satisfy the following conditions:

- For each pair i, j such that $K_i \cap K_j \ne \emptyset$, we define $\mathring{V}_{p_j,p_i} \subset \mathring{V}_{p_i}$ by $\mathring{V}_{p_j,p_i} = \mathring{V}_{p_i} \cap \phi_{p_j,p_i}^{-1}(\mathring{V}_{p_j})$. Then $\mathring{V}_{p_j,p_i}$ is relatively compact in V_{p_j,p_i}.
- For any triple j_1, j_2, j_3 such that $K_{j_1} \cap K_{j_2} \cap K_{j_3} \ne \emptyset$, $\mathring{V}_{p_{j_2},p_{j_1}} \cap \mathring{V}_{p_{j_3},p_{j_1}} \subset V_{p_{j_3},p_{j_2},p_{j_1}}$.
- For any triple j_1, j_2, j_3, if $K_{j_1} \cap K_{j_2} \cap K_{j_3} = \emptyset$, $K_{j_1} \cap K_{j_2} \ne \emptyset$ and $K_{j_1} \cap K_{j_3} \ne \emptyset$, then $\mathring{V}_{p_{j_2},p_{j_1}} \cap \mathring{V}_{p_{j_3},p_{j_1}} = \emptyset$.

(These conditions are satisfied if $\mathring{V}_{p_j} \Subset V_{p_j}$ are sufficiently small open neighborhoods of $\psi_{p_j}^{-1}(K_j)$.) We define an equivalence relation $\sim$ on $\coprod_j \mathring{V}_{p_j}$ as follows. For $x \in \mathring{V}_{p_i}$ and $y \in \mathring{V}_{p_j}$, $x \sim y$ if $K_i \cap K_j \ne \emptyset$ and $y = \phi_{p_j,p_i}(x)$. It is easy to check that the above conditions imply that this is indeed an equivalence relation.

We define a manifold V_{q_-,q_+} by $V_{q_-,q_+} = \coprod_j \mathring{V}_{p_j} / \sim$. First we check the Hausdorffness of V_{q_-,q_+}. For any $x \in \mathring{V}_{p_i}$ and $y \in \mathring{V}_{p_j}$ such that $x \nsim y$, if $K_i \cap K_j = \emptyset$, then $\mathring{V}_{p_i}$ and $\mathring{V}_{p_j}$ are their disjoint open neighborhoods in V_{q_-,q_+}. If $K_i \cap K_j \ne \emptyset$ and $x \notin \overline{\mathring{V}_{p_j,p_i}}$, then $\mathring{V}_{p_i} \setminus \overline{\mathring{V}_{p_j,p_i}}$ and $\mathring{V}_{p_j}$ are their disjoint open neighborhoods in V_{q_-,q_+}. If $x \in \overline{\mathring{V}_{p_j,p_i}}$ and $y \in \overline{\mathring{V}_{p_i,p_j}}$, let $U_1, U_2 \subset V_{p_j,p_i}$ be disjoint open neighborhoods of $x, \phi_{p_i,p_j}(y) \in V_{p_j,p_i}$, respectively. Then $U_1 \cap \mathring{V}_{p_1}$ and $\phi_{p_j,p_i}(U_2) \cap \mathring{V}_{p_2}$ are disjoint open neighborhoods of x and y in V_{q_-,q_+}. Hence V_{q_-,q_+} is a Hausdorff space. Since ϕ_{p_j,p_i} are diffeomorphisms, the smooth structures of $\mathring{V}_{p_j}$ define the smooth structure of V_{q_-,q_+}.

The maps s_{p_j} induce a smooth map $s_{q_-,q_+} : V_{q_-,q_+} \to E_{q_-,q_+}$. Similarly, the homeomorphisms ψ_{p_j} induce $\psi_{q_-,q_+} : s^{-1}_{q_-,q_+}(0) \to \overline{\mathcal{M}}_0(q_-,q_+)$. Then $(V_{q_-,q_+}, E_{q_-,q_+}, s_{q_-,q_+}, \psi_{q_-,q_+})$ is a global Kuranishi neighborhood of $\overline{\mathcal{M}}_0(q_-,q_+)$.

For each decreasing sequence $P_0 = (q_0 = q_-, q_1, \ldots, q_m = q_+)$ of critical points, define the submanifold $V^{P_0}_{q_-,q_+} \subset V_{q_-,q_+}$ by $V^{P_0}_{q_-,q_+} = \coprod_j \mathring{V}^{P_0}_{p_j} / \sim$. The argument in Section 8 implies that if we shrink V_{q_-,q_+} by the induction of $f(q_-) - f(q_+)$, then we can construct a natural open embedding

$$\phi_{P_0} : V^{P_0}_{q_-,q_+} \to V_{q_-,q_1} \times V_{q_1,q_2} \times \cdots \times V_{q_{m-1},q_+}.$$

It is easy to check that these open embeddings satisfy the conditions in Proposition 1.2.

10. Some remarks

We can construct the global Kuranishi neighborhoods of marked stable trajectories $\overline{\mathcal{M}}_\mu(q_-,q_+)$ similarly. In this case, for the construction of a Kuranishi neighborhood of a point $p_0 = (\ell = \overline{\mathbb{R}}_1 \cup \cdots \cup \overline{\mathbb{R}}_m, t^0, \gamma_0)$, we use the coordinate $[0,1)^{m-1} \times \mathcal{U}_{t^0 \cup t^{0,+}} \to \overline{\mathcal{M}}^{\mathrm{SL}}_{\mu + \#\mathcal{A}_{q_-,q_+}}$ instead of $[0,1)^{m-1} \times \mathcal{U}_{t^{0,+}} \to \overline{\mathcal{M}}^{\mathrm{SL}}_{\#\mathcal{A}_{q_-,q_+}}$. We can also construct open embeddings for the corners of global Kuranishi neighborhoods similarly.

For the space of marked stable trajectories, we can also define the forgetful map. For a subset $C \subset \{1, \ldots, \mu\}$, we define the forgetful map $\mathrm{forget}_C : \overline{\mathcal{M}}_\mu(q_-,q_+) \to \overline{\mathcal{M}}_{\mu - \#C}(q_-,q_+)$ by forgetting marked points t_i $(i \in C)$ and collapsing all $\overline{\mathbb{R}}_i$ on which the map γ is constant and which does not contain any marked points. We can also define the forgetful map for the global Kuranishi neighborhoods similarly. This map is smooth. This is because the forgetful maps for the space of stable lines are smooth.

Next we remark about the construction of the global Kuranishi neighborhoods of class C^∞. If we use the strong differential structure of the space of stable lines defined by the coordinate change $\rho_i = \exp(-\hat{\rho}_i^{-1})$ instead of $\rho_i = \hat{\rho}_i^\beta$, then we can prove that (10) is of class C^∞, and we obtain the global Kuranishi neighborhoods of class C^∞ for $\overline{\mathcal{M}}_\mu(q_-,q_+)$. However, in this case, the forgetful maps lose the smoothness. For example, consider the forgetful map $\mathrm{forget}_2 : \overline{\mathcal{M}}^{\mathrm{SL}}_3 \to \overline{\mathcal{M}}^{\mathrm{SL}}_2$ on a neighborhood of the point $(\overline{\mathbb{R}}_1 \cup \overline{\mathbb{R}}_2 \cup \overline{\mathbb{R}}_3, (t_i = 0_i)_{i=1,2,3})$. We regard $\overline{\mathbb{R}}_i$ as $\overline{\mathbb{R}}_i = (-\infty_i, 0]_i \cup [-T,T] \cup [0, +\infty_i)$ for some $T > 0$, and we let $[0,1)^2 \to \overline{\mathcal{M}}^{\mathrm{SL}}_3$ and $[0,\infty)^1 \to \overline{\mathcal{M}}^{\mathrm{SL}}_2$ be the coordinates on neighborhoods of $(\overline{\mathbb{R}}_1 \cup \overline{\mathbb{R}}_2 \cup \overline{\mathbb{R}}_3, (t_i = 0_i)_{i=1,2,3})$ and $(\overline{\mathbb{R}}_1 \cup \overline{\mathbb{R}}_3, (t_1 = 0_1, t_3 = 0_2))$. Then under these coordinates, the forgetful map is written as $(\rho_1, \rho_2) \mapsto \rho = e^{2T} \rho_1 \rho_2$. If we use the coordinate change $\rho_i = \hat{\rho}_i^\beta$, then it becomes $(\hat{\rho}_1, \hat{\rho}_2) \mapsto \hat{\rho} = e^{2T/\beta} \hat{\rho}_1 \hat{\rho}_2$, which is also smooth. However, if we use the coordinate change $\rho_i = \exp(-\hat{\rho}_i^{-1})$, then it becomes

$$(\hat{\rho}_1, \hat{\rho}_2) \mapsto \hat{\rho} = \frac{\hat{\rho}_1 \hat{\rho}_2}{\hat{\rho}_1 + \hat{\rho}_2 + 2T\hat{\rho}_1\hat{\rho}_2},$$

and this is not smooth.

In general, the coordinate change $\rho_i = \exp(-\hat{\rho}_i^{-1})$ does not work well if we need to use stabilization of the domain curve. (In the above example, we need to collapse $\overline{\mathbb{R}}_2$.) In the case of the construction of symplectic field theory, we also need stabilization of the domain curve to study the relation of the moduli space of disconnected holomorphic buildings and those for their connected components. Hence also in this case, we can only construct a Kuranishi structure of class C^N for an arbitrary fixed constant $N \geq 1$.

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