Divisorial contractions to cDV points with discrepancy greater than 1

Yuki Yamamoto

Abstract We study 3-dimensional divisorial contractions to cDV points with discrepancy greater than 1 which are of exceptional type. We show that every 3-dimensional divisorial contraction is obtained as a weighted blowup.

1. Introduction

Let $P \in X$ be a germ of a 3-dimensional terminal singularity defined over $\mathbb{C}$. A projective birational morphism $f\colon Y \to X$ is called a *divisorial contraction* if

(i) $-K_Y$ is f-ample,
(ii) Y has only terminal singularities, and
(iii) the exceptional locus E of f is an irreducible divisor.

In this situation, we write $K_Y = f^*K_X + a(E,X)E$ with $a(E,X) \in \mathbb{Q}$. The coefficient $a(E,X)$ is called the *discrepancy* of E over X. When $f(E) = P$, that is, $f_{Y\setminus E}\colon Y \setminus E \to X \setminus \{P\}$ is an isomorphism, we write $f\colon (Y \supset E) \to (X \ni P)$.

It is a fundamental problem in 3-dimensional birational geometry to find all divisorial contractions $f\colon (Y \supset E) \to (X \ni P)$. In this article, I finish the classification of 3-dimensional divisorial contractions which contract an irreducible divisor to a point. The classification of all divisorial contractions to a point tells us that they are obtained as weighted blowups.

THEOREM 1.1

Let $f\colon Y \to X$ be a 3-dimensional divisorial contraction whose exceptional divisor E contracts to a point P. Then f is a weighted blowup of the singularity $P \in X$ embedded into a cyclic quotient 5-fold.

A detailed version of our main results in Theorem 1.1 shall be given in Section 2. The classification of all divisorial contractions to a non-Gorenstein point $P \in X$ in Theorem 1.1 has already been settled by [2]–[4], [10], [11], and [13]. For a Gorenstein point $P \in X$, several cases of divisorial contractions to P have already

Kyoto Journal of Mathematics, Vol. 58, No. 3 (2018), 529–567
First published online June 12, 2018.
DOI 10.1215/21562261-2017-0028,
Received July 7, 2015. Accepted October 17, 2016.
2010 Mathematics Subject Classification: Primary 14E05; Secondary 14E30.

Table 1. Divisorial contraction of exceptional type

Type	Terminal P	a	E^3	Non-Gorenstein terminal on Y
$e1$	cA_2[1], cD	4	$1/r$	$\frac{1}{r}(1,-1,8)$; $r \equiv \pm 3 \pmod 8$[1]
	cD	2	$2/r$	$\frac{1}{r}(1,-1,4)$
$e2$	cD, $cE_{6,7}$	2	$1/r$	cA/r or $cD/3$ deforming to $2 \times \frac{1}{r}(1,-1,2)$; $cD/3$ for $cE_{6,7}$
$e3$	cA_2, cD, cE_6	3	$1/4$	$cAx/4$ deforming to $\frac{1}{2}(1,1,1)$, $\frac{1}{4}(1,3,3)$
$e5$	cE_7	2	$1/7$	$\frac{1}{7}(1,6,6)$
$e9$	$cE_{7,8}$	2	$1/15$	$\frac{1}{3}(1,2,2)$ and $\frac{1}{5}(1,4,4)$

been classified. Kawakita [7] showed that f is obtained as a suitable weighted blowup in the case of a nonsingular point P, and Kawakita [8] classified divisorial contractions to a cA_1 point. Kawakita [10] also classified all divisorial contractions to a point into two types: the ordinary type and the exceptional type. We know that all divisorial contractions of ordinary type are classified by [10, Theorem 1.2]. Hayakawa [5], [6] classified divisorial contractions to points of type cD, cE with discrepancy 1. As a result, the remaining cases in Theorem 1.1 are divisorial contractions of exceptional type with discrepancy greater than 1, which are listed in Table 1. The main aim in this article is to finish the classification of all divisorial contractions listed in Table 1.

Chen, Hayakawa, and Kawakita found several examples of exceptional type listed in Table 1. There are several examples of type $e1$, $e2$, $e3$, and $e9$ which are weighted blowups by [10]. Chen has examples of type $e1$ with P of type cD and discrepancy 4, and there is an example of type $e5$ in [1].

In this article, we describe divisorial contractions to a Gorenstein point, and we show that every divisorial contraction listed in Table 1 is obtained as a weighted blowup if it exists. Our method of classification is to study the structure of the graded ring $\bigoplus_j f_*\mathcal{O}(-jE)/f_*\mathcal{O}(-(j+1)E)$. We find local coordinates at P to meet this structure and verify that f should be a certain weighted blowup. In certain cases, there are some choices of local coordinates unlike in the non-Gorenstein cases. So we should compute weighted blowups in detail, and in several cases, there is no suitable local coordinate. There are no divisorial contractions of type $e1$ with P of type cA_2 and discrepancy 4, type $e2$ with type cE_7, and type $e3$ with type cE_6.

We shall give the results in Section 2, and their proofs shall be given in Section 4. We explain terminal singularity, weighted blowup, and the singular Riemann–Roch theorem in Section 3.

2. Main results

We consider the divisorial contractions $f\colon (Y \supset E) \to (X \ni P)$ listed in Table 1. Our main results show that such contractions are obtained as weighted blowups

[1]The new case and the condition given by the erratum [12].

embedded into $\mathbb{C}^4$ or $\mathbb{C}^5$ if they exist. The following is a detailed version of our main results. Proofs shall be given in Section 4.

THEOREM 2.1

There is no divisorial contraction of type e1 *which contracts to a* cA_2 *point with discrepancy* 4.

THEOREM 2.2

Suppose that f *is a divisorial contraction of type* e1 *which contracts to a* cD *point with discrepancy* 4. *Then* f *is the weighted blowup with* $\mathrm{wt}(x_1,x_2,x_3,x_4,x_5) = (\frac{r+1}{2}, \frac{r-1}{2}, 4, 1, r)$ *with* $r \geq 7$, $r \equiv \pm 3 \pmod 8$, *after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + \lambda x_2 x_3^k + x_4 x_5 + p(x_3,x_4) = 0, \\ x_2^2 + 2x_1 q_1(x_3,x_4) + q_2(x_3,x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover, the equations defining X *satisfy the following conditions.*

(i) $\lambda \in \mathbb{C}$, $k > \frac{r+3}{8}$, $\mathrm{wt}\, p \geq r+1$, $\mathrm{wt}\, q_1 = \frac{r-3}{2}$, $\mathrm{wt}\, q_2 = r-1$, *and* q_1, q_2 *are weighted homogeneous for the weights distributed above.*

(ii) q_2 *is not square if* $q_1 = 0$.

(iii) *If* $r \equiv 3 \pmod 8$ *(resp.,* $r \equiv -3 \pmod 8$*), then* $x_3^{\frac{r+1}{4}} \in p$ *(resp.,* $x_3^{\frac{r-1}{4}} \in q_2$*).*

THEOREM 2.3

Suppose that f *is a divisorial contraction of type* e1 *which contracts to a* cD *point with discrepancy* 2. *Then* f *is the weighted blowup with* $\mathrm{wt}(x_1,x_2,x_3,x_4,x_5) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1, r)$ *with* $r \geq 5$ *after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + \lambda x_2 x_3^k + x_4 x_5 + p(x_3,x_4) = 0, \\ x_2^2 + 2x_1 q_1(x_3,x_4) + q_2(x_3,x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover, the equations defining X *satisfy the following conditions.*

(i) $\lambda \in \mathbb{C}$, $k > \frac{r+1}{4}$, $\mathrm{wt}\, p \geq r+1$, $\mathrm{wt}\, q_1 = \frac{r-3}{2}$, $\mathrm{wt}\, q_2 = r-1$, *and* q_1, q_2 *are weighted homogeneous for the weights distributed above.*

(ii) q_2 *is not square if* $q_1 = 0$.

(iii) $x_3^{\frac{r+1}{2}} \in p$.

THEOREM 2.4

Suppose that f *is a divisorial contraction of type* e2 *which contracts to a* cD *point with discrepancy* 2. *Then one of the following holds.*

(i) f *is the weighted blowup with* $\mathrm{wt}(x_1,x_2,x_3,x_4) = (r,r,2,1)$ *after an identification of* $P \in X$ *with*

$$o \in \left(x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3,x_4) + \lambda x_2 x_3^k + q(x_3,x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover, the equation defining X *satisfies the following conditions.*

(1) $\lambda \in \mathbb{C}$, $k > \frac{r}{2}$, $\operatorname{wt} q \geq 2r$, *and* p *is weighted homogeneous of weight* $r-1$ *for the weights distributed above.*

(2) $p \neq 0$ *or* $q_{wt=2r} \neq 0$, *and* $q_{\mathrm{wt}=2r}$ *is not square if* $p = 0$.

(3) $x_3^r \in q$.

The non-Gorenstein singularity of Y *is of type* cA/r.

(ii) f *is the weighted blowup with* $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 3, 1, 2)$ *after an identification of* $P \in X$ *with*

$$o \in \left(x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + x_2 x_3^3 + q(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}.$$

Moreover, the equation defining X *satisfies the following conditions.*

(1) $\operatorname{wt} q \geq 6$, *and* p *is weighted homogeneous of weight* 2 *for the weights distributed above.*

(2) $x_4^3 \in q$.

The non-Gorenstein singularity of Y *is of type* $cD/3$, *and* P *is of type* cD_4.

THEOREM 2.5

Suppose that f *is a divisorial contraction of type* $e2$ *which contracts to a* cE_6 *point with discrepancy* 2. *Then* f *is the weighted blowup with* $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 3, 2, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \left(x_1^2 + \{x_2 - p(x_3, x_4)\}^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}.$$

Moreover, the equation defining X *satisfies the following conditions.*

(i) $\operatorname{wt} g \geq 3$, $\operatorname{wt} h \geq 6$, *and* p *is weighted homogeneous of weight* 2 *for the weights distributed above.*

(ii) $\deg g \geq 3$ *and* $\deg h \geq 4$.

(iii) $x_3 \in p$ *and* $x_4^3 \in g$.

There is no divisorial contraction of type $e2$ *which contracts to a* cE_7 *point with discrepancy* 2.

THEOREM 2.6

Suppose that f *is a divisorial contraction of type* $e3$ *which contracts to a* cA_2 *point with discrepancy* 3. *Then* f *is the weighted blowup with* $\operatorname{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \big(x_1^2 + x_2^2 + 2c x_1 x_2 + 2x_1 p(x_3, x_4) + 2c x_2 p_{\mathrm{wt}=3}(x_3, x_4) + x_3^3 + g(x_3, x_4) = 0\big) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}.$$

Moreover, the equation defining X *satisfies the following conditions.*

(i) $c \neq \pm 1$, $\operatorname{wt} g \geq 6$, *and* p *contains only monomials with weight* 2 *and* 3 *for the weights distributed above.*

(ii) $x_4^2 \in p$ *and* $\deg g(x_3, 1) \leq 2$.

THEOREM 2.7

Suppose that f is a divisorial contraction of type $e3$ which contracts to a cD_4 point with discrepancy 3. Then f is the weighted blowup with $\mathrm{wt}(x_1,x_2,x_3,x_4) = (3,4,2,1)$ *after an identification of $P \in X$ with*

$$o \in \left(x_1^2 + x_2^2x_4 + 2x_2x_4p(x_3,x_4) + \lambda x_2x_3^k + q(x_3,x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover, the equation defining X satisfies the following conditions.

(i) $\lambda \in \mathbb{C}$, $k > 2$, $\mathrm{wt}\, q \geq 6$, *and p contains only monomials with weight at most 3 for the weights distributed above.*

(ii) $x_4 \in p$ *and* $x_3^3 \in q$.

For any $n > 5$, there is no divisorial contraction of type $e3$ which contracts to a cD_n point with discrepancy 3.

THEOREM 2.8

There is no divisorial contraction of type $e3$ which contracts to a cE_6 point with discrepancy 3.

THEOREM 2.9

Suppose that f is a divisorial contraction of type $e5$ which contracts to a cE_7 point with discrepancy 2. Then f is the weighted blowup with $\mathrm{wt}(x_1,x_2,x_3,x_4,x_5) = (5,3,2,2,7)$ *after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + x_2x_5 + p(x_3,x_4) = 0, \\ x_2^2 + q(x_3,x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover, the equations defining X satisfy the following conditions.

(i) $\mathrm{wt}\, p \geq 10$, $\mathrm{wt}\, q \geq 6$ *for the weights distributed above.*

(ii) $\gcd(p_5, q_3) = 1$.

THEOREM 2.10

Suppose that f is a divisorial contraction of type $e9$ which contracts to a $cE_{7,8}$ point with discrepancy 2. Then f is the weighted blowup with $\mathrm{wt}(x_1,x_2,x_3,x_4) = (7,5,3,2)$ *after an identification of $P \in X$ with*

$$o \in \left(x_1^2 + x_2^3 + \lambda x_2^2x_4^2 + x_2g(x_3,x_4) + h(x_3,x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover, the equation defining X satisfies the following conditions.

(i) $\lambda \in \mathbb{C}$ *and* $\mathrm{wt}\, g \geq 9$, $\mathrm{wt}\, h \geq 14$ *for the weights distributed above.*

(ii) *If P is of type cE_7 (resp., cE_8), then $x_3^3 \in g$ (resp., x_3^5 or $x_3^4x_4 \in h$).*

(iii) $x_4^7 \in h$.

We can show that every 3-dimensional divisorial contraction to a Gorenstein point is obtained as a weighted blowup by [4]–[9], and the above theorems. Therefore, we can prove Theorem 1.1 by [11]. Proofs of these theorems shall be given in Section 4.

NOTATION

(i) We denote $\mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$ by $\mathbb{C}^n_{x_1 \ldots x_n}$.

(ii) We define the action of a cyclic group μ_m of order m on $\mathbb{C}^n_{x_1 \ldots x_n}$ by

$$(x_1, \ldots, x_n) \mapsto (\zeta^{a_1} x_1, \ldots, \zeta^{a_n} x_n),$$

where ζ is a primitive mth root of unity. The quotient space is denoted by $\mathbb{C}^n_{x_1 \ldots x_n}/\frac{1}{m}(a_1, \ldots, a_n)$, $\mathbb{C}^n/\frac{1}{m}(a_1, \ldots, a_n)$, or simply $\frac{1}{m}(a_1, \ldots, a_n)$.

(iii) For $\mathrm{wt}(x_3, x_4) = (a, b)$ and $g(x_3, x_4) = \sum p_{ij} x_3^i x_4^j \in \mathbb{C}\{x_3, x_4\}$, we define

$$\mathrm{wt}\big(g(x_3, x_4)\big) = \inf\{ai + bj \mid p_{ij} \neq 0\}.$$

For a positive integer n, we define

$$g_{\mathrm{wt}=n}(x_3, x_4) = \sum_{ai+bj=n} p_{ij} x_3^i x_4^j,$$

$$g_{\mathrm{wt}\geq n}(x_3, x_4) = \sum_{ai+bj\geq n} p_{ij} x_3^i x_4^j.$$

(iv) Let $\mathbb{C}\{x_1, \ldots, x_n\}$ be the ring of convergent power series in variables $x_1, \ldots, x_n$. For $f \in \mathbb{C}\{x_1, \ldots, x_n\}$, we denote by f_m the homogeneous part of degree m of f.

(v) We say that a monomial, for example, x^n, appears in a power series f or f contains x^n if there exists a monomial x^n with nonzero coefficient in the power series expansion of f, and we denote it by $x^n \in f$.

3. Preliminaries

3.1. Classification of terminal singularities

It is known that a 3-dimensional Gorenstein terminal singularity is an isolated cDV hypersurface singularity, that is, a singularity with local equation of the form

$$f(x_1, x_2, x_3) + x_4 g(x_1, x_2, x_3, x_4) = 0$$

for some $f(x_1, x_2, x_3)$ defining a Du Val (equivalently rational double point) singularity. If $P \in X$ is a 3-dimensional Gorenstein terminal singularity, then according to the type of $f(x_1, x_2, x_3)$, we have that $P \in X \simeq o \in (\varphi = 0) \subset \mathbb{C}^4$ for some φ belongs to one of the following:

(i) type cA: $(x_1 x_2 + g(x_3, x_4) = 0) \subset \mathbb{C}^4$ with $g(x_3, x_4) \in \mathfrak{m}^2$,

(ii) type cD: $(x_1^2 + x_2^2 x_4 + \lambda x_2 x_3^l + g(x_3, x_4) = 0) \subset \mathbb{C}^4$ with $\lambda \in \mathbb{C}$, $l \geq 2$, $g(x_3, x_4) \in \mathfrak{m}^3$,

(iii) type cE: $(x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0) \subset \mathbb{C}^4$ with $g(x_3, x_4) \in \mathfrak{m}^3$, $h(x_3, x_4) \in \mathfrak{m}^4$,

where $\mathfrak{m}$ denotes the maximal ideal of $o \in \mathbb{C}^4$. In the cE case, it is of type cE_6 (resp., cE_7, cE_8) if $h_4 \neq 0$ (resp., $h_4 = 0$ and $g_3 \neq 0$, $h_4 = g_3 = 0$ and $h_5 \neq 0$).

To prove Theorems 2.1 and 2.6, we need to construct a standard identification.

LEMMA 3.1

Let $P \in X$ be a germ of a 3-dimensional Gorenstein terminal singularity. If P is of type cA_2, then there is an identification

$$P \in X \simeq o \in \big(x_1x_2 + x_3^3 + g(x_3,x_4) = 0\big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}$$
$$\simeq o \in \big(x_1^2 + x_2^2 + x_3^3 + g(x_3,x_4) = 0\big) \subset \mathbb{C}^4_{x_1x_2x_3x_4},$$

where $\deg g(x_3,1) \le 2$.

Proof

By definition, there is an identification

$$P \in X \simeq o \in \big(x_1^2 + x_2^2 + x_3^3 + x_4F(x_1,x_2,x_3,x_4) = 0\big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}$$

for some $F(x_1,x_2,x_3,x_4) \in \mathfrak{m}^2$. By using the Weierstrass preparation theorem and completing a square, we may assume that

$$P \in X \simeq o \in \big(x_1^2 + x_2^2 + x_3^3 + x_4F'(x_3,x_4) = 0\big)$$

for $F'(x_3,x_4) \in \mathfrak{m}^2$. We may assume that $\deg F'(x_3,1) \le 2$ by the Weierstrass preparation for x_3. Thus, we get the desired forms by the automorphism $x_1 + ix_2 \mapsto x_1$ and $x_1 - ix_2 \mapsto x_2$ if necessary. □

Mori [15] classified that a 3-dimensional terminal singularity $P \in X$ with index $r > 1$ is isomorphic to a cyclic quotient of an isolated cDV singularity, and Kollár and Shepherd-Barron [14] showed that these isolated cDV's quotients are terminal singularities.

THEOREM 3.2

There exists an identification

$$P \in X \simeq o \in (\varphi = 0) \subset \mathbb{C}^4_{x_1x_2x_3x_4}/\mu_r,$$

where μ_r denotes the cyclic group of order r and x_1,x_2,x_3,x_4,φ are μ_r-semi-invariant. Furthermore, φ and the action of μ_r have one of the following forms:

(i) *type cA/r*: $(x_1x_2 + g(x_3^r,x_4) = 0) \subset \mathbb{C}^4/\frac{1}{r}(a,-a,1,0)$ *with* $g(x_3,x_4) \in \mathfrak{m}^2$, $\gcd(a,r) = 1$*;*

(ii) *type $cAx/2$*: $(x_1^2 + x_2^2 + g(x_3,x_4) = 0) \subset \mathbb{C}^4/\frac{1}{2}(0,1,1,1)$ *with* $g(x_3,x_4) \in \mathfrak{m}^3$*;*

(iii) *type $cAx/4$*: $(x_1^2 + x_2^2 + g(x_3,x_4) = 0) \subset \mathbb{C}^4/\frac{1}{4}(1,3,1,2)$ *with* $g(x_3,x_4) \in \mathfrak{m}^3$*;*

(iv) *type $cD/3$*: $(\varphi = 0) \subset \mathbb{C}^4/\frac{1}{3}(0,2,1,1)$, *where φ has one of the following forms:*

(1) $x_1^2 + x_2^3 + x_3^3 + x_4^3$,
(2) $x_1^2 + x_2^3 + x_3^2x_4 + x_2g(x_3,x_4) + h(x_3,x_4)$ *with* $g \in \mathfrak{m}^4$, $h \in \mathfrak{m}^6$,
(3) $x_1^2 + x_2^3 + x_3^3 + x_2g(x_3,x_4) + h(x_3,x_4)$ *with* $g \in \mathfrak{m}^4$, $h \in \mathfrak{m}^6$*;*

(v) *type $cD/2$*: $(\varphi = 0) \subset \mathbb{C}^4/\frac{1}{2}(1,0,1,1)$, *where φ has one of the following forms:*

(1) $x_1^2+x_2^3+x_2x_3x_4+g(x_3,x_4)$ *with* $g\in\mathfrak{m}^4$,
(2) $x_1^2+x_2x_3x_4+x_2^n+g(x_3,x_4)$ *with* $n\le 4$, $g\in\mathfrak{m}^4$,
(3) $x_1^2+x_2x_3^2+x_2^n+g(x_3,x_4)$ *with* $n\le 3$, $g\in\mathfrak{m}^4$,

(vi) *type* $cE/2$: $(x_1^2+x_2^3+x_2g(x_3,x_4)+h(x_3,x_4)=0)\subset\mathbb{C}^4/\frac{1}{2}(1,0,1,1)$ *with* $g,\ h\in\mathfrak{m}^4$, $h_4\neq 0$.

Conversely, if φ *as above defines an isolated singularity and the action of* μ_r *on* $\varphi=0$ *is free outside the origin, then* P *is a terminal singularity.*

3.2. Weighted blowup

We recall the construction of weighted blowups by using the toric language. Let $N=\mathbb{Z}^d$ be a free Abelian group, called a *lattice*, of rank d with standard basis $\{e_1,\dots,e_d\}$. Let M be the dual lattice of N. Let σ be the cone in $N\otimes\mathbb{R}$ generated by the standard basis $e_1,\dots,e_d$, and let Δ be the fan which consists of σ and all the faces of σ. We consider

$$T_N(\Delta):=\operatorname{Spec}\mathbb{C}[\sigma^\vee\cap M]=\mathbb{C}^d.$$

Let $v=(a_1,\dots,a_d)$ be a primitive vector in N, that is, the vector which has no element in N between 0 and v. We assume that $a_i\in\mathbb{Z}_{\ge 0}$ and $\gcd(a_1,\dots,a_d)=1$. For any i with $a_i>0$, let σ_i be the cone generated by $\{e_1,\dots,e_{i-1},v,e_{i+1},\dots,e_d\}$, and let $\Delta(v)$ be the fan consisting of all σ_i's and their all faces. $\Delta(v)$ is called the *star-shaped decomposition* for v. Then

$$T_N\big(\Delta(v)\big)=\bigcup_{a_i>0}\operatorname{Spec}\mathbb{C}[\sigma_i^\vee\cap M].$$

If $a_i>0$ for all i, the natural map $\pi\colon T_N(\Delta(v))\to T_N(\Delta)$ is called the *weighted blowup* over $o\in T_N(\Delta)$ with weight $v=(a_1,\dots,a_d)$. In each affine chart $\mathcal{U}_i:=\operatorname{Spec}\mathbb{C}[\sigma_i^\vee\cap M]$, the natural map $\mathcal{U}_i\to T_N(\Delta)$ is given by

$$\begin{cases} x_j\mapsto x_jx_i^{a_j} & \text{if } j\neq i,\\ x_i\mapsto x_i^{a_i}. \end{cases}$$

The exceptional divisor $\mathcal{E}$ of π is isomorphic to $\mathbb{P}(a_1,\dots,a_d)$.

Let $X:=(\varphi(x_1,\dots,x_d)=0)\subset T_N(\Delta)$ be a hypersurface, and let Y be the birational transform on $T_N(\Delta(v))$ of X. We also call the induced map $\pi'\colon Y\to X$ the *weighted blowup* of X with weight v. The affine chart $U_i:=\mathcal{U}_i\cap Y$ can be expressed as

$$\big(\varphi(x_1x_i^{a_1},\dots,x_{i-1}x_i^{a_{i-1}},x_i^{a_i},x_{i+1}x_i^{a_{i+1}},\dots,x_dx_i^{a_d})x_i^{-\operatorname{wt}\varphi}=0\big)\subset\mathcal{U}_i$$

for each i. The exceptional divisor of π' is denoted by $E:=\mathcal{E}\cap Y$. If E is irreducible and reduced and we have $\dim(T_N(\Delta(v))\cap Y)\le 1$, then we have the adjunction formula

$$K_Y=\pi'^*K_X+\Big(\sum_i a_i-\operatorname{wt}\varphi-1\Big)E.$$

We define weighted blowups of the complete intersection similarly.

3.3. The singular Riemann–Roch formula

As we shall use the method in [10] and [11], we recall the singular Riemann–Roch formula.

THEOREM 3.3 ([16, Theorem 10.2])

Let X be a projective 3-fold with canonical singularities, and let D be a divisor on X such that $D \sim e_P K_X$ with $e_P \in \mathbb{Z}$ at each $P \in X$.

(i) *There is a formula of the form*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12}D(D-K_X)(2D-K_X) + \frac{1}{12}D \cdot c_2(X) + \sum_P c_P(D),$$

where the summation takes place over the singularities on X, and $c_P(D) \in \mathbb{Q}$ is a contribution due to the singularity at P, depending only on the local analytic type of P and $\mathcal{O}_X(D)$.

(ii) *If $P \in X$ is a terminal cyclic quotient singularity of type $\frac{1}{r_P}(1,-1,b_P)$, then*

$$c_P(D) = -\overline{i_P}\frac{r_P^2-1}{12r_P} + \sum_{l=1}^{\overline{i_P}-1} \frac{\overline{lb_P}(r_P - \overline{lb_P})}{2r_P},$$

where $\bar{i} = i - \lfloor \frac{i}{r_P} \rfloor r_P$ denotes the residue of i modulo r_P. (The sum $\sum_{l=1}^{\overline{i_P}-1}$ is zero by convention if $\overline{i_P} = 0$ or 1.)

(iii) *For an arbitrary terminal singularity P,*

$$c_P(D) = \sum_Q c_Q(D_Q),$$

where $\{(Q, D_Q)\}$ is a flat deformation of (P, D) to the basket of terminal cyclic quotient singularities Q.

4. Proofs of main results

In this section we prove the main theorem by using the method in [10] and [11]. Our strategy for the classification is to determine the exceptional divisor in the sense of valuation by applying Lemma 4.1 or Lemma 4.2 (see [9, Lemma 6.1], [10, Lemma 6.1]).

LEMMA 4.1

Let $f\colon (Y \supset E) \to (X \ni P)$ be a germ of a 3-dimensional divisorial contraction to a cDV point P. We identify $P \in X$ with

$$P \in X \simeq o \in (\varphi = 0) \subset \bar{X} := \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Let a denote the discrepancy of f, and let m_i denote the multiplicity of x_i along E, that is, the largest integer such that $x_i \in f_*\mathcal{O}_Y(-m_iE)$. Suppose that

(m_1, m_2, m_3, m_4) is primitive in $\mathbb{Z}^4$. Let d denote the weighted order of φ with respect to weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1, m_2, m_3, m_4)$, and decompose φ as

$$\varphi = \varphi_d(x_1, x_2, x_3, x_4) + \varphi_{>d}(x_1, x_2, x_3, x_4),$$

where φ_d is the weighted homogeneous part of weight d and $\varphi_{>d}$ is the part of weight greater than d. Set $c := m_1 + m_2 + m_3 + m_4 - 1 - d$. Let $\bar{g}\colon (\bar{Z} \supset \bar{F}) \to (\bar{X} \ni o)$ be the weighted blowup with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1, m_2, m_3, m_4)$ and with $\bar{F}$ its exceptional divisor. Let Z denote the birational transform on $\bar{Z}$ of X, and let $g\colon Z \to X$ be the induced morphism. If we have the four conditions

(i) *$\bar{F} \cap Z$ defines an irreducible and reduced 2-cycle F,*
(ii) *Z is smooth at the generic point of F,*
(iii) *$\dim(\mathrm{Sing}\, \bar{Z} \cap Z) \leq 1$, and*
(iv) *$c = a$,*

then we have $f \simeq g$ over X.

We shall apply the following extension of Lemma 4.1 to several cases.

LEMMA 4.2

Let $f\colon (Y \supset E) \to (X \ni P)$ be a germ of a 3-dimensional divisorial contraction to a cDV point P. We identify $P \in X$ with

$$P \in X \simeq o \in \begin{pmatrix} \varphi = 0, \\ \psi = 0 \end{pmatrix} \subset \bar{X} := \mathbb{C}^5_{x_1 x_2 x_3 x_4 x_5}.$$

Let a denote the discrepancy of f, and let m_i denote the multiplicity of x_i along E. Suppose that $(m_1, m_2, m_3, m_4, m_5)$ is primitive in $\mathbb{Z}^5$. Let d (resp., e) denote the weighted order of φ (resp., ψ) with respect to weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (m_1, m_2, m_3, m_4, m_5)$, and decompose φ and ψ as

$$\varphi = \varphi_d(x_1, x_2, x_3, x_4, x_5) + \varphi_{>d}(x_1, x_2, x_3, x_4, x_5),$$

$$\psi = \psi_e(x_1, x_2, x_3, x_4, x_5) + \psi_{>e}(x_1, x_2, x_3, x_4, x_5),$$

where φ_d (resp., ψ_e) is the weighted homogeneous part of weight d (resp., e) and $\varphi_{>d}$ (resp., $\psi_{>e}$) is the part of weight greater than d (resp., e). Set $c := m_1 + m_2 + m_3 + m_4 + m_5 - 1 - d - e$. Let $\bar{g}\colon (\bar{Z} \supset \bar{F}) \to (\bar{X} \ni o)$ be the weighted blowup with weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (m_1, m_2, m_3, m_4, m_5)$ and with $\bar{F}$ its exceptional divisor. Let Z denote the birational transform on $\bar{Z}$ of X, and let $g\colon Z \to X$ be the induced morphism. If we have the four conditions

(i) *$\bar{F} \cap Z$ defines an irreducible and reduced 2-cycle F,*
(ii) *Z is smooth at the generic point of F,*
(iii) *$\dim(\mathrm{Sing}\, \bar{Z} \cap Z) \leq 1$, and*
(iv) *$c = a$,*

then we have $f \simeq g$ over X.

Table 2

Type	J	Type	J
$e1$	$(r,2)$	$e5$	$(7,3)$
$e2$	$(r,1)$, $(r,1)$	$e9$	$(5,2)$, $(3,1)$
$e3$	$(2,1)$, $(4,1)$		

Now we study 3-dimensional divisorial contractions to cDV points. We let

$$f\colon (Y \supset E) \to (X \ni P)$$

be a germ of a 3-dimensional divisorial contraction whose exceptional divisor E contracts to a singular point P of index 1, and we let a denote its discrepancy. Let $I_0 := \{Q \text{ of type } (1/r_Q)(1,-1,b_Q)\}$ denote the basket of fictitious singularities on Y, and let e_Q for $Q \in I_0$ be the smallest positive integer such that $E \sim e_Q K_Y$ at Q. By replacing b_Q with $r_Q - b_Q$ if necessary, we may assume that $v_Q := \overline{e_Q b_Q} \leq r_Q/2$, where $\bar{\cdot}$ denotes the residue modulo r_Q. We set $I := \{Q \in I_0 \mid v_Q \neq 0\}$ and $J := \{(r_Q, v_Q)\}_{Q \in I}$. We can compute J for each case in Table 1, and we give the results in Table 2.

We shall prove the main results as follows.

Step 1. For an integer j, we compute the dimension of the vector space

$$V_j := f_*\mathcal{O}_Y(-jE)/f_*\mathcal{O}_Y\bigl(-(j+1)E\bigr).$$

This space is regarded as the space of functions on X vanishing with multiplicity j along E. For a function h on X, we let $\operatorname{mult}_E h$ denote the multiplicity of h along E.

Step 2. We find the basis of V_j by starting with an arbitrary identification

$$P \in X \simeq o \in (\varphi = 0) \subset \mathbb{C}^4_{x_1x_2x_3x_4}, \tag{1}$$

and we compute the favorite weights $\operatorname{wt}(x_1, x_2, x_3, x_4)$.

Step 3. In order to apply Lemma 4.1 or Lemma 4.2, we follow these procedures.

(i) Determine $\operatorname{wt}(x_1, x_2, x_3, x_4)$, and rewrite φ.

(ii) Let $f'\colon Z \to X$ be the weighted blowup with $\operatorname{wt} x_i = \operatorname{mult} x_i$. Find the condition that the exceptional locus of f' is irreducible and reduced.

(iii) Verify the assumption of Lemma 4.1, and find the condition that every singular point in Z is terminal.

Step 4. Then we can apply Lemma 4.1 or Lemma 4.2 and show that f coincides with f'.

We note that $\dim V_j$ and the basis of V_j are dependent only on the type of f but not on the type of P. So we shall show the main theorems according to the type of f.

We compute $\dim V_j$ by using the singular Riemann–Roch formula. For each j, there is a natural exact sequence

$$0 \to \mathcal{O}_Y\bigl(-(j+1)E\bigr) \to \mathcal{O}_Y(-jE) \to \mathcal{O}_E(-jE|_E) \to 0.$$

So we have a long exact sequence

$$\begin{aligned} 0 &\to f_*\mathcal{O}_Y\bigl(-(j+1)E\bigr) \to f_*\mathcal{O}_Y(-jE) \to f_*\mathcal{O}_E(-jE|_E) \\ &\to R^1 f_*\mathcal{O}_Y\bigl(-(j+1)E\bigr) \to R^1 f_*\mathcal{O}_Y(-jE) \to R^1 f_*\mathcal{O}_E(-jE|_E) \\ &\to \cdots. \end{aligned}$$

Since P is terminal, we have $R^i f_*\mathcal{O}_Y(-(j+1)E) = 0$ and $R^i f_*\mathcal{O}_Y(-jE) = 0$ for any $i \geq 1$, j by the Kawamata–Viehweg theorem, and $R^i f_*\mathcal{O}_E(-jE|_E) = H^i(E, \mathcal{O}_E(-jE|_E))$ for any i, j. Then

$$\begin{aligned} \dim_{\mathbb{C}} V_j &= \dim_{\mathbb{C}} f_*\mathcal{O}_E(-jE|_E) \\ &= \dim_{\mathbb{C}} H^0\bigl(E, \mathcal{O}_E(-jE|_E)\bigr) = \chi\bigl(\mathcal{O}_E(-jE|_E)\bigr) \\ &= \chi\bigl(\mathcal{O}_Y(-jE)\bigr) - \chi\bigl(\mathcal{O}_Y\bigl(-(j+1)E\bigr)\bigr). \end{aligned}$$

Applying the singular Riemann–Roch formula, we have

$$(*) \qquad \begin{aligned} \dim V_j = {} & \frac{1}{12}\bigl(6j(j+a+1) + (a+1)(a+2)\bigr)E^3 \\ & + \frac{1}{12} E \cdot c_2(Y) + A_j - A_{j+1}. \end{aligned}$$

Here the contribution term A_j is given by $A_j := \sum_{Q \in I} A_Q(\overline{-je_Q})$, where

$$(**) \qquad A_Q(k) := -k\frac{r_Q^2 - 1}{12 r_Q} + \sum_{l=1}^{k-1} \frac{\overline{lb_Q}(r_Q - \overline{lb_Q})}{2 r_Q}.$$

For $j < 0$, we have $V_j = 0$. Now we compute $\dim V_j$ explicitly and show that f is a weighted blowup in each case. Since we shall use similar procedures in each case, we start with easy cases and proceed to complicated cases.

4.1. Case $e9$ with discrepancy 2

In this section, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e9$, and its discrepancy a is 2. In this case, Y has two non-Gorenstein singular points. One point Q_1 is of type $\frac{1}{3}(1,2,2)$, and another point Q_2 is of type $\frac{1}{5}(1,4,4)$. Set $N_j := \{(l_1, l_2, l_3, l_4) \in \mathbb{Z}^4_{\geq 0} \mid 7l_1 + 5l_2 + 3l_3 + 2l_4 = j, l_1 \leq 1\}$.

LEMMA 4.3

We have that $\dim V_j = \# N_j$.

Proof

By Tables 1 and 2, we see that $(r_{Q_1}, b_{Q_1}, v_{Q_1}) = (3,2,1)$, $(r_{Q_2}, b_{Q_2}, v_{Q_2}) = (5,4,2)$, and $E^3 = 1/15$. We also have $e_{Q_1} = 2$, $e_{Q_2} = 3$. So

$$
\begin{aligned}
\dim V_j = {} & \frac{1}{30}j(j+3) + \frac{1}{15} + \frac{1}{12}E\cdot c_2(Y) \\
& - (\bar{j} - \overline{j+1})\frac{2}{9} + \Big(\sum_{l=1}^{\bar{j}-1} - \sum_{l=1}^{\overline{j+1}-1}\Big)\frac{\overline{2l}(3-\overline{2l})}{6} \\
& - \big(\overline{2j}' - \overline{2(j+1)}'\big)\frac{2}{5} + \Big(\sum_{l=1}^{\overline{2j}'-1} - \sum_{l=1}^{\overline{2(j+1)}'-1}\Big)\frac{\overline{4l}'(5-\overline{4l}')}{10}.
\end{aligned}
$$

Here $\bar{\cdot}$ denotes the residue modulo 3, and $\bar{\cdot}'$ denotes the residue modulo 5. Since $\dim V_0 = 1$, we have

$$\frac{1}{15} + \frac{1}{12}E\cdot c_2(Y) = \frac{17}{45}.$$

Now we consider

$$
\begin{aligned}
\dim V_j - \dim V_{j-5} = {} & \frac{1}{3}(j-1) - \frac{2}{9}(\bar{j} - 2\overline{j+1} + \overline{j+2}) \\
& + \Big(\sum_{l=1}^{\bar{j}-1} - 2\sum_{l=1}^{\overline{j+1}-1} + \sum_{l=1}^{\overline{j+2}-1}\Big)\frac{\overline{2l}(3-\overline{2l})}{6}
\end{aligned}
$$

for any $j \geq 5$. We have

$$\dim V_j - \dim V_{j-5} = \begin{cases} j/3 & \text{if } k \equiv 0 \pmod 3, \\ (j-1)/3 & \text{if } k \equiv 1 \pmod 3, \\ (j-2)/3 & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

On the other hand, we have a decomposition

$$N_j = \{(l_1, 0, l_3, l_4) \in N_j\} \sqcup \{\vec{l} + (0,1,0,0) \mid \vec{l} \in N_{j-5}\}.$$

Hence, for any $j \geq 5$,

$$\#N_j - \#N_{j-5} = \#\{(l_1, 0, l_3, l_4) \in N_j\}.$$

So we have

$$\#N_j - \#N_{j-5} = \begin{cases} j/3 & \text{if } k \equiv 0 \pmod 3, \\ (j-1)/3 & \text{if } k \equiv 1 \pmod 3, \\ (j-2)/3 & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Therefore, we have $\dim V_j - \dim V_{j-5} = \#N_j - \#N_{j-5}$ for any $j \geq 5$. We can compute $\dim V_j = \#N_j$ for $j \leq 4$. Then we have $\dim V_j = \#N_j$ for any j. □

LEMMA 4.4

(i) *There exist some* $1 \leq k,\ l \leq 4$ *with* $\operatorname{mult}_E x_k = 2$ *and* $\operatorname{mult}_E x_l = 3$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *Moreover,* $\operatorname{mult}_E x_k \geq 4$ *for* $k = 1, 2$.

(ii) *If* $j < 5$, *the monomials* $x_3^{l_3} x_4^{l_4}$ *for* $(0,0,l_3,l_4) \in N_j$ *form a basis of* V_j. *In particular, for* $k = 1, 2$, $\operatorname{mult}_E \bar{x}_k \geq 5$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$ *and summation over* $(0,0,l_3,l_4) \in \bigcup_{j<5} N_j$.

(iii) *There exists some* $k=1,2$ *with* $\operatorname{mult}_E \bar{x}_k = 5$ *such that the monomials* $\bar{x}_k$ *and* $x_3^{l_3}x_4^{l_4}$ *for* $(0,0,l_3,l_4)\in N_5$ *form a basis of* V_5*. By permutation, we may assume that* $\bar{x}_k=\bar{x}_2$*.*

(iv) *The monomials* $\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(0,l_2,l_3,l_4)\in N_6$ *form a basis of* V_6*, and we have* $\operatorname{mult}\hat{x}_1 \geq 7$ *for* $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *with some* $c_{l_2l_3l_4}\in\mathbb{C}$ *and summation over* $(0,l_2,l_3,l_4)\in N_6$*.*

(v) *We have* $\operatorname{mult}_E \hat{x}_1 = 7$*, and for* $j<14$*, the monomials* $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4)\in N_j$ *form a basis of* V_j*.*

(vi) *Set* $\tilde{N}_j = \{(l_1,l_2,l_3,l_4)\in\mathbb{Z}^4_{\geq 0} \mid 7l_1+5l_2+3l_3+2l_4 = j\}$*. The monomials* $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4)\in N_{14}$ *have one nontrivial relation, say,* ψ*, in* V_{14}*. The natural exact sequence*

$$0\to\mathbb{C}\psi\to\bigoplus_{(l_1,l_2,l_3,l_4)\in\tilde{N}_{14}}\mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\to V_{14}\to 0$$

is exact.

Proof

We have $\dim V_1 = 0$, $\dim V_2 = \dim V_3 = 1$ by Lemma 4.3. This implies (i). By permutation, we may assume that $\operatorname{mult}_E x_4 = 2$, $\operatorname{mult}_E x_3 = 3$. To prove (ii), we shall show that the monomials $x_3^{l_3}x_4^{l_4}$ for $(0,0,l_3,l_4)\in N_j$ are linearly independent in V_j for any j. Suppose $0=\sum_{(0,0,l_3,l_4)\in N_j} c_{l_3l_4}x_3^{l_3}x_4^{l_4}\in V_j$, $c_{l_3l_4}\in\mathbb{C}$. We shall show that $c_{l_3l_4}=0$ for any $(0,0,l_3,l_4)\in N_j$. We set $j=6k+\alpha$, where $0\leq k\in\mathbb{Z}$ and $0\leq\alpha\leq 5$. We study the case $j=6k$ for $0\leq k\in\mathbb{Z}$. So, we can write

$$\sum_{(0,0,l_3,l_4)\in N_j} c_{l_3l_4}x_3^{l_3}x_4^{l_4} = \sum_{l=0}^{k} c_l x_3^{2l}x_4^{3(k-l)}$$

for $c_l\in\mathbb{C}$. Since $\mathbb{C}$ is an algebraically closed field, we factorize

$$\sum_{l=0}^{k} c_l x_3^{2l}x_4^{3(k-l)} = (d_1x_3^2+d_2x_4^3)\Big(\sum_{l=1}^{k} c'_l x_3^{2(l-1)}x_4^{3(k-l)}\Big)$$

for $c'_l, d_1, d_2\in\mathbb{C}$. Hence, we have $c_l=0$ for all $0\leq l\leq k$ by induction on k. We can show that $c_{l_3l_4}=0$ for any other case similarly. We set $W(j) := \langle x_3^{l_3}x_4^{l_4} \mid (0,0,l_3,l_4)\in N_j\rangle\subset V_j$ for each j. Then $\dim W(j) = \#N_j$ for $j<5$, and thus we obtain (ii) by Lemma 4.3. Since $\dim V_5 = \dim W(5)+1$ by Lemma 4.3, we obtain (iii). By permutation, we may assume that $\bar{x}_2$ forms a basis of $V_5/W(5)\simeq\mathbb{C}$. Since the monomials $x_3^{l_3}x_4^{l_4}$ for $(0,0,l_3,l_4)\in N_j$ are linearly independent in V_j for any j, and $\dim V_7 = W(7)+2$ by Lemma 4.3, we obtain (iv) and $\operatorname{mult}_E\hat{x}_1 = 7$. For any $j<14$, we have $\dim V_j = \#\tilde{N}_j$ by Lemma 4.3. This implies (v). Since $\dim V_{14} = \#N_{14} = \#\tilde{N}_{14}-1$, we have a nontrivial relation, say, ψ in V_{14}, and we obtain the natural exact sequence in (vi). □

COROLLARY 4.5

We distribute weights $\operatorname{wt}(\hat{x}_1,\bar{x}_2,x_3,x_4) = (7,5,3,2)$ *to the coordinates* $\hat{x}_1$*,* $\bar{x}_2$*,* x_3*,*

x_4 obtained in Lemma 4.4. Then φ is of form

$$\varphi = c\psi + \varphi_{>14}(\hat{x}_1, \bar{x}_2, x_3, x_4)$$

with $c \in \mathbb{C}$ and a function $\varphi_{>14}$ of weighted order greater than 14, where ψ in (1) is the one in Lemma 4.4(vi).

Proof

Decompose $\varphi = \varphi_{\leq 14} + \varphi_{>14}$ into the part $\varphi_{\leq 14}$ of weighted order at most 14 and the part $\varphi_{>14}$ of weighted order greater than 14. Then $\mathrm{mult}_E \varphi_{\leq 14} = \mathrm{mult}_E \varphi_{>14} > 14$, so $\varphi_{\leq 14}$ is mapped to zero by the natural homomorphism

$$\bigoplus_{(l_1,l_2,l_3,l_4)\in\bigcup_{j\leq 14}\tilde{N}_j} \mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4} \to \mathcal{O}_X/f_*\mathcal{O}_Y(-15E),$$

whose kernel is $\mathbb{C}\psi$ by Lemmas 4.4(v) and 4.4(vi). □

Proof of Theorem 2.10

The $cE_{7,8}$ point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$ and $h \in \mathfrak{m}^4$. If P is of type cE_7 (resp., cE_8), then $g_3 \neq 0$ (resp., $g_3 = 0$, $h_5 \neq 0$).

(i) We shall show that we distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (7, 5, 3, 2)$, and that φ can write

$$\varphi = x_1^2 + x_2^3 + \lambda x_2^2 x_4^2 + x_2 g(x_3, x_4) + h(x_3, x_4),$$

with $\lambda \in \mathbb{C}$, $g \in \mathfrak{m}^3$, and $h \in \mathfrak{m}^4$. By Corollary 4.5, we have $\mathrm{wt}\,\varphi = 14$. So we can show that we distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (7, 5, 3, 2)$ easily. We obtain a quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where $c \in \mathbb{C}$, p, and q are as in Lemma 4.4; that is, p (resp., q) contains only monomials with weight at most 6 (resp., at most 4).

Then we rewrite φ as

$$\begin{aligned}\varphi &= (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^3 + (\bar{x}_2 - q)g + h \\ &= \hat{x}_1^2 - 2p\hat{x}_1 - 2c\hat{x}_1\bar{x}_2 + \bar{x}_2^3 + (c^2 - 3q)\bar{x}_2^2 \\ &\quad + (2cp + 3q^2 + g)\bar{x}_2 + (p^2 - q^3 - qg + h).\end{aligned}$$

Since $\mathrm{wt}\,\varphi = 14$, we can show that $c = p = 0$, $\mathrm{wt}\,q = 4$, $\mathrm{wt}(3q^2 + g) \geq 9$, and $\mathrm{wt}(-q^3 - qg + h) \geq 14$. We also have $q = \lambda x_4^2$ with $\lambda \in \mathbb{C}$. Moreover, if P is of type cE_7 (resp., cE_8), then we have $x_3^3 \in g$ (resp., x_3^5 or $x_3^4 x_4 \in h$). Replacing $3q^2 + g$ with g and $-q^3 - qg + h$ with h and replacing variables, we have the desired expression in (i).

(ii) Let $f' \colon Z \to X$ be the weighted blowup with $\mathrm{wt}\,x_i = \mathrm{mult}_E\,x_i$. If P is of type cE_7, it is obvious that the exceptional locus F of f' is irreducible and reduced. If P is of type cE_8, we need the condition that $x_3 x_4^3 \in g$ or $x_4^7 \in h$ if $\lambda = 0$ and $x_3^4 x_4 \notin h$, which is equivalent to F being irreducible and reduced.

(iii) We shall show that φ has the condition $x_4^7 \in h$ if and only if every singular point in Z is terminal. The x_4-chart U_4 of the weighted blowup f' can be expressed as

$$U_4 = \Big(x_1'^2 + x_2'^3 x_4' + \lambda x_2'^2 + x_2' \frac{1}{x_4'^9} g(x_3' x_4'^3, x_4'^2) + \frac{1}{x_4'^{14}} h(x_3' x_4'^3, x_4'^2) = 0\Big) \Big/ \frac{1}{2}(1,1,1,1).$$

If the origin o is contained in U_4, then this point is not terminal, since this equation has only even degree terms. So we need the condition $o \notin U_4$, which is equivalent to the condition $x_4^7 \in h$. Hence, Z is covered by U_1, U_2, and U_3. We study U_2 and U_3:

$$U_2 = \Big(x_1'^2 + x_2' + \lambda x_4'^2 + \frac{1}{x_2'^9} g(x_2'^3 x_3', x_2'^2 x_4') + \frac{1}{x_2'^{14}} h(x_2'^3 x_3', x_2'^2 x_4') = 0\Big) \Big/ \frac{1}{5}(4,3,1,4),$$

$$U_3 = \Big(x_1'^2 + x_2'^3 x_3' + \lambda x_2'^2 x_4'^2 + x_2' \frac{1}{x_3'^9} g(x_3'^3, x_3'^2 x_4') + \frac{1}{x_3'^{14}} h(x_3'^3, x_3'^2 x_4') = 0\Big) \Big/ \frac{1}{3}(1,2,2,2).$$

The origin of U_2 is of type $\frac{1}{5}(1,4,4)$, and the origin of U_3 is of type $\frac{1}{3}(1,2,2)$. We shall check that U_3 has only isolated singularities. Every singular point in U_3 lies only on the hyperplane $(x_3' = 0)$ since F is contracted to P by f'. So, it is enough to study terms of degree at most 1 with respect to x_3':

terms of degree 0: $x_1'^2 + x_2' g_{\mathrm{wt}=9}(1, x_4') + h_{\mathrm{wt}=14}(1, x_4')$;

terms of degree 1: $x_2'^3 + x_2' g_{\mathrm{wt}=10}(1, x_4') + h_{\mathrm{wt}=15}(1, x_4')$.

Therefore, we can check that U_3 has only isolated singularities. Similarly, we can check that U_1 and U_2 have only isolated singularities. Thus, the proof of (iii) is finished.

Therefore, we can apply Lemma 4.1, and f should coincide with f'. The proof of Theorem 2.10 is completed. □

4.2. Case $e2$ with discrepancy 2

In this section, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e2$, and its discrepancy a is 2. In this case, Y has one non-Gorenstein singular point. This point deforms to two points Q_1 and Q_2 which are of type $\frac{1}{r}(1,-1,2)$. Set $N_j := \{(l_1,l_2,l_3,l_4) \in \mathbb{Z}_{\geq 0}^4 \mid rl_1 + rl_2 + 2l_3 + l_4 = j, l_1 l_2 = 0\}$.

LEMMA 4.6

We have that $\dim V_j = \# N_j$.

Proof

By Tables 1 and 2, we see that $(r_{Q_i}, b_{Q_i}, v_{Q_i}) = (r, 2, 1)$ for $i = 1$, 2 and $E^3 = 1/r$. We also have $e_{Q_i} = (r+1)/2$. So

$$\dim V_j = \frac{1}{2r}j(j+3) + \frac{1}{r} + \frac{1}{12}E \cdot c_2(Y) - \Big(\overline{j\frac{r-1}{2}} - \overline{(j+1)\frac{r-1}{2}}\Big)\frac{r^2-1}{12r} + \Big(\sum_{l=1}^{\overline{j\frac{r-1}{2}}-1} - \sum_{l=1}^{\overline{(j+1)\frac{r-1}{2}}-1}\Big)\frac{\overline{2l}(r-\overline{2l})}{2r}.$$

Here $\bar{\cdot}$ denotes the residue modulo r. Since $\dim V_0 = 1$, we have

$$\frac{1}{r} + \frac{1}{12}E \cdot c_2(Y) = 1 - \frac{r-1}{2} \cdot \frac{r^2-1}{12r} + \sum_{l=1}^{\frac{r-1}{2}-1} \frac{\overline{2l}(r-\overline{2l})}{2r}.$$

Now we can compute

$$\dim V_j - \dim V_{j-2} = \frac{1}{r}(2j+1) + \frac{\overline{j+1}(r-\overline{j+1}) - \bar{j}(r-\bar{j})}{2r}$$

for any $j \geq 2$. We can show $\dim V_j - \dim V_{j-2} = \#N_j - \#N_{j-2}$ as Lemma 4.3. □

LEMMA 4.7

(i) *There exist some* $1 \leq k, l \leq 4$ *with* $\operatorname{mult}_E x_k = 1$ *and* $\operatorname{mult}_E x_l = 2$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *Moreover,* $\operatorname{mult}_E x_k \geq 3$ *for* $k = 1$, 2.

(ii) *If* $j < r$, *then the monomials* $x_3^{l_3}x_4^{l_4}$ *for* $(0,0,l_3,l_4) \in N_j$ *form a basis of* V_j. *In particular, for* $k = 1, 2$, $\operatorname{mult}_E \bar{x}_k \geq r$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4}x_3^{l_3}x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$ *and summation over* $(0,0,l_3,l_4) \in \bigcup_{j<r} N_j$.

(iii) *We have* $\operatorname{mult}_E \bar{x}_k = r$ *for* $k = 1, 2$, *and if* $j < 2r$, *then the monomials* $\bar{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4) \in N_j$ *form a basis of* V_j.

(iv) *Set* $\tilde{N}_j = \{(l_1,l_2,l_3,l_4) \in \mathbb{Z}^4_{\geq 0} \mid rl_1 + rl_2 + 2l_3 + l_4 = j\}$. *The monomials* $\bar{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4) \in N_{2r}$ *have one nontrivial relation, say,* ψ, *in* V_{2r}. *The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4)\in\tilde{N}_{2r}} \mathbb{C}\bar{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4} \to V_{2r} \to 0$$

is exact.

Proof

We follow the proof of Lemma 4.4 using the computation of Lemma 4.6. Statement (i) follows from $\dim V_1 = 1$ and $\dim V_2 = 2$. By permutation, we may assume that $\operatorname{mult}_E x_4 = 1$, $\operatorname{mult}_E x_3 = 2$. To prove (ii), we shall show that the monomials $x_3^{l_3}x_4^{l_4}$ for $(0,0,l_3,l_4) \in N_j$ are linearly independent in V_j for any j. Suppose $0 = \sum_{(0,0,l_3,l_4)\in N_j} c_{l_3l_4}x_3^{l_3}x_4^{l_4} \in V_j$, $c_{l_3l_4} \in \mathbb{C}$. We shall show that $c_{l_3l_4} = 0$ for any $(0,0,l_3,l_4) \in N_j$. We study the case $j = 2k$ for $0 \leq k \in \mathbb{Z}$. So we can write

$$\sum_{(0,0,l_3,l_4)\in N_j} c_{l_3 l_4} x_3^{l_3} x_4^{l_4} = \sum_{l=0}^{k} c_l x_3^l x_4^{2(k-l)}$$

for $c_l \in \mathbb{C}$. We factorize

$$\sum_{l=0}^{k} c_l x_3^l x_4^{2(k-l)} = (d_1 x_3 + d_2 x_4^2)\Big(\sum_{l=1}^{k} c'_l x_3^{l-1} x_4^{2(k-l)}\Big)$$

for $c'_l, d_1, d_2 \in \mathbb{C}$. Hence, we have $c_l = 0$ for all $0 \le l \le k$ by induction on k. We can show that $c_{l_3 l_4} = 0$ for the case j is odd similarly. We set $W(j) := \langle x_3^{l_3} x_4^{l_4} \mid (0,0,l_3,l_4) \in N_j \rangle \subset V_j$ for each j. Then $\dim W(j) = \# N_j$ for $j < r$, and thus we obtain (ii). Since $\dim V_r = \dim W(r) + 2$, by permutation, we may assume that $\bar{x}_2$ and $\bar{x}_1$ form a basis of $V_r / W(r) \simeq \mathbb{C}^2$, and we have $\operatorname{mult}_E \bar{x}_1 = \operatorname{mult}_E \bar{x}_2 = r$. The monomials $x_3^{l_3} x_4^{l_4}$ for $(0,0,l_3,l_4) \in N_j$ are linearly independent in V_j for any j, and we have $\dim V_j = \dim W(j) + 2\# N_{j-r} = \# \tilde{N}_j$ for any $j < 2r$. This implies (iii). Since $\dim V_{2r} = \# N_{2r} = \# \tilde{N}_{2r} - 1$, we have a nontrivial relation, say, ψ, in V_{2r}, and we obtain the natural exact sequence in (iv). □

COROLLARY 4.8

We distribute weights $\operatorname{wt}(\bar{x}_1, \bar{x}_2, x_3, x_4) = (r, r, 2, 1)$ *to the coordinates* $\bar{x}_1$, $\bar{x}_2$, x_3, x_4 *obtained in Lemma 4.7. Then* φ *is of the form*

$$\varphi = c\psi + \varphi_{>2r}(\bar{x}_1, \bar{x}_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>2r}$ *of weighted order greater than* $2r$, *where* ψ *in (1) is the one in Lemma 4.7(iv).*

Proof of Theorem 2.4

The cD point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2 x_4 + \lambda x_2 x_3^k + g(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $\lambda \in \mathbb{C}$, and $k \ge 2$.

(i) By Corollary 4.8, we have $\operatorname{wt}\varphi = 2r$. So we have $\operatorname{wt} x_1$, $\operatorname{wt} x_2 = r$. We obtain a quartuple $(\bar{x}_1, \bar{x}_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where p and q are as in Lemma 4.7. Then we rewrite φ as

$$\begin{aligned}\varphi &= (\bar{x}_1 - p)^2 + (\bar{x}_2 - q)^2 x_4 + \lambda(\bar{x}_2 - q) x_3^k + g \\ &= (\bar{x}_1 - p)^2 + \bar{x}_2^2 x_4 - 2\bar{x}_2 x_4 q + \lambda \bar{x}_2 x_3^k + (q^2 x_4 - \lambda q x_3^k + g).\end{aligned}$$

Since $\operatorname{wt}\varphi = 2r$, we can show that $p = 0$, $\operatorname{wt}(q^2 x_4 - \lambda q x_3^k + g) \ge 2r$, and q contains only monomials with weight $r-2$ and $r-1$. So, by replacing variables, we can rewrite φ as

$$\varphi = x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + \lambda x_2 x_3^k + q(x_3, x_4),$$

with $\lambda \in \mathbb{C}$, $k \ge 2$, $\operatorname{wt} q \ge 2r$, and p contains only monomials with weight $r-2$ and $r-1$.

• Suppose that $\mathrm{wt}(x_1,x_2,x_3,x_4)=(r,r,2,1)$.

In this case, we have $k>r/2$, and p is weighted homogeneous of weight $r-1$ for the weights distributed above. Let $f'\colon Z\to X$ be the weighted blowup with $\mathrm{wt}(x_1,x_2,x_3,x_4)=(r,r,2,1)$.

(ii) We have the two conditions below if and only if the exceptional locus F of f' is irreducible and reduced.

(1) $p\neq 0$ or $q_{\mathrm{wt}=2r}\neq 0$.
(2) $q_{\mathrm{wt}=2r}$ is not square if $p=0$.

If $x_3^r\in q$, then either (1) or (2) holds.

(iii) We shall show that φ has the condition $x_3^r\in q$ if and only if every singular point in Z is terminal. The x_3-chart U_3 of the weighted blowup f' can be expressed as

$$\Big(x_1'^2+x_2'^2x_3'x_4'+2x_2'x_4'p+\lambda x_2'x_3'^{2k-r}+\frac{1}{x_3'^{2r}}q(x_3'^2,x_3'x_4')=0\Big)\Big/\frac{1}{2}(1,1,1,1).$$

If the origin o is contained in U_3, then this point is not terminal, since this equation has only even degree terms. So we need the condition $o\notin U_3$, which is equivalent to the condition $x_3^r\in q$. Hence, Z is covered by U_1, U_2, and U_4. The origin of U_2 is of type cA/r. We can check that Z has only isolated singularities as in the proof of Theorem 2.10. Therefore, we can apply Lemma 4.1, and f should coincide with f'.

• Suppose that $\mathrm{wt}(x_1,x_2,x_3,x_4)=(r,r,1,2)$.

In this case, we have $k\geq r$. Let $f'\colon Z\to X$ be the weighted blowup with $\mathrm{wt}(x_1,x_2,x_3,x_4)=(r,r,1,2)$. We shall show that $r=3$, $\lambda\neq 0$, and $k=3$. The x_2-chart U_2 of weighted blowup f' can be expressed as

$$\Big(x_1'^2+x_2'^2x_4'+2x_4'\frac{1}{x_2'^{r-2}}p(x_2'x_3',x_2'^2x_4') \\ +\lambda x_2'^{k-r}x_3'^k+\frac{1}{x_2'^{2r}}q(x_2'x_3',x_2'^2x_4')=0\Big)\Big/\frac{1}{r}\Big(0,\frac{r-1}{2},-\frac{r-1}{2},1\Big).$$

It is impossible that the origin of U_2 is of type cA/r. So it is necessary that the origin be of type $cD/3$, and we need $r=3$, $\lambda\neq 0$, and $k=3$. Moreover, we have $\mathrm{wt}\,p=2$. Replacing variables, we can rewrite φ as

$$\varphi=x_1^2+x_2^2x_4+2x_2x_4p(x_3,x_4)+x_2x_3^3+q(x_3,x_4),$$

where $\mathrm{wt}\,q\geq 6$ and p is weighted homogeneous of weight 2.

(ii′) The exceptional locus F of f' is irreducible and reduced if and only if $q_{\mathrm{wt}=6}$ is not square.

(iii′) We shall show that φ has the condition $x_4^3\in q$ if and only if every singular point in Z is terminal. The x_4-chart U_4 of the weighted blowup f' can be expressed as

$$\Big(x_1'^2 + x_2'^2 x_4'^2 + 2x_2' \frac{1}{x_4'} p(x_3'x_4', x_4'^2) \\ + x_2' x_3'^3 + \frac{1}{x_4'^6} q(x_3'x_4', x_4'^2) = 0\Big) \Big/ \frac{1}{2}(1,1,1,1).$$

If the origin o is contained in U_4, then this point is not terminal, since this equation has only even degree terms. So we have the condition $o \notin U_4$, which is equivalent to the condition $x_4^3 \in q$. Hence, Z is covered by U_1, U_2, and U_3. The origin of U_2 is of type $cD/3$. We can check that Z has only isolated singularities as in the proof of Theorem 2.10. Therefore, we can apply Lemma 4.1, and f should coincide with f'. The proof of Theorem 2.4 is completed. □

Proof of Theorem 2.5

The $cE_{6,7}$ point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$ and $h \in \mathfrak{m}^4$. If P is of type cE_6 (resp., cE_7), then $h_4 \neq 0$ (resp., $h_4 = 0$, $g_3 \neq 0$).

(i) We shall show that we distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 3, 2, 1)$ and that φ can be written as

$$\varphi = x_1^2 + \big\{x_2 - p(x_3, x_4)\big\}^3 + x_2 g(x_3, x_4) + h(x_3, x_4),$$

where $g \in \mathfrak{m}^3$, $h \in \mathfrak{m}^4$, and p is weighted homogeneous of weight 2 for the weights distributed above. By Table 1, Y has $cD/3$ at which E is not Cartier, so we have $r = 3$. By Corollary 4.8, we have $\mathrm{wt}\,\varphi = 6$. So we can distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 3, 2, 1)$. We obtain a quartuple $(\bar{x}_1, \bar{x}_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where p and q are as in Lemma 4.7. Then we rewrite φ as

$$\begin{aligned}\varphi &= (\bar{x}_1 - p)^2 + (\bar{x}_2 - q)^3 + (\bar{x}_2 - q) g(x_3, x_4) + h(x_3, x_4) \\ &= (\bar{x}_1 - p)^2 + (\bar{x}_2 - q)^3 + \bar{x}_2 g + (-qg + h).\end{aligned}$$

Since $\mathrm{wt}\,\varphi = 6$, we can show that $p = 0$, $\mathrm{wt}\,g \geq 3$, $\mathrm{wt}(-qg + h) \geq 6$, and q is weighted homogeneous of weight 2. Replacing $\bar{x}_1$, $\bar{x}_2$, q, and h, we have the desired expression in (i).

(ii) Let $f' \colon Z \to X$ be the weighted blowup with $\mathrm{wt}\,x_i = \mathrm{mult}_E\, x_i$. We can show that the exceptional locus F of f' is irreducible and reduced in (iii).

(iii) We shall show that φ has the condition $x_4^3 \in g$ and $x_3 \in p$ if and only if every singular point in Z is terminal. The x_2-chart U_2 of the weighted blowup f' can be expressed as

$$\Big(x_1'^2 + \big\{x_2' - p(x_3', x_4')\big\}^3 \\ + \frac{1}{x_2'^3} g(x_2'^2 x_3', x_2' x_4') + \frac{1}{x_2'^6} h(x_2'^2 x_3', x_2' x_4') = 0\Big) \Big/ \frac{1}{3}(0,1,1,2).$$

It is necessary that the origin be of type $cD/3$. So we need $x_4^3 \in g$. Moreover, we show that the exceptional locus F of f' is irreducible and reduced. The x_3-chart U_3 of the weighted blowup f' can be expressed as

$$\Big(x_1'^2 + \big\{x_2'x_3' - p(1,x_4')\big\}^3 + \frac{x_2'}{x_3'^3} g(x_3'^2, x_3'x_4') + \frac{1}{x_3'^6} h(x_3'^2, x_3'x_4') = 0\Big) \Big/ \frac{1}{2}(1,1,1,1).$$

If the origin o is contained in U_3, then this point is not terminal, since this equation has only even degree terms. So we have the condition $o \notin U_3$, which is equivalent to the condition $x_3 \in p$. We can check that Z has only isolated singularities as in the proof of Theorem 2.10. Therefore, we can apply Lemma 4.1, and f should coincide with f'.

Let $\bar{x}_2 = x_2 - p$. Then we have

$$\varphi = x_1^2 + \bar{x}_2^3 + \bar{x}_2 g(x_3, x_4) + \big(p(x_3,x_4)g(x_3,x_4) + h(x_3,x_4)\big).$$

If P is of type cE_7, then h should contain $x_3x_4^3$, since $x_3 \in p$ and $x_4^3 \in g$. This is a contradiction to $\operatorname{wt} h \geq 6$. So P is of type cE_6. Therefore, the proof of Theorem 2.5 is completed. □

4.3. Case $e5$ with discrepancy 2

In this section, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e5$, and its discrepancy a is 2. In this case, Y has one non-Gorenstein singular point. This point Q is of type $\frac{1}{7}(1,6,6)$. Set $N_j := \{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}_{\geq 0}^5 \mid 5l_1 + 3l_2 + 2l_3 + 2l_4 + 7l_5 = j, l_1, l_2 \leq 1\}$.

LEMMA 4.9

We have that $\dim V_j = \#N_j$.

Proof

By Tables 1 and 2, we see that $(r_Q, b_Q, v_Q) = (7,3,6)$ and $E^3 = 1/7$. We also have $e_Q = 4$. So

$$\dim V_j = \frac{1}{14} j(j+3) + \frac{1}{7} + \frac{1}{12} E \cdot c_2(Y) - \big(\overline{3j} - \overline{3(j+1)}\big)\frac{4}{7} + \Big(\sum_{l=1}^{\overline{3j}-1} - \sum_{l=1}^{\overline{3(j+1)}-1}\Big)\frac{\overline{6l}(7-\overline{6l})}{14}.$$

Here $\bar{\cdot}$ denotes the residue modulo 7. Since $\dim V_0 = 1$, we have

$$\frac{1}{7} + \frac{1}{12} E \cdot c_2(Y) = \frac{3}{7}.$$

Now we consider

$$\dim V_j - \dim V_{j-7} = j - 2$$

for any $j \geq 7$. We can show $\dim V_j - \dim V_{j-7} = \#N_j - \#N_{j-7}$ as Lemma 4.3. □

LEMMA 4.10

(i) *There exist some* $1 \le k, l \le 4$ *with* $\operatorname{mult}_E x_k = \operatorname{mult}_E x_l = 2$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *Moreover, there exists some* $k = 1, 2$ *with* $\operatorname{mult}_E x_k = 3$. *By permutation, we may assume that* $x_k = x_2$.

(ii) *If* $j < 5$, *then the monomials* $x_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(0, l_2, l_3, l_4, 0) \in N_j$ *form a basis of* V_j. *In particular,* $\operatorname{mult}_E \bar{x}_1 \ge 5$ *for* $\bar{x}_1 := x_1 + \sum c_{l_2 l_3 l_4} x_2^{l_2} x_3^{l_3} x_4^{l_4}$ *with some* $c_{l_2 l_3 l_4} \in \mathbb{C}$ *and summation over* $(0, l_2, l_3, l_4, 0) \in \bigcup_{j<5} N_j$.

(iii) $\operatorname{mult}_E \bar{x}_1 = 5$, *and the monomials* $\bar{x}_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4, 0) \in N_5$ *form a basis of* V_5.

(iv) *Set* $\tilde{N}_j = \{(l_1, l_2, l_3, l_4, l_5) \in \mathbb{Z}^5_{\ge 0} \mid 5l_1 + 3l_2 + 2l_3 + 2l_4 + 7l_5 = j\}$. *The monomials* $\bar{x}_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4, 0) \in N_6$ *have one nontrivial relation, say,* ψ, *in* V_6. *The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,0)\in\tilde{N}_6} \mathbb{C}\bar{x}_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} \to V_6 \to 0$$

is exact.

(v) *We have* $\operatorname{mult}_E \psi = 7$. *The natural exact sequences*

$$0 \to \mathbb{C}x_3\psi \oplus \mathbb{C}x_4\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,l_5)\in\tilde{N}_8} \mathbb{C}\bar{x}_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} \psi^{l_5} \to V_8 \to 0,$$

$$0 \to \mathbb{C}x_2\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,l_5)\in\tilde{N}_9} \mathbb{C}\bar{x}_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} \psi^{l_5} \to V_9 \to 0$$

are exact.

Proof

We follow the proof of Lemma 4.4 using the computation of Lemma 4.9. Statement (i) follows from $\dim V_1 = 0$ and $\dim V_2 = 2$. Now (ii)–(iv) follow from the same argument as in Lemma 4.4. Since $\psi = 0$ in $V_6 = f_*\mathcal{O}_Y(-6E)/f_*\mathcal{O}_Y(-7E)$, we have $\operatorname{mult}_E \psi = 7$. We also obtain the sequences in (v), which are exact possibly except for the middle. Their exactness is verified by comparing dimensions. □

COROLLARY 4.11

We distribute weights $\operatorname{wt}(\bar{x}_1, x_2, x_3, x_4) = (5, 3, 2, 2)$ *to the coordinates* $\bar{x}_1$, x_2, x_3, x_4 *obtained in Lemma 4.10. Then* φ *is of the form*

$$\varphi = cx_2\psi + \varphi_{>9}(\bar{x}_1, x_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>9}$ *of weighted order greater than* 9, *where* ψ *in (1) is the one in Lemma 4.10(iv).*

Proof of Theorem 2.9

The cE_7 point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $h \in \mathfrak{m}^5$, and $g_3 \neq 0$.

(i) We shall show that we distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (5, 3, 2, 2)$ and that φ and ψ can be rewritten as

$$\varphi = x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4),$$
$$\psi = x_2^2 + g_{\mathrm{wt}=6}(x_3, x_4),$$

where $\mathrm{wt}\, g \geq 6$ and $\mathrm{wt}\, h \geq 10$.

By Corollary 4.11, we have $\mathrm{wt}\, \varphi = 9$. So we show that we distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (5, 3, 2, 2)$. We obtain a quartuple $(\bar{x}_1, x_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + cx_2 + p(x_3, x_4)$, where $c \in \mathbb{C}$ and p are as in Lemma 4.10. Then we rewrite φ as

$$\varphi = (\bar{x}_1 - cx_2 - p)^2 + x_2^3 + x_2 g + h.$$

Since $\mathrm{wt}\, \varphi = 9$, we can show that $c = p = 0$, $\mathrm{wt}\, g \geq 6$, and $\mathrm{wt}\, h \geq 10$. By Corollary 4.11, we have $\psi = x_2^2 + g_{\mathrm{wt}=6}(x_3, x_4)$. Replacing $\bar{x}_1$ with x_1, we have the desired expression in (i). By setting $x_5 := -(\psi + g_{\mathrm{wt}\geq 7})$ and replacing $x_2 \mapsto -x_2$, we rewrite φ as

$$\begin{cases} \varphi = x_1^2 + x_2 x_5 + p(x_3, x_4) = 0, \\ x_2^2 + q(x_3, x_4) + x_5 = 0, \end{cases}$$

with $\mathrm{wt}\, p \geq 10$ and $\mathrm{wt}\, q \geq 6$.

(ii) Let $f' \colon Z \to X$ be the weighted blowup with $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (5, 3, 2, 2, 7)$. It is obvious that the exceptional locus F of f' is irreducible and reduced.

(iii) We shall show that we have the condition that $\gcd(p_5, q_3) = 1$ if and only if every singular point in Z is terminal. The x_3-chart U_3 of the weighted blowup f' can be expressed as

$$\text{(A)} \qquad U_3 = \begin{pmatrix} x_1'^2 + x_2' x_5' + \frac{1}{x_3'^{10}} p(x_3'^2, x_3'^2 x_4') = 0, \\ x_2'^2 + \frac{1}{x_3'^6} q(x_3'^2, x_3'^2 x_4') + x_3' x_5' = 0 \end{pmatrix} \Big/ \frac{1}{2}(1, 1, 1, 0, 1).$$

If the origin o is contained in U_3, then this point is not terminal since U_3 is not embedded in 4-dimensional quotient space. So we need the condition $o \notin U_3$, which is equivalent to the condition $x_3^5 \in p$ or $x_3^3 \in q$. Moreover, the action on equations (A) is free outside the points $(0, 0, 0, x_4', 0)$, which satisfy the equations

$$\text{(B)} \qquad \begin{cases} p_{\mathrm{wt}=10}(1, x_4') = 0, \\ q_{\mathrm{wt}=6}(1, x_4') = 0. \end{cases}$$

Since such points are of type $\frac{1}{2}(1, 1, 1, 1)$, there is no solution on (B). Similarly, we have the condition $x_4^5 \in p$ or $x_4^3 \in q$, and there is no solution on

$$\text{(C)} \qquad \begin{cases} p_{\mathrm{wt}=10}(x_3', 1) = 0, \\ q_{\mathrm{wt}=6}(x_3', 1) = 0. \end{cases}$$

It is easy to show that the following four conditions,

- $x_3^5 \in p$ or $x_3^3 \in q$,
- there is no solution on (B),
- $x_4^5 \in p$ or $x_4^3 \in q$, and
- there is no solution on (C),

are equivalent to the condition $\gcd(p_5, q_3) = 1$. We can check that Z has only isolated singularities by using the Jacobian criterion. Thus, the proof of (iii) is finished. Therefore, we can apply Lemma 4.2, and f should coincide with f'. The proof of Theorem 2.9 is completed. □

4.4. Case $e1$ with discrepancy 2

In this section, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e1$, and its discrepancy a is 2. In this case, Y has one non-Gorenstein singular point. This point Q is of type $\frac{1}{r}(1,-1,4)$. Set $N_j := \{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\geq 0} \mid \frac{r+1}{2}l_1 + \frac{r-1}{2}l_2 + 2l_3 + l_4 + rl_5 = j, l_1, l_2 \leq 1\}$ and $M_j := \{(l_1,l_2,l_3,l_4) \in \mathbb{Z}^4_{\geq 0} \mid 3l_1 + 2l_2 + l_3 + l_4 = j, l_2 \leq 1\}$.

LEMMA 4.12

We have that $\dim V_j = \begin{cases} \#N_j & \text{if } r \geq 5, \\ \#M_j & \text{if } r = 3. \end{cases}$

Proof

By Tables 1 and 2, we see that $(r_Q, b_Q, v_Q) = (r, 4, 2)$ and $E^3 = 2/r$. We also have $e_Q = (r+1)/2$. So

$$\dim V_j = \frac{1}{r}j(j+3) + \frac{2}{r} + \frac{1}{12}E \cdot c_2(Y) - \Big(\overline{j\frac{r-1}{2}} - \overline{(j+1)\frac{r-1}{2}}\Big)\frac{r^2-1}{12r} + \Big(\sum_{l=1}^{\overline{j\frac{r-1}{2}}-1} - \sum_{l=1}^{\overline{(j+1)\frac{r-1}{2}}-1}\Big)\frac{\overline{4l}(r-\overline{4l})}{2r}.$$

Here $\bar{\cdot}$ denotes the residue modulo r. Since $\dim V_0 = 1$, we have

$$\frac{2}{r} + \frac{1}{12}E \cdot c_2(Y) = 1 - \frac{r-1}{2} \cdot \frac{r^2-1}{12r} - \sum_{l=1}^{\frac{r-1}{2}-1} \frac{\overline{4l}(r-\overline{4l})}{2r}.$$

If $r \geq 5$, we consider

$$\dim V_j - \dim V_{j-2} = \frac{2}{r}(2j+1) + \frac{\overline{2(j+1)}(r - \overline{2(j+1)}) - \overline{2j}(r-\overline{2j})}{2r}$$

for any $j \geq 2$. We can show $\dim V_j - \dim V_{j-2} = \#N_j - \#N_{j-2}$ as Lemma 4.3. If $r = 3$, we consider

$$\dim V_j - \dim V_{j-3} = 2j$$

for any $j \geq 3$. We can show $\dim V_j - \dim V_{j-3} = \#M_j - \#M_{j-3}$ as Lemma 4.3. □

LEMMA 4.13

If $r\geq 5$, then we have the following conditions.

(i) *There exist some $1\leq k,l\leq 4$ with $\operatorname{mult}_E x_k=1$, $\operatorname{mult}_E x_l=2$. By permutation, we may assume that $x_k=x_4$, $x_l=x_3$.*

(ii) *If $j<\frac{r-1}{2}$, then the monomials $x_3^{l_3}x_4^{l_4}$ for $(0,0,l_3,l_4,0)\in N_j$ form a basis of V_j. In particular, for $k=1,2$, $\operatorname{mult}_E\bar{x}_k\geq\frac{r-1}{2}$ for $\bar{x}_k:=x_k+\sum c_{kl_3l_4}x_3^{l_3}x_4^{l_4}$ with some $c_{kl_3l_4}\in\mathbb{C}$ and summation over $(0,0,l_3,l_4,0)\in\bigcup_{j<\frac{r-1}{2}}N_j$.*

(iii) *There exists some $k=1,2$ with $\operatorname{mult}_E\bar{x}_k=\frac{r-1}{2}$ such that the monomials $\bar{x}_k$ and $x_3^{l_3}x_4^{l_4}$ for $(0,0,l_3,l_4)\in N_{\frac{r-1}{2}}$ form a basis of $V_{\frac{r-1}{2}}$. By permutation, we may assume that $\bar{x}_k=\bar{x}_2$; then $\operatorname{mult}\hat{x}_1\geq\frac{r+1}{2}$ for $\hat{x}_1:=\bar{x}_1+\sum c_{l_2l_3l_4}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ with some $c_{l_2l_3l_4}\in\mathbb{C}$ and summation over $(0,l_2,l_3,l_4)\in N_{\frac{r-1}{2}}$.*

(iv) *We have $\operatorname{mult}_E\hat{x}_1=\frac{r+1}{2}$, and if $j<r-1$, then the monomials $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ for $(l_1,l_2,l_3,l_4)\in N_j$ form a basis of V_j.*

(v) *Set $\tilde{N}_j=\{(l_1,l_2,l_3,l_4,l_5)\in\mathbb{Z}^5_{\geq 0}\mid\frac{r+1}{2}l_1+\frac{r-1}{2}l_2+2l_3+l_4+rl_5=j\}$. The monomials $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ for $(l_1,l_2,l_3,l_4,0)\in N_{r-1}$ have one nontrivial relation, say, ψ, in V_{r-1}. The natural exact sequence*

$$0\to\mathbb{C}\psi\to\bigoplus_{(l_1,l_2,l_3,l_4,0)\in\tilde{N}_{r-1}}\mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\to V_{r-1}\to 0$$

is exact.

(vi) $\operatorname{mult}_E\psi=r$. *The natural exact sequence*

$$0\to\mathbb{C}x_4\psi\to\bigoplus_{(l_1,l_2,l_3,l_4,l_5)\in\tilde{N}_r}\mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\psi^{l_5}\to V_r\to 0$$

is exact.

Proof

We follow the proof of Lemma 4.10 using the computation of Lemma 4.12. Statement (i) follows from $\dim V_1=1$ and $\dim V_2=2$. Now (ii)–(vi) follow from the same argument as in Lemma 4.10. ☐

COROLLARY 4.14

We distribute weights $\operatorname{wt}(\hat{x}_1,\bar{x}_2,x_3,x_4)=(\frac{r+1}{2},\frac{r-1}{2},2,1)$ to the coordinates $\hat{x}_1$, $\bar{x}_2$, x_3, x_4 obtained in Lemma 4.13. Then φ is of the form

$$\varphi=cx_4\psi+\varphi_{>r}(\hat{x}_1,\bar{x}_2,x_3,x_4)$$

with $c\in\mathbb{C}$ and a function $\varphi_{>r}$ of weighted order greater than r, where ψ in (1) is the one in Lemma 4.13(v).

LEMMA 4.15

If $r=3$, then we have the following conditions.

(i) *There exist some* $1 \le k, l \le 4$ *with* $\operatorname{mult}_E x_k = \operatorname{mult}_E x_l = 1$*. By permutation, we may assume that* $x_k = x_4$*,* $x_l = x_3$*. Moreover, there exists some* $k = 1, 2$ *with* $\operatorname{mult}_E x_k = 2$*. By permutation, we may assume that* $x_k = x_2$*.*

(ii) *The monomials* $x_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(0, l_2, l_3, l_4, 0) \in N_2$ *form a basis of* V_2*. In particular,* $\operatorname{mult}_E \bar{x}_1 \ge 3$ *for* $\bar{x}_1 := x_1 + \sum c_{l_2 l_3 l_4} x_2^{l_2} x_3^{l_3} x_4^{l_4}$ *with some* $c_{l_2 l_3 l_4} \in \mathbb{C}$ *and summation over* $(0, l_2, l_3, l_4, 0) \in \bigcup_{j<2} N_j$*.*

(iii) $\operatorname{mult}_E \bar{x}_1 = 3$*, and the monomials* $\bar{x}_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4, 0) \in N_3$ *form a basis of* V_3*.*

(iv) *Set* $\tilde{N}_j = \{(l_1, l_2, l_3, l_4) \in \mathbb{Z}_{\ge 0}^4 \mid 3l_1 + 2l_2 + l_3 + l_4 = j\}$*. The monomials* $\bar{x}_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4) \in N_4$ *have one nontrivial relation, say,* ψ*, in* V_4*. The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4) \in \tilde{N}_4} \mathbb{C}\bar{x}_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} \to V_4 \to 0$$

is exact.

COROLLARY 4.16

We distribute weights $\operatorname{wt}(\bar{x}_1, x_2, x_3, x_4) = (4, 3, 2, 1)$ *to the coordinates* $\bar{x}_1$*,* x_2*,* x_3*,* x_4 *obtained in Lemma 4.15. Then* φ *is of the form*

$$\varphi = c\psi + \varphi_{>4}(\bar{x}_1, x_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>4}$ *of weighted order greater than* 4*, where* ψ *in (1) is the one in Lemma 4.15(iv).*

Proof of Theorem 2.3

The cD point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2 x_4 + \lambda x_2 x_3^k + g(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $\lambda \in \mathbb{C}$, and $k \ge 2$. We shall show that $r \ge 5$. Suppose $r = 3$. By Corollary 4.16, we have $\operatorname{wt}\varphi = 4$. So it is possible to distribute weight $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 2, 1, 1)$, $(3, 1, 1, 2)$, $(2, 3, 1, 1)$, or $(2, 1, 1, 3)$.

We suppose $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 2, 1, 1)$. Then we obtain a quartuple $(\bar{x}_1, x_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + cx_2 + p(x_3, x_4)$, where $c \in \mathbb{C}$ and p are as in Lemma 4.15. Thus, we rewrite φ as

$$\varphi = (\bar{x}_1 - cx_2 - p)^2 + x_2^2 x_4 + \lambda x_2 x_3^k + g.$$

We replace $\bar{x}_1$ with x_1. Let $f' \colon Z \to X$ be the weighted blowup with $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 2, 1, 1)$. The x_1-chart U_1 of the weighted blowup f' can be expressed as

$$\Bigl(\Bigl(x_1' - cx_2' - \frac{1}{x_1'^2} p(x_1' x_3', x_1' x_4')\Bigr)^2 + x_1' x_2'^2 x_4' + \lambda x_1'^{k-2} x_2' x_3'^k + \frac{1}{x_1'^2} g(x_1' x_3', x_1' x_4') = 0\Bigr) \Big/ \frac{1}{3}(1, 1, 2, 2).$$

It is necessary that $o \in U_1$ be of type $\frac{1}{3}(1,1,-1)$, but this is impossible. So we have a contradiction. Similarly, we have a contraction in any other case. Therefore, we have $r \geq 5$.

(i) We shall show that we distribute $\mathrm{wt}(x_1,x_2,x_3,x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1)$ and that φ can be rewritten as

$$\varphi = x_1^2 + \lambda x_2 x_3^k + x_4\psi + p(x_3,x_4),$$

$$\psi = x_2^2 + 2x_1 q_1(x_3,x_4) + q_2(x_3,x_4),$$

where $\lambda \in \mathbb{C}$, $k > \frac{r+1}{4}$, $\mathrm{wt}\, p \geq r+1$, $\mathrm{wt}\, q_1 = \frac{r-3}{2}$, $\mathrm{wt}\, q_2 = r-1$, and q_1, q_2 are weighted homogeneous for the weights distributed above.

By Corollary 4.14, we have $\mathrm{wt}\,\varphi = r$. So we can distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1)$. We obtain a quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3,x_4)$, $\bar{x}_2 = x_2 + q(x_3,x_4)$, where $c \in \mathbb{C}$, p, and q are as in Lemma 4.13. Then we rewrite φ as

$$\varphi = (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^2 x_4 + \lambda(\bar{x}_2 - q)x_3^k + g.$$

Since $\mathrm{wt}\,\varphi = r$, we can show that $c = 0$, $k > \frac{r+1}{4}$, $q = 0$, $\mathrm{wt}(p^2 + g) \geq r$, and p is weighted homogeneous of weight $\frac{r-1}{2}$. So by replacing variables, we can rewrite φ as

$$\varphi = x_1^2 + 2x_1 p(x_3,x_4) + x_2^2 x_4 + \lambda x_2 x_3^k + g(x_3,x_4),$$

where $\lambda \in \mathbb{C}$, $k > \frac{r+1}{4}$, $\mathrm{wt}\, g \geq r$, and p is weighted homogeneous of weight $\frac{r-1}{2}$. We can write ψ as

$$\psi = x_2^2 + 2x_1 \frac{1}{x_4} p(x_3,x_4) + \frac{1}{x_4} g_{\mathrm{wt}=r}(x_3,x_4).$$

Therefore, we have the desired expression in (i).

(ii) Set $x_5 = \psi$. Let $f' \colon Z \to X$ be the weighted blowup with $\mathrm{wt}\, x_i = \mathrm{mult}\, x_i$. We have the condition that q_2 is not square if $q_1 = 0$, which is equivalent to the condition that the exceptional locus F of f' be irreducible and reduced.

(iii) We shall show that φ has the condition $x_3^{\frac{r+1}{2}} \in p$ if and only if every singular point in Z is terminal. The x_3-chart U_3 of the weighted blowup f' can be expressed as

$$\begin{pmatrix} x_1'^2 + \lambda x_2' x_3'^{2k-\frac{r+3}{2}} + x_4' x_5' + \frac{1}{x_3'^{r+1}} p(x_3'^2, x_3' x_4') = 0, \\ x_2'^2 + 2x_1' q_1(1,x_4') + q_2(1,x_4') + x_3' x_5' = 0 \end{pmatrix} \Big/ \frac{1}{2}\Big(-\frac{r-3}{2}, \frac{r-5}{2}, 1,1,1\Big).$$

If the origin o is contained in U_3, then this point is not terminal since U_3 is not embedded in 4-dimensional quotient space. So we need the condition $o \notin U_3$, which is equivalent to the condition $x_3^{\frac{r+1}{2}} \in p$. Hence, Z is covered by U_1, U_2, U_4, and U_5. The origin of U_5 is of type $\frac{1}{r}(1,-1,4)$. We can check that Z has only isolated singularities as in the proof of Theorem 2.9. Therefore, we can apply Lemma 4.2, and f should coincide with f'. The proof of Theorem 2.3 is completed. □

4.5. Case $e1$ with discrepancy 4

In this section, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e1$, and its discrepancy a is 4. In this case, Y has one non-Gorenstein singular point. This point Q is of type $\frac{1}{r}(1,-1,8)$. Set $N_j := \{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\geq 0} \mid \frac{r+1}{2}l_1 + \frac{r-1}{2}l_2 + 4l_3 + l_4 + rl_5 = j, l_1, l_2 \leq 1\}$, $M_j := \{(l_1,l_2,l_3,l_4) \in \mathbb{Z}^4_{\geq 0} \mid 5l_1 + 3l_2 + 2l_3 + l_4 = j, l_2 \leq 1\}$, and $L_j := \{(l_1,l_2,l_3) \in \mathbb{Z}^3_{\geq 0} \mid 3l_1 + l_2 + l_3 = j\}$.

LEMMA 4.17

We have that

$$\dim V_j = \begin{cases} \#N_j & \text{if } r > 5, \\ \#M_j & \text{if } r = 5, \\ \#L_j & \text{if } r = 3. \end{cases}$$

Proof

By Tables 1 and 2, we see that $(r_Q, b_Q, v_Q) = (r, 8, 2)$ and $E^3 = 1/r$. We also have $e_Q = (r+1)/4$ (resp., $e_Q = (3r+1)/4$) if $r \equiv 3 \pmod 8$ (resp., $r \equiv -3 \pmod 8$). So

$$\begin{aligned}\dim V_j &= \frac{1}{2r} j(j+5) + \frac{5}{2r} + \frac{1}{12} E \cdot c_2(Y) \\ &\quad - \left(\overline{-je_Q} - \overline{-(j+1)e_Q}\right)\frac{r^2-1}{12r} + \Big(\sum_{l=1}^{\overline{-je_Q}-1} - \sum_{l=1}^{\overline{-(j+1)e_Q}-1}\Big)\frac{\overline{8l}(r-\overline{8l})}{2r}.\end{aligned}$$

Here $\bar{\cdot}$ denotes the residue modulo r. Since $\dim V_0 = 1$, we have

$$\frac{5}{2r} + \frac{1}{12} E \cdot c_2(Y) = 1 - \overline{-e_Q} \cdot \frac{r^2-1}{12r} + \sum_{l=1}^{\overline{-e_Q}-1} \frac{\overline{8l}(r-\overline{8l})}{2r}.$$

If $r > 5$, we consider

$$\begin{aligned}\dim V_j - \dim V_{j-4} &= \frac{2}{r}(2j+1) \\ &\quad - \left(\overline{-je_Q} - \overline{-(j+1)e_Q} - \overline{-(j-4)e_Q} + \overline{(j-3)e_Q}\right) \\ &\quad + \sum \frac{\overline{8l}(r-\overline{8l})}{2r}\end{aligned}$$

for any $j \geq 4$. We can show $\dim V_j - \dim V_{j-4} = \#N_j - \#N_{j-4}$ as Lemma 4.3. If $r = 5$ (resp., $r = 3$), we consider

$$\dim V_j - \dim V_{j-5} = j (\text{resp., } \dim V_j - \dim V_{j-3} = j+1)$$

for any $j \geq 5$ (resp., $j \geq 3$). We can show $\dim V_j = \#M_j$ (resp., $\dim V_j = \#L_j$) as Lemma 4.3. □

LEMMA 4.18

If $r > 5$, then we have the following conditions.

(i) *There exist some* $1 \le k, l \le 4$ *with* $\mathrm{mult}_E\, x_k = 1$, $\mathrm{mult}_E\, x_l = 4$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$.

(ii) *If* $j < \frac{r-1}{2}$, *then the monomials* $x_3^{l_3} x_4^{l_4}$ *for* $(0,0,l_3,l_4,0) \in N_j$ *form a basis of* V_j. *In particular, for* $k = 1,2$, $\mathrm{mult}_E\, \bar{x}_k \ge \frac{r-1}{2}$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$ *and summation over* $(0,0,l_3,l_4,0) \in \bigcup_{j<\frac{r-1}{2}} N_j$.

(iii) *There exists some* $k = 1,2$ *with* $\mathrm{mult}_E\, \bar{x}_k = \frac{r-1}{2}$ *such that the monomials* $\bar{x}_k$ *and* $x_3^{l_3} x_4^{l_4}$ *for* $(0,0,l_3,l_4) \in N_{\frac{r-1}{2}}$ *form a basis of* $V_{\frac{r-1}{2}}$. *By permutation, we may assume that* $\bar{x}_k = \bar{x}_2$; *then* $\mathrm{mult}\, \hat{x}_1 \ge \frac{r+1}{2}$ *for* $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *with some* $c_{l_2l_3l_4} \in \mathbb{C}$ *and summation over* $(0,l_2,l_3,l_4) \in N_{\frac{r-1}{2}}$.

(iv) *We have* $\mathrm{mult}_E\, \hat{x}_1 = \frac{r+1}{2}$, *and if* $j < r-1$, *then the monomials* $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4) \in N_j$ *form a basis of* V_5.

(v) *Set* $\tilde{N}_j = \{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\ge 0} \mid \frac{r+1}{2} l_1 + \frac{r-1}{2} l_2 + 4l_3 + l_4 + rl_5 = j\}$. *The monomials* $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4,0) \in N_{r-1}$ *have one nontrivial relation, say,* ψ, *in* V_{r-1}. *The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,0) \in \tilde{N}_{r-1}} \mathbb{C}\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \to V_{r-1} \to 0$$

is exact.

(vi) *We have* $\mathrm{mult}_E\, \psi = r$. *The natural exact sequence*

$$0 \to \mathbb{C}x_4\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,l_5) \in \tilde{N}_r} \mathbb{C}\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \psi^{l_5} \to V_r \to 0$$

is exact.

COROLLARY 4.19

We distribute weights $\mathrm{wt}(\hat{x}_1, \bar{x}_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 4, 1)$ *to the coordinates* $\hat{x}_1$, $\bar{x}_2$, x_3, x_4 *obtained in Lemma 4.18. Then* φ *is of the form*

$$\varphi = cx_4\psi + \varphi_{>r}(\hat{x}_1, \bar{x}_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>r}$ *of weighted order greater than* r, *where* ψ *in (1) is the one in Lemma 4.18(v).*

LEMMA 4.20

If $r = 5$, *then we have the following conditions.*

(i) *There exist some* $1 \le k, l \le 4$ *with* $\mathrm{mult}_E\, x_k = 1$ *and* $\mathrm{mult}_E\, x_l = 2$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *The monomials* $x_3^{l_3} x_4^{l_4}$ *for* $(0,0,l_3,l_4) \in M_2$ *form a basis of* V_2. *In particular, for* $k = 1,2$, $\mathrm{mult}_E\, \bar{x}_k \ge 3$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$ *and summation over* $(0,0,l_3,l_4) \in \bigcup_{j<3} M_j$.

(ii) *There exists some* $k = 1,2$ *with* $\mathrm{mult}_E\, \bar{x}_k = 3$ *such that the monomials* $\bar{x}_k^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(0,l_2,l_3,l_4) \in M_j$ *form a basis of* V_j *if* $j < 5$. *By permutation, we assume that* $\bar{x}_k = \bar{x}_2$. *Then* $\mathrm{mult}\, \hat{x}_1 \ge 5$ *for* $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *with some* $c_{l_2l_3l_4} \in \mathbb{C}$ *and summation over* $(0,l_2,l_3,l_4) \in \bigcup_{j<5} M_j$.

(iii) $\mathrm{mult}_E\,\hat{x}_1 = 5$, *and the monomials* $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4)\in M_5$ *form a basis of* V_5.

(iv) *Set* $\tilde{M}_j = \{(l_1,l_2,l_3,l_4)\in \mathbb{Z}^4_{\geq 0} \mid 5l_1+3l_2+2l_3+l_4=j\}$. *The monomials* $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4)\in M_6$ *have one nontrivial relation, say,* ψ, *in* V_6. *The natural exact sequence*

$$0\to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4)\in \tilde{M}_6} \mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\to V_6\to 0$$

is exact.

COROLLARY 4.21

We distribute weights $\mathrm{wt}(\hat{x}_1,\bar{x}_2,x_3,x_4)=(5,3,2,1)$ *to the coordinates* $\hat{x}_1$, $\bar{x}_2$, x_3, x_4 *obtained in Lemma 4.20. Then* φ *is of the form*

$$\varphi = c\psi + \varphi_{>6}(\hat{x}_1,\bar{x}_2,x_3,x_4)$$

with $c\in\mathbb{C}$ *and a function* $\varphi_{>6}$ *of weighted order greater than* 6, *where* ψ *in (1) is the one in Lemma 4.20(iv).*

If $r=3$, then we have the following conditions.

(i) There exist some $1\leq k,l\leq 4$ with $\mathrm{mult}_E\,x_k = \mathrm{mult}_E\,x_l = 1$. By permutation, we may assume that $x_k = x_2$, $x_l = x_3$. The monomials $x_2^{l_2}x_3^{l_3}$ for $(0,l_2,l_3)\in L_2$ form a basis of V_2. In particular, for $k=1,4$, $\mathrm{mult}_E\,\bar{x}_k\geq 3$ for $\bar{x}_k := x_k + \sum c_{kl_2l_3}x_2^{l_2}x_3^{l_3}$ with some $c_{kl_2l_3}\in\mathbb{C}$ and summation over $(0,l_2,l_3)\in \bigcup_{j<3} L_j$.

(ii) There exists some $k=1,4$ with $\mathrm{mult}_E\,\bar{x}_k=3$ such that the monomials $\bar{x}_k^{l_1}x_2^{l_2}x_3^{l_3}$ for $(l_1,l_2,l_3)\in L_j$ form a basis of V_j for any j. By permutation, we assume that $\bar{x}_k=\bar{x}_1$.

So we have $\bigoplus_{(l_1,l_2,l_3)\in L_j}\mathbb{C}\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3}\simeq V_j$ for any j. This means that $\varphi\in\mathbb{C}\{x_1,x_2,x_3\}$. This is a contradiction that P is cDV. Therefore, we have $r\geq 5$.

Proof of Theorem 2.1

The cA_2 point $P\in X$ has an identification such that

$$\varphi = x_1^2+x_2^2+x_3^3+g(x_3,x_4)=0 \qquad \text{or} \tag{2}$$

$$\varphi = x_1x_2+x_3^3+g(x_3,x_4)=0, \tag{3}$$

where $g\in\mathfrak{m}^2$ and $\deg g(x_3,1)\leq 2$. We shall show that there is no suitable weight $\mathrm{wt}(x_1,x_2,x_3,x_4)$ in each case.

Case (2). If $r=5$, we can show that $\mathrm{wt}(x_1,x_2,x_3,x_4)=(5,3,2,1)$ by Corollary 4.21. We obtain a quartuple $(\hat{x}_1,\bar{x}_2,x_3,x_4)$ by $\hat{x}_1 = x_1+c\bar{x}_2+p(x_3,x_4)$, $\bar{x}_2 = x_2+q(x_3,x_4)$, where $c\in\mathbb{C}$, p, and q as in Lemma 4.20. Then we rewrite φ as

$$\varphi = (\hat{x}_1 - c\bar{x}_2-p)^2+(\bar{x}_2-q)^2+x_3^3+g(x_3,x_4).$$

By replacing variables, we rewrite φ as

$$\begin{aligned}\varphi &= x_1^2 + 2cx_1x_2 + (c^2+1)x_2^2 \\ &\quad + 2x_1p(x_3,x_4) + 2cx_2p(x_3,x_4) + x_3^3 + q(x_3,x_4),\end{aligned}$$

where $c \in \mathbb{C}$, $\operatorname{wt} q \geq 6$, and p contains only monomials with weight 3 and 4.

Let $f'\colon Z \to X$ be the weighted blowup with $\operatorname{wt} x_i = \operatorname{mult}_E x_i$. Then the x_1-chart U_1 of the weighted blowup f' can be expressed as

$$\begin{aligned}&\Big(x_1'^4 + 2cx_1'^2x_2' + (c^2+1)x_2'^2 + 2\frac{1}{x_1'}p(x_1'^2x_3', x_1'x_4') \\ &\quad + 2c\frac{x_2'}{x_1'^3}p(x_1'^2x_3', x_1'x_4') + x_3'^3 + \frac{1}{x_1'^6}q(x_1'^2x_3', x_1'x_4') = 0\Big)\Big/\frac{1}{5}(1,-3,3,-1).\end{aligned}$$

The origin is a nonhidden singularity which is not of type $\frac{1}{5}(1,-1,3)$. It is a contradiction by Table 1.

If $r > 5$, there is no suitable weight $\operatorname{wt}(x_1,x_2,x_3,x_4)$ by Corollary 4.19.

Case (3). If $r = 5$, we can distribute weights $\operatorname{wt}(x_1,x_2,x_3,x_4) = (5,2,3,1)$, $(5,3,2,1)$. Let $f'\colon Z \to X$ be the weighted blowup with $\operatorname{wt} x_i = \operatorname{mult}_E x_i$. As in the proof of case (2), the origin of the x_1-chart U_1 of the weighted blowup f' is not a nonhidden singularity which is not of type $\frac{1}{5}(1,-1,3)$. It is a contradiction.

If $r > 5$, by Lemma 4.18, we show that $r = 11$ and $\operatorname{wt}(x_1,x_2,x_3,x_4) = (6,5,4,1)$. However, since $\operatorname{wt}(x_1x_2) = 11$, it is impossible that φ forms as in Corollary 4.19.

Therefore, there is no divisorial contraction of type $e1$ which contracts to a cA_2 point with discrepancy 4. The proof of Theorem 2.1 is completed. ☐

Proof of Theorem 2.2
The cD point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2x_4 + \lambda x_2x_3^k + g(x_3,x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $\lambda \in \mathbb{C}$, and $k \geq 2$. We can show that $r \neq 5$ as in the proof of Theorem 2.3.

(i) As in the proof of Theorem 2.3, we can show that $\operatorname{wt}(x_1,x_2,x_3,x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 4, 1)$ and that φ can be written as

$$\begin{aligned}\varphi &= x_1^2 + \lambda x_2x_3^k + x_4\psi + p(x_3,x_4), \\ \psi &= x_2^2 + 2x_1q_1(x_3,x_4) + q_2(x_3,x_4),\end{aligned}$$

where $\lambda \in \mathbb{C}$, $k > \frac{r+3}{4}$, $\operatorname{wt} p \geq r+1$, $\operatorname{wt} q_1 = \frac{r-3}{2}$, $\operatorname{wt} q_2 = r-1$, and q_1, q_2 are weighted homogeneous for the weights distributed above.

(ii) Set $x_5 = -\psi$, and replace x_4 with $-x_4$. Let $f'\colon Z \to X$ be the weighted blowup with $\operatorname{wt} x_i = \operatorname{mult} x_i$. We have the condition that q_2 is not square if $q_1 = 0$, which is equivalent to the condition that the exceptional locus F of f' is irreducible and reduced.

(iii) We shall show the condition below if and only if every singular point in Z is terminal:

- $x_3^{\frac{r+1}{4}} \in p$ if $r \equiv 3 \pmod 8$,
- $x_3^{\frac{r-1}{4}} \in q_2$ if $r \equiv -3 \pmod 8$.

The x_3-chart U_3 of the weighted blowup f' can be expressed as

$$\begin{pmatrix} x_1'^2 + \lambda x_2' x_3'^{4k-\frac{r+3}{2}} + x_4' x_5' + \frac{1}{x_3'^{r+1}} p(x_3'^4, x_3' x_4') = 0, \\ x_2'^2 + 2x_1' q_1(1, x_4') + q_2(1, x_4') + x_3' x_5' = 0 \end{pmatrix}$$
$$\Big/ \frac{1}{4}\Big(\frac{7-r}{2}, \frac{9-r}{2}, 1, 3, 4-r\Big).$$

If $o \in U_3$, then the origin is not terminal since U_3 is not embedded in 4-dimensional quotient space. So we have the condition $o \notin U_3$, which is equivalent to the condition $x_3^{\frac{r+1}{4}} \in p$ (resp., $x_3^{\frac{r-1}{4}} \in q_2$) if $r \equiv 3 \pmod 8$ (resp., $r \equiv 5 \pmod 8$). Hence, Z is covered by U_1, U_2, U_4, and U_5. The origin of U_5 is of type $\frac{1}{r}(1, -1, 8)$. We can check that Z has only isolated singularities as in the proof of Theorem 2.9.

Therefore, we can apply Lemma 4.2, and f should coincide with f'. The proof of Theorem 2.2 is completed. □

4.6. Case $e3$ with discrepancy 3

In this section, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e3$, and its discrepancy a is 3. In this case, Y has one non-Gorenstein singular point. This point deforms to two points: Q_1 of type $\frac{1}{2}(1,1,1)$ and Q_2 of type $\frac{1}{4}(1,3,3)$. Set $N_j := \{(l_1, l_2, l_3, l_4) \in \mathbb{Z}^4_{\geq 0} \mid 4l_1 + 3l_2 + 2l_3 + l_4 = j, l_1 l_3 = 0\}$.

LEMMA 4.22

We have that $\dim V_j = \# N_j$.

Proof

By Tables 1 and 2, we can see that $(r_{Q_1}, b_{Q_1}, v_{Q_1}) = (2,1,1)$, $(r_{Q_2}, b_{Q_2}, v_{Q_2}) = (4,3,1)$, and $E^3 = 1/4$. We also have $e_{Q_1} = 1$, $e_{Q_2} = 3$. So

$$\begin{aligned} \dim V_j &= \frac{1}{8} j(j+4) + \frac{5}{12} + \frac{1}{12} E \cdot c_2(Y) \\ &\quad - (\overline{j} - \overline{j+1})\frac{1}{8} - (\overline{j}' - \overline{j+1}')\frac{5}{16} + \Big(\sum_{l=1}^{\overline{j}'-1} - \sum_{l=1}^{\overline{j+1}'-1}\Big)\frac{\overline{3l}'(4-\overline{3l}')}{8}. \end{aligned}$$

Here $\overline{\cdot}$ denotes the residue modulo 2, and $\overline{\cdot}'$ denotes the residue modulo 4. Since $\dim V_0 = 1$, we have

$$\frac{5}{12} + \frac{1}{12} E \cdot c_2(Y) = \frac{9}{16}.$$

Now we consider

$$\dim V_j - \dim V_{j-3} = \frac{3}{8}(2j+1) - \frac{1}{4}(\overline{j} - \overline{j+1}) - \frac{5}{16}(\overline{j}' - 2\overline{j+1}' + \overline{j+2}') + \sum \frac{\overline{3l}'(4-\overline{3l}')}{8}$$

for any $j \geq 3$. We can show $\dim V_j - \dim V_{j-3} = \#N_j - \#N_{j-3}$ as Lemma 4.3. □

LEMMA 4.23

(i) *There exist some* $1 \leq k, l \leq 4$ *with* $\mathrm{mult}_E x_k = 1$ *and* $\mathrm{mult}_E x_l = 2$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *The monomials* $x_3^{l_3} x_4^{l_4}$ *for* $(0,0,l_3,l_4) \in N_2$ *form a basis of* V_2. *In particular, for* $k = 1, 2$, $\mathrm{mult}_E \bar{x}_k \geq 3$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$ *and summation over* $(0,0,l_3,l_4) \in \bigcup_{j<3} N_j$.

(ii) *There exists some* $k = 1, 2$ *with* $\mathrm{mult}_E \bar{x}_k = 3$ *such that the monomials* $\bar{x}_k$ *and* $x_3^{l_3} x_4^{l_4}$ *for* $(0,0,l_3,l_4) \in N_3$ *form a basis of* V_3. *By permutation,* $\bar{x}_k = \bar{x}_2$. *Then* $\mathrm{mult}\, \hat{x}_1 \geq 4$ *for* $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *with some* $c_{l_2l_3l_4} \in \mathbb{C}$ *and summation over* $(0,l_2,l_3,l_4) \in N_4$.

(iii) *We have* $\mathrm{mult}_E \hat{x}_1 = 4$. *If* $j < 6$, *then the monomials* $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4) \in N_j$ *form a basis of* V_j.

(iv) *Set* $\tilde{N}_j = \{(l_1,l_2,l_3,l_4) \in \mathbb{Z}^4_{\geq 0} \mid 4l_1 + 3l_2 + 2l_3 + l_4 = j\}$. *The monomials* $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4) \in N_6$ *have one nontrivial relation, say,* ψ, *in* V_6. *The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4) \in \tilde{N}_6} \mathbb{C}\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \to V_6 \to 0$$

is exact.

COROLLARY 4.24

We distribute weights $\mathrm{wt}(\hat{x}_1, \bar{x}_2, x_3, x_4) = (4,3,2,1)$ *to the coordinates* $\hat{x}_1$, $\bar{x}_2$, x_3, x_4 *obtained in Lemma 4.23. Then* φ *is of the form*

$$\varphi = c\psi + \varphi_{>6}(\hat{x}_1, \bar{x}_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>6}$ *of weighted order greater than* 6, *where* ψ *in (1) is the one in Lemma 4.23(iv).*

Proof of Theorem 2.6

The cA_2 point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2 + x_3^3 + g(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^2$ and $\deg g(x_3, 1) \leq 2$.

(i) We shall show that we distribute weight $\mathrm{wt}(x_1,x_2,x_3,x_4) = (4,3,2,1)$ and that φ can be written as

$$\begin{aligned}\varphi &= x_1^2 + x_2^2 + 2cx_1x_2 + 2x_1p(x_3,x_4)\\ &\quad + 2cx_2p_{\mathrm{wt}=3}(x_3,x_4) + x_3^3 + g(x_3,x_4)\\ &= 0,\end{aligned}$$

where $c \neq \pm 1$, $2 \le \operatorname{wt} p \le 3$, $\operatorname{wt} g \ge 6$, and $\deg g(x_3,1) \le 2$. By Corollary 4.24, we can distribute weight $\operatorname{wt}(x_1,x_2,x_3,x_4) = (4,3,2,1)$. We obtain a quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3,x_4)$, $\bar{x}_2 = x_2 + q(x_3,x_4)$, where $c \in \mathbb{C}$, p, and q are as in Lemma 4.23. Then we rewrite φ as

$$\varphi = (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^2 + x_3^3 + g.$$

Since $\operatorname{wt}\varphi = 6$, we have $cp_{\mathrm{wt}\le 2} = -q$, and p contains only monomials with weight 2 and 3. Moreover, since $P \in X$ is of type cA_2, we have $c^2 + 1 \neq 0$. So by replacing variables, we have the desired expression in (i).

(ii) Let $f'\colon Z \to X$ be the weighted blowup with $\operatorname{wt} x_i = \operatorname{mult}_E x_i$. We have the condition that g is not square if $p_{\mathrm{wt}=2} = 0$, which is equivalent to the condition that the exceptional locus F of f' is irreducible and reduced.

(iii) We shall show that φ needs the condition $x_4^2 \in p$ and that every singular point in Z is terminal. The x_1-chart U_1 of the weighted blowup f' can be expressed as

$$\begin{aligned}&\Big(x_1'^2 + x_2'^2 + 2cx_1'x_2' + 2\frac{1}{x_1'^2}p(x_1'^2x_3', x_1'x_4')\\ &\quad + 2cx_2'p_{\mathrm{wt}=3}(x_3',x_4') + x_3'^3 + \frac{1}{x_1'^6}g(x_1'^2x_3', x_1'x_4') = 0\Big)\Big/\frac{1}{4}(1,1,2,3).\end{aligned}$$

It is necessary that the origin be of type $cAx/4$. So we have the condition $x_4^2 \in p$. We can check that Z has only isolated singularities as in the proof of Theorem 2.10.

Therefore, we can apply Lemma 4.1, and f should coincide with f'. The proof of Theorem 2.6 is completed. □

Proof of Theorem 2.7

The cD point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2x_4 + \lambda x_2x_3^k + g(x_3,x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $\lambda \in \mathbb{C}$, and $k \ge 2$. Since $\operatorname{wt}\varphi = 6$, we can distribute weight $\operatorname{wt}(x_1, x_2,x_3,x_4) = (4,3,2,1)$, $(4,3,1,2)$, $(4,2,1,3)$, $(3,4,2,1)$, $(3,4,1,2)$, $(3,2,1,4)$, or $(3,1,2,4)$.

- At first, we suppose $\operatorname{wt}(x_1,x_2,x_3,x_4) = (3,4,2,1)$.

(i) We shall show that φ can be written as

$$\begin{aligned}\varphi &= x_1^2 + x_2^2x_4 + 2x_2x_4p(x_3,x_4) + c^2x_1^2x_4 + \lambda x_2x_3^k\\ &\quad + c\big(2x_1x_2x_4 + 2x_1x_4p(x_3,x_4) + \lambda x_1x_3^k\big) + g(x_3,x_4)\\ &= 0,\end{aligned}$$

where c, $\lambda \in \mathbb{C}$, $k > 2$, $\operatorname{wt} g \geq 6$, and p contains only monomials with weight at most 3.

We obtain the quartuple $(\bar{x}_1, \hat{x}_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + p(x_3, x_4)$, $\hat{x}_2 = x_2 + c\bar{x}_1 + q(x_3, x_4)$, where $c \in \mathbb{C}$, p, and q are as in Lemma 4.23. Then we rewrite φ as

$$\varphi = (\bar{x}_1 - q)^2 + (\hat{x}_2 - \bar{x}_1 - p)^2 x_4 + \lambda(\hat{x}_2 - c\bar{x}_1 - p)x_3^k + g(x_3, x_4).$$

Since $\operatorname{wt}\varphi = 6$, we can assume $q = 0$. Moreover, we have $\operatorname{wt}(p^2 x_4 - \lambda p x_3^k + g) \geq 6$, and p contains only monomials with weight at most 3. So replacing variables, we have the desired expression in (i).

(ii) Let $f' \colon Z \to X$ be the weighted blowup with $\operatorname{wt} x_i = \operatorname{mult}_E x_i$. We have the condition that g is not square if $p_{\mathrm{wt}=1} = 0$, which is equivalent to the condition that the exceptional locus F of f' is irreducible and reduced. If $x_4 \in p$, then F is irreducible and reduced.

(iii) We shall show that φ has the conditions $c = 0$, $x_4 \in p$, and $x_3^3 \in g$ if and only if every singular point in Z is terminal and Z has a nonhidden terminal of type $cAx/4$. The x_2-chart U_2 of the weighted blowup f' can be expressed as

$$\begin{aligned}\Big(&x_1'^2 + x_2'^3 x_4' + 2\frac{x_4'}{x_2'} p(x_2'^2 x_3', x_2' x_4') + c^2 x_1'^2 x_2' x_4' \\ &+ c\Big(2x_1' x_2'^2 x_4' + 2x_1' x_4' \frac{1}{x_2'^2} p(x_2'^2 x_3, x_2' x_4) + \lambda x_1' x_2'^{2k-3} x_3'^k\Big) \\ &+ \lambda x_2'^{2k-2} x_3'^k + \frac{1}{x_2'^6} g(x_2'^2 x_3', x_2' x_4') = 0\Big) \Big/ \frac{1}{4}(1,1,2,3).\end{aligned}$$

The origin of U_2 is of type $cAx/4$. So we have the conditions $x_4 \in p$ and $c = 0$. Moreover, since the equation is free outside the origin, we have $g_{\mathrm{wt}=6}(x_3, 0) \neq 0$, which is equivalent to the condition $x_3^3 \in g$. Thus, φ can be written as

$$\varphi = x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + \lambda x_2 x_3^k + g(x_3, x_4),$$

and P is of type cD_4. We can check that Z has only isolated singularities as in the proof of Theorem 2.10. Therefore, we can apply Lemma 4.1, and f should coincide with f' if $P \in X$ is cD_4.

• Next, we shall show that there is no weighted blowup of type $e3$ which contracts to a cD point with $\operatorname{wt} x_1 = 4$.

We select $\operatorname{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$. We obtain the quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where $c \in \mathbb{C}$, p, and q are as in Lemma 4.23. Then we rewrite φ as

$$\varphi = (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^2 x_4 + \lambda(\bar{x}_2 - q)x_3^k + g(x_3, x_4).$$

We replace $\hat{x}_1 \mapsto x_1$ and $\bar{x}_2 \mapsto x_2$. Let $f' \colon Z \to X$ be the weighted blowup with $\operatorname{wt} x_i = \operatorname{mult}_E x_i$. Then the x_1-chart U_1 of the weighted blowup f' can be expressed as

$$\begin{aligned}\Big(\Big(x_1' - cx_2' - \frac{1}{x_1'^3} p(x_1'^2 x_3', x_1' x_4')\Big)^2 \\ + \Big(x_2' - \frac{1}{x_1'^3} q(x_1'^2 x_3', x_1' x_4')\Big)^2 x_1' x_4' \\ + \lambda\Big(x_2' - \frac{1}{x_1'^3} q(x_1'^2 x_3', x_1' x_4')\Big) x_1'^{2k-3} x_3'^k \\ + \frac{1}{x_1'^6} g(x_1'^2 x_3', x_1' x_4') = 0\Big) \Big/ \frac{1}{4}(1,1,2,3).\end{aligned}$$

It is necessary that the origin be of type $cAx/4$. So we have $x_4^2 \in p$, and moreover $c = 0$. Now the x_2-chart U_2 of the weighted blowup f' can be expressed as

$$\begin{aligned}\Big(\Big(x_1' x_2' - \frac{1}{x_2'^3} p(x_2'^2 x_3', x_2' x_4')\Big)^2 \\ + \Big(1 - \frac{1}{x_2'^3} q(x_2'^2 x_3', x_2' x_4')\Big)^2 x_2' x_4' \\ + \lambda\Big(1 - \frac{1}{x_2'^3} q(x_2'^2 x_3', x_2' x_4')\Big) x_2'^{2k-3} x_3'^k \\ + \frac{1}{x_2'^6} g(x_2'^2 x_3', x_2' x_4') = 0\Big) \Big/ \frac{1}{3}(2,1,1,2).\end{aligned}$$

The origin is a nonhidden singularity. It is a contradiction by Table 1. Similarly, we have a contradiction in any other case. Therefore, there is no weighted blowup of type $e2$ which contracts to a cD point with $\operatorname{wt} x_1 = 4$.

• Finally, we shall show that there is no weighted blowup of type $e3$ which contracts to a cD_n point with $\operatorname{wt} x_1 = 3$ for any $n \geq 5$. We can show that P is of type cD_4 with the weight $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 4, 1, 2)$ as in the proof with the weight $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 4, 2, 1)$. We select $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 2, 1, 4)$ or $(3, 1, 2, 4)$. We obtain the quartuple $(\bar{x}_1, x_2, x_3, \hat{x}_4)$ by $\hat{x}_4 = x_4 + c\bar{x}_1 + p(x_2, x_3)$, $\bar{x}_1 = x_1 + q(x_2, x_3)$, where $c \in \mathbb{C}$, p, and q are as in Lemma 4.23. Then we rewrite φ as

$$\varphi = (\bar{x}_1 - q)^2 + x_2^2(\hat{x}_4 - c\bar{x}_1 - p) + \lambda x_2 x_3^k + g(x_3, \hat{x}_4 - c\bar{x}_1 - p).$$

Replacing variables, we can rewrite φ as

$$\varphi = x_1^2 + x_2^2\big(x_4 + cx_1 + p(x_2, x_3)\big) + \lambda x_2 x_3^k + g(x_1, x_2, x_3, x_4),$$

where $c \in \mathbb{C}$, $k \geq 2$ $\operatorname{wt} g \geq 6$, and p contains only monomials with weight at most 3. Let $f' \colon Z \to X$ be the weighted blowup with $\operatorname{wt} x_i = \operatorname{mult}_E x_i$. If $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 2, 1, 4)$, then the x_4-chart U_4 of the weighted blowup f' can be expressed as

$$\begin{aligned}\Big(x_1'^2 + x_2'^2\Big(x_4'^2 + cx_1' x_4' + \frac{1}{x_4'^2} p(x_2' x_4'^2, x_3' x_4')\Big) \\ + \lambda x_2' x_3'^k x_4'^{k-4} + \frac{1}{x_4'^6} g(x_1' x_4'^3, x_2' x_4'^2, x_3' x_4', x_4'^4) = 0\Big) \Big/ \frac{1}{4}(1,2,3,1).\end{aligned}$$

It is necessary that the origin be of type $cAx/4$. So we have the condition $x_3^2x_4 \in g$. This means that P is of type cD_4.

If $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 1, 2, 4)$, we have $c = 0$, and we can assume $p = 0$ by replacing g if necessary. The x_3-chart U_3 of the weighted blowup f' can be expressed as

$$\Big(x_1'^2 + x_2'^2x_4' + \lambda x_2'x_3'^{2k-5} + \frac{1}{x_3'^6}g(x_1'x_3'^3, x_2'x_3', x_3'^2, x_3'^4x_4') = 0\Big)\Big/\frac{1}{2}(1,1,1,0).$$

We need the condition $o \notin U_3$, which is equivalent to the condition $x_3^3 \in g$. Then P is of type cD_4. Therefore, there is no divisorial contraction of type $e3$ which contracts to a cD_n point with discrepancy 3 for any $n \geq 5$. The proof of Theorem 2.7 is completed. □

Proof of Theorem 2.8
The cE_6 point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^3 + x_2g(x_3, x_4) + h(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $h \in \mathfrak{m}^4$, and $h_4 \neq 0$. By Corollary 4.24, we have $\mathrm{wt}\,\varphi = 6$. So we can distribute weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$, $(4, 2, 3, 1)$, $(3, 4, 2, 1)$, or $(3, 2, 4, 1)$. Suppose $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$. Then we obtain a quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where $c \in \mathbb{C}$, p, and q are as in Lemma 4.23. We rewrite φ as

$$\varphi = (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^3 + (\bar{x}_2 - q)g + h.$$

We replace $\hat{x}_1$ with x_1 and $\bar{x}_2$ with x_2. Since $\mathrm{wt}\,\varphi = 6$, we can rewrite φ as

$$\begin{aligned}\varphi &= x_1^2 + x_2^3 + p(x_3, x_4)x_2^2 + 2cx_1x_2 \\ &\quad + 2q(x_3, x_4)x_1 + x_2g(x_3, x_4) + h(x_3, x_4) \\ &= 0,\end{aligned}$$

where $g \in \mathfrak{m}^3$, $h \in \mathfrak{m}^4$, $h_4 \neq 0$, $c \in \mathbb{C}$, and p (resp., q) contains only monomials with weight 1 and 2 (resp., 2 and 3).

Let $f' \colon Z \to X$ be the weighted blowup with $\mathrm{wt}\,x_i = \mathrm{mult}_E\,x_i$. The x_1-chart U_1 of the weighted blowup f' can be expressed as

$$\begin{aligned}&\Big(x_1'^2 + x_1'^3x_2'^3 + p(x_1'^2x_3', x_1'x_4')x_2'^2 \\ &\quad + 2cx_1'x_2' + 2\frac{1}{x_1'^2}q(x_1'^2x_3', x_1'x_4') \\ &\quad + x_2'\frac{1}{x_1'^3}g(x_1'^2x_3', x_1'x_4') + \frac{1}{x_1'^6}h(x_1'^2x_3', x_1'x_4') = 0\Big)\Big/\frac{1}{4}(1,1,2,3).\end{aligned}$$

It is necessary that the origin be of type $cAx/4$. So we need $x_4^2 \in q$ and $x_3 \notin q$. Moreover, we need that the action is free outside the origin, which is equivalent to the condition that $x_3^3 \in h$. This is a contradiction. Similarly, we have a contradiction in any other case. Therefore, there is no divisorial contraction of type

$e3$ which contracts to a cE_6 point with discrepancy 3. The proof of Theorem 2.8 is completed. □

Acknowledgments. I was motivated to write this article by some questions of Professor T. Hayakawa. I am grateful to him for his useful advice and helpful comments. I would like to thank Professor M. Kawakita for answering my question earnestly and sending me the erratum [12].

References

[1] J. A. Chen, "Threefold divisorial contractions to singularities of cE type" in *Recent Advances in Algebraic Geometry*, London Math. Soc. Lecture Note Ser. **417**, Cambridge Univ. Press, Cambridge, 2015, 102–122. MR 3380446.

[2] T. Hayakawa, *Blowing ups of* 3*-dimensional terminal singularities*, Publ. Res. Inst. Math. Sci. **35** (1999), 515–570. MR 1710753. DOI 10.2977/prims/1195143612.

[3] ———, *Blowing ups of* 3*-dimensional terminal singularities, II*, Publ. Res. Inst. Math. Sci. **36** (2000), 423–456. MR 1781436. DOI 10.2977/prims/1195142953.

[4] ———, *Divisorial contractions to* 3*-dimensional terminal singularities with discrepancy one*, J. Math. Soc. Japan **57** (2005), 651–668. MR 2139726.

[5] ———, *Divisorial contractions to cD points*, preprint.

[6] ———, *Divisorial contractions to cE points*, preprint.

[7] M. Kawakita, *Divisorial contractions in dimension three which contract divisors to smooth points*, Invent. Math. **145** (2001), 105–119. MR 1839287. DOI 10.1007/s002220100144.

[8] ———, *Divisorial contractions in dimension three which contract divisors to compound* A_1 *points*, Compos. Math. **133** (2002), 95–116. MR 1918291. DOI 10.1023/A:1016334006624.

[9] ———, *General elephants of three-fold divisorial contractions*, J. Amer. Math. Soc. **16** (2003), 331–362. MR 1949163. DOI 10.1090/S0894-0347-02-00416-2.

[10] ———, *Three-fold divisorial contractions to singularities of higher indices*, Duke Math. J. **130** (2005), 57–126. MR 2176548. DOI 10.1215/S0012-7094-05-13013-7.

[11] ———, *Supplement to classification of threefold divisorial contractions*, Nagoya Math. J. **208** (2012), 67–73. MR 2926484.

[12] ———, *Erratum to "General elephants of three-fold divisorial contractions,"* preprint, December 2014, http://www.kurims.kyoto-u.ac.jp/~masayuki/Website/Documents/erratum-03.pdf.

[13] Y. Kawamata, "Divisorial contractions to 3-dimensional terminal quotient singularities" in *Higher-Dimensional Complex Varieties (Trento, 1994)*, de Gruyter, Berlin, 1996, 241–246. MR 1463182.

[14] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformation of surface singularities*, Invent. Math. **91** (1988), 299–338. MR 0922803. DOI 10.1007/BF01389370.

[15] S. Mori, *On 3-dimensional terminal singularities*, Nagoya Math. J. **98** (1985), 43–66. MR 0792770.

[16] M. Reid, "Young person's guide to canonical singularities" in *Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, Proc. Sympos. Pure Math. **46**, Amer. Math. Soc., Providence, 1987, 345–414. MR 0927963.

Division of Mathematical and Physical Sciences, Graduate School of Natural Science and Technology, Kanazawa University, Kakuma-machi, Kanazawa, Japan; colormocchi@gmail.com