Bayer–Macrì decomposition on Bridgeland moduli spaces over surfaces

Wanmin Liu

Abstract We find a decomposition formula of the local Bayer–Macrì map for the nef line bundle theory on the Bridgeland moduli space over a surface. If there is a global Bayer–Macrì map, then such a decomposition gives a precise correspondence from Bridgeland walls to Mori walls. As an application, we compute the nef cone of the Hilbert scheme $S^{[n]}$ of n-points over special kinds of a fibered surface S of Picard rank 2.

1. Introduction

Let S be a smooth projective surface over $\mathbb{C}$. Let M be the Gieseker moduli space of semistable sheaves with fixed Chern character ch over S. One viewpoint for studying the birational geometry of M is to study the classification of line bundles on M and different cones inside its real Néron–Severi group $N^1(M)$, such as the nef cone $\mathrm{Nef}(M)$ and pseudoeffective cone $\overline{\mathrm{Eff}}(M)$. Another viewpoint is introduced by Bridgeland [10] with the idea of enlarging the category of coherent sheaves $\mathrm{Coh}(S)$ to its bounded derived version $\mathrm{D^b}(S)$ and studying moduli spaces of Bridgeland semistable objects in $\mathrm{D^b}(S)$. Let σ be a Bridgeland stability condition, and let $M_\sigma(\mathrm{ch})$ be the moduli space of σ-semistable objects in $\mathrm{D^b}(S)$ with the same invariant ch. The collection of all stability conditions forms an interesting parameter space $\mathrm{Stab}(S)$, which is a $\mathbb{C}$-manifold. Bridgeland identified M as $M_\sigma(\mathrm{ch})$, for σ in a special chamber inside $\mathrm{Stab}(S)$. He envisioned that the wall-chamber structures of $\mathrm{Stab}(S)$ will recover birational models of M.

When S is the projective plane $\mathbb{P}^2$ and $\mathrm{ch} = (1, 0, -n)$, the moduli space M is the Hilbert scheme $\mathbb{P}^{2[n]}$ of n-points over $\mathbb{P}^2$. Arcara, Bertram, Coskun, and Huizenga [2] found a precise relation between Bridgeland walls inside $\mathrm{Stab}(\mathbb{P}^2)$ with respect to $(1, 0, -n)$ and Mori walls inside $\overline{\mathrm{Eff}}(\mathbb{P}^{2[n]})$ (see Corollary 4.14). Bertram and Coskun [8] generalized the speculation to other rational surfaces. Bayer and Macrì [6], [5] linked the two viewpoints by establishing a line bundle theory on Bridgeland moduli spaces. Let $\sigma = \sigma_{\omega,\beta}$ be the stability condition constructed by Arcara and Bertram [1], which depends on an ample line bundle ω and another line bundle β over S. Assume that σ is in a chamber $\mathtt{C}$. Bayer and

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Macrì constructed a map by sending σ to a nef line bundle ℓ_σ on $M_\sigma(\mathrm{ch})$, which is called the local Bayer–Macrì map. The line bundle ℓ_σ only depends on the chamber $\mathtt{C}$. When S is a K3 surface and ch is primitive, they constructed a global Bayer–Macrì map (by gluing the local Bayer–Macrì maps; see Definition 3.6),

$$\ell : \mathrm{Stab}^\dagger(S) \to N^1(M),$$

sending a stability condition σ to a line bundle ℓ_σ on M. The existence of the global Bayer–Macrì map is also known for the projective plane $\mathbb{P}^2$ with primitive Chern character ch by Li and Zhao [20].

In this article, we find a decomposition of the line bundle $\ell_{\sigma_{\omega,\beta}}$. The decomposition is classified into two cases according to the given Chern character ch. The case for objects supported in dimension 1 is given in Lemma 4.5. The case for objects supported in dimension 2 is given in Lemma 4.8. In this case, an equivalent decomposition is also obtained by Bolognese, Huizenga, Lin, Riedl, Schmidt, Woolf, and Zhao [9, Proposition 3.8]. If there is a global Bayer–Macrì map, we then obtain the precise correspondences from Bridgeland walls to Mori walls for two such cases (see, resp., Theorems 4.10 and 4.13). By Mori walls, we mean the walls that appear on the stable base locus decomposition of the pseudoeffective cone $\overline{\mathrm{Eff}}(M)$.

As an application of the main theorem, we compute the nef cone of the Hilbert scheme $S^{[n]}$ of n-points over special kinds of a fibered surface S in Theorem 5.2. Here S is either the Hirzebruch surface or an elliptic surface over $\mathbb{P}^1$ with a global section of Picard rank 2. The example suggests that, to obtain the extremal nef line bundle, we cannot assume that ω is parallel to β.

Some of the techniques discussed in this article have been partially generalized by Coskun and Huizenga [13] to compute the nef cone of certain Gieseker moduli spaces.

Outline of the article. Section 2 is a brief review of the notion of Bridgeland stability conditions. Section 3 is a brief review of Bayer and Macrì's line bundle theory on Bridgeland moduli spaces. The main theorems on the Bayer–Macrì decomposition are given in Section 4. In Section 5, we provide an application of the main theorem. Some background on the large-volume limit is given in Appendix 5. Some parallel computations by using $\hat{Z}_{\omega,\beta}$ (see (A.6)) for a K3 surface are given in Appendix 5.

2. Bridgeland stability conditions

Let S be a smooth projective surface over $\mathbb{C}$, and let $\mathrm{D^b}(S)$ be the bounded derived category of coherent sheaves on S. Denote the Grothendieck group of $\mathrm{D^b}(S)$ by $K(S)$. A *Bridgeland stability condition* (see [10, Proposition 5.3]) $\sigma = (Z, \mathcal{A})$ on $\mathrm{D^b}(S)$ consists of a pair $(Z, \mathcal{A})$, where $Z : K(S) \to \mathbb{C}$ is a group homomorphism (called the *central charge*) and $\mathcal{A} \subset \mathrm{D^b}(S)$ is the heart of a *bounded t-structure* satisfying the following three properties.

(1) *Positivity*. For any $0 \neq E \in \mathcal{A}$, the central charge $Z(E)$ lies in the semi-closed upper half-plane $\mathbb{R}_{>0} \cdot e^{(0,1] \cdot i\pi}$.

Let $E \in \mathcal{A} \setminus \{0\}$. Define the *Bridgeland slope* (might be $+\infty$-valued) and the *phase* of E as

$$\mu_\sigma(E) := \frac{-\Re(Z(E))}{\Im(Z(E))}, \qquad \phi(E) := \frac{1}{\pi}\arg\big(Z(E)\big) \in (0,1].$$

For nonzero $E, F \in \mathcal{A}$, we have the equivalent relation:

$$\mu_\sigma(F) < (\leq) \mu_\sigma(E) \Longleftrightarrow \phi(F) < (\leq) \phi(E).$$

For $0 \neq E \in \mathcal{A}$, we say E is *Bridgeland (semi)stable* if for any subobject $0 \neq F \subsetneq E$ ($0 \neq F \subseteq E$) we have $\mu_\sigma(F) < (\leq) \mu_\sigma(E)$.

(2) *Harder–Narasimhan property*. By this property every object $E \in \mathcal{A}$ has a Harder–Narasimhan filtration $0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_n = E$ such that the quotients E_i/E_{i-1} are Bridgeland semistable with $\mu_\sigma(E_1/E_0) > \mu_\sigma(E_2/E_1) > \cdots > \mu_\sigma(E_n/E_{n-1})$.

(3) *Support property*. There is a constant $C > 0$ such that, for all Bridgeland-semistable objects $E \in \mathcal{A}$, we have $\|E\| \leq C|Z(E)|$, where $\|\cdot\|$ is a fixed norm on $K(X) \otimes \mathbb{R}$.

2.1. Bridgeland stability conditions on surfaces

Let S be a smooth projective surface. Fix $\omega, \beta \in N^1(S) := \mathrm{NS}(S)_\mathbb{R}$ with ω ample. Define

$$Z_{\omega,\beta}(E) := -\int_S e^{-(\beta+\sqrt{-1}\omega)}.\,\mathrm{ch}(E).$$

For $E \in \mathrm{Coh}(S)$, denote its *Mumford slope* by

$$\mu_\omega(E) := \begin{cases} \frac{\omega \cdot \mathrm{ch}_1(E)}{\mathrm{ch}_0(E)} & \text{if } \mathrm{ch}_0(E) \neq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{T}_{\omega,\beta} \subset \mathrm{Coh}(S)$ be the subcategory of coherent sheaves whose Harder–Narasimhan (HN) factors (with respect to Mumford stability) are of Mumford slope strictly greater than $\omega.\beta$. Let $\mathcal{F}_{\omega,\beta} \subset \mathrm{Coh}(S)$ be the subcategory of coherent sheaves whose HN-factors (with respect to Mumford stability) are of Mumford slope less than or equal to $\omega.\beta$. Then $(\mathcal{T}_{\omega,\beta}, \mathcal{F}_{\omega,\beta})$ is a torsion pair of $\mathrm{Coh}(S)$ (see [1]). Define the heart $\mathcal{A}_{\omega,\beta}$ as the tilt of this torsion pair:

$$\mathcal{A}_{\omega,\beta} := \big\{E \in \mathrm{D^b}(S) : H^{-1}(E) \in \mathcal{F}_{\omega,\beta}, H^0(E) \in \mathcal{T}_{\omega,\beta}, H^p(E) = 0 \text{ otherwise}\big\}.$$

LEMMA 2.1 ([1, Corollary 2.1])

Fix $\omega, \beta \in \mathrm{NS}(S)_\mathbb{R}$ with ω ample. Then $\sigma_{\omega,\beta} := (Z_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$ is a Bridgeland stability condition.

2.2. Logarithm Todd class

Let X be a smooth projective variety over $\mathbb{C}$. Let us introduce a formal variable t and write

$$\begin{aligned}\mathrm{td}(X)(t) := 1 + \Big(-\frac{1}{2}\mathrm{K}_X\Big)t + \frac{1}{12}\Big(\frac{3}{2}\mathrm{K}_X^2 - \mathrm{ch}_2(X)\Big)t^2 \\ + \Big(-\frac{1}{24}\mathrm{K}_X.\Big(\frac{1}{2}\mathrm{K}_X^2 - \mathrm{ch}_2(X)\Big)\Big)t^3 + \text{higher order of } t^4.\end{aligned}$$

Taking the logarithm with respect to t and expressing it in the power series of t, we obtain

$$\ln \mathrm{td}(X)(t) = -\frac{1}{2}\mathrm{K}_X t - \frac{1}{12}\mathrm{ch}_2(X)t^2 + 0 \cdot t^3 + \text{higher order of } t^4.$$

In particular, the *logarithm Todd class* of a smooth projective surface S or a smooth projective threefold X is given, respectively, by

$$\begin{aligned}\ln \mathrm{td}(S) &:= \Big(0, -\frac{1}{2}\mathrm{K}_S, -\frac{1}{12}\mathrm{ch}_2(S)\Big) \qquad \text{or} \\ \ln \mathrm{td}(X) &:= \Big(0, -\frac{1}{2}\mathrm{K}_X, -\frac{1}{12}\mathrm{ch}_2(X), 0\Big).\end{aligned}$$

2.3. The Mukai pairing

We refer to [16, Section 5.2] for the details. Let X still be a smooth projective variety of dimension n over $\mathbb{C}$. Define the Mukai vector of an object $E \in \mathrm{D^b}(X)$ by

$$v(E) := \mathrm{ch}(E).e^{\frac{1}{2}\ln \mathrm{td}(X)} \in \bigoplus H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q}) =: H^*_{\mathrm{alg}}(X, \mathbb{Q}).$$

Let $A(X)$ be the Chow ring of X. The Chern character gives a mapping $\mathrm{ch}: K(X) \to A(X) \otimes \mathbb{Q}$. There is a natural involution $^*: A(X) \to A(X)$,

$$v = (v_0, \ldots, v_i, \ldots, v_n) \mapsto v^* := \big(v_0, \ldots, (-1)^i v_i, \ldots, (-1)^n v_n\big).$$

We call v^* the *Mukai dual* of v. Denote $E^\vee := R\mathcal{H}\mathrm{om}(E, \mathcal{O}_S)$. We have

$$\mathrm{ch}(E^\vee) = \big(\mathrm{ch}(E)\big)^*, \qquad v(E^\vee) = \big(v(E)\big)^*.e^{-\frac{1}{2}\mathrm{K}_X}.$$

Define the *Mukai pairing* for two Mukai vectors w and v by

$$\langle w, v\rangle_X := -\int_X w^*.v.e^{-\frac{1}{2}\mathrm{K}_X}. \tag{2.1}$$

The Hirzebruch–Riemann–Roch theorem gives

$$\chi(F, E) = \int_X \mathrm{ch}(F^\vee).\mathrm{ch}(E).\mathrm{td}(X) = -\big\langle v(F), v(E)\big\rangle_X.$$

For a smooth projective surface S, the Mukai vector of $E \in \mathrm{D^b}(S)$ is

$$\begin{aligned}v(E) &= \big(v_0(E), v_1(E), v_2(E)\big) \\ &= \Big(\mathrm{ch}_0, \mathrm{ch}_1 - \frac{1}{4}\mathrm{ch}_0 \mathrm{K}_S, \mathrm{ch}_2 - \frac{1}{4}\mathrm{ch}_1 .\mathrm{K}_S + \frac{1}{2}\mathrm{ch}_0\Big(\chi(\mathcal{O}_S) - \frac{1}{16}\mathrm{K}_S^2\Big)\Big). \qquad (2.2)\end{aligned}$$

By (2.1) the Mukai pairing of $w = (w_0, w_1, w_2)$ and $v = (v_0, v_1, v_2)$ is

$$\langle w, v\rangle_S = w_1.v_1 - w_0\Big(v_2 - \frac{1}{2}v_1.\mathrm{K}_S\Big) - v_0\Big(w_2 + \frac{1}{2}w_1.\mathrm{K}_S\Big) - \frac{1}{8}w_0v_0\mathrm{K}_S^2. \tag{2.3}$$

2.4. Central charge in terms of the Mukai pairing

LEMMA 2.2

The central charge $Z_{\omega,\beta}$ has the expression

$$Z_{\omega,\beta}(E) = \big\langle \mho_{Z_{\omega,\beta}}, v(E)\big\rangle_S, \quad \textit{where } \mho_{Z_{\omega,\beta}} := e^{\beta - \frac{3}{4}\mathrm{K}_S + \sqrt{-1}\omega + \frac{1}{24}\mathrm{ch}_2(S)}. \tag{2.4}$$

Moreover, the vector $\mho_{Z_{\omega,\beta}}$ (or simply $\mho_Z$) is given by

$$\begin{aligned}\mho_{Z_{\omega,\beta}} = &\Big(1, \beta - \frac{3}{4}\mathrm{K}_S, -\frac{1}{2}\omega^2 + \frac{1}{2}\Big(\beta - \frac{3}{4}\mathrm{K}_S\Big)^2 - \frac{1}{2}\Big(\chi(\mathcal{O}_S) - \frac{1}{8}\mathrm{K}_S^2\Big)\Big)\\ &+ \sqrt{-1}\Big(0, \omega, \Big(\beta - \frac{3}{4}\mathrm{K}_S\Big).\omega\Big).\end{aligned} \tag{2.5}$$

Proof

We have

$$\begin{aligned}Z_{\omega,\beta}(E) &= -\int_S e^{-(\beta+\sqrt{-1}\omega)}.\mathrm{ch}(E)\\ &= -\int_S e^{-(\beta+\ln\mathrm{td}(S)+\sqrt{-1}\omega)}.\sqrt{\mathrm{td}(S)}.\mathrm{ch}(E).\sqrt{\mathrm{td}(S)}.\end{aligned}$$

Denote $\mathrm{ch}(F^\vee) := e^{-(\beta+\ln\mathrm{td}(S)+\sqrt{-1}\omega)}$. Then

$$\begin{aligned}\mathrm{ch}(F)^* = \mathrm{ch}(F^\vee) &= e^{-(\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega)+\frac{1}{12}\mathrm{ch}_2(S)}\\ &= \Big(1, -\Big(\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega\Big), \frac{1}{2}\Big(\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega\Big)^2 + \frac{1}{12}\mathrm{ch}_2(S)\Big).\end{aligned}$$

So

$$\begin{aligned}\mathrm{ch}(F) &= \Big(1, \Big(\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega\Big), \frac{1}{2}\Big(\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega\Big)^2 + \frac{1}{12}\mathrm{ch}_2(S)\Big)\\ &= e^{(\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega)+\frac{1}{12}\mathrm{ch}_2(S)}.\end{aligned}$$

Therefore,

$$Z_{\omega,\beta}(E) = -\int_S \mathrm{ch}(F^\vee).\sqrt{\mathrm{td}(S)}.\mathrm{ch}(E).\sqrt{\mathrm{td}(S)} = \big\langle v(F), v(E)\big\rangle_S.$$

So

$$\mho_{Z_{\omega,\beta}} = v(F) = \mathrm{ch}(F).e^{\frac{1}{2}\ln\mathrm{td}(S)} = e^{\beta-\frac{3}{4}\mathrm{K}_S+\sqrt{-1}\omega+\frac{1}{24}\mathrm{ch}_2(S)}.$$

By using Noether's formula

$$-\frac{1}{12}\mathrm{ch}_2(S) = \chi(\mathcal{O}_S) - \frac{1}{8}\mathrm{K}_S^2$$

and direct computation, we get the concrete expression of $\mho_Z$. □

Denote by $\mathrm{Stab}(S)$ the collection of all Bridgeland stability conditions. It is a $\mathbb{C}$-manifold of dimension $K_{\mathrm{num}}(S)\otimes\mathbb{C}$, with two group actions: a left action by

$\mathrm{Aut}(\mathrm{D^b}(S))$ and a right action by $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ (see [10, Lemma 8.2]). The stability σ is said to be *geometric* if all skyscraper sheaves $\mathcal{O}_x$, $x \in S$, are σ-stable of the same phase. We can set the phase to be 1 by a right group action. Denote by $\mathrm{Stab}^\dagger(S) \subset \mathrm{Stab}(S)$ the connected component containing the geometric stability conditions. The stability σ is said to be *numerical* if the central charge Z takes the form $Z(E) = \langle \pi(\sigma), v(E)\rangle_S$ for some vector $\pi(\sigma) \in K_{\mathrm{num}}(S) \otimes \mathbb{C}$. As in [17, Remark 4.33], we further assume the numerical Bridgeland stability factors through $K_{\mathrm{num}}(S)_{\mathbb{Q}} \otimes \mathbb{C} \to H^*_{\mathrm{alg}}(S,\mathbb{Q}) \otimes \mathbb{C}$. Therefore, $\pi(\sigma) \in H^*_{\mathrm{alg}}(S,\mathbb{Q}) \otimes \mathbb{C}$. For a numerical geometric stability condition with skyscraper sheaves of phase 1, the heart $\mathcal{A}$ must be of the form $\mathcal{A}_{\omega,\beta}$ (see [10, Proposition 10.3] and Huybrechts [17, Theorem 4.39]). Therefore, Lemma 2.2 gives

$$\pi(\sigma_{\omega,\beta}) = \mho_{Z_{\omega,\beta}} \in H^*_{\mathrm{alg}}(S,\mathbb{Q}) \otimes \mathbb{C}. \tag{2.6}$$

2.5. Bertram's nested wall theorem

We follow the notation from [23, Section 2] (but we use H instead of ω therein). Fix an ample divisor H and another divisor $\gamma \in H^\perp$, that is, $H.\gamma = 0$. Denote

$$g := H^2, \qquad -d := \gamma^2.$$

It is known by the *Hodge index theorem* that $d \geq 0$ and that $d = 0$ if and only if $\gamma = 0$. Let $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ be of Bogomolov type, that is, $\mathrm{ch}_1^2 - 2\,\mathrm{ch}_0\,\mathrm{ch}_2 \geq 0$. Write it as

$$\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) := (x, y_1 H + y_2\gamma + \delta, z),$$

where y_1, y_2 are real coefficients and $\delta \in \{H,\gamma\}^\perp$. Write the potential destabilizing Chern character as

$$\mathrm{ch}' = (\mathrm{ch}'_0, \mathrm{ch}'_1, \mathrm{ch}'_2) := (r, c_1 H + c_2\gamma + \delta', \chi),$$

where c_1, c_2 are real coefficients and $\delta' \in \{H,\gamma\}^\perp$. A *potential wall* is defined as

$$W(\mathrm{ch},\mathrm{ch}') := \big\{\sigma \in \mathrm{Stab}(S) \;\big|\; \mu_\sigma(\mathrm{ch}) = \mu_\sigma(\mathrm{ch}')\big\}.$$

A potential wall $W(\mathrm{ch},\mathrm{ch}')$ is a *Bridgeland wall* if there are a $\sigma \in \mathrm{Stab}(S)$ and objects $E, F \in \mathcal{A}_\sigma$ such that $\mathrm{ch}(E) = \mathrm{ch}$, $\mathrm{ch}(F) = \mathrm{ch}'$, and $\mu_\sigma(E) = \mu_\sigma(F)$. There is a wall-chamber structure on $\mathrm{Stab}(S)$ with respect to ch (see [10], [11], [27]). Bridgeland walls are $\mathbb{R}$-codimension 1 in $\mathrm{Stab}(S)$ and separate $\mathrm{Stab}(S)$ into chambers. Let E be an object that is σ_0-stable for a stability condition σ_0 in some chamber $\mathtt{C}$. Then E is σ-stable for any $\sigma \in \mathtt{C}$. Choose

$$\begin{cases} \omega := tH, \\ \beta := sH + u\gamma, \end{cases} \tag{2.7}$$

for some real numbers t, s, u, with t positive. With a sign choice of γ, we further assume $u \geq 0$. There is a half-three-space of stability conditions

$$\Omega_{\omega,\beta} = \Omega_{tH,sH+u\gamma} := \{\sigma_{tH,sH+u\gamma} \mid t > 0, u \geq 0\} \subset \mathrm{Stab}^\dagger(S),$$

which should be considered to be the u-indexed family of half-planes:

$$\Pi_{(H,\gamma,u)} := \{\sigma_{tH,sH+u\gamma} \mid t>0, u \text{ is fixed}\}.$$

DEFINITION 2.3

A *frame* with respect to the triple (H,γ,u) is a choice of an ample divisor H on S, another divisor $\gamma \in H^{\perp}$, and a nonnegative number u such that the stability conditions $\sigma_{\omega,\beta}$ are on the half-plane $\Pi_{(H,\gamma,u)}$ with (s,t)-coordinates as in (2.7). We simply call this fixing a frame (H,γ,u), and we write $\sigma_{s,t} := \sigma_{tH,sH+u\gamma}$.

THEOREM 2.4 (Bertram's nested wall theorem in (s,t)-model [23, Section 2])

Fix a frame (H,γ,u). The potential walls $W(\mathrm{ch},\mathrm{ch}')$ (for the fixed ch and different potential destabilizing Chern characters ch') in the (s,t)-half-plane $\Pi_{(H,\gamma,u)}$ ($t>0$) are given by nested semicircles with center $(C,0)$ and radius $R=\sqrt{D+C^2}$:

$$(s-C)^2+t^2 = D+C^2, \tag{2.8}$$

where $C=C(\mathrm{ch},\mathrm{ch}')$ and $D=D(\mathrm{ch},\mathrm{ch}')$ are given by

$$C(\mathrm{ch},\mathrm{ch}') := \frac{x\chi - rz + ud(xc_2-ry_2)}{g(xc_1-ry_1)}, \tag{2.9}$$

$$D(\mathrm{ch},\mathrm{ch}') := \frac{2zc_1-2c_2udy_1-xu^2dc_1+2y_2udc_1-2\chi y_1+ru^2dy_1}{g(xc_1-ry_1)}. \tag{2.10}$$

- *If $\mathrm{ch}_0 = x\neq 0$, we have*

$$D = -\frac{2y_1}{x}C + \frac{ud(2y_2-ux)+2z}{gx} \tag{2.11}$$

$$= -\frac{2y_1}{x}C + \Big(\frac{y_1^2}{x^2} - F\Big), \tag{2.12}$$

where $F=F(\mathrm{ch})$ is independent of ch',

$$F(\mathrm{ch}) := \frac{d}{g}\Big(u-\frac{y_2}{x}\Big)^2 + \frac{1}{x^2g}(y_1^2g - y_2^2d - 2xz). \tag{2.13}$$

Moreover, if ch is of Bogomolov type, that is, $\mathrm{ch}_1^2 - 2\,\mathrm{ch}_0\,\mathrm{ch}_2 \geq 0$, then $F(\mathrm{ch})\geq 0$ for all u.

- *If $\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1 .H>0$, that is, $x=0$ and $y_1>0$, then $\mathrm{ch}_0' = r\neq 0$ and $C=\frac{z+duy_2}{gy_1}$ is independent of ch'. We have*

$$D = -\frac{2c_1}{r}C + \frac{ud(2c_2-ur)+2\chi}{gr} \tag{2.14}$$

$$= -\frac{2c_1}{r}C + \Big(\frac{c_1^2}{r^2} - F'\Big), \tag{2.15}$$

where $F'=F'(\mathrm{ch}')$ is independent of ch,

$$F'(\mathrm{ch}') := \frac{d}{g}\Big(u-\frac{c_2}{r}\Big)^2 + \frac{1}{r^2g}(c_1^2g - c_2^2d - 2r\chi). \tag{2.16}$$

Moreover, if ch' is of Bogomolov type, then $F(\mathrm{ch}')\geq 0$ for all u.

Proof

We refer to Maciocia [23, Section 2]. The only unproved parts are (2.14) and (2.15). It is an easy exercise to check them. □

2.6. From (s,t)-model to (s,q)-model

We follow the ideas of Li–Zhao [20] and consider a $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$-action on $\sigma_{\omega,\beta}$. The potential walls in the (s,q)-plane are semilines.

DEFINITION 2.5

Fix a frame (H,γ,u). Define $\sigma'_{\omega,\beta}=(Z'_{\omega,\beta},\mathcal{A}'_{\omega,\beta})$ as the right action of $\left(\begin{smallmatrix}1 & 0\\ -\frac{s}{t} & \frac{1}{t}\end{smallmatrix}\right)$ on $\sigma_{\omega,\beta}$, that is, $\mathcal{A}'_{\omega,\beta}=\mathcal{A}_{\omega,\beta}$ and

$$Z'_{\omega,\beta}(E):=\Big(\Re Z_{\omega,\beta}(E)-\frac{s}{t}\Im Z_{\omega,\beta}(E)\Big)+\frac{1}{t}i\Im Z_{\omega,\beta}(E). \tag{2.17}$$

LEMMA 2.6

Fix a frame (H,γ,u). The above right action does not change the potential walls $W(\mathrm{ch},\mathrm{ch}')$ in the (s,t)-plane $\Pi_{(H,\gamma,u)}$.

Proof

This is a direct computation because the potential wall relation for $Z'_{\omega,\beta}$ is equivalent to the potential wall relation for $Z_{\omega,\beta}$ by using (2.17):

$$\begin{aligned}&\Re Z'(\mathrm{ch}')\Im Z'(\mathrm{ch})-\Re Z'(\mathrm{ch})\Im Z'(\mathrm{ch}')=0\\ \Leftrightarrow\quad &\Re Z(\mathrm{ch}')\Im Z(\mathrm{ch})-\Re Z(\mathrm{ch})\Im Z(\mathrm{ch}')=0.\end{aligned}$$ □

DEFINITION 2.7

Fix a frame (H,γ,u). We change the (s,t)-plane $\Pi_{(H,\gamma,u)}$ to the (s,q)-plane $\Sigma_{(H,\gamma,u)}$ by keeping the same s and defining

$$q:=\frac{s^2+t^2}{2}. \tag{2.18}$$

Denote $\sigma_{s,q}:=\sigma'_{tH,sH+u\gamma}$. The central charge (2.17) becomes

$$\begin{aligned}Z_{s,q}(E)=&\big(-\mathrm{ch}_2(E)+\mathrm{ch}_0(E)H^2q\big)+\Big(-\frac{1}{2}\mathrm{ch}_0(E)\gamma^2u^2+u\,\mathrm{ch}_1(E).\gamma\Big)\\&+i\big(\mathrm{ch}_1(E).H-\mathrm{ch}_0(E)H^2s\big).\end{aligned}$$

COROLLARY 2.8 (Bertram's nested wall theorem in (s,q)-model)

Fix a frame (H,γ,u), and use the notation as above. The potential walls $W(\mathrm{ch},\mathrm{ch}')$ in the (s,q)-plane $\Sigma_{(H,\gamma,u)}$ are given by semilines

$$q=Cs+\frac{1}{2}D\quad\Big(q>\frac{s^2}{2}\Big).$$

- *If $x \neq 0$, then the potential walls are given by semilines passing through a fixed point $(\frac{y_1}{x}, \frac{1}{2}(\frac{y_1^2}{x^2} - F))$ with slope $C = C(\mathrm{ch}, \mathrm{ch}')$:*

$$q = C\Big(s - \frac{y_1}{x}\Big) + \frac{1}{2}\Big(\frac{y_1^2}{x^2} - F\Big) \quad \Big(q > \frac{s^2}{2}\Big), \tag{2.19}$$

where $F = F(\mathrm{ch})$ as in (2.13) is independent of ch'.

- *If $x = 0$ and $y_1 > 0$, then $r \neq 0$. The potential walls are given by parallel semilines with constant slope $C = \frac{z+duy_2}{gy_1}$:*

$$q = C\Big(s - \frac{c_1}{r}\Big) + \frac{1}{2}\Big(\frac{c_1^2}{r^2} - F'\Big) \quad \Big(q > \frac{s^2}{2}\Big), \tag{2.20}$$

where $F' = F'(\mathrm{ch}')$ as in (2.16) is independent of ch.

Proof
This is a direct computation by using (2.8) and (2.18). □

REMARK 2.9
In the case of $\mathbb{P}^2$, the condition $q > \frac{s^2}{2}$ is relaxed, q could be a little negative, and the boundary is given by a fractal curve (see [20]).

2.7. Duality induced by derived dual

LEMMA 2.10 ([24, Theorem 3.1])
The functor $\Phi(\cdot) := R\mathcal{H}\mathrm{om}(\cdot, \mathcal{O}_S)[1]$ induces an isomorphism between the Bridgeland moduli spaces $M_{\omega,\beta}(\mathrm{ch})$ and $M_{\omega,-\beta}(-\mathrm{ch}^*)$ provided these moduli spaces exist and $Z_{\omega,\beta}(\mathrm{ch})$ belongs to the open upper half-plane.

Proof
This is a variation of Martinez's duality theorem [24, Theorem 3.1], where the duality functor is taken as $R\mathcal{H}\mathrm{om}(\cdot, \omega_S)[1]$. □

COROLLARY 2.11
Fix the Chern character $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. Assume that $Z_{\omega,\beta}(\mathrm{ch})$ belongs to the open upper half-plane. The wall-chamber structures of $\sigma_{\omega,\beta}$ with respect to ch are dual to the wall-chamber structures of $\Phi(\sigma_{\omega,\beta})$ with respect to $\Phi(\mathrm{ch}) = -\mathrm{ch}^* = (-\mathrm{ch}_0, \mathrm{ch}_1, -\mathrm{ch}_2)$ in the sense that

$$\Phi(\sigma_{\omega,\beta}) = \sigma_{\omega,-\beta}.$$

Applying Φ again, we have $\Phi \circ \Phi(\sigma_{\omega,\beta}) = \sigma_{\omega,\beta}$. Moreover, if we fix a frame (H, γ, u), then $\sigma_{\omega,\beta} \in \Pi_{(H,\gamma,u)}$ with coordinates (s,t) is dual to $\Phi(\sigma_{\omega,\beta}) \in \Pi_{(H,-\gamma,u)}$ with coordinates $(-s,t)$.

- *If $\sigma_{\omega,\beta} \in \mathtt{C}$, where $\mathtt{C}$ is a chamber with respect to ch in $\Pi_{(H,\gamma,u)}$, then we have $\Phi(\sigma_{\omega,\beta}) \in \mathtt{DC}$, where $\mathtt{DC}$ is the corresponding chamber with respect to $\Phi(\mathrm{ch})$ in $\Pi_{(H,-\gamma,u)}$.*
- *If $\sigma := \sigma_{\omega,\beta} \in W(\mathrm{ch}, \mathrm{ch}')$ holds in $\Pi_{(H,\gamma,u)}$, then it also holds that $\Phi(\sigma) \in W(-\mathrm{ch}^*, -\mathrm{ch}'^*)$ in $\Pi_{(H,-\gamma,u)}$, and there are relations $\mu_{\Phi(\sigma)}(-\mathrm{ch}^*) = -\mu_\sigma(\mathrm{ch})$,*

$C_{\Phi(\sigma)}(-\mathrm{ch}^*, -\mathrm{ch}'^*) = -C_\sigma(\mathrm{ch}, \mathrm{ch}')$, $D_{\Phi(\sigma)}(-\mathrm{ch}^*, -\mathrm{ch}'^*) = D_\sigma(\mathrm{ch}, \mathrm{ch}')$, *and* $R_{\Phi(\sigma)}(-\mathrm{ch}^*, -\mathrm{ch}'^*) = R_\sigma(\mathrm{ch}, \mathrm{ch}')$.

Proof
The proof is a direct computation. □

REMARK 2.12
The assumption that $Z_{\omega,\beta}(\mathrm{ch})$ belongs to the open upper half-plane means exactly that we exclude the case $\Im Z_{\omega,\beta}(\mathrm{ch}) = 0$, which is equivalent to one of the following three subcases:

- $\mathrm{ch} = (0, 0, n)$ for some positive integer n;
- $\mathrm{ch}_0 > 0$ and $\Im Z_{\omega,\beta}(\mathrm{ch}) = 0$; or
- $\mathrm{ch}_0 < 0$ and $\Im Z_{\omega,\beta}(\mathrm{ch}) = 0$.

We call the first subcase the trivial chamber, the second subcase the Uhlenbeck wall, and the third subcase the dual Uhlenbeck wall (see Definition A.4).

3. Bayer–Macrì's nef line bundle theory

3.1. The local Bayer–Macrì map

Let S be a smooth projective surface over $\mathbb{C}$. Let $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(S)$ be a stability condition, and let $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ be a choice of Chern character. Assume that we are given a *flat family* (see [6, Definition 3.1]) $\mathcal{E} \in \mathrm{D^b}(M \times S)$ of σ-semistable objects of class ch parameterized by a proper algebraic space M of finite type over $\mathbb{C}$. Denote by $N^1(M) = \mathrm{NS}(M)_\mathbb{R}$ the group of real Cartier divisors modulo numerical equivalence. Write $N_1(M)$ as the group of real 1-cycles modulo numerical equivalence with respect to the intersection pairing with Cartier divisors. Bayer–Macrì's numerical Cartier divisor class $\ell_{\sigma,\mathcal{E}} \in N^1(M) = \mathrm{Hom}(N_1(M), \mathbb{R})$ is defined as follows: for any projective integral curve $C \subset M$,

$$\ell_{\sigma,\mathcal{E}}\big([C]\big) = \ell_{\sigma,\mathcal{E}}.C := \Im\Big(-\frac{Z(\Phi_\mathcal{E}(\mathcal{O}_C))}{Z(\mathrm{ch})}\Big) = \Im\Big(-\frac{Z((p_S)_*\mathcal{E}|_{C\times S})}{Z(\mathrm{ch})}\Big), \tag{3.1}$$

where $\Phi_\mathcal{E} : \mathrm{D^b}(M) \to \mathrm{D^b}(S)$ is the Fourier–Mukai functor with kernel $\mathcal{E}$ and $\mathcal{O}_C$ is the structure sheaf of C.

THEOREM 3.1 ([6, Theorem 1.1])
The divisor class $\ell_{\sigma,\mathcal{E}}$ is nef on M. In addition, we have $\ell_{\sigma,\mathcal{E}}.C = 0$ if and only if, for two general points $c, c' \in C$, the corresponding objects $\mathcal{E}_c, \mathcal{E}_{c'}$ are S-equivalent.

Here two semistable objects are *S-equivalent* if their Jordan–Hölder filtrations into stable factors of the same phase have identical stable factors.

DEFINITION 3.2
Let $\mathtt{C}$ be a Bridgeland chamber with respect to ch. Assume the existence of the moduli space $M_\sigma(\mathrm{ch})$ for $\sigma \in \mathtt{C}$ with a universal family $\mathcal{E}$. Then $M_\mathtt{C}(\mathrm{ch}) := M_\sigma(\mathrm{ch})$

is constant for $\sigma \in \mathtt{C}$. Theorem 3.1 yields a map

$$\ell : \overline{\mathtt{C}} \longrightarrow \operatorname{Nef}\big(M_{\mathtt{C}}(\mathrm{ch})\big),$$
$$\sigma \mapsto \ell_{\sigma,\mathcal{E}},$$

which is called the *local Bayer–Macrì map* for the chamber $\mathtt{C}$ with respect to ch.

For any $\sigma \in \operatorname{Stab}^{\dagger}(S)$, after a $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$-action, we assume that $\sigma = \sigma_{\omega,\beta}$, that is, skyscraper sheaves are stable of phase 1. Denote

$$\mathbf{v} := v(\mathrm{ch}) = \mathrm{ch} \cdot e^{\frac{1}{2}\ln \mathrm{td}(S)}.$$

The local Bayer–Macrì map is the composition of the following three maps:

$$\operatorname{Stab}^{\dagger}(S) \xrightarrow{\pi} H^*_{\mathrm{alg}}(S,\mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\mathcal{I}} \mathbf{v}^{\perp} \xrightarrow{\theta_{\mathtt{C},\mathcal{E}}} N^1\big(M_{\mathtt{C}}(\mathrm{ch})\big).$$

- The map π forgets the heart: $\pi(\sigma_{\omega,\beta}) := \mho_{Z_{\omega,\beta}}$ as in (2.6).
- For any $\mho \in H^*_{\mathrm{alg}}(S,\mathbb{Q}) \otimes \mathbb{C}$, define $\mathcal{I}(\mho) := \Im \frac{\mho}{-\langle \mho, \mathbf{v}\rangle_S}$. One can check that $\mathcal{I}(\mho) \in \mathbf{v}^{\perp}$ (this also follows from Lemma 3.4), where the perpendicular relation is with respect to the Mukai pairing:

$$\mathbf{v}^{\perp} := \big\{ w \in H^*_{\mathrm{alg}}(S,\mathbb{Q}) \otimes \mathbb{R} \;\big|\; \langle w, \mathbf{v}\rangle_S = 0 \big\}. \tag{3.2}$$

- The third map $\theta_{\mathtt{C},\mathcal{E}}$ is the *algebraic Mukai morphism*. More precisely, for a fixed Mukai vector $w \in \mathbf{v}^{\perp}$ and an integral curve $C \subset M_{\mathtt{C}}(\mathrm{ch})$,

$$\theta_{\mathtt{C},\mathcal{E}}(w).[C] := \big\langle w, v\big(\Phi_{\mathcal{E}}(\mathcal{O}_C)\big)\big\rangle_S.$$

DEFINITION 3.3

Define $w_{\sigma_{\omega,\beta}}(\mathrm{ch}) := -\Im(\overline{\langle \mho_Z, \mathbf{v}\rangle_S} \cdot \mho_Z)$. We simply write it as $w_{\omega,\beta}$ or w_σ.

LEMMA 3.4

Fix the Chern character $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. *The line bundle class* $\ell_{\sigma_{\omega,\beta}} \in N^1(M_{\sigma_{\omega,\beta}}(\mathrm{ch}))$ *(if it exists) is given by*

$$\ell_{\sigma_{\omega,\beta}} \overset{\mathbb{R}_+}{=\!=} \theta_{\sigma,\mathcal{E}}(w_{\omega,\beta}), \tag{3.3}$$

where $w_{\omega,\beta} \in \mathbf{v}^{\perp}$ *is given by*

$$w_{\omega,\beta} = \big(\Im Z(\mathrm{ch})\big)\Re\mho_Z - \big(\Re Z(\mathrm{ch})\big)\Im\mho_Z. \tag{3.4}$$

Proof

By the definition, $w_{\omega,\beta} = |Z(E)|^2 \mathcal{I}(\mho_Z)$. Applying the Mukai morphism, we get (3.3). Taking the complex conjugate of (2.4) we get $\overline{\langle \mho_Z, \mathbf{v}\rangle_S} = \Re Z(\mathrm{ch}) - \sqrt{-1}\Im Z(\mathrm{ch})$. The relation (3.4) thus follows from the definition of $w_{\omega,\beta}$. By the definition of $\mho_Z$, we have $\langle \Re\mho_Z, \mathbf{v}\rangle_S + \sqrt{-1}\langle \Im\mho_Z, \mathbf{v}\rangle_S = \langle \mho_Z, \mathbf{v}\rangle_S = \Re Z(\mathrm{ch}) + \sqrt{-1}\Im Z(\mathrm{ch})$. We then obtain the perpendicular relation $\langle w_\sigma, \mathbf{v}\rangle_S = (\Im Z(\mathrm{ch})) \times \langle \Re\mho_Z, \mathbf{v}\rangle_S - (\Re Z(\mathrm{ch}))\langle \Im\mho_Z, \mathbf{v}\rangle_S = 0$. □

The Mukai morphism is the dual version of the Donaldson morphism (see [6, Proposition 4.4, Remark 5.5]). The surjectivity of the Mukai morphism is not known in general. We will compute the image of the local Bayer–Macrì map in Theorem 4.13.

Let $\mathcal{E}$ be a universal family over $M_\sigma(\mathrm{ch})$. Denote by $\mathcal{F}$ the dual universal family over $M_{\Phi(\sigma)}(-\mathrm{ch}^*)$. Then $w_{\omega,-\beta}(-\mathrm{ch}^*) \in v(-\mathrm{ch}^*)^\perp$.

LEMMA 3.5

Fix the Chern character ch. *Let* $\sigma := \sigma_{\omega,\beta}$, *and assume that* $Z_{\omega,\beta}(\mathrm{ch})$ *belongs to the open upper half-plane (Remark 2.12). Then*

$$\ell_\sigma \cong \ell_{\Phi(\sigma)}, \quad \text{that is, } \theta_{\sigma,\mathcal{E}}\big(w_{\omega,\beta}(\mathrm{ch})\big) \cong \theta_{\Phi(\sigma),\mathcal{F}}\big(w_{\omega,-\beta}(-\mathrm{ch}^*)\big).$$

Proof

This is a consequence of the isomorphism of the moduli spaces

$$M_{\sigma_{\omega,\beta}}(\mathrm{ch}) \cong M_{\sigma_{\omega,-\beta}}(-\mathrm{ch}^*)$$

induced by the duality functor $\Phi(\cdot) = R\mathcal{H}\mathrm{om}(\cdot, \mathcal{O}_S)[1]$. □

3.2. The global Bayer–Macrì map

Let σ be in a chamber $\mathtt{C}$. The line bundle $\ell_{\sigma,\mathcal{E}}$ is only defined locally, that is, $\ell_{\sigma,\mathcal{E}} \in N^1(M_{\mathtt{C}}(\mathrm{ch}))$. If we take another chamber $\mathtt{C}'$, we cannot say $\ell_{\sigma,\mathcal{E}} \in N^1(M_{\mathtt{C}'}(\mathrm{ch}))$ directly. We want to associate to $\ell_{\sigma,\mathcal{E}}$ the global meaning in the following way.

Let $\sigma \in \mathtt{C}$ and $\tau \in \mathtt{C}'$ be two generic numerical stability conditions in different chambers with respect to ch. Assume that $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$ are nonempty and irreducible with universal families $\mathcal{E}$ and $\mathcal{F}$, respectively. And assume that there is a birational map between $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$, induced by a derived autoequivalence Ψ of $\mathrm{D^b}(S)$ in the following sense: there exists a common open subset U of $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$, with complements of *codimension at least* 2, such that, for any $u \in U$, the corresponding objects $\mathcal{E}_u \in M_\sigma(\mathrm{ch})$ and $\mathcal{F}_u \in M_\tau(\mathrm{ch})$ are related by $\mathcal{F}_u = \Psi(\mathcal{E}_u)$. Then the Néron–Severi groups of $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$ can canonically be identified. So for a Mukai vector $w \in \mathbf{v}^\perp$, the two line bundles $\theta_{\mathtt{C},\mathcal{E}}(w)$ and $\theta_{\mathtt{C}',\mathcal{F}}(w)$ are identified.

DEFINITION 3.6

Fix a base geometric numerical stability condition σ in a chamber. A global Bayer–Macrì map

$$\ell : \mathrm{Stab}^\dagger(S) \to N^1\big(M_\sigma(\mathrm{ch})\big)$$

is glued by the local Bayer–Macrì map by the above identification.

THEOREM 3.7 ([5, Theorem 1.2] for K3, [20, Theorem 0.2] for $\mathbb{P}^2$)

Let S be a K3 surface or the projective plane $\mathbb{P}^2$. Let ch *be a primitive character over S. There is a global Bayer–Macrì map.*

4. Bayer–Macrì decomposition

In this section, we give an intrinsic decomposition of the Mukai vector $w_{\omega,\beta}(\mathrm{ch})$ in Lemmas 4.5 and 4.8, respectively, according to the dimension of the support of objects with invariants ch. In particular, each component is in $\mathbf{v}^{\perp}$. So we can apply $\theta_{\sigma,\mathcal{E}}$ and obtain the intrinsic decomposition of $\ell_{\sigma_{\omega,\beta}}$. We call such a decomposition of $w_{\omega,\beta}$ or $\ell_{\sigma_{\omega,\beta}}$ the *Bayer–Macrì decomposition*.

4.1. Preliminary computation by using $\mho_Z$

LEMMA 4.1

If $\Im Z(\mathrm{ch})=0$, *then* $w_\sigma \overset{\mathbb{R}_+}{=\!=} \Im\mho_Z$. *If* $\Im Z(\mathrm{ch})>0$, *then*

$$\begin{aligned} w_\sigma &\overset{\mathbb{R}_+}{=\!=} \mu_\sigma(\mathrm{ch})\Im\mho_Z+\Re\mho_Z \\ &= \Big(0,\mu_\sigma(\mathrm{ch})\omega+\beta,-\frac{3}{4}\mathrm{K}_S.\big(\mu_\sigma(\mathrm{ch})\omega+\beta\big)\Big) \\ &\quad+\Big(1,-\frac{3}{4}\mathrm{K}_S,-\frac{1}{2}\chi(\mathcal{O}_S)+\frac{11}{32}\mathrm{K}_S^2\Big)\Big) \\ &\quad+\Big(0,0,\beta.\big(\mu_\sigma(\mathrm{ch})\omega+\beta\big)-\frac{1}{2}(\omega^2+\beta^2)\Big). \end{aligned} \tag{4.1}$$

Proof

The case for $\Im Z(\mathrm{ch})=0$ follows from (3.4). If $\Im Z(\mathrm{ch})>0$, we divide (3.4) by this positive number and obtain (4.1). The concrete formula is then derived by (2.5). □

LEMMA 4.2

Fix a frame (H,γ,u). *We have relations*

$$\mu_\sigma(\mathrm{ch})\omega+\beta=C(\mathrm{ch},\mathrm{ch}')H+u\gamma, \tag{4.2}$$

$$\beta.\big(\mu_\sigma(\mathrm{ch})\omega+\beta\big)-\frac{1}{2}(\omega^2+\beta^2)=-\frac{g}{2}D(\mathrm{ch},\mathrm{ch}')-\frac{d}{2}u^2, \tag{4.3}$$

where the numbers $C(\mathrm{ch},\mathrm{ch}')$ *and* $D(\mathrm{ch},\mathrm{ch}')$ *are given by (2.9) and (2.10).*

Proof

The proof is a direct computation by using Maciocia's Theorem 2.4. For the reader's convenience, we give the details. For (4.2), we only need to check that

$$\mu_\sigma(\mathrm{ch})t+s=C(\mathrm{ch},\mathrm{ch}').$$

Recall that the wall equation is $(s-C)^2+t^2=D+C^2$. Now

$$\begin{aligned} \mu_\sigma(\mathrm{ch})t+s &= \frac{z-sy_1g+uy_2d+\frac{x}{2}(s^2g-u^2d-t^2g)}{(y_1-xs)g}+s \\ &= \frac{z+uy_2d-\frac{x}{2}u^2d-\frac{xg}{2}(s^2+t^2)}{(y_1-xs)g} \end{aligned}$$

$$= \frac{z + uy_2 d - \frac{x}{2}u^2 d - \frac{xg}{2}(2sC + D)}{(y_1 - xs)g} \quad \text{by using the wall equation.}$$

So we only need to check that

$$z + uy_2 d - \frac{x}{2}u^2 d - \frac{xg}{2}(2sC + D) = (y_1 - xs)gC. \tag{4.4}$$

If $x = 0$, then (4.4) is true since $C = \frac{z+duy_2}{gy_1}$. If $x \neq 0$, then (4.4) is still true by using (2.11).

Let us prove (4.3). We have that

$$\begin{aligned} \text{LHS of (4.3)} &= (sH + u\gamma).(CH + u\gamma) - \frac{g}{2}\Big(t^2 + s^2 - \frac{d}{g}u^2\Big) \\ &= sCg - u^2 d - \frac{g}{2}\Big(2sC + D - \frac{d}{g}u^2\Big) = \text{RHS of (4.3)}. \end{aligned}$$

□

DEFINITION 4.3

Fix a frame (H, γ, u). Define the vector $\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}, \mathrm{ch}')$ as

$$\Big(1, CH + u\gamma - \frac{3}{4}\mathrm{K}_S, -\frac{3}{4}\mathrm{K}_S.(CH + u\gamma) - \frac{1}{2}\chi(\mathcal{O}_S) + \frac{11}{32}\mathrm{K}_S^2\Big),$$

where the center $C = C(\mathrm{ch}, \mathrm{ch}')$ is as in (2.9).

LEMMA 4.4

If $\Im Z_{\omega,\beta}(\mathrm{ch}) > 0$, *then*

$$w_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=} \Big(\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2}u^2\Big)(0, 0, -1) + \mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}, \mathrm{ch}'). \tag{4.5}$$

Proof

This is a direct computation by using (4.1), (4.2), and (4.3). □

4.2. The local Bayer–Macrì decomposition

We decompose w_σ into three cases according to the dimension of the support of objects with invariants ch. Assume that there is a flat family $\mathcal{E} \in \mathrm{D^b}(M_\sigma(\mathrm{ch}) \times S)$, and denote the Mukai morphism by $\theta_{\sigma,\mathcal{E}}$.

4.2.1. Supported in dimension 0

Fix $\mathrm{ch} = (0, 0, n)$, with n a positive integer. Fix a frame (H, γ, u). Since $t > 0$ is the trivial chamber and there is no wall on $\Pi_{(H,\gamma,u)}$, we obtain $w_\sigma \overset{\mathbb{R}_+}{=\!=} \Im\mho_Z \overset{\mathbb{R}_+}{=\!=} (0, H, (\beta - \frac{3}{4}\mathrm{K}_S).H)$, and the nef line bundle $\ell_\sigma = \theta_{\sigma,\mathcal{E}}(0, H, (sH - \frac{3}{4}\mathrm{K}_S).H)$ on the moduli space $M_\sigma(\mathrm{ch}) \cong \mathrm{Sym}^n(S)$ (see [22, Lemma 2.10]), which is independent of s.

4.2.2. Supported in dimension 1

Fix a frame (H,γ,u). We assume that $\mathrm{ch}=(0,\mathrm{ch}_1,\mathrm{ch}_2)$ with $\mathrm{ch}_1 .H>0$. Now the center is given by $C=\frac{z+duy_2}{gy_1}$, which is independent of ch'. So the vector

$$\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}):=\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch},\mathrm{ch}')$$

is also independent of ch'. There is another special vector

$$w_{\infty H,\beta}\xlongequal{\mathbb{R}_+}(0,0,-1).$$

We get two well-defined line bundles in the following theorem:

$$\mathcal{S}:=\theta_{\sigma,\mathcal{E}}(0,0,-1),\qquad \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}):=-\theta_{\sigma,\mathcal{E}}\big(\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch})\big). \tag{4.6}$$

LEMMA 4.5 (The local Bayer–Macrì decomposition in dimension 1)

Fix a frame (H,γ,u)*. Assume that* $\mathrm{ch}=(0,\mathrm{ch}_1,\mathrm{ch}_2)$ *with* $\mathrm{ch}_1 .H>0$.

(a) *There is a decomposition*

$$w_{\sigma\in W(\mathrm{ch},\mathrm{ch}')}\xlongequal{\mathbb{R}_+}\Big(\frac{g}{2}D(\mathrm{ch},\mathrm{ch}')+\frac{d}{2}u^2\Big)(0,0,-1)+\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}),$$

where $(0,0,-1),\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch})\in\mathbf{v}^\perp$*. Moreover,* $r=\mathrm{ch}_0'\neq 0$*, and the coefficient before* $(0,0,-1)$ *is expressed in terms of the potential destabilizing Chern character* $\mathrm{ch}'=(r,c_1H+c_2\gamma+\delta',\chi)$*:*

$$\frac{g}{2}D(\mathrm{ch},\mathrm{ch}')+\frac{d}{2}u^2=\frac{\chi-gCc_1+udc_2}{r}. \tag{4.7}$$

(b) *Assume that there is a flat family* $\mathcal{E}\in\mathrm{D^b}(M_\sigma(\mathrm{ch})\times S)$*. Then the Bayer–Macrì nef line bundle on the moduli space* $M_\sigma(\mathrm{ch})$ *has a decomposition*

$$\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')}\xlongequal{\mathbb{R}_+}\Big(\frac{g}{2}D(\mathrm{ch},\mathrm{ch}')+\frac{d}{2}u^2\Big)\mathcal{S}-\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}). \tag{4.8}$$

Proof

Part (a) follows from computation. The Mukai vector $\mathbf{v}$ is given by

$$\mathbf{v}=\Big(0,\mathrm{ch}_1,\mathrm{ch}_2-\frac{1}{4}\mathrm{ch}_1 .\mathrm{K}_S\Big).$$

So $(0,0,-1)\in\mathbf{v}^\perp$ by the definition in (3.2) and the formula (2.3). To show $\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch})\in\mathbf{v}^\perp$, we can either directly compute the Mukai pairing

$$\big\langle\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}),\mathbf{v}\big\rangle_S=\Big(CH+u\gamma-\frac{3}{4}\mathrm{K}_S\Big)\cdot\mathrm{ch}_1-\Big(\mathrm{ch}_2-\frac{1}{4}\mathrm{ch}_1 .\mathrm{K}_S-\frac{1}{2}\mathrm{ch}_1 .\mathrm{K}_S\Big)=0$$

or note the relation (4.5) and the fact that $w_{\sigma\in W(\mathrm{ch},\mathrm{ch}')}\in\mathbf{v}^\perp$, $(0,0,-1)\in\mathbf{v}^\perp$. Then $\mathcal{S}$ and $\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})$ are well defined in (4.6). Recall (2.10) for $D(\mathrm{ch},\mathrm{ch}')$. Since $x=\mathrm{ch}_0=0$, we obtain $r\neq 0$. The relation (4.7) is then derived by using (2.14). Part (b) follows from part (a) by applying the Mukai morphism $\theta_{\sigma,\mathcal{E}}$. $\square$

4.2.3. Supported in dimension 2

Assume that $\mathrm{ch}_0 \neq 0$. If $\mathrm{ch}_0 < 0$, then we observe

$$w_{\infty H,\beta} \overset{\mathbb{R}_+}{=\!=} \Big(0, H, \Big(\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \frac{3}{4}\mathrm{K}_S\Big)H\Big) \overset{\mathbb{R}_+}{=\!=} w_{\sigma\in\mathtt{DUW}}.$$

DEFINITION 4.6

Fix $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ with $\mathrm{ch}_0 \neq 0$, and define

$$\begin{aligned}
\mathbf{w}(\mathrm{ch}) &:= \Big(1, -\frac{3}{4}\mathrm{K}_S, -\frac{\mathrm{ch}_2}{\mathrm{ch}_0} - \frac{1}{2}\chi(\mathcal{O}_S) + \frac{11}{32}\mathrm{K}_S^2\Big),\\
\mathbf{m}(L,\mathrm{ch}) &:= \Big(0, L, \Big(\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \frac{3}{4}\mathrm{K}_S\Big).L\Big), \quad \text{where } L \in N^1(S),\\
\mathbf{u}(\mathrm{ch}) &:= \mathbf{w}(\mathrm{ch}) + \mathbf{m}\Big(\frac{1}{2}\mathrm{K}_S, \mathrm{ch}\Big)\\
&= \Big(1, -\frac{1}{4}\mathrm{K}_S, -\frac{\mathrm{ch}_2}{\mathrm{ch}_0} + \frac{\mathrm{ch}_1 .\mathrm{K}_S}{2\,\mathrm{ch}_0} - \frac{1}{2}\chi(\mathcal{O}_S) - \frac{1}{32}\mathrm{K}_S^2\Big).
\end{aligned}$$

LEMMA 4.7

We have the following three perpendicular relations for Mukai vectors:

$$\mathbf{m}(L,\mathrm{ch}), \qquad \mathbf{w}(\mathrm{ch}), \qquad \mathbf{u}(\mathrm{ch}) \in \mathbf{v}^{\perp}.$$

Proof

The perpendicular relations can be checked directly by (3.2), (2.2), and (2.3). □

LEMMA 4.8 (The local Bayer–Macrì decomposition in dimension 2)

(a) *If* $\mathrm{ch}_0 \neq 0$ *and* $\Im Z(\mathrm{ch}) > 0$, *then there is a decomposition (up to a positive scalar)*

$$w_{\omega,\beta}(\mathrm{ch}) \overset{\mathbb{R}_+}{=\!=} \mu_\sigma(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch}) + \mathbf{m}(\beta,\mathrm{ch}) + \mathbf{w}(\mathrm{ch}) \tag{4.9}$$

$$= \mu_\sigma(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch}) + \mathbf{m}(\alpha,\mathrm{ch}) + \mathbf{u}(\mathrm{ch}), \tag{4.10}$$

where $\mathbf{m}(\omega,\mathrm{ch})$*,* $\mathbf{m}(\beta,\mathrm{ch})$*,* $\mathbf{m}(\alpha,\mathrm{ch})$*,* $\mathbf{w}(\mathrm{ch})$*,* $\mathbf{u}(\mathrm{ch}) \in \mathbf{v}^{\perp}$*.*

(b) *Assume that there is a flat family* $\mathcal{E}$*. Then the Bayer–Macrì line bundle class* $\ell_{\sigma_{\omega,\beta}}$ *has a decomposition in* $N^1(M_\sigma(\mathrm{ch}))$*:*

$$\ell_{\sigma_{\omega,\beta}} \overset{\mathbb{R}_+}{=\!=} \mu_\sigma(\mathrm{ch})\theta_{\sigma,\mathcal{E}}\big(\mathbf{m}(\omega,\mathrm{ch})\big) + \theta_{\sigma,\mathcal{E}}\big(\mathbf{m}(\beta,\mathrm{ch})\big) + \theta_{\sigma,\mathcal{E}}\big(\mathbf{w}(\mathrm{ch})\big). \tag{4.11}$$

Proof

Recall (4.1). To show (4.9), we only need to check that

$$\frac{\mathrm{ch}_1}{\mathrm{ch}_0}.\big(\mu_\sigma(\mathrm{ch})\omega + \beta\big) - \frac{\mathrm{ch}_2}{\mathrm{ch}_0} = \beta.\big(\mu_\sigma(\mathrm{ch})\omega + \beta\big) - \frac{1}{2}(\omega^2 + \beta^2). \tag{4.12}$$

By the definition of the Bridgeland slope, we have

$$\mu_\sigma(\mathrm{ch}) = \frac{\mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_0(\omega^2 - \beta^2) - \mathrm{ch}_1 .\beta}{\omega.(\mathrm{ch}_1 - \mathrm{ch}_0\,\beta)}.$$

So

$$\mu_\sigma(\mathrm{ch})\omega.\Big(\frac{\mathrm{ch}_1}{\mathrm{ch}_0}-\beta\Big)=\frac{\mathrm{ch}_2}{\mathrm{ch}_0}-\frac{1}{2}(\omega^2-\beta^2)-\frac{\mathrm{ch}_1}{\mathrm{ch}_0}.\beta.$$

Therefore, we have (4.12). Then (4.10) follows from (4.9) and the relation $\alpha = \beta - \frac{1}{2}\mathrm{K}_S$. Part (b) follows directly by applying the Mukai morphism $\theta_{\sigma,\mathcal{E}}$. □

REMARK 4.9

An equivalent decomposition of $w_{\omega,\beta}(\mathrm{ch})$ in (4.9) is also obtained by Bolognese, Huizenga, Lin, Riedl, Schmidt, Woolf, and Zhao [9, Proposition 3.8].

4.3. The global Bayer–Macrì decomposition

Assume the existence of the global Bayer–Macrì map. Assume that $W(\mathrm{ch},\mathrm{ch}')$ is an actual Bridgeland wall.

4.3.1. Supported in dimension 1

THEOREM 4.10

Fix a frame (H,γ,u), and assume that $\mathrm{ch}=(0,\mathrm{ch}_1,\mathrm{ch}_2)$ with $\mathrm{ch}_1 .H>0$. Assume the existence of the global Bayer–Macrì map, with the fixed base stability condition in the Simpson chamber $\mathtt{SC}$, that is, $M_{\sigma\in\mathtt{SC}}(\mathrm{ch})\cong M_{(\alpha,\omega)}(\mathrm{ch})$. Then there is a correspondence from the Bridgeland wall $W(\mathrm{ch},\mathrm{ch}')$ as semicircle (2.8) in the half-plane $\Pi_{(H,\gamma,u)}$ with fixed center C (or, equivalently, as semiline (2.20) in the plane $\Sigma_{(H,\gamma,u)}$ with fixed slope C) to the Mori wall inside the pseudoeffective cone $\overline{\mathrm{Eff}}(M_{(\alpha,\omega)})(\mathrm{ch})$:

$$\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')}\overset{\mathbb{R}_+}{=\!=}\Big(\frac{g}{2}D(\mathrm{ch},\mathrm{ch}')+\frac{d}{2}u^2\Big)\mathcal{S}-\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}) \tag{4.13}$$

$$=\frac{\chi-gCc_1+udc_2}{r}\mathcal{S}-\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}). \tag{4.14}$$

Proof

The center $C=\frac{z+duy_2}{gy_1}$ is independent of ch'. So is $\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})$. The number $D(\mathrm{ch},\mathrm{ch}')$ is given by (2.15). Since the base stability condition $\sigma\in\mathtt{SC}$, we obtain

$$\mathcal{S}=\theta_{\mathtt{SC},\mathcal{E}}\big((0,0,-1)\big)\in N^1\big(M_{(\alpha,\omega)}(\mathrm{ch})\big),$$

$$\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})=-\theta_{\mathtt{SC},\mathcal{E}}\big(\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch})\big)\in N^1\big(M_{(\alpha,\omega)}(\mathrm{ch})\big).$$

Then (4.13) follows from (4.8) by fixing the above two line bundles $\mathcal{S}$ and $\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})$ in the Simpson moduli space. We obtain (4.14) by using (4.7). □

By using the Donaldson morphism $\lambda_{\mathcal{E}}$, we have [18, Example 8.1.3]

$$\mathcal{S}=\lambda_{\mathcal{E}}(0,0,1)=(p_M)_*\big(\det(\mathcal{E})|_{M\times\{s\}}\big).$$

The line bundle $\mathcal{S}$ is conjectured to induce the support morphism, which maps $E\in M_{\sigma\in\mathtt{SC}}(\mathrm{ch})$ to $\mathrm{Supp}(E)$. This is proved for the case in which $S=\mathbb{P}^2$ (see [28]) or S is a K3 surface (see [5, Lemma 11.3]).

4.3.2. Supported in dimension 2

By applying the derived dual functor if necessary, we further assume that $\mathrm{ch}_0 > 0$ in Lemma 4.8. Recall Lemma 3.5 and Corollary 2.11. We obtain

$$
\begin{aligned}
w_{\omega,-\beta}(-\mathrm{ch}^*) &\overset{\mathbb{R}_+}{=\!=} \mu_{\Phi(\sigma)}(-\mathrm{ch}^*)\mathbf{m}(\omega,-\mathrm{ch}^*) + \mathbf{m}(-\beta,\mathrm{ch}) + \mathbf{w}(-\mathrm{ch}^*),\\
\ell_\sigma \cong \ell_{\Phi(\sigma)} &\overset{\mathbb{R}_+}{=\!=} \mu_{\Phi(\sigma)}(-\mathrm{ch}^*)\theta_{\Phi(\sigma),\mathcal{F}}\big(\mathbf{m}(\omega,-\mathrm{ch}^*)\big)\\
&\qquad + \theta_{\Phi(\sigma),\mathcal{F}}\big(\mathbf{m}(-\beta,-\mathrm{ch}^*)\big) + \theta_{\Phi(\sigma),\mathcal{F}}\big(\mathbf{w}(-\mathrm{ch}^*)\big).
\end{aligned}
$$

Since $\mu_{\Phi(\sigma)}(-\mathrm{ch}^*) = -\mu_\sigma(\mathrm{ch})$, we get

$$\theta_{\Phi(\sigma),\mathcal{F}}\big(\mathbf{w}(-\mathrm{ch}^*)\big) \cong \theta_{\sigma,\mathcal{E}}\big(\mathbf{w}(\mathrm{ch})\big), \qquad \theta_{\Phi(\sigma),\mathcal{F}}\big(\mathbf{m}(\omega,-\mathrm{ch}^*)\big) \cong -\theta_{\sigma,\mathcal{E}}\big(\mathbf{m}(\omega,\mathrm{ch})\big).$$

NOTATION 4.11

Assume that $\mathrm{ch}_0 > 0$. Let L be a line bundle on S. Denote

$$\widetilde{L} := \theta_{\Phi(\sigma),\mathcal{F}}\big(\mathbf{m}(L,-\mathrm{ch}^*)\big) \cong -\theta_{\sigma,\mathcal{E}}\big(\mathbf{m}(L,\mathrm{ch})\big), \qquad \mathcal{B}_0 := -\theta_{\sigma,\mathcal{E}}\big(\mathbf{u}(\mathrm{ch})\big).$$

Then we have

$$\theta_{\sigma,\mathcal{E}}\big(\mathbf{w}(\mathrm{ch})\big) = \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \tag{4.15}$$

Recall $\alpha = \beta - \frac{1}{2}\mathrm{K}_S$. Denote

$$\mathcal{B}_\alpha := \widetilde{\beta} - \theta_{\Phi(\sigma),\mathcal{F}}\big(\mathbf{w}(-\mathrm{ch}^*)\big) \cong \widetilde{\beta} - \theta_{\sigma,\mathcal{E}}\big(\mathbf{w}(\mathrm{ch})\big) = \widetilde{\alpha} + \mathcal{B}_0.$$

Note that $\widetilde{L}$, $\mathcal{B}_\alpha$, and $\mathcal{B}_0$ are line bundles on $M_\sigma(\mathrm{ch})$.

ASSUMPTION 4.12

The Chern character $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ satisfies *condition* (C) if the following three assumptions hold:

- $\mathrm{ch}_0 > 0$;
- (Bogomolov type) $\mathrm{ch}_1^2 - 2\,\mathrm{ch}_0\,\mathrm{ch}_2 \geq 0$; and
- $\gcd(\mathrm{ch}_0, \mathrm{ch}_1.H, \mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_1.\mathrm{K}_S) = 1$ for a fixed ample line bundle H (see [18, Corollary 4.6.7]).

THEOREM 4.13

Fix ch*, and assume that it satisfies condition* (C)*. Assume the existence of the global Bayer–Macrì map, with the fixed base stability condition in the Gieseker chamber* $\mathtt{GC}$*, that is,* $M_{\sigma\in\mathtt{GC}}(\mathrm{ch}) \cong M_{(\alpha,\omega)}(\mathrm{ch})$*. Then the following conclusions hold.*

(a) *There is a Bayer–Macrì decomposition for the line bundle* $\ell_{\sigma_{\omega,\beta}}$*:*

$$\ell_{\sigma_{\omega,\beta}} \overset{\mathbb{R}_+}{=\!=} \big(-\mu_{\sigma_{\omega,\beta}}(\mathrm{ch})\big)\widetilde{\omega} - \mathcal{B}_\alpha = \big(-\mu_{\sigma_{\omega,\beta}}(\mathrm{ch})\big)\widetilde{\omega} - \widetilde{\alpha} - \mathcal{B}_0.$$

(b) *The line bundle* $\widetilde{\omega}$ *induces the Gieseker–Uhlenbeck morphism from the* (α,ω)*-Gieseker semistable moduli space* $M_{(\alpha,\omega)}(\mathrm{ch})$ *to the Uhlenbeck space* $U_\omega(\mathrm{ch})$*.*

(c) *If* $\mathrm{ch}_0 = 2$ *and* $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$*, the divisor* $\mathcal{B}_\alpha$ *is the* α*-twisted boundary divisor of the induced Gieseker–Uhlenbeck morphism. In particular, in the case of* $\alpha = 0$*, the divisor* $\mathcal{B}_0$ *is the (untwisted) boundary divisor from the* ω*-semistable Gieseker moduli space* $M_\omega(\mathrm{ch})$ *to the Uhlenbeck space* $U_\omega(\mathrm{ch})$*.*

(d) *Fix a frame* (H,γ,u)*. Then there is a correspondence from the Bridgeland wall* $W(\mathrm{ch},\mathrm{ch}')$ *as semicircle (2.8) in the half-plane* $\Pi_{(H,\gamma,u)}$ *with center* $C(\mathrm{ch},\mathrm{ch}')$ *(or, equivalently, as semiline (2.19) in the plane* $\Sigma_{(H,\gamma,u)}$ *with slope* $C(\mathrm{ch},\mathrm{ch}')$*) to the effective line bundle on the moduli space* $M_{(\alpha,\omega)}(\mathrm{ch})$*:*

$$\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=} -C(\mathrm{ch},\mathrm{ch}')\widetilde{H} - u\widetilde{\gamma} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \tag{4.16}$$

Proof

We identify $\widetilde{\omega}$ and $\mathcal{B}_\alpha$ as line bundles on $M_{(\alpha,\omega)}(\mathrm{ch})$. Then (4.11) implies part (a). Since $-\mu_{\sigma\in\mathtt{UW}}(\mathrm{ch}) = \mu_{\Phi(\sigma)\in\mathtt{DUW}}(-\mathrm{ch}^*) = +\infty$, we obtain

$$\ell_{\sigma\in\mathtt{UW}} \overset{\mathbb{R}_+}{=\!=} \widetilde{\omega}.$$

Part (b) follows from [18, Theorem 8.2.8]. If $\mathrm{ch}_0 = 2$ and $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$, then the Gieseker–Uhlenbeck morphism is a divisorial contraction by [18, Lemma 9.2.1], and $\mathcal{B}_\alpha$ is the boundary divisor from the α-twisted moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ to the Uhlenbeck space $U_\omega(\mathrm{ch})$. In particular, $\mathcal{B}_0$ is the untwisted boundary divisor. This shows part (c). Then (4.16) follows from a direct computation by using (4.11), (4.2), and (4.15):

$$\begin{aligned}\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} &\overset{\mathbb{R}_+}{=\!=} \theta_{\sigma,\mathcal{E}}\big(\mathbf{m}\big(\mu_\sigma(\mathrm{ch})\omega + \beta, \mathrm{ch}\big)\big) + \theta_{\sigma,\mathcal{E}}\big(\mathbf{w}(\mathrm{ch})\big) \\ &= -C(\mathrm{ch},\mathrm{ch}')\widetilde{H} - u\widetilde{\gamma} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \qquad \square\end{aligned}$$

There are two line bundles $\mathcal{L}_0$, $\mathcal{L}_1$ on Gieseker moduli space introduced by Le Potier. We follow the notation from [18, Definition 8.1.9]. Then

$$\mathcal{L}_0 = -\mathrm{ch}_0\,\mathcal{B}_0, \qquad \mathcal{L}_1 = \mathrm{ch}_0\,\widetilde{H}.$$

Arcara, Bertram, Coskun, and Huizenga [2] studied the Hilbert scheme of n-points on the projective plane $\mathbb{P}^2$ and gave a precise conjecture between the Bridgeland walls and Mori walls, which was one of the motivations of Bayer–Macrì's line bundle theory. This conjecture was proved by Li and Zhao [19]. The relation still holds for a more general primitive character (see [20]).

COROLLARY 4.14 (see [20])

Let $S = \mathbb{P}^2$*, and denote by* H *the hyperplane divisor on* $\mathbb{P}^2$*. Fix* ch *primitive with* $\mathrm{ch}_0 > 0$*. Assume that* $M_H(\mathrm{ch}) \neq \emptyset$*. Then there is a relation*

$$\ell_{\sigma\in W(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=} -\Big(C(\mathrm{ch},\mathrm{ch}') + \frac{3}{2}\Big)\widetilde{H} - \mathcal{B}_0.$$

Proof

The existence of the global Bayer–Macrì map was proved by Li and Zhao [20, Theorem 0.2]. We then apply (4.16) with $\gamma = 0$ and $\mathrm{K}_S = -3H$. $\square$

EXAMPLE 4.15

Assume that the irregularity of the surface is 0. If $\mathrm{ch} = (1, 0, -n)$, then the Gieseker–Uhlenbeck morphism is the Hilbert–Chow morphism $h : S^{[n]} \to S^{(n)}$, which maps the Hilbert scheme of n-points on S to the symmetric product $S^{(n)}$. In particular,

$$\widetilde{H} = \mathcal{L}_1 = h^*\big(\mathcal{O}_{S^{(n)}}(1)\big),$$

which induces the Hilbert–Chow morphism (see [18, Example 8.2.9]). The boundary divisor of the Hilbert–Chow morphism is

$$\mathrm{B} := \big\{\xi \in S^{[n]} : \big|\mathrm{Supp}(\xi)\big| < n\big\}. \tag{4.17}$$

It is known from [7, Appendix] that $\frac{1}{2}\mathrm{B}$ is an integral divisor and $\mathcal{B}_0 = -\mathcal{L}_0 = \frac{1}{2}\mathrm{B}$.

5. A toy model: Fibered surface over $\mathbb{P}^1$ with a global section

We compute the nef cone of the Hilbert scheme $S^{[n]}$ of n-points over S by using Theorem 4.13. Here let $\pi : S \to \mathbb{P}^1$ be either a $\mathbb{P}^1$-fibered or an elliptic-fibered surface over $\mathbb{P}^1$ with a global section E whose self-intersection number is $-e$. We assume that all fibers are reduced and irreducible, and the Picard group of S is generated by E and F, where F is the generic fiber class. We have the intersection numbers

$$E.E = -e, \qquad E.F = 1, \qquad F^2 = 0.$$

- $\mathbb{P}^1$ fibration. In this case, $F \cong \mathbb{P}^1$ and S is the *Hirzebruch surface* Σ_e with integer $e \geq 0$. Then $\mathrm{K}_S = -2(E + eF) + (e-2)F$. Here Σ_0 is the surface $\mathbb{P}^1 \times \mathbb{P}^1$.
- Elliptic fibration. In this case, the generic fiber F is an elliptic curve. We denote the surface by S_e and further assume that $e \geq 2$. Then S_e has the unique section E and $\mathrm{K}_S = (e-2)F$ (see [26]).

Since the nef cone of S is generated by the two extremal nef line bundles $E + eF$ and F, any ample line bundle H, after rescaling, can be written as

$$H := \lambda(E + eF) + (1 - \lambda)F, \quad 0 < \lambda < 1.$$

Take γ such that $H.\gamma = 0$ and $H^2 = -\gamma^2$. Basic computation shows that $\gamma = \pm(-\lambda(E + eF) + (1 - \lambda + e\lambda)F)$. An (H, γ, u)-frame, with $u \geq 0$, is fixed by the choice

$$\gamma := -\lambda(E + eF) + (1 - \lambda + e\lambda)F.$$

The two numbers λ and u are regarded as the *initial values.*

Fix $\mathrm{ch} = (1, 0, -n)$, with integer $n \geq 2$. The potential walls are given by $(s - C)^2 + t^2 = C^2 + D$ with $t > 0$, where

$$C = C(\mathrm{ch}, \mathrm{ch}') = \frac{\mathrm{ch}_2' + \mathrm{ch}_0' n - u\,\mathrm{ch}_1'.\gamma}{\mathrm{ch}_1'.H}, \qquad D = D(\mathrm{ch}, \mathrm{ch}') = -u^2 - \frac{2n}{H^2}.$$

Recall $s_0 := \frac{\mathrm{ch}_1 .H}{\mathrm{ch}_0\, H^2} = 0$. The $\mathtt{UW}$ is given by $s = s_0 = 0$. Therefore, $C < 0$.

One type of nef line bundle on $S^{[n]}$ is $\widetilde{\omega}$ for $\omega \in \mathrm{Amp}(S)$, which induces a Gieseker–Uhlenbeck morphism. By taking ω to be extremal, that is, $\omega = E + eF$ or F, we obtain two extremal nef line bundles on $S^{[n]}$:

$$(\widetilde{E+eF}), \qquad \widetilde{F}. \tag{5.1}$$

To find the nef cone of $S^{[n]}$, we need to find the biggest nontrivial wall, that is, the smallest value of C. Let us call such a wall the *Gieseker wall*.

LEMMA 5.1

If the Gieseker wall is given by a rank 1 wall, then

$$\mathrm{ch}' = (1, -F, 0) \qquad \textit{or} \qquad \mathrm{ch}' = \Big(1, -E, \frac{-e}{2}\Big).$$

Proof

The idea is the same as in [2] (see also [8]). Any destabilizing subsheaf of I_Z of rank 1 has the form $L \otimes I_W$ with Chern character $\mathrm{ch}' = (1, L, \frac{L^2}{2} - w)$ for some line bundle L and some ideal sheaf I_W of length $w \geq 0$. Then $C(\mathrm{ch}, \mathrm{ch}') = \frac{\frac{L^2}{2} + n - uL.\gamma}{L.H} + \frac{w}{-L.H}$. To guarantee $L \otimes I_W$ as an object in the heart, we need $L.H < 0$. Then we write $L = -(mF + kE)$, with two nonnegative integers m and k, and $(m, k) \neq (0, 0)$. To get the biggest nontrivial wall, we must take $w = 0$. Denote the line bundle on $S^{[n]}$ corresponding to the destabilizing line bundle $-(mF + kE)$ by $\ell(m, k)$. The locus contracted by $\ell(0, 1)$ is $\{Z \in S^{[n]} \mid Z \subset E\}$. The locus contracted by $\ell(1, 0)$ is $\{Z \in S^{[n]} \mid Z \subset F, Z \text{ is linear equivalent to } n(E \cap F)\}$. Assume that the smallest value is obtained by taking $(m, k) \neq (1, 0)$ or $(0, 1)$. Recall that the walls are nested. But the loci contracted by $\ell(1, 0)$ or $\ell(0, 1)$ are also contracted by $\ell(m, k)$, which is a contradiction. $\square$

The line bundles $\ell(1, 0)$ and $\ell(0, 1)$ depend on the initial values λ and u. By (4.16), we have

$$\ell(0,1)_{\lambda,u} = n(\widetilde{E+eF}) + \Big(\Big(\frac{1-\lambda}{\lambda}\Big)n - u\big(2(1-\lambda) + e\lambda\big)\Big)\widetilde{F} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0,$$

$$\ell(1,0)_{\lambda,u} = \Big(\Big(n - \frac{1}{2}e\Big)\Big(\frac{\lambda}{1-\lambda}\Big) + u\Big(\lambda + \Big(\frac{\lambda}{1-\lambda}\Big)(e\lambda + 1 - \lambda)\Big)\Big)(\widetilde{E+eF}) + \Big(n - \frac{1}{2}e\Big)\widetilde{F} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0.$$

By taking λ to 0^+ or 1^-, respectively, we get two nef boundaries as in (5.1):

$$\ell(0,1)_{0^+,u} \overset{\mathbb{R}_+}{=\!=} \widetilde{F}, \qquad \ell(1,0)_{1^-,u} \overset{\mathbb{R}_+}{=\!=} (\widetilde{E+eF}).$$

Moreover, $\ell(0,1)_{\lambda,u}$ is decreasing and $\ell(1,0)_{\lambda,u}$ is increasing with respect to λ. The two types of loci are simultaneously contracted if and only if

$$\ell(0,1)_{\lambda,u} = \ell(1,0)_{\lambda,u}.$$

The solutions are given by

$$u = U(\mathrm{ch}, \lambda) := \frac{\frac{n}{\lambda} - \frac{n - \frac{e}{2}}{1\lambda}}{2 + e \cdot \frac{\lambda}{1-\lambda}} \quad \text{and} \quad 0 < \lambda < 1. \tag{5.2}$$

Moreover, with $0 < \lambda < 1$, we have

$$\ell(0,1)_{\lambda, U(\mathrm{ch},\lambda)} = \ell(1,0)_{\lambda, U(\mathrm{ch},\lambda)} = n\widetilde{(E+eF)} + \Big(n - \frac{e}{2}\Big)\widetilde{F} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \tag{5.3}$$

THEOREM 5.2

(a) *The nef cone* $\mathrm{Nef}(\Sigma_e^{[n]})$ *($e \geq 0$, $n \geq 2$) is generated by the nonnegative combinations of* $\widetilde{(E+eF)}$, $\widetilde{F}$, *and* $(n-1)\widetilde{(E+eF)} + (n-1)\widetilde{F} - \frac{1}{2}\mathrm{B}$. *(See [8, Theorems 1(2), 1(3)].)*

(b) *The nef cone* $\mathrm{Nef}(S_e^{[n]})$ *($e \geq 2$, $n \geq 2$) is generated by the nonnegative combinations of* $\widetilde{(E+eF)}$, $\widetilde{F}$, *and* $n\widetilde{(E+eF)} + (n-1)\widetilde{F} - \frac{1}{2}\mathrm{B}$.

Proof

We only need to show that the Gieseker wall is not a higher rank wall in the case of (5.2). Note that the bundle in (5.3) is independent of the λ. So it is enough to check for the case $\lambda = \frac{1}{2}$. Now we have $u = U(\mathrm{ch}, \frac{1}{2}) = \frac{e}{e+2}$. The two walls given by Lemma 5.1 coincide with the center $C = -2n + \frac{e}{e+2}$. By the estimation formula from [8, Section 5], the center C_k of a rank k wall ($k \geq 2$) is bounded by

$$C_k^2 \leq \Big(u^2 + \frac{2n}{H^2}\Big)\frac{(2k-1)^2}{(2k-1)^2 - 1} \leq \Big(u^2 + \frac{2n}{H^2}\Big)\frac{9}{8}.$$

Now $H^2 = \frac{e}{4u}$, and $u = \frac{e}{e+2}$. It is easy to check that

$$\Big(u^2 + \frac{8nu}{e}\Big)\frac{9}{8} < (-2n + u)^2.$$

So $C_k^2 < C^2$. Therefore, higher rank walls are strictly inside the wall given by center C. By (4.16), the extremal nef line bundle corresponding to C is

$$n\widetilde{(E+eF)} + \Big(n - \frac{e}{2}\Big)\widetilde{F} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0.$$

The nef cone of $S^{[n]}$ is generated by the nonnegative combinations of

$$\widetilde{(E+eF)}, \qquad \widetilde{F}, \qquad n\widetilde{(E+eF)} + \Big(n - \frac{e}{2}\Big)\widetilde{F} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0.$$

Recall that $\mathcal{B}_0 = \frac{1}{2}\mathrm{B}$ in Example 4.15. The proof is completed by using $\mathrm{K}_S = -2(E+eF) + (e-2)F$ or $(e-2)F$, respectively. $\square$

The above computation suggests that the number u plays an important role in order to find the extremal nef line bundle, and in general $u \neq 0$, that is, ω is not parallel to β. The nef cone of $\Sigma_e^{[n]}$ has been obtained by Bertram and Coskun [8]. The nef cone of $S_2^{[n]}$ ($n \geq 2$) has been obtained by J. Li and W.-P. Li [21]. Both of the results use the notion of k-very ample line bundles (see [7]).

Appendix A. Twisted Gieseker stability and the large-volume limit

DEFINITION A.3 ([14, Definition 3.4], [15, Definition 4.1], [25, Definition 3.2])

Let $\omega, \alpha \in \mathrm{NS}(S)_{\mathbb{Q}}$ with ω ample. For $E \in \mathrm{Coh}(S)$, we denote the leading coefficient of $\chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m})$ with respect to m by a_d. A coherent sheaf E of dimension d is said to be α-twisted ω-Gieseker-(semi)stable if E is pure and, for all $0 \neq F \subsetneq (\subseteq) E$,

$$\frac{\chi(F \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{a_d(F)} < (\leq) \frac{\chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{a_d(E)} \quad \text{for } m \gg 0. \tag{A.4}$$

We also write them as (α,ω)-Gieseker (semi)stability. Denote $M_{(\alpha,\omega)}(\mathrm{ch})$ (if it exists) as the moduli space of (α,ω)-semistable sheaves E with $\mathrm{ch}(E) = \mathrm{ch}$.

PROPOSITION-DEFINITION A.4 ([22, Theorem 1.2])

Fix $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. Fix a frame (H,γ,u), and consider $\sigma_{\omega,\beta}$ on the (s,t)-half-plane $\Pi_{(H,\gamma,u)}$ (Definition 2.3). Denote $s_0 := \frac{\mathrm{ch}_1 . H}{\mathrm{ch}_0\, H^2}$ if $\mathrm{ch}_0 \neq 0$. We always fix the relation

$$\alpha = \beta - \frac{1}{2}\mathrm{K}_S. \tag{A.5}$$

$\mathtt{TC}$. If $\mathrm{ch} = (0,0,n)$ with positive integer n, then there is *no wall*, and $t > 0$ is the *trivial chamber* in $\Pi_{(H,\gamma,u)}$. Additionally,

$$M_{\sigma_{\omega,\beta}}(\mathrm{ch}) = M_{(\alpha,\omega)}(\mathrm{ch}) = \mathrm{Sym}^n(S).$$

$\mathtt{SC}$. If $\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1 . H > 0$, we define the chamber for $t \gg 0$ as the *Simpson chamber with respect to* (H,γ,u). Then

$$M_{\sigma_{\omega,\beta} \in \mathtt{SC}}(\mathrm{ch}) = M_{(\alpha,\omega)}(\mathrm{ch}).$$

Additionally, the (α,ω)-Gieseker semistability is the Simpson semistability defined by the slope $\frac{\mathrm{ch}_2(E) - \mathrm{ch}_1(E).\beta}{\omega.\mathrm{ch}_1(E)}$.

$\mathtt{GC}$. If $\mathrm{ch}_0 > 0$, we define the chamber for $t \gg 0$ and $s < s_0$ as the *Gieseker chamber with respect to* (H,γ,u). If ch satisfies condition (C), then

$$M_{\sigma_{\omega,\beta} \in \mathtt{GC}}(\mathrm{ch}) \cong M_{(\alpha,\omega)}(\mathrm{ch}).$$

$\mathtt{UW}$. If $\mathrm{ch}_0 > 0$, we define the wall $t > 0$ and $s = s_0$, that is, $\Im Z(\mathrm{ch}) = 0$ as the *Uhlenbeck wall with respect to* (H,γ,u).

$\mathtt{DGC}$. If $\mathrm{ch}_0 < 0$, we define the chamber for $t \gg 0$ and $s > s_0$ as the *dual Gieseker chamber with respect to* (H,γ,u). If $-(\mathrm{ch})^*$ satisfies condition (C), then by Lemma 2.10,

$$M_{\sigma_{\omega,\beta} \in \mathtt{DGC}}(\mathrm{ch}) \cong M_{\sigma_{\omega,-\beta} \in \mathtt{GC}}\big(-(\mathrm{ch})^*\big).$$

$\mathtt{DUW}$. If $\mathrm{ch}_0 < 0$, we define the wall $t > 0$ and $s = s_0$, that is, $\Im Z(\mathrm{ch}) = 0$ as the *dual Uhlenbeck wall with respect to* (H,γ,u). If $-(\mathrm{ch})^*$ satisfies condition (C), then

$$M_{\sigma_{\omega,\beta} \in \mathtt{DUW}}(\mathrm{ch}) \cong U_\omega\big(-(\mathrm{ch})^*\big),$$

where $U_\omega(-(\mathrm{ch})^*) = U_H(-(\mathrm{ch})^*)$ is the Uhlenbeck compactification of the moduli space $M^{\mathrm{lf}}_\omega(-(\mathrm{ch})^*)$ of locally free sheaves with invariant $-(\mathrm{ch})^*$.

Appendix B. Bayer–Macrì decomposition on K3 surfaces by using $\hat{\sigma}_{\omega,\beta}$

Let S be a smooth projective surface. By some physical hints (e.g., [3, Section 6.2.3]), the central charge is often taken as (e.g., [11], [6])

$$\hat{Z}_{\omega,\beta}(E) := -\int_S e^{-(\beta+\sqrt{-1}\omega)}.\mathrm{ch}(E).\sqrt{\mathrm{td}\,(S)}. \tag{A.6}$$

Similarly to Lemma 2.2, one can check that

$$\hat{Z}_{\omega,\beta}(E) = \langle \mho_{\hat{Z}}, v(E)\rangle_S, \quad \text{where } \mho_{\hat{Z}_{\omega,\beta}} := e^{\beta-\frac{1}{2}\mathrm{K}_S+\sqrt{-1}\omega}. \tag{A.7}$$

Write $\mho_{\hat{Z}_{\omega,\beta}}$ as $\mho_{\hat{Z}}$. Basic computation shows that

$$\langle \mho_Z, \mho_Z\rangle_S = \chi(\mathcal{O}_S) - \frac{1}{4}\mathrm{K}_S^2, \qquad \langle \mho_{\hat{Z}}, \mho_{\hat{Z}}\rangle_S = -\frac{1}{8}\mathrm{K}_S^2.$$

Recall from [12] and [4] that a numerical stability condition σ is called *reduced* if the corresponding $\pi(\sigma)$ satisfies $\langle \pi(\sigma), \pi(\sigma)\rangle_S = 0$.

In the following, we always assume that S is a smooth projective K3 surface and assume that $\hat{Z}_{\omega,\beta}(F) \notin \mathbb{R}_{\leq 0}$ for all spherical sheaves $F \in \mathrm{Coh}(S)$. Then $\hat{\sigma}_{\omega,\beta} = (\hat{Z}_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$ is a reduced numerical geometric Bridgeland stability condition (see [11, Lemma 6.2]). Let $\mathbf{v} = v(\mathrm{ch}) \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$ be a primitive class with $\langle \mathbf{v}, \mathbf{v}\rangle_S > 0$. Define $\hat{w}_{\omega,\beta} := \hat{w}_{\hat{\sigma}} := -\Im(\overline{\langle \mho_{\hat{Z}}, \mathbf{v}\rangle_S} \cdot \mho_{\hat{Z}})$. Define $\ell_{\hat{\sigma},\mathcal{E}}$ similarly to that in (3.1) but use $\hat{Z}$ instead. Then

$$\ell_{\hat{\sigma}_{\omega,\beta}} \overset{\mathbb{R}_+}{=\!=} \theta_{\hat{\sigma},\mathcal{E}}(\hat{w}_{\omega,\beta}).$$

Fix a frame (H,γ,u). The potential walls $\hat{W}(\mathrm{ch},\mathrm{ch}')$ in the (s,t)-model are given by semicircles (or in the (s,q)-model are given by semilines)

$$(s-C)^2 + t^2 = C^2 + D + \frac{2}{H^2} \quad \left(\text{or } q = Cs + \frac{1}{2}D + \frac{1}{H^2}\right), \tag{A.8}$$

where C and D are defined in Theorem 2.4. There is a global Bayer–Macrì map (see [5, Theorem 1.2]).

THEOREM A.5 (Bayer–Macrì decomposition on K3 surfaces)
Use the notation and assumptions as above.

- *If* $\mathrm{ch}_0 = 0$ *and* $\mathrm{ch}_1.H > 0$*, then the Bayer–Macrì line bundle has a decomposition*

$$\ell_{\hat{\sigma}\in\hat{W}(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=} \left(\frac{g}{2}D(\mathrm{ch},\mathrm{ch}') + \frac{d}{2}u^2\right)\mathcal{S} - \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}). \tag{A.9}$$

The line bundle $\mathcal{S}$ *induces the support morphism.*

- *If* $\mathrm{ch}_0 > 0$*, then the Bayer–Macrì line bundle has a decomposition*

$$\ell_{\hat{\sigma}\in\hat{W}(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=} -C\widetilde{H} - u\widetilde{\gamma} - \mathcal{B}_0. \tag{A.10}$$

The line bundle $\widetilde{\omega}$ *(or* $\widetilde{H}$*) induces the Gieseker–Uhlenbeck morphism.*

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Center for Geometry and Physics, Institute for Basic Science, Pohang 37673, Republic of Korea; wanminliu@gmail.com