Hamiltonian C^0-continuity of Lagrangian capacity on the cotangent bundle

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Abstract Partially motivated by the study of topological Hamiltonian dynamics, we prove the following C^0-continuity of the Lagrangian capacity function γ^{lag}:

$$\gamma^{\mathrm{lag}}\big(\phi_H^1(o_N)\big) := \rho^{\mathrm{lag}}(H;1) - \rho^{\mathrm{lag}}\big(H;[pt]^\#\big) \to 0,$$

as $\phi_H^1 \to id$, *provided* the H's satisfy $\operatorname{supp} X_H \subset D^R(T^*N) \setminus o_B$ for some $R > 0$ and a closed subset $B \subset N$ with nonempty interior. We also provide an estimate of the capacity in terms of the C^0-distance of $d_{C^0}(\phi_H^1, id)$ and the subset $B \subset N$ relative to T^*N.

1. Introduction

We always assume that the ambient manifolds M and N are connected throughout the entire article.

1.1. Weak Hamiltonian topology of $\mathrm{Ham}(M,\omega)$

The author and Müller [14] introduced the notion of Hamiltonian topology on the subset of the space $\mathcal{P}(\mathrm{Homeo}(M), id)$ of continuous paths on $\mathrm{Homeo}(M)$ consisting of Hamiltonian paths $\lambda : [0,1] \to \mathrm{Symp}(M,\omega)$ with $\lambda(t) = \phi_H^t$ for some time-dependent Hamiltonian H. We denote this subset by

$$\mathcal{P}^{\mathrm{Ham}}\big(\mathrm{Symp}(M,\omega), id\big).$$

We would like to emphasize that we do *not* assume that H is normalized unless otherwise stated explicitly. This is because we need to consider both compactly supported and mean-normalized Hamiltonians and suitably transform one to the other in the course of the proofs of the various theorems of this article.

In this section, we first recall the definition from [14] of the Hamiltonian topology mostly restricted on the open manifold T^*N. While [14] considers *strong* Hamiltonian topology, except in [14, Remark 3.27], the more relevant topology in the present article will be the *weak* Hamiltonian topology. We first recall its definition.

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For a given continuous function $h : M \to \mathbb{R}$, we denote

$$\operatorname{osc}(h) = \max h - \min h.$$

We define the C^0-distance $\overline{d}$ on $\operatorname{Homeo}(M)$ by the symmetrized C^0-distance

$$\overline{d}(\phi, \psi) = d_{C^0}(\phi, \psi) + d_{C^0}(\phi^{-1}, \psi^{-1})$$

and the C^0-distance on $\mathcal{P}(\operatorname{Homeo}(M), id)$, again denoted by $\overline{d}$, by

$$\overline{d}(\lambda, \mu) = \max_{t \in [0,1]} \overline{d}\big(\lambda(t), \mu(t)\big).$$

This induces the corresponding C^0-distance on $\mathcal{P}^{\mathrm{Ham}}(\operatorname{Symp}(M, \omega), id)$. The Hofer length of the Hamiltonian path $\lambda = \phi_H$ is defined by

$$\operatorname{leng}(\lambda) = \int_0^1 \operatorname{osc}(H_t)\, dt = \|H\|.$$

Following the notation of [14], we denote by ϕ_H the Hamiltonian path

$$\phi_H : t \mapsto \phi_H^t; \qquad [0,1] \to \operatorname{Ham}(M, \omega).$$

DEFINITION 1.1

Let (M, ω) be an open symplectic manifold. Let λ, μ be smooth Hamiltonian paths with compact support in $\operatorname{Int} M$. The weak Hamiltonian topology is the metric topology induced by the metric

$$d_{\mathrm{Ham}}(\lambda, \mu) := \overline{d}\big(\lambda(1), \mu(1)\big) + \operatorname{leng}(\lambda^{-1}\mu). \tag{1.1}$$

1.2. Hamiltonian C^0-topology on $\mathfrak{Iso}_B(o_N; T^*N)$

Let N be a closed smooth manifold. We equip the cotangent bundle T^*N with the Liouville one-form θ defined by

$$\theta_x(\xi_x) = p\big(d\pi(\xi_x)\big), \quad x = (q, p) \in T^*N.$$

The canonical symplectic form ω_0 on T^*N is defined by

$$\omega_0 = -d\theta = \sum_{k=1}^n dq^k \wedge dp_k, \tag{1.2}$$

where $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ are the canonical coordinates of T^*N associated to the coordinates $(q^1, \ldots, q^n)$ of N.

Consider the Hamiltonian $H = H(t, x)$ such that H_t is asymptotically constant, that is, the one whose Hamiltonian vector field X_H is compactly supported. We define

$$\operatorname{supp}_{\mathrm{asc}} H = \operatorname{supp} X_H := \bigcup_{t \in [0,1]} \operatorname{supp} X_{H_t}.$$

For each given $K, R \in \mathbb{R}_+$, we define

$$\mathcal{P}C^\infty_{R,K} = \big\{ H \in C^\infty\big([0,1] \times T^*N, \mathbb{R}\big) \;\big|\; \operatorname{supp}_{\mathrm{asc}} H \subset D^R(T^*N), \|H\| \le K \big\} \tag{1.3}$$

which provides a natural filtration of the space $C^\infty([0,1]\times T^*N,\mathbb{R})$. We also denote

$$\mathcal{P}C^\infty_R = \bigcup_{K\in\mathbb{R}_+} \mathcal{P}C^\infty_{R,K}, \qquad \mathcal{P}C^\infty_{\mathrm{asc}} = \bigcup_{R\geq 0} \mathcal{P}C^\infty_R. \tag{1.4}$$

By definition, each element H_t is independent of $x=(q,p)$ if $|p|$ is sufficiently large and so carries a smooth function $c_\infty : [0,1]\to\mathbb{R}$ defined by

$$c_\infty(t) = H(t,\infty).$$

Therefore, we have the natural evaluation map

$$\pi_\infty : \mathcal{P}C^\infty_{\mathrm{asc}} \to C^\infty\big([0,1],\mathbb{R}\big).$$

For each given smooth function $c:[0,1]\to\mathbb{R}$, we denote

$$\mathcal{P}C^\infty_{\mathrm{asc};c} := \pi_\infty^{-1}(c). \tag{1.5}$$

We then introduce the space of Hamiltonian deformations of the zero section and denote

$$\mathfrak{Iso}(o_N;T^*N) = \big\{\phi^1_H(o_N)\;\big|\; H\in\mathcal{P}C^\infty_{\mathrm{asc}}\big\},$$

following the terminology of [22], and

$$\begin{aligned}\mathfrak{Iso}\big(o_N;D^R(T^*N)\big) &= \big\{\phi^1_H(o_N)\;\big|\; H\in\mathcal{P}C^\infty_R\big\},\\ \mathfrak{Iso}^K\big(o_N;D^R(T^*N)\big) &= \big\{\phi^1_H(o_N)\;\big|\; H\in\mathcal{P}C^\infty_{R,K}\big\}.\end{aligned} \tag{1.6}$$

Now we equip a topology with $\mathfrak{Iso}(o_N;T^*N)$. One needs to pay some attention in finding the correct definition of the topology suitable for the study of the Hamiltonian geometry of the set $\mathfrak{Iso}(o_N;T^*N)$. For this purpose, we introduce the following measurement of the C^0-fluctuation of the Hamiltonian diffeomorphism of ϕ^1_F along the zero section $o_N\subset T^*N$:

$$\mathrm{osc}_{C^0}(\phi^1_F;o_N) := \max\big\{\max_{x\in o_N} d\big(\phi^1_F(x),x\big), \max_{x\in o_N} d\big((\phi^1_F)^{-1}(x),x\big)\big\}.$$

Using this measurement, we introduce the following restricted C^0-distance.

DEFINITION 1.2

Let $L_0,L_1\in\mathfrak{Iso}(o_N;T^*N)$ with $L_0=\phi^1_{F^0}(o_N), L_1=\phi^1_{F^1}(o_N)$. We define the distance function

$$\begin{aligned}d^{\mathrm{Ham}}_{C^0}(L_0,L_1) = \inf_{\{H;\phi^1_H(L_0)=L_1\}} &\max\big\{\mathrm{osc}_{C^0}\big((\phi^1_{F^1})^{-1}\phi^1_H\phi^1_{F^0};o_N\big),\\ &\mathrm{osc}_{C^0}\big((\phi^1_{F^0})^{-1}(\phi^1_H)^{-1}\phi^1_{F^1};o_N\big)\big\}\end{aligned} \tag{1.7}$$

on $\mathfrak{Iso}^K(o_N;D^R(T^*N))$, which induces the metric topology theorem. We equip with $\mathfrak{Iso}(o_N;T^*N)$ the direct limit topology of $\mathfrak{Iso}^K(o_N;D^R(T^*N))$ as $R,K\to\infty$ and call it the *Hamiltonian C^0-topology* of $\mathfrak{Iso}(o_N;T^*N)$.

For the main theorem proved in the present article, we will also need to consider the following subset of Hamiltonian functions H. Let $B\subset N$ be a given closed subset, and let $o_B \subset o_N$ be the corresponding subset of the zero section. Denote by T an open neighborhood of o_B in T^*N. We define

$$\mathcal{P}C^\infty_{\mathrm{asc};B} = \bigl\{H \in C^\infty\bigl([0,1]\times T^*N, \mathbb{R}\bigr) \bigm| \operatorname{supp} X_H \subset (T^*N \setminus B) \text{ is compact}\bigr\}. \tag{1.8}$$

We have the filtration

$$\mathcal{P}C^\infty_{\mathrm{asc};B} = \bigcup_{T\supset B} \bigcup_{R>0} \mathcal{P}C^\infty_{R;T}$$

over the set of open neighborhoods T of B and the positive numbers $R > 0$, where

$$\mathcal{P}C^\infty_T = \{H \in \mathcal{P}C^\infty_{\mathrm{asc};B} \mid \phi^1_H \equiv id \text{ on } T\}. \tag{1.9}$$

Here we would like to emphasize that the support condition on $T \supset o_B$ is imposed only for the time-one map ϕ^1_H, but not for the whole path ϕ_H. This shows the relevance of the following discussion to the weak Hamiltonian topology described above.

In a way similar to $\mathcal{P}C^\infty_{R,K}$ above, we define $\mathcal{P}C^\infty_{R,K;T}$. We define $\mathfrak{Iso}_B(o_N; T^*N)$ to be the subset

$$\mathfrak{Iso}_B(o_N; T^*N) = \bigl\{\phi^1_H(o_N) \bigm| H \in \mathcal{P}C^\infty_{\mathrm{asc};B}\bigr\}.$$

This has the filtration

$$\mathfrak{Iso}_B(o_N; T^*N) = \bigcup_{K\ge 0} \bigcup_{T\supset B} \mathfrak{Iso}^K_T(o_N; T^*N; T),$$

where

$$\mathfrak{Iso}^K_T\bigl(o_N; D^R(T^*N)\bigr) = \bigl\{\phi^1_H(o_N) \bigm| H \in \mathcal{P}C^\infty_{R,K;T}\bigr\}.$$

DEFINITION 1.3

Equip with $\mathfrak{Iso}^K_T(o_N; T^*N)$ the subspace topology of the Hamiltonian C^0-topology of $\mathfrak{Iso}(o_N; T^*N)$. We then put on $\mathfrak{Iso}_B(o_N; T^*N)$ the direct limit topology of $\mathfrak{Iso}^K_T(o_N; T^*N)$ over $T \supset o_B$ and $K \ge 0$. We call this topology the Hamiltonian C^0-topology of $\mathfrak{Iso}_B(o_N; T^*N)$.

Unraveling the definition, we can rephrase the meaning of the convergence $L_i \to L$ in $\mathfrak{Iso}_B(o_N; T^*N)$ into the existence of $R, K > 0$, $T \supset o_B$, and a sequence H_i such that $L_i = \phi^1_{H_i}(L)$ and

(1) $\|H_i\| \le K$ for all i,
(2) $\operatorname{supp} X_{H_i} \subset D^R(T^*N) \setminus o_B$ for all i,
(3) $\phi^1_{H_i} \equiv id$ on T for all i,
(4) $d^{\mathrm{Ham}}_{C^0}(L_i, L) \to 0$ as $i \to \infty$.

REMARK 1.4 (1) We refer to the proof of Lemma 3.6 and Remark 3.7 for the reason for using support hypotheses (2) and (3) imposed in our definition of the Hamiltonian C^0-topology of $\mathfrak{Iso}_B(o_N; T^*N)$. This topology may be regarded as the Lagrangian analogue to the above-mentioned weak Hamiltonian topology and seems to be the weakest possible topology with respect to which one can prove the C^0-continuity of spectral capacity γ^{lag} which is stated in Main Theorem below.

(2) In the Lagrangianization $\operatorname{Graph}\phi_F^1$ of Hamiltonian $F:[0,1]\times M\to\mathbb{R}$ with $\operatorname{supp}F\subset M\setminus B$, there exists an open neighborhood $U\supset B$ such that $\operatorname{supp}F\supset M\setminus U$. Therefore, provided the C^0-distance of $\overline{d}(\phi_F,id)=:\epsilon$ is so small that its graph is contained in a Weinstein neighborhood of the diagonal, such a graph will automatically satisfy

$$\phi_{\mathbb{F}}^1(o_{\Delta_B})\subset T_\epsilon,\qquad \mathbb{F}(t,\mathbf{x}):=F(t,x),\quad \mathbf{x}=(x,y),$$

where T_ϵ is the ϵ-neighborhood of o_{Δ_B} in $T^*\Delta$ and, hence, is automatically contained in $\mathfrak{Iso}_{o_{\Delta_B}}(o_\Delta,T^*\Delta)$.

1.3. Statement of main results

By considering the moduli space of solutions of the perturbed Cauchy–Riemann equation

$$\begin{cases}\frac{\partial u}{\partial\tau}+J\big(\frac{\partial u}{\partial t}-X_H(u)\big)=0,\\ u(\tau,0),u(\tau,1)\in o_N,\end{cases}\tag{1.10}$$

and applying a chain-level Floer mini-max theory to the classical action functional

$$\mathcal{A}_H^{\text{cl}}(\gamma)=\int\gamma^*\theta-\int_0^1 H\big(t,\gamma(t)\big)\,dt$$

on the set

$$\mathcal{P}(T^*N;o_N)=\big\{\gamma:[0,1]\to T^*N\bigm|\gamma(0),\gamma(1)\in o_N\big\},$$

the author [9] defined a homologically essential critical value, denoted by $\rho(H;a)$, associated to each cohomology class $a\in H^*(N)$.

REMARK 1.5

A similar construction using the generating function method was earlier given by Viterbo [20], and it is shown in [5] and [6] that both invariants coincide *modulo a normalization constant.* Indeed, it may be worthwhile to mention that for the construction of Viterbo's [20] invariant $c(u,L)$ for the cohomology class u, this normalization should be fixed uniformly over $u\in H^*(X)$. This was implicitly done in [20] by assuming the associated Lagrangian submanifold $L\subset T^*X$ coincides with the zero section of T^*X on its nonempty open subset as in the condition required above in Definition 1.3. All the main applications of these invariants to symplectic topology carried out in [20] occur on $\mathbb{R}^{2n}$ through the one-point compactification $S^{2n}=\mathbb{R}^{2n}\cup\{\infty\}$ of *compactly supported Hamiltonian flows on* $\mathbb{R}^{2n}$ and Lagrangianization of the graphs of such Hamiltonian flows. Such a graph then as a Lagrangian submanifold of T^*S^{2n} automatically satisfies the above-mentioned hypothesis.

The number $\rho(H;a)$ depends on H, not just on $L_H=\phi_H^1(o_N)$.

We are particularly interested in two spectral invariants $\rho^{\mathrm{lag}}(F;1)$, $\rho^{\mathrm{lag}}(F;[pt]^\#)$ and their difference $\rho^{\mathrm{lag}}(F;1)-\rho^{\mathrm{lag}}(F;[pt]^\#)$. This difference does not depend on the choice of normalization mentioned above. Therefore, we can define a function

$$\gamma^{\mathrm{lag}}:\mathfrak{Iso}(o_N;T^*N)\to\mathbb{R}$$

unambiguously by setting

$$\gamma^{\mathrm{lag}}(L;o_N):=\rho^{\mathrm{lag}}(F;1)-\rho\big(F;[pt]^\#\big) \tag{1.11}$$

for $L=\phi_F^1(o_N)$. We call this function the *spectral capacity* of L (relative to the zero section o_N; see [20], [9]).

We denote by γ_B^{lag} the restriction of γ^{lag} to the subset $\mathfrak{Iso}_B(o_N;T^*N)$. The following Hamiltonian C^0-continuity result is the Lagrangian analogue to Seyfaddini's result [18, Corollary 1.2].

MAIN THEOREM (THEOREM 3.9)

Let N be a closed manifold. Then the function γ_B^{lag} is continuous on $\mathfrak{Iso}_B(o_N;T^*N)$ with respect to the Hamiltonian C^0-topology defined above.

The following is a very interesting open question on the Hamiltonian C^0-topology.

QUESTION 1.6

Is the full function $\gamma^{\mathrm{lag}}:\mathfrak{Iso}(o_N;T^*N)\to\mathbb{R}$ continuous (without restricting to $\mathfrak{Iso}_B(o_N;T^*N)$ with B having nonempty interior)?

The question seems to be an important matter to understand in C^0-symplectic topology. Indeed, the affirmative answer to the question is a key ingredient in relation to Viterbo's [21] symplectic homogenization program. The question is sometimes called *Viterbo's conjecture*. We refer to Theorem 3.9 for a more precise statement on the relationship between the Hamiltonian C^0-distance $d_{C^0}^{\mathrm{Ham}}$ and the spectral capacity $\gamma_B^{\mathrm{lag}}(\phi_F^1(o_N))$ and support conditions (2) and (3) of the Hamiltonian path ϕ_F given in Definition 1.3.

The research performed in this article is partially motivated by the study of topological Hamiltonian dynamics and its applications to the problem of simpleness on the area-preserving homeomorphism group of the 2-disk. We anticipate that these studies will play some important role in the study of the homotopy invariance of the Hamiltonian spectral invariant function $\phi_F\mapsto\rho(F;a)$ for a topological Hamiltonian path ϕ_F in the sense of [14] and [12] on any closed symplectic manifolds (M,ω). It should also be regarded as a natural continuation of the author's study of Lagrangian spectral invariants performed in [8] and [9].

The content of the current article is extracted from the preprint [13], which has been circulated since the year 2012 under the title "Geometry of generating functions and Lagrangian spectral invariants."

Notation and conventions

We follow the conventions of [11] and [12] for the definitions of Hamiltonian vector fields, action functional, and others appearing in the Hamiltonian Floer theory and in the construction of spectral invariants on a general closed symplectic manifold. They are different from, for example, those used in [16] and [1] one way or the other, but coincide with those used in [18].

(1) We usually use the letter M to denote a symplectic manifold and N to denote a general smooth manifold.

(2) The Hamiltonian vector field X_H is defined by $dH = \omega(X_H, \cdot)$.

(3) The flow of X_H is denoted by $\phi_H : t \mapsto \phi_H^t$, and its time-one map is denoted by $\phi_H^1 \in \mathrm{Ham}(M, \omega)$.

(4) We denote by $z_H^q(t) = \phi_H^t(q)$ the Hamiltonian trajectory associated to the initial point q.

(5) We denote by $z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x))$ the Hamiltonian trajectory associated to the final point x.

(6) $\overline{H}(t,x) = -H(t, \phi_H^t(x))$ is the Hamiltonian generating the inverse path $(\phi_H^t)^{-1}$.

(7) The canonical symplectic form on the cotangent bundle T^*N is denoted by $\omega_0 = -d\theta$, where θ is the Liouville one-form which is given by $\theta = \sum_i p_i \, dq^i$ in the canonical coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$.

(8) The classical Hamilton's action functional on the space of paths in T^*N is given by

$$\mathcal{A}_H^{\mathrm{cl}}(\gamma) = \int \gamma^*\theta - \int_0^1 H\big(t, \gamma(t)\big)\, dt.$$

(9) We denote by o_N the zero section of T^*N.

(10) We denote by $\rho^{\mathrm{lag}}(H; a)$ the Lagrangian spectral invariant on T^*N (relative to the zero section o_N) defined in [8] for an asymptotically constant Hamiltonian H on T^*N.

(11) We denote by f_H the basic phase function and denote its associated Lagrangian selector by $\sigma_H : N \to T^*N$, which is given by $\sigma_H(q) = df_H(q)$ at which $df_H(q)$ exists.

2. Lagrangian Floer homology and Lagrangian spectral invariants

In this section, we first briefly recall the construction of Lagrangian spectral invariants $\rho^{\mathrm{lag}}(H; a)$ for $L_H = \phi_H^1(o_N)$ performed by the author [9]. A priori, this invariant may depend on H, not just on L_H itself. The author [9] proves that

$$\rho^{\mathrm{lag}}(H; a) = \rho^{\mathrm{lag}}(F; a) \tag{2.1}$$

for all $a \in H^*(N; \mathbb{Z})$ if $L_H = L_F$, but modulo the addition of a constant and then by using a somewhat ad hoc normalization to remove this ambiguity of the constant.

2.1. Definition of Lagrangian spectral invariants

For any given time-dependent Hamiltonian $H = H(t,x)$, the classical action functional on the space

$$\mathcal{P}(T^*N) := C^\infty\big([0,1], T^*N\big)$$

is defined by

$$\mathcal{A}_H^{\mathrm{cl}}(\gamma) = \int \gamma^*\theta - \int_0^1 H\big(t,\gamma(t)\big)\,dt.$$

We define the subset $\mathcal{P}(T^*N; o_N)$ by

$$\mathcal{P}(T^*N; o_N) = \big\{\gamma : [0,1] \to T^*N \;\big|\; \gamma(0) \in o_N\big\}.$$

The assignment $\gamma \mapsto \pi(\gamma(1))$ defines a fibration

$$\mathcal{P}(T^*N; o_N) \to o_N \cong N$$

with fiber at $q \in N$ given by

$$\mathcal{P}(T^*N; o_N, T_q^*N) := \big\{\gamma : [0,1] \to T^*N \;\big|\; \gamma(0) \in o_N, \gamma(1) \in T_q^*N\big\}.$$

For given $x \in L_H$, we denote the Hamiltonian trajectory

$$z_x^H(t) = \phi_H^t\big((\phi_H^1)^{-1}(x)\big),$$

which is a Hamiltonian trajectory such that, by definition,

$$z_x^H(0) \in o_N, \qquad z_x^H(1) = x. \tag{2.2}$$

We denote $L_H = \phi_H^1(o_N)$ and denote by $i_H : L_H \hookrightarrow T^*N$ the inclusion map.

Motivated by Weinstein's observation that the action functional

$$\mathcal{A}_H^{\mathrm{cl}} : \mathcal{P}(T^*N; o_N) \to \mathbb{R}$$

can be interpreted as the canonical generating function of L_H, the author in [8] and [9] constructed a family of spectral invariants of L_H by performing a mini-max theory via the chain-level Floer homology theory. Indeed, the function defined by

$$h_H(x) = \mathcal{A}_H^{\mathrm{cl}}(z_x^H) \tag{2.3}$$

is a canonical generating function of L_H in that

$$i_H^*\theta = dh_H. \tag{2.4}$$

We call h_H the *basic generating function* of L_H. As a function on N not on L_H, it is a multivalued function. Similarly, one may regard $N \to \phi_H^1(o_N)$ as a multivalued section of T^*N.

Consider the zero section o_N and the space

$$\mathcal{P}(o_N, o_N) = \big\{\gamma : [0,1] \to T^*N \;\big|\; \gamma(0), \gamma(1) \in o_N\big\}.$$

The set of generators of $CF(H; o_N, o_N)$ is that of solutions

$$\dot z = X_H\big(t, z(t)\big), \quad z(0), z(1) \in o_N,$$

and its Floer differential is defined by counting the number of solutions of

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0, \\ u(\tau,0), u(\tau,1) \in o_N. \end{cases} \tag{2.5}$$

An element $\alpha \in CF(H; o_N, o_N)$ is expressed as a finite sum

$$\alpha = \sum_{z \in \mathcal{C}\mathrm{hord}(H; o_N, o_N)} a_z[z], \quad a_z \in \mathbb{Z}.$$

We define the level of the chain α by

$$\lambda_H(\alpha) := \max_{z \in \operatorname{supp} \alpha} \{\mathcal{A}_H^{\mathrm{cl}}(z)\}. \tag{2.6}$$

For a given nonzero cohomology class $a \in H^*(N,\mathbb{Z})$, we consider its Poincaré dual $[a]^\flat := PD(a) \in H_*(N,\mathbb{Z})$ and its image under the canonical isomorphism (see [2], [3], [8] for the construction of this isomorphism)

$$\Phi : H_*(N,\mathbb{Z}) \to HF_*(H, J; o_N, o_N).$$

DEFINITION 2.1

Let (H,J) be a Floer regular pair relative to (o_N, o_N), and let $(CF(H), \partial_{(H,J)})$ be its associated Floer complex. For any $0 \neq a \in H^*(N,\mathbb{Z})$, we define

$$\rho^{\mathrm{lag}}(H;a) = \inf_{\alpha \in \Phi(a^\flat)} \{\lambda_H(\alpha)\}. \tag{2.7}$$

One important result is the following basic property, called *spectrality* in [11], which is not explicitly stated in [8] but can be easily derived by a compactness argument (see the proof in [11] given in the Hamiltonian context).

PROPOSITION 2.2

Let $H = H(t,x)$ be any (not necessarily nondegenerate) smooth Hamiltonian. Then for any $0 \neq a \in H^*(N,\mathbb{Z})$, there exists a point $x \in L_H \cap o_N$ such that

$$\mathcal{A}_H^{\mathrm{cl}}(z_x^H) = \rho^{\mathrm{lag}}(H;a).$$

In particular, $\rho^{\mathrm{lag}}(H;a) \in \mathrm{Spec}(H;N)$.

2.2. Triangle inequality for Lagrangian spectral invariants

We recall from [17] and [11] that the triangle inequality of the Hamiltonian spectral invariants

$$\rho^{\mathrm{Ham}}(H\#F; a\cdot b) \le \rho^{\mathrm{Ham}}(H;a) + \rho^{\mathrm{Ham}}(F;b)$$

for the product Hamiltonian $H\#F$ relies on the homotopy invariance property of spectral invariants, which in turn relies on the existence of the canonical normalization procedure of Hamiltonians on closed (M,ω), which is nothing but the mean normalization. On the other hand, one can directly prove

$$\rho^{\mathrm{Ham}}(H * F; a\cdot b) \le \rho^{\mathrm{Ham}}(H;a) + \rho^{\mathrm{Ham}}(F;b)$$

more easily for the concatenated Hamiltonian (see, e.g., [4] for the proof). Once we have the latter inequality, we can derive the former from the latter again by the homotopy invariance property of $\rho^{\mathrm{Ham}}(\cdot;a)$ for the *mean-normalized Hamiltonians.*

When one attempts to assign an invariant of the Lagrangian submanifold $\phi_H^1(o_N)$ itself out of the spectral invariant $\rho^{\mathrm{lag}}(H;a)$, one has to choose a normalization of the Hamiltonian relative to the Lagrangian submanifold. Since there is no canonical normalization, unlike the Hamiltonian case, the invariance property of Lagrangian spectral invariants, and thus the triangle inequality, is somewhat more nontrivial than the case of Hamiltonian spectral invariants. In this section, we clarify these issues of the invariance property and of the triangle inequality.

The following parameterization independence follows immediately from the construction of Lagrangian spectral invariants and the $L^{(1,\infty)}$-continuity of $H \mapsto \rho^{\mathrm{lag}}(H;a)$.

LEMMA 2.3

Let $H = H(t,x)$ be any, not necessarily nondegenerate, smooth Hamiltonian, and let $\chi : [0,1] \to [0,1]$ be a reparameterization function with $\chi(0) = 0$ and $\chi(1) = 1$. Then

$$\rho^{\mathrm{lag}}(H;a) = \rho^{\mathrm{lag}}(H^\chi;a),$$

where $H^\chi(t,x) = \chi'(t)H(\chi(t),x)$.

We first recall the following triangle inequality, which was essentially proved in [9, Theorem 6.4 and Lemma 6.5]. In [9], the cohomological version of the Floer complex was considered, and hence, the opposite inequality is stated. Other than that, the same proof can be applied here.

PROPOSITION 2.4

Let $H, F \in \mathcal{PC}^\infty_{\mathrm{asc}}(T^*N;\mathbb{R})$, and assume that F is autonomous. Then we have

$$\rho^{\mathrm{lag}}(H\#F;ab) \le \rho^{\mathrm{lag}}(H;a) + \rho^{\mathrm{lag}}(F;b). \tag{2.8}$$

Monzner, Vichery, and Zapolsky [7] proved the following form of the triangle inequality, which uses the concatenated Hamiltonian $H * F$ instead of the product Hamiltonian $H\#F$.

PROPOSITION 2.5 ([7, PROPOSITION 2.4])

Let H, K be compactly supported. Suppose that $H(1,x) \equiv F(0,x)$, and let $H * F$ be the concatenated Hamiltonian. Then

$$\rho^{\mathrm{lag}}(H * F;ab) \le \rho^{\mathrm{lag}}(H;a) + \rho^{\mathrm{lag}}(F;b) \tag{2.9}$$

for all $a, b \in H^*(N)$.

In particular, this proposition applies to all pairs (H, F) which are compactly supported and boundary flat.

REMARK 2.6
We suspect that (2.8) holds even for the nonautonomous F as in the Hamiltonian case, but we did not check this, since it is not needed in the present article.

2.3. Assigning spectral invariants to Lagrangian submanifolds

In this section, we identify a class, denoted by $\mathcal{PC}^\infty_{(B;e)}$, of Hamiltonians H among those satisfying $\phi_H^1(o_N) = \phi_F^1(o_N)$ such that the equality

$$\rho^{\mathrm{lag}}(H;a) = \rho^{\mathrm{lag}}(F;a)$$

holds for all $H, F \in \mathcal{PC}^\infty_{\mathrm{asc};B}$. As the notation suggests, the class depends on the subset $B \subset N$.

We start with the following proposition. The proof closely follows that of [7, Lemma 2.6], which uses Proposition 2.5 in a significant way. We need to modify their proof to obtain a somewhat stronger statement, which replaces the condition "$\phi_H^1 = \phi_F^1$" used in [7] by the conditions put in this proposition.

PROPOSITION 2.7 (CF. [7, LEMMA 2.6])
Let $H, F \in \mathcal{PC}^\infty_{\mathrm{asc}}(T^*N;\mathbb{R})$ be boundary flat. Suppose in addition that H, F satisfy the following:

(1) $\phi_H^1(o_N) = \phi_F^1(o_N)$,
(2) $H \equiv c(t)$, $F \equiv d(t)$ *on a tubular neighborhood $T \supset B$ in T^*N of a closed ball $B \subset o_N$, where $c(t)$, $d(t)$ are independent of $x \in T$, and*
(3) *they satisfy*

$$\int_0^1 c(t)\,dt = \int_0^1 d(t)\,dt.$$

Then $\rho^{\mathrm{lag}}(H;a) = \rho^{\mathrm{lag}}(F;a)$ holds for all $a \in H^*(N,\mathbb{Z})$ without the ambiguity of a constant.

Proof
We consider the Hamiltonian path $\phi_G : t \mapsto \phi_G^t$ with $G = \widetilde{F} * H$ with $\widetilde{F}(t,x) = -F(1-t,x)$. This defines a loop of a Lagrangian submanifold

$$t \mapsto \phi_G^t(o_N), \quad \phi_G^1(o_N) = o_N$$

and satisfies $\phi_G^t|_B \equiv id$ and

$$G(t,q) = \begin{cases} -c(1-2t) & 0 \le t \le 1/2, \\ d(2t-1) & 1/2 \le t \le 1, \end{cases}$$

for all $q \in B \subset T$ by the definition $G = \widetilde{F} * H$.

We claim that $\rho^{\mathrm{lag}}(G;a) = 0$ for all $0 \neq a \in H^*(N)$. This will be an immediate consequence of the following lemma and the spectrality of numbers $\rho^{\mathrm{lag}}(G;a)$.

LEMMA 2.8

The value $\mathcal{A}^{\mathrm{cl}}_G(z)$ does not depend on the Hamiltonian chord $z \in \mathcal{C}\mathrm{hord}(G; o_N, o_N)$. In particular, $\mathcal{A}^{\mathrm{cl}}_G(z) = 0$.

Proof

Recall that any Hamiltonian chord in $\mathcal{C}\mathrm{hord}(G; o_N, o_N)$ has the form

$$z(t) = z^q_G(t)$$

for some $q \in o_N$. Here we use the hypothesis $\phi^1_G(o_N) = o_N$. Consider any smooth path $\alpha : [0,1] \to o_N$ with $\alpha(0) = q, \alpha(1) = q'$. Then

$$\mathcal{A}^{\mathrm{cl}}_G(z^{q'}_G) - \mathcal{A}^{\mathrm{cl}}_G(z^q_G) = \int_0^1 \frac{d}{du} \mathcal{A}^{\mathrm{cl}}_G(z^{\alpha(u)}_G)\, du.$$

Recall the definition of the classical action functional

$$\mathcal{A}^{\mathrm{cl}}_G(\gamma) = \int \gamma^*\theta - \int_0^1 G\big(t, \gamma(t)\big)\, dt$$

on the space $\mathcal{P}(T^*N)$ of paths $\gamma : [0,1] \to T^*N$, and recall its first variation formula

$$\begin{aligned} d\mathcal{A}_G(\gamma)(\xi) = &\int_0^1 \omega\big(\dot\gamma - X_G\big(t, \gamma(t)\big), \xi(t)\big)\, dt \\ &- \big\langle \theta\big(\gamma(0)\big), \xi(0)\big\rangle + \big\langle \theta\big(\gamma(1)\big), \xi(1)\big\rangle. \end{aligned} \tag{2.10}$$

Using this, a straightforward calculation shows that

$$\frac{d}{du} \mathcal{A}^{\mathrm{cl}}_G(z^{\alpha(u)}_G) = \Big\langle \theta, \frac{\partial}{\partial u}\big(\phi_G\big(\alpha(u)\big)\big)\Big\rangle - \Big\langle \theta, \frac{\partial}{\partial u}\big(\alpha(u)\big)\Big\rangle = 0 - 0 = 0,$$

since $\phi_G(\alpha(u)), \alpha(u) \in o_N$.

For the second statement, we only have to consider the constant path $z \equiv c_q \in B$ for which

$$\begin{aligned} \mathcal{A}^{\mathrm{cl}}_G(c_q) &= -\int_0^1 G(t,q)\, dt = \int_0^{1/2} c(1-2t)\, dt - \int_{1/2}^1 d(2t-1)\, dt \\ &= \int_0^1 c(t)\, dt - \int_0^1 d(t)\, dt = 0. \end{aligned}$$

This proves the lemma. □

Once we have the lemma, we can apply the triangle inequality (2.9)

$$\rho^{\mathrm{lag}}(H; a) \le \rho^{\mathrm{lag}}(F; a) + \rho^{\mathrm{lag}}(G; 1) = \rho^{\mathrm{lag}}(F; a)$$

for any given $a \in H^*(N)$. By changing the role of H and F in the proof of the above lemma, we also obtain $\rho^{\mathrm{lag}}(\widetilde{G}; 1) = 0$ and then obtain $\rho^{\mathrm{lag}}(F; a) \le \rho^{\mathrm{lag}}(H; a)$ by the triangle inequality. This finishes the proof of the proposition. □

This proposition motivates us to introduce the following definitions.

DEFINITION 2.9

For each given $B \subset N$, we define

$$\mathfrak{Iso}_B(o_N;T^*N) = \big\{L \in \mathfrak{Iso}(o_N;T^*N) \mid o_N \cap L \supset o_B\big\}.$$

When a function $c:[0,1] \to \mathbb{R}$ is given in addition, we define

$$\mathcal{PC}^\infty_{(B;e)} = \Big\{H \in \mathcal{PC}^\infty_{\mathrm{asc}} \,\Big|\, H_t \equiv c(t) \text{ on a neighborhood of } o_B \text{ in } T^*N \\ \text{and } \int_0^1 c(t)\,dt = e\Big\}.$$

With these definitions, the proposition enables us to unambiguously define the following spectral invariant attached to L.

DEFINITION 2.10

Suppose $L \in \mathfrak{Iso}_B(o_N;T^*N)$, and let $e \in \mathbb{R}$ be given. For each such e, we define a spectral invariant of $L \in \mathfrak{Iso}_{(B;e)}(o_N;T^*N)$ by

$$\rho^{(B;e)}(L;a) := \rho^{\mathrm{lag}}(H;a), \quad L = \phi^1_H(o_N)$$

for any $H \in \mathcal{PC}^\infty_{(B;e)}$.

With this definition, we have the following obvious lemma.

LEMMA 2.11

Let $H \in \mathcal{PC}^\infty_{(B;e)}$. *Then* $\widetilde{H}, \overline{H} \in \mathcal{PC}^\infty_{(B;-e)}$.

Then we prove the following duality statement of $\rho^{(B;e)}$.

PROPOSITION 2.12

Let $H \in \mathcal{PC}^\infty_{(B;e)}$, *and let* $L = \phi^1_H(o_N)$. *We denote* $\widetilde{L} = \phi^1_{\widetilde{H}}(o_N) = \phi^1_{\overline{H}}(o_N)$. *Then*

$$\rho^{(B;-e)}(\widetilde{L};1) = -\rho^{(B;e)}\big(L;[pt]^\#\big). \tag{2.11}$$

Proof

By the above lemma, $\widetilde{H} \in \mathcal{PC}^\infty_{(B;-e)}$ and so $\rho^{(B;-e)}(\widetilde{L};1)$ is given by

$$\rho^{(B;-e)}(\widetilde{L};1) = \rho^{\mathrm{lag}}(\widetilde{H};1)$$

by definition. But it was proven in [20], [8], and [9] that

$$\rho^{\mathrm{lag}}(\widetilde{H};1) = -\rho^{\mathrm{lag}}\big(H;[pt]^\#\big), \tag{2.12}$$

which follows from the Poincaré duality argument, by studying the time-reversal flow $\widetilde{u}$ of the Floer equation (2.5) which is defined by $\widetilde{u}(\tau,t) = u(-\tau,1-t)$. The map $\widetilde{u}$ satisfies the equation

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial \tau} + \widetilde{J}\big(\frac{\partial \widetilde{u}}{\partial t} - X_{\widetilde{H}}(\widetilde{u})\big) = 0, \\ \widetilde{u}(\tau,0), \widetilde{u}(\tau,1) \in o_N. \end{cases}$$

Furthermore, this equation is compatible with the involution of the path space

$$\iota : \Omega(o_N, o_N) \to \Omega(o_N, o_N)$$

defined by $\iota(\gamma)(t) = \widetilde{\gamma}(t)$ with $\widetilde{\gamma}(t) = \gamma(1-t)$ and the action functional identity

$$\mathcal{A}^{\rm cl}_{\widetilde{H}}(\widetilde{\gamma}) = -\mathcal{A}^{\rm cl}_H(\gamma).$$

We refer to [9] for the details of the duality argument in the Floer theory used in the derivation of (2.12).

On the other hand, by definition,

$$\rho^{\rm lag}\big(H; [pt]^\#\big) = \rho^{(B;e)}\big(L; [pt]^\#\big)$$

since $H \in \mathcal{P}C^\infty_{(B;e)}$. This finishes the proof. $\square$

The following special case with $e = 0$ is worthwhile to separately mention in that it alerts readers to the importance of the normalization condition $H \equiv 0$ on a neighborhood of B in T^*N in the construction of spectral invariants if one would like to make this kind of duality statement hold in terms of Lagrangian submanifolds $\phi^1_H(o_N)$ rather than in terms of the associated Hamiltonian H.

COROLLARY 2.13

Let L and $\widetilde{L}$ be as in Proposition 2.12. Consider the spectral invariants $\rho^{\rm lag}_B(L;\cdot) := \rho^{(B;0)}(L;\cdot)$. Then

$$\rho^{\rm lag}_B(\widetilde{L}; 1) = -\rho^{\rm lag}_B\big(L; [pt]^\#\big).$$

3. A Hamiltonian C^0-continuity of spectral Lagrangian capacity

We first recall the definition of the function $\gamma^{\rm lag}_B : \mathfrak{Iso}_B(o_N; T^*N) \to \mathbb{R}$ defined by

$$\gamma^{\rm lag}_B(L) = \rho^{\rm lag}_B(H; 1) - \rho^{\rm lag}_B\big(H; [pt]^\#\big)$$

for $L = \phi^1_H(o_N)$ with $H \in \mathcal{P}C^\infty_{{\rm asc};B}$.

In this section, we prove the following Hamiltonian C^0-continuity result of the function which is the Lagrangian analogue to [18, Theorem 1].

THEOREM 3.1

The function $\gamma^{\rm lag}_B : \mathfrak{Iso}_B(o_N; T^*N) \to \mathbb{R}$ is continuous with respect to the Hamiltonian C^0-topology in the sense of Definition 1.2.

The triangle inequality of $\gamma^{\rm lag}$ stated in Section 2.2 implies the inequalities

$$\big|\gamma^{\rm lag}_B(L_1) - \gamma^{\rm lag}_B(L_2)\big| \le \max\big\{\gamma^{\rm lag}_B\big(\phi^{-1}_{H^2}\big(\phi_{H^1}(o_N)\big)\big), \gamma^{\rm lag}_B\big(\phi^{-1}_{H^1}\big(\phi_{H^2}(o_N)\big)\big)\big\}.$$

We also note that for $L_k = \phi^1_{H^k}(o_N) \in \mathfrak{Iso}_B(o_N; T^*N)$ for $k = 1, 2$

$$\max\big\{d_{C^0}(\phi^{-1}_{H^2}\phi_{H^1}|_{o_N}, i_{o_N}), d_{C^0}(\phi^{-1}_{H^1}\phi_{H^2}|_{o_N}, i_{o_N})\big\} \to 0$$

for the inclusion map $i_{o_N} : o_N \to T^*N$ if and only if

$$\max\big\{d_{C^0}(\phi_{H^1}|_{o_N}, \phi_{H^2}|_{o_N}), d_{C^0}(\phi^{-1}_{H^1}|_{o_N}; \phi^{-1}_{H^2}|_{o_N})\big\} \to 0,$$

provided we assume that $\operatorname{supp}\phi_{H^k}$ is compact and so $\operatorname{supp}\phi_{H^k} \subset D^R(T^*N)\setminus T$, $k=1,2$, for some $R>0$ and $T\supset B$. The latter assumption is already embedded in the definition of Hamiltonian topology given in Definition 1.2. Therefore, to prove the above theorem, it is enough to prove the continuity of γ_B^{lag} at the zero section o_N in $\mathfrak{Iso}_B(o_N;T^*N)$.

By unravelling the definition of the Hamiltonian C^0-topology on $\mathfrak{Iso}_B(o_N;T^*N)$ given in Definition 1.2, we now rephrase the continuity statement at the zero section o_N more explicitly. For this purpose, we introduce the notation

$$\operatorname{osc}_{C^0}(\phi_H^1;o_N) := \max\Big\{\max_{x\in o_N} d\big(\phi_H^1(x),x\big), \max_{x\in o_N} d\big((\phi_H^1)^{-1}(x),x\big)\Big\}.$$

Then it is easy to see that this continuity at o_N is equivalent to the following.

THEOREM 3.2

Let $\lambda_i=\phi_{H_i}$ where $H_i\in\mathcal{PC}^\infty_{\mathrm{asc}}$ is a sequence such that

(1) *$H_i\in\mathcal{PC}^\infty_{R,K}$ for some $R,K>0$ for all i and $s\in[0,1]$;*

(2) *there exists a closed ball $B\subset N$ such that $\phi_{H_i}^t\equiv id$ on B for all $t\in[0,1]$ and for all i;*

(3) *there exists a uniform neighborhood $T\supset o_B$ in T^*N such that $\phi_{H_i}^1\equiv id$ on T for all i;*

(4) *$\lim_{i\to\infty}\operatorname{osc}_{C^0}(\phi_{H_i}^1;o_N)=0$.*

Then

$$\lim_{i\to\infty}\big(\rho^{\mathrm{lag}}(H_i;1)-\rho^{\mathrm{lag}}\big(H_i;[pt]^\#\big)\big)=0.$$

The proof of this theorem is an adaptation to the Lagrangian context of the one used by Seyfaddini in his proof of [18, Theorem 1 (or rather Corollary 1.3)]. The proof is also a variation of Ostrover's [15] scheme and is an adaptation thereof. In our proof, however, we use the Lagrangian analogue to the notion of "ε-shiftability" introduced by Seyfaddini [18], instead of the notion of "displaceability" used in [15] and in other literature such as [1] and [19]. In the Lagrangian context here, the ε-shiftable domain is realized as the graph of df of a function f having no critical points on the corresponding domain. In this regard, it appears to the author that the notion of ε-shiftability becomes more geometric and intuitive in the Lagrangian context than in the Hamiltonian context.

3.1. ε-Shifting of the zero section by the differential of the function

Fix a Riemannian metric g and the Levi-Civita connection on N. They naturally induce a metric via the splitting $T(T^*N)\cong TN\oplus T^*N$, sometimes called a *Sasakian metric*, on T^*N. Denote the latter metric on T^*N by $\widetilde{g}$, and denote the corresponding distance function by $\widetilde{d}(x,y)$ for $x,y\in T^*N$. We denote by $D^r(T^*N)$ the disk bundle of T^*N of radius r.

The following are well-known facts on this metric $\widetilde{g}$, which can be easily checked, so we omit the proofs.

LEMMA 3.3

The metric $\widetilde{g}$ carries the following properties.

(1) *$\widetilde{g}$ is invariant under the reflection $(q,p) \mapsto (q,-p)$, and in particular, o_N is totally geodesic.*

(2) *There exists a sufficiently small $r = r(N,g) > 0$ depending only on (N,g) such that*

(a) *for all $d(q,q') < r$, $\widetilde{d}(o_q, o_{q'}) = d(q,q')$,*

(b) *for all $x \in D^r(T^*N)$, which we denote $x = (q(x), p(x))$,*

$$d(o_{q(x)}, x) \geq \max\{|p(x)|, d(q, q(x))\} \geq |p(x)|, \tag{3.1}$$

where $|p(x)|$ is the norm on $T^*_{q(x)}N$.

From now on, we will drop the tilde from $\widetilde{d}$ and just denote it by d even for the distance function of $\widetilde{g}$ on T^*N, which should not confuse readers.

Consider the subset

$$C^\infty_{\mathrm{crit}}(N;B) = \{f \in C^\infty(N) \mid \mathrm{Crit}\, f \subset \mathrm{Int}\, B\}.$$

The set $C^\infty_{\mathrm{crit}}(N;B) \subset C^\infty(N)$ has the filtration

$$C^\infty_{\mathrm{crit}}(N;B) = \bigcup_T C^\infty_{\mathrm{crit}}(N;B,T),$$

where $C^\infty_{\mathrm{crit}}(N;B,T)$ is the subset of $C^\infty_{\mathrm{crit}}(N;B)$ that consists of the f's satisfying

$$\mathrm{Graph}(df|_B) \subset T. \tag{3.2}$$

It is easy to check that $C^\infty_{\mathrm{crit}}(N;B,T) \neq \emptyset$ for any such $T \supset o_B$ by considering λf for a sufficiently small $\lambda > 0$ for any given Morse function f with $\mathrm{Crit}\, f \subset \mathrm{Int}\, B$.

We now introduce the collection, denoted by $\mathcal{T}_{(B;r)}$, of the pairs (T,f) consisting of a tubular neighborhood $T \supset o_B$ in T^*N and a Morse function $f \in C^\infty_{\mathrm{crit}}(N;B,T)$ such that

$$\mathrm{Graph}\, df \subset D^r(T^*N) \tag{3.3}$$

for the constant $r = r(N,g)$ given in Lemma 3.3. By the choice of the pair $(T,f) \in \mathcal{T}_{(B;r)}$, we have

$$\min\{\min_{p \in N \setminus B} |df(p)|, d_{\mathrm{H}}(N \setminus B, \mathrm{Crit}\, f)\} > 0,$$

where $d_{\mathrm{H}}(N \setminus B, \mathrm{Crit}\, f)$ is the Hausdorff distance.

DEFINITION 3.4

We define a positive constant

$$C_{(f;B,T)} := \min\{\min_{p \in N \setminus B} |df(p)|, d_{\mathrm{H}}(N \setminus B, \mathrm{Crit}\, f)\}. \tag{3.4}$$

By the definition of $C_{(f;B,T)}$, if $q \in N \setminus B$, we have

$$|df(q)|, d(q, \mathrm{Crit}\, f) \geq C_{(f;B,T)} > 0. \tag{3.5}$$

LEMMA 3.5

For any $f \in C^\infty_{\mathrm{crit}}(N;B,T)$,

$$C_{(\delta f;B,T)} = \min_{p\in N\setminus B}\bigl|d(\delta f)(p)\bigr|$$

whenever $\delta > 0$ *is so small that*

$$\min_{p\in N\setminus B}\bigl|d(\delta f)(p)\bigr| < d_H(N\setminus T, B).$$

In particular, for such $\delta > 0$,

$$\lambda C_{(\delta f;B,T)} = C_{(\lambda\delta f;B,T)} \tag{3.6}$$

for any $\lambda \leq 1$.

Proof

First note that the distance $d_{\mathrm{H}}(N\setminus B, \mathrm{Crit}(\delta f))$ does not depend on δ and that

$$\min_{p\in N\setminus B}\bigl|\delta\, df(p)\bigr| = \delta \min_{p\in N\setminus B}\bigl|df(p)\bigr| \to 0$$

as $\delta \to 0$. Therefore, the minimum in the definition

$$C_{(\delta f;B,T)} = \min\bigl\{\min_{p\in N\setminus B}\bigl|d(\delta f)(p)\bigr|, d_{\mathrm{H}}\bigl(N\setminus B, \mathrm{Crit}(\delta f)\bigr)\bigr\}$$

is realized by $\min_{p\in N\setminus B}|d(\delta f)(p)|$ for all sufficiently small $\delta > 0$. Then the lemma follows. □

Now we consider the Hamiltonians H adapted to the triple $(f;B,T)$ as in the definition of the Hamiltonian C^0-topology of $\mathfrak{Iso}_B(o_N;T^*N)$.

LEMMA 3.6

Let $T \supset o_B$ *in* T^*N *and* $H \in \mathcal{P}C^\infty_{\mathrm{asc};B}$ *satisfy*

$$\phi^1_H \equiv id \tag{3.7}$$

on T. *Then we have*

$$L_f \cap o_N = \phi^1_H(L_f) \cap o_N$$

whenever H *satisfies*

$$\mathrm{osc}_{C^0}(\phi^1_H; o_N) < C_{(f;B,T)}. \tag{3.8}$$

In particular, all the Hamiltonian trajectories of $H\#(f\circ\pi)$ *are constants that are equal to* o_p *for some point* $p \in \mathrm{Crit}\, f$ *for such a Hamiltonian* H.

Proof

In the proof, we will denote $p \in N$ and denote the corresponding point in the zero section of T^*N by o_p for notational consistency. Obviously we have $\mathrm{Crit}\, f = L_f \cap o_B \subset \phi^1_H(L_f) \cap o_N$, since we assume that $\phi^1_H \equiv id$ on a neighborhood T of $o_B \supset \mathrm{Crit}\, f$. We will now prove the opposite inclusion $\phi^1_H(L_f) \cap o_N \subset L_f \cap o_B$. Suppose $o_p \in \phi^1_H(L_f) \cap o_N$. Then we have $(\phi^1_H)^{-1}(o_p) \in L_f$.

Consider first the case $p \in B$. In this case since we assume that $\phi_H^1 = id$ on a neighborhood of o_B, it in particular implies $o_p = (\phi_H^1)^{-1}(o_p)$ for all i and, hence, $o_p \in o_B \cap L_f \cong \mathrm{Crit}\, f$.

Now we will show that p cannot lie in $N \setminus B$. Suppose $p \in N \setminus B$ to the contrary, and write

$$(\phi_H^1)^{-1}(o_p) = df(p')$$

for some $p' \in N$. Therefore,

$$d\big(o_p, df(p')\big) = d\big(o_p, (\phi_H^1)^{-1}(o_p)\big) \le \mathrm{osc}_{C^0}(\phi_H^1; o_N).$$

Furthermore, we also have $|df(p')| \le d(o_p, df(p'))$ by Lemma 3.3 since $\mathrm{Graph}\, df \subset D^r(T^*N)$. Therefore, we have shown

$$\big|df(p')\big| \le \mathrm{osc}_{C^0}(\phi_H^1; o_N) < C_{(f;B,T)}. \tag{3.9}$$

This in particular implies that $(\phi_H^1)^{-1}(o_p) = df(p')$ must lie in $\mathrm{Graph}\, df|_B \subset T$, because otherwise $|df(p')| \ge C_{(f;B,T)}$ by the definition of $C_{(f;B,T)}$, which would contradict (3.9).

This in turn implies $(\phi_H^1)^{-1}(o_p) \in T$. But ϕ_H^1 is assumed to be the identity map on T and, hence, follows

$$o_p = (\phi_H^1)^{-1}(o_p) = df(p').$$

In particular, $df(p') \in o_N$, so $p' \in \mathrm{Crit}\, f$ and, hence, $o_{p'} = df(p')$. This implies $p = p'$ and so $d(p, \mathrm{Crit}\, f) = 0$, that is, $p \in \mathrm{Crit}\, f \subset B$, a contradiction to the hypothesis $p \in N \setminus B$. Therefore, p cannot lie in $N \setminus B$, and this proves $o_p \in o_B \cap L_f \cong \mathrm{Crit}\, f$ for any $o_p \in \phi_H^1(L_f) \cap o_N$. This then finishes the proof of the first statement

$$L_f \cap o_N = \phi_H^1(L_f) \cap o_N. \tag{3.10}$$

To prove the second statement, the first statement of the lemma implies that all the Hamiltonian trajectories of $H \# f \circ \pi$ ending at a point in $\phi_H^1(L_f) \cap o_N$ have the form

$$z_p^{H\#f\circ\pi}(t) = \phi_{H\#f\circ\pi}^t\big((\phi_{H\#f\circ\pi}^1)^{-1}(o_p)\big)$$

for some intersection point $o_p \in \phi_H^1(L_f) \cap o_N = L_f \cap o_N$. By definition, we have $z_p^{H\#f\circ\pi}(1) = o_p$.

But we also have $df(p) = 0$ and $(\phi_H^1)^{-1}(o_p) = o_p$, since

$$o_p \in \phi_H^1(L_f) \cap o_N = L_f \cap o_N \subset o_B \cap \mathrm{Crit}\, f$$

and $\phi_H^1 \equiv id$ near p. Therefore,

$$(\phi_{H\#f\circ\pi}^1)^{-1}(o_p) = (\phi_{f\circ\pi}^1)^{-1}(\phi_H^1)^{-1}(o_p) = o_p.$$

Therefore,

$$\begin{aligned} z_p^{H\#f\circ\pi}(t) &= \phi_{H\#f\circ\pi}^t\big((\phi_{H\#f\circ\pi}^1)^{-1}(o_p)\big) = \phi_{H\#f\circ\pi}^t(o_p) \\ &= \phi_H^t\big(\phi_{f\circ\pi}^t(o_p)\big) = \phi_H^t(o_p) = o_p, \end{aligned}$$

since $df(p) = 0$ and $\phi_H^t(o_p) = o_p$ for all $t \in [0,1]$. The last statement follows since we assume that $\operatorname{supp}\phi_H \cap o_B = \emptyset$. By the compactness of $\operatorname{supp}\phi_H$ and the closeness of B, $\operatorname{supp}\phi_H \cap o_B = \emptyset$ implies $\phi_H^t \equiv id$ for all $t \in [0,1]$ on a neighborhood $T' \supset o_B$ in T^*N. This finishes the proof. □

REMARK 3.7

We would like to mention that, in the above proof, the choice of the neighborhood $T' \supset B$ is allowed to vary depending on the H's. This is because our Hamiltonian C^0-topology requires only $\operatorname{supp}\phi_H^t \cap o_B = \emptyset$ for $t \in [0,1]$, not the existence of a uniform neighborhood $T \supset o_B$ independent of H. It only requires the existence of such a uniform neighborhood for the time-one map ϕ_H^1.

REMARK 3.8

In fact, all the discussion in this section can be generalized by replacing the differential df by any closed one-form α and replacing $\operatorname{Crit} f$ by the zero set of α. But we restrict to the exact case since the discussion in the next section seems to require the exactness of the form.

3.2. Lagrangian capacity versus Hamiltonian C^0-fluctuation

In fact, Theorem 3.2 is an immediate consequence of the following comparison result between the Lagrangian capacity $\gamma_B^{\mathrm{lag}}(L) = \rho^{\mathrm{lag}}(H;1) - \rho^{\mathrm{lag}}(H;[pt]^\#)$ and the Hamiltonian C^0-fluctuation $\operatorname{osc}_{C^0}(\phi_H^1; o_N)$ for $L = \phi_H^1(o_N)$ for $H \in \mathcal{P}C^\infty_{\mathrm{asc};B}$, which itself has some independent interest in its own right.

THEOREM 3.9

Let $B \subset N$ be a closed ball, and let $(T,f) \in \mathcal{T}_{(B;r)}$. Consider the set of Hamiltonians H satisfying $\operatorname{supp}\phi_H \cap o_B = \emptyset$, and assume that

$$\operatorname{osc}_{C^0}(\phi_H^1; o_N) < C_{(f;B,T)}.$$

Then we have

$$\frac{\gamma_B^{\mathrm{lag}}(L)}{\operatorname{osc}_{C^0}(\phi_H^1; o_N)} \le \frac{2\operatorname{osc} f}{C_{(f;B,T)}} \tag{3.11}$$

for $L = \phi_H^1(o_N)$.

We would like to mention that the right-hand side of (3.11) does not depend on the scale change of f to δf for $\delta > 0$.

The following question is interesting in regard to the precise estimate of the upper bound in this theorem and Question 1.6.

QUESTION 3.10

For given H satisfying the condition in Theorem 3.9, what is an optimal estimate of the constant $\frac{2\operatorname{osc} f}{C_{(f;B,T)}}$ in terms of B, T, and H? For example, can we obtain an upper bound independent of B or T?

The rest of the section is occupied by the proof of Theorem 3.9. The following proposition is a crucial ingredient of the proof, which is a variation of [15, Proposition 2.6], [1, Proposition 3.3], [19, Proposition 3.1], and [18, Proposition 2.3].

PROPOSITION 3.11

Let $H \in \mathcal{P}C^\infty_{\mathrm{asc}}$ in T^*N be such that

$$\operatorname{supp}\phi_H \cap o_B = \emptyset. \tag{3.12}$$

Take any $f \in C^\infty_{\mathrm{crit}}(N;B)$ such that (3.8) holds. Then

$$\rho^{\mathrm{lag}}(H;1) - \rho^{\mathrm{lag}}\big(H;[pt]^\#\big) \le 2\operatorname{osc} f. \tag{3.13}$$

Proof

Denote $L_f := \operatorname{Graph} df$, $L_t = \phi_H^t(L_f) = \phi_H^t(\operatorname{Graph} df)$. Note that the condition (3.12) implies

$$H_t|_B \equiv c_B(t) \tag{3.14}$$

for a function $c_B = c_B(t)$ depending only on t but not on $x \in B$.

The following lemma is the analogue of [15, Lemma 5.1].

LEMMA 3.12

We have that

$$\rho^{\mathrm{lag}}(H\#f;1) - \rho^{\mathrm{lag}}\big(H\#f;[pt]^\#\big) \le \operatorname{osc} f. \tag{3.15}$$

Proof

By the spectrality of $\rho^{\mathrm{lag}}(\cdot,1)$ in general, we have

$$\rho^{\mathrm{lag}}(H\#f\circ\pi;1) = \mathcal{A}^{\mathrm{cl}}_{(H\#f\circ\pi)}(z^{H\#f\circ\pi}_{p_-}),$$
$$\rho^{\mathrm{lag}}\big(H\#f\circ\pi;[pt]^\#\big) = \mathcal{A}^{\mathrm{cl}}_{(H\#f\circ\pi)}(z^{H\#f\circ\pi}_{p_+}),$$

for some $p_\pm \in L_f \cap o_N$. Using the second statement of Lemma 3.6, we compute

$$\begin{aligned}
&\mathcal{A}^{\mathrm{cl}}_{(H\#f\circ\pi)}(z^{H\#f\circ\pi}_{p_+}) - \mathcal{A}^{\mathrm{cl}}_{(H\#f\circ\pi)}(z^{H\#f\circ\pi}_{p_-})\\
&\quad = -\int_0^1 (H\#f\circ\pi)(t,p_+)\,dt + \int_0^1 (H\#f\circ\pi)(t,p_-)\,dt\\
&\quad = -\int_0^1 c_B(t)\,dt - f(p_+) + \int_0^1 c_B(t)\,dt + f(p_-)\\
&\quad = -f(p_+) + f(p_-) \le \max f - \min f = \operatorname{osc} f.
\end{aligned}$$

Here for the equality in the penultimate line, we use the identity

$$(H\#f\circ\pi)(t,p_\pm) = H(t,p_\pm) + f\big(\phi_H^t(p_\pm)\big) = c_B(t) + f(p_\pm).$$

This finishes the proof. □

On the other hand, we have

$$\phi_H^1(L_f) = \phi_H^1\big(\phi_{f\circ\pi}^1(o_N)\big) = \phi_{H\#f\circ\pi}^1(o_N),$$

and so by the triangle inequality (Proposition 2.4),

$$\rho^{\mathrm{lag}}\big(H\#(f\circ\pi);1\big) \geq \rho^{\mathrm{lag}}(H;1) - \rho^{\mathrm{lag}}(-f\circ\pi;1),$$
$$\rho^{\mathrm{lag}}\big(H\#(f\circ\pi);[pt]^{\#}\big) \leq \rho^{\mathrm{lag}}\big(H;[pt]^{\#}\big) + \rho^{\mathrm{lag}}(f\circ\pi;1).$$

(One can also use Proposition 2.5 with the concatenation $H*(f\circ\pi)$ instead. Here $f\circ\pi$ is not boundary flat, which is required in Proposition 2.5, but one can always reparameterize the flow $t\mapsto\phi_{f\circ\pi}^t$ by multiplying $\chi'(t)$ to $f\circ\pi$ so that the perturbation is as small as we want in the $L^{(1,\infty)}$-topology, which in turn perturbs ρ slightly. See [10, Lemma 5.2] and [7, Remark 2.5] for a precise statement on this approximation procedure. This enables us to apply the triangle inequality from Proposition 2.5 in the current context.)

Therefore, subtracting the second inequality from the first and using the identity (see [9] for its proof)

$$\rho^{\mathrm{lag}}(-f\circ\pi;1) = \max f, \qquad \rho^{\mathrm{lag}}(f\circ\pi;1) = -\min f,$$

we obtain

$$\begin{aligned}&\rho^{\mathrm{lag}}\big(H\#(f\circ\pi);1\big) - \rho^{\mathrm{lag}}\big(H\#(f\circ\pi);[pt]^{\#}\big)\\ &\quad\geq \rho^{\mathrm{lag}}(H;1) - \rho^{\mathrm{lag}}\big(H;[pt]^{\#}\big) - (\max f - \min f),\end{aligned}$$

which in turn gives rise to

$$\begin{aligned}\rho^{\mathrm{lag}}(H;1) - \rho^{\mathrm{lag}}\big(H;[pt]^{\#}\big) &\leq \rho^{\mathrm{lag}}\big(H\#(f\circ\pi);1\big) - \rho^{\mathrm{lag}}\big(H\#(f\circ\pi);[pt]^{\#}\big)\\ &\qquad + (\max f - \min f)\\ &\leq 2\operatorname{osc} f.\end{aligned}$$

We have finished the proof of the proposition. □

We now go back to the proof of Theorem 3.9.

Let $H\in\mathcal{PC}^{\infty}_{\mathrm{asc};B}$ and $T\supset o_B$ be such that $\phi_H^1\equiv id$ on T, and assume (3.8). If $\operatorname{osc}_{C^0}(\phi_H^1;o_N)=0$, we have $\phi_H^1(o_N)=o_N$ and so $\rho^{\mathrm{lag}}(H;1)-\rho^{\mathrm{lag}}(H;[pt]^{\#})=0$, for which (3.13) obviously holds. Therefore, we assume that $\operatorname{osc}_{C^0}(\phi_H^1;o_N)\neq 0$.

Recall from Lemma 3.6 that the choice of f depends only on the ball B and the neighborhood $T\supset o_B$ in T^*N. Then we choose $\lambda>0$ such that

$$\operatorname{osc}_{C^0}(\phi_H^1;o_N) = \lambda C_{(f;B,T)};$$

that is,

$$\lambda = \frac{\operatorname{osc}_{C^0}(\phi_H^1;o_N)}{C_{(f;B,T)}} < 1.$$

Obviously we have

$$\operatorname{osc}_{C^0}(\phi_H^1;o_N) < (\lambda+\varepsilon)C_{(f;B,T)}$$

for all $\varepsilon > 0$. We note that both $d_{\mathrm{H}}(N \setminus B, \mathrm{Crit}(\delta f))$ and the ratio $\frac{2\,\mathrm{osc}\, f}{C_{(f;B,T)}}$ do not depend on the choice of $\delta > 0$.

Therefore, we can replace f by δf for a sufficiently small $\delta > 0$, if necessary, so that

$$\min_{p \in N \setminus B} \big| d\big(\lambda(\delta f)\big)(p)\big| < d_{\mathrm{H}}\big(N \setminus B, \mathrm{Crit}(\delta f)\big), \tag{3.16}$$

which in turn implies

$$\lambda C_{(\delta f;B,T)} = C_{(\lambda \delta f;B,T)}$$

by Lemma 3.5. From now on, we assume that

$$\min_{p \in N \setminus B} \big| d(\lambda f)(p)\big| < d_{\mathrm{H}}(N \setminus B, \mathrm{Crit}\, f) \tag{3.17}$$

without loss of generality.

Lemma 3.5 also implies that

$$(\lambda + \varepsilon) C_{(f;B,T)} = C_{((\lambda+\varepsilon) f;B,T)}$$

for all small $\varepsilon > 0$ such that $\lambda + \varepsilon < 1$ and

$$\min_{p \in N \setminus B} \big| (\lambda + \varepsilon)\, df(p)\big| < d(N \setminus B, \mathrm{Crit}\, f).$$

For example, we can choose any $\varepsilon > 0$ so that

$$0 < \varepsilon < \frac{d(N \setminus B, \mathrm{Crit}\, f)}{\min_{p \in N \setminus B} |df(p)|}. \tag{3.18}$$

Since (3.13) holds for any pair (H, f) that satisfy (3.12) and (3.8), applying it to the pair $(H, (\lambda + \varepsilon) f)$ for $T \supset B$ chosen above independently of the i's, we derive

$$\begin{aligned} \rho^{\mathrm{lag}}(H;1) - \rho^{\mathrm{lag}}\big(H;[pt]^{\#}\big) &\leq 2\,\mathrm{osc}\big((\lambda+\varepsilon) f\big) = 2(\lambda+\varepsilon)\,\mathrm{osc}\, f \\ &= 2\Big(\frac{\mathrm{osc}_{C^0}(\phi_H^1; o_N)}{C_{(f;B,T)}} + \varepsilon\Big)\,\mathrm{osc}\, f. \end{aligned}$$

Since this holds for all $\varepsilon > 0$ satisfying (3.18), it follows that

$$0 \leq \rho^{\mathrm{lag}}(H;1) - \rho^{\mathrm{lag}}\big(H;[pt]^{\#}\big) \leq 2\Big(\frac{\mathrm{osc}\, f}{C_{(f;B,T)}}\Big)\,\mathrm{osc}_{C^0}(\phi_H^1; o_N), \tag{3.19}$$

letting $\varepsilon \to 0$. This finishes the proof of Theorem 3.9.

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References

[1] M. Entov and L. Polterovich, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. IMRN **2003**, no. 30, 1635–1676. MR 1979584. DOI 10.1155/S1073792803210011.

[2] A. Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28** (1988), 513–547. MR 0965228.

[3] ———, *Witten's complex and infinite-dimensional Morse theory*, J. Differential Geom. **30** (1989), 207–221. MR 1001276.

[4] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Spectral invariants with bulk, quasimorphisms and Lagrangian Floer theory*, to appear in Mem. Amer. Math. Soc., preprint, arXiv:1105.5123v3 [math.SG].

[5] D. Milinković, *On equivalence of two constructions of invariants of Lagrangian submanifolds*, Pacific J. Math. **195** (2000), 371–415. MR 1782171.

[6] D. Milinković and Y.-G. Oh, "Generating functions versus action functional: Stable Morse theory versus Floer theory" in *Geometry, Topology, and Dynamics (Montreal, PQ, 1995)*, CRM Proc. Lecture Notes **15**, Amer. Math. Soc., Providence, 1998, 107–125. MR 1619127.

[7] A. Monzner, N. Vichery, and F. Zapolsky, *Partial quasimorphisms and quasistates on cotangent bundles, and symplectic homogenization*, J. Mod. Dyn. **6** (2012), 205–249. MR 2968955. DOI 10.3934/jmd.2012.6.205.

[8] Y.-G. Oh, *Symplectic topology as the geometry of action functional, I: Relative Floer theory on the cotangent bundle*, J. Differential Geom. **46** (1997), 499–577. MR 1484890.

[9] ———, *Symplectic topology as the geometry of action functional, II: Pants product and cohomological invariants*, Comm. Anal. Geom. **7** (1999), 1–54. MR 1674121. DOI 10.4310/CAG.1999.v7.n1.a1.

[10] ———, *Chain level Floer theory and Hofer's geometry of the Hamiltonian diffeomorphism group*, Asian J. Math. **6** (2002), 579–624. MR 1958084. DOI 10.4310/AJM.2002.v6.n4.a1. *Erratum*, Asian J. Math. **7** (2003), 447–448.

[11] ———, "Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds" in *The Breadth of Symplectic and Poisson Geometry*, Progr. Math. **232**, Birkhäuser, Boston, 2005, 525–570. MR 2103018. DOI 10.1007/0-8176-4419-9_18.

[12] ———, "The group of Hamiltonian homeomorphisms and continuous Hamiltonian flows" in *Symplectic Topology and Measure Preserving Dynamical Systems*, Contemp. Math. **512**, Amer. Math. Soc., Providence, 2010, 149–177. MR 2605316. DOI 10.1090/conm/512/10062.

[13] ———, *Geometry of generating functions and Lagrangian spectral invariants*, preprint, arXiv:1206.4788v2 [math.SG].

[14] Y.-G. Oh and S. Müller, *The group of Hamiltonian homeomorphisms and C^0-symplectic topology*, J. Symplectic Geom. **5** (2007), 167–219. MR 2377251.

[15] Y. Ostrover, *A comparison of Hofer's metrics on Hamiltonian diffeomorphisms and Lagrangian submanifolds*, Commun. Contemp. Math. **5** (2003), 803–811. MR 2017719. DOI 10.1142/S0219199703001154.

[16] L. Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms*, Lectures in Math., ETH Zürich, Birkhäuser, Basel, 2001. MR 1826128. DOI 10.1007/978-3-0348-8299-6.

[17] M. Schwarz, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. **193** (2000), 419–461. MR 1755825. DOI 10.2140/pjm.2000.193.419.

[18] S. Seyfaddini, *C^0-limits of Hamiltonian paths and the Oh-Schwarz spectral invariants*, Int. Math. Res. Not. IMRN **2013**, no. 21, 4920–4960. MR 3123670.

[19] M. Usher, *The sharp energy-capacity inequality*, Commun. Contemp. Math. **12** (2010), 457–473. MR 2661273.

[20] C. Viterbo, *Symplectic topology as the geometry of generating functions*, Math. Ann. **292** (1992), 685–710. MR 1157321.

[21] ———, *Symplectic homogenization*, preprint, arXiv:0801.0206v3 [math.SG].

[22] A. Weinstein, *Connections of Berry and Hannay type for moving Lagrangian submanifolds*, Adv. Math. **82** (1990), 133–159. MR 1063955. DOI 10.1016/0001-8708(90)90086-3.

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