The moment map on symplectic vector space and oscillator representation

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Abstract Let G denote $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{U}(p,q)$, or $\mathrm{O}^*(2n)$. The main aim of this article is to show that the canonical quantization of the moment map on a symplectic G-vector space (W,ω) naturally gives rise to the oscillator (or Segal–Shale–Weil) representation of $\mathfrak{g} := \mathrm{Lie}(G) \otimes \mathbb{C}$. More precisely, after taking a complex Lagrangian subspace V of the complexification of W, we assign an element of the Weyl algebra for V to $\langle \mu, X\rangle$ for each $X \in \mathfrak{g}$, which we denote by $\langle \widehat{\mu}, X\rangle$. Then we show that the map $X \mapsto \mathrm{i}\langle \widehat{\mu}, X\rangle$ gives a representation of $\mathfrak{g}$. With a suitable choice of V in each case, the representation coincides with the oscillator representation of $\mathfrak{g}$.

1. Introduction

Let (W,ω) be a symplectic vector space, and let $\mathrm{Sp}(W)$ be the group of linear symplectic isomorphisms of W. Then it is well known that each component of the moment map, that is, the Hamiltonian function H_X on W, is quadratic in the coordinate functions for any $X \in \mathfrak{sp}(W)$ (see, e.g., [2]). Therefore, taking account of the fact that the commutators among the quantized operators corresponding to the coordinate functions are central, one can see that the canonical quantization gives a representation of $\mathfrak{sp}(W)$, since the map $X \mapsto H_X$ is a Lie algebra homomorphism from $\mathfrak{sp}(W)$ into $C^\infty(W)$, where the latter is regarded as a Lie algebra of infinite dimension with respect to the Poisson bracket.

The main aim of this article is to show that, for real reductive Lie groups $G = \mathrm{Sp}(n,\mathbb{R})$, $\mathrm{U}(p,q)$, and $\mathrm{O}^*(2n)$, the canonical quantization of the moment map on the real symplectic G-vector space (W,ω) gives rise to the oscillator (or Segal–Shale–Weil) representation of the complexified Lie algebra $\mathfrak{g}$ of $\mathfrak{g}_0 := \mathrm{Lie}\,(G)$ in a natural way. Here, we understand that the canonical quantization is to construct a mapping from the space of smooth functions on W into the ring of polynomial coefficient differential operators on a complex Lagrangian subspace V of the complexification $W_{\mathbb{C}}$ of W, the so-called Weyl algebra for V, that induces a Lie algebra homomorphism from $\mathfrak{g}$ into the Weyl algebra. We remark that a different choice of a Lagrangian subspace results in a different quantization and, hence, a different representation of the Lie algebra. In fact, when $G = \mathrm{U}(p,q)$

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and $\mathrm{O}^*(2n)$, we will find in Sections 3–5 that one choice of a Lagrangian subspace produces finite-dimensional irreducible representations of $\mathfrak{g}$, while another produces infinite-dimensional ones (i.e., the oscillator representations).

The oscillator representations have been extensively studied in relation to the Howe duality and the minimal representations. Note that each Lie group G we consider in this article is a counterpart of Howe's reductive dual pair (G, G') with G' compact, that is, G and G' are centralizers of each other in a symplectic group $\mathrm{Sp}(N, \mathbb{R})$ for some N. One can obtain the oscillator representations by embedding G into $\mathrm{Sp}(N, \mathbb{R})$ for $G = \mathrm{U}(p, q)$ and $\mathrm{O}^*(2n)$ (see [13], [4], [10], [11], [9], [3], etc.).

As another approach to a construction of the oscillator representations, we should mention Hilgert, Kobayashi, Möllers, and Ørsted [8], in which they construct the oscillator representations via Jordan algebras when G is an arbitrary Hermitian Lie group of tube type.

It was shown in [6] that, for the classical Hermitian symmetric pairs $(G, K) = (\mathrm{SU}(p, q), \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q)))$, $(\mathrm{Sp}(n, \mathbb{R}), \mathrm{U}(n))$, and $(\mathrm{SO}^*(2n), \mathrm{U}(n))$, one obtains generating functions of the principal symbols of $K_{\mathbb{C}}$-invariant differential operators on G/K in terms of the determinant or Pfaffian of a certain $\mathfrak{g}$-valued matrix whose entries are the total symbols of the differential operators corresponding to the holomorphic discrete series representations realized via Borel–Weil theory, where $K_{\mathbb{C}}$ denotes a complexification of K. We note that the $K_{\mathbb{C}}$-invariant differential operators play a prominent role in the Capelli identity (see [12]). Moreover, the author [6] also clarified that the $\mathfrak{g}$-valued matrix mentioned above can be regarded as the twisted moment map μ_λ on the cotangent bundle of G/K which reduces to the moment map μ on the cotangent bundle when $\lambda \to 0$, where λ is an element of $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$, that parameterizes the representations. In summary, one can say that the moment map relates noncommutative objects (representation operators which are realized as differential operators) to commutative ones (symbols of the differential operators). Now in this article, we will proceed in the reverse direction: from commutative objects to noncommutative ones.

In the remainder of this section, we briefly review a few relevant notions from symplectic geometry, and we state our main result. Let (M, ω) be a real symplectic manifold. For $f \in C^\infty(M)$, the space of smooth $\mathbb{R}$-valued functions on M, let ξ_f denote the vector field on M satisfying $\iota(\xi_f)\omega = \mathrm{d}f$, where ι stands for the contraction. Then we define the Poisson bracket by

$$\{f, g\} := \omega(\xi_g, \xi_f) \quad \big(f, g \in C^\infty(M)\big), \tag{1.1}$$

which we extend to the space of smooth $\mathbb{C}$-valued functions by linearity. If we denote the quantum observable corresponding to a classical observable $f \in C^\infty(M)$ by $\widehat{f}$, then the quantization principles require in particular that

$$\text{if } \{f_1, f_2\} = f_3, \quad \text{then } [\widehat{f_1}, \widehat{f_2}] = -\mathrm{i}\hbar \widehat{f_3}, \tag{1.2}$$

where $\hbar$ is the Planck constant (see, e.g., [1], [16]); we set $\hbar = 1$ for simplicity in what follows.

Suppose that a Lie group G acts on M symplectically, that is, $g^*\omega = \omega$ for all $g \in G$. A smooth map $\mu : M \to \mathfrak{g}_0^*$ is called the moment map if the following conditions hold: μ is G-equivariant, and it satisfies

$$\mathrm{d}\langle \mu, X\rangle = \iota(X_M)\omega \quad \text{for all } X \in \mathfrak{g}_0, \tag{1.3}$$

where $\mathfrak{g}_0^*$ is the dual space of $\mathfrak{g}_0$ and X_M denotes the vector field on M defined by

$$X_M(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(-tX).p \quad (p \in M). \tag{1.4}$$

We often identify $\mathfrak{g}_0^*$ with $\mathfrak{g}_0$ via the nondegenerate symmetric invariant bilinear form B defined by

$$B(X,Y) = \begin{cases} \frac{1}{2}\operatorname{tr}(XY) & \text{if } \mathfrak{g}_0 = \mathfrak{sp}(n,\mathbb{R}) \text{ or } \mathfrak{o}^*(2n), \\ \operatorname{tr}(XY) & \text{if } \mathfrak{g}_0 = \mathfrak{u}(p,q), \end{cases} \tag{1.5}$$

which extends to the one on $\mathfrak{g} = \mathfrak{sp}_n$, $\mathfrak{o}_{2n}$, or $\mathfrak{gl}_{p+q}$, the complexification of $\mathfrak{g}_0 = \mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{o}^*(2n)$, or $\mathfrak{u}(p,q)$. If there is no risk of confusion, we denote the composition of μ and the isomorphism $\mathfrak{g}_0^* \simeq \mathfrak{g}_0$ also by μ. Our symplectic G-manifold (M,ω) will be a real symplectic G-vector space.

The main result of this article is the following, which we prove case by case.

THEOREM

Let $G = \mathrm{Sp}(n,\mathbb{R})$, $\mathrm{U}(p,q)$, and $\mathrm{O}^*(2n)$, and let (W,ω) be the real symplectic G-vector spaces $W = \mathbb{R}^{2n}$, $(\mathbb{C}^{p+q})_{\mathbb{R}}$, and $(\mathbb{C}^{2n})_{\mathbb{R}}$ equipped with ω given by

$$\omega(v,w) = \begin{cases} {}^t v J_n w & \text{if } W = \mathbb{R}^{2n}, \\ \operatorname{Im}(v^* I_{p,q} w) & \text{if } W = (\mathbb{C}^{p+q})_{\mathbb{R}}, \\ \operatorname{Im}(v^* I_{n,n} w) & \text{if } W = (\mathbb{C}^{2n})_{\mathbb{R}} \end{cases}$$

for $v, w \in W$, where $J_n = \left[\begin{smallmatrix} & 1_n \\ -1_n & \end{smallmatrix}\right]$ and $I_{p,q} = \left[\begin{smallmatrix} 1_p & \\ & -1_q \end{smallmatrix}\right]$. Then, with a certain choice of the complex Lagrangian subspace of the complexification $W_{\mathbb{C}}$ of W, the canonical quantization of the moment map $\mu : W \to \mathfrak{g}_0^*$ given by (2.9), (5.7), and (4.19) below yields the oscillator representations of $\mathfrak{g} = \mathfrak{sp}_n$, $\mathfrak{gl}_{p+q}$, and $\mathfrak{o}_{2n}$, respectively.

The rest of this article is organized as follows. In Section 2, we consider the case where $G = \mathrm{Sp}(n,\mathbb{R})$, which is the most fundamental case in this article in the sense that a choice of a complex Lagrangian subspace is the key to obtain the oscillator representation. The original motivation for this project started from this case with $n = 1$. In Section 3, we turn to the case where $G = \mathrm{U}(p,q)$ and show that the canonical quantization of the moment map with a certain choice of a complex Lagrangian subspace yields irreducible finite-dimensional representations of $\mathfrak{gl}_{p+q}$. We postpone showing that another choice leads to the oscillator representations of $\mathfrak{gl}_{p+q}$ until Section 5. In Section 4, we treat the case $G = \mathrm{O}^*(2n)$, in which the moment map can be expressed in two ways due to the fact that the quaternionic vector space $\mathbb{H}^n$ is $\mathbb{C}$-isomorphic to $\mathbb{C}^{2n}$ and to $\mathrm{Mat}_{n\times 2}(\mathbb{C})$. In

Section 5, we take complex Lagrangian subspaces different from the ones considered in Sections 3 and 4 in the cases of $\mathrm{U}(p,q)$ and $\mathrm{O}^*(2n)$: one leading to finite-dimensional irreducible representations when $\mathfrak{g}=\mathfrak{o}_{2n}$, and one leading to the oscillator representation when $\mathfrak{g}=\mathfrak{gl}_{p+q}$. Finally, we note a relation between the moment map and the associated variety of the corresponding irreducible $\mathfrak{g}$-modules occurring in the irreducible decomposition of the space of polynomials on the Lagrangian subspace under the joint action of the dual pairs $(\mathfrak{g},G')$.

1.1. Notation

(i) Throughout the article, we fix a Cartan involution θ to be given by $\theta X=-X^*$. Let $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$ denote the Cartan decomposition for $\mathfrak{g}_0$, and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ denote the corresponding complexified Cartan decomposition for $\mathfrak{g}=\mathfrak{g}_0\otimes\mathbb{C}$.

For a given basis $\{X_\alpha\}_\alpha$ for $\mathfrak{g}_0$ (resp., $\mathfrak{g}$), let us denote by $\{X_\alpha^\vee\}$ its dual basis with respect to B, that is, the basis for $\mathfrak{g}_0$ (resp., $\mathfrak{g}$) satisfying

$$B(X_\alpha,X_\beta^\vee)=\delta_{\alpha,\beta},$$

where $\delta_{\alpha,\beta}$ is Kronecker's delta, that is, is equal to 1 if $\alpha=\beta$ and 0 otherwise.

(ii) For a positive integer i, we set

$$\bar{\imath}:=\begin{cases} n+i & \text{if } \mathfrak{g}=\mathfrak{sp}_n \text{ or } \mathfrak{o}_{2n},\\ p+i & \text{if } \mathfrak{g}=\mathfrak{gl}_{p+q},\end{cases}$$

where $\mathfrak{sp}_n$, $\mathfrak{o}_{2n}$, and $\mathfrak{gl}_{p+q}$ denote the complexified Lie algebras of $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{o}^*(2n)$, and $\mathfrak{u}(p,q)$, respectively.

2. Reductive dual pair $(\mathfrak{sp}(n,\mathbb{R}),\mathrm{O}_k)$

In this section, let G denote the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ of rank n over $\mathbb{R}$ which we realize as

$$\mathrm{Sp}(n,\mathbb{R})=\left\{g\in\mathrm{GL}_{2n}(\mathbb{R});{}^tgJ_ng=J_n\right\}$$

with $J_n=\left[\begin{smallmatrix} & 1_n\\ -1_n & \end{smallmatrix}\right]$. Set $\mathfrak{g}_0=\mathfrak{sp}(n,\mathbb{R})$, the Lie algebra of G, and take a basis for $\mathfrak{g}_0$ as

$$\begin{aligned} X_{i,j}^0&=E_{i,j}-E_{\bar{\jmath},\bar{\imath}} \quad (1\le i,j\le n),\\ X_{i,j}^+&=E_{i,\bar{\jmath}}+E_{j,\bar{\imath}} \quad (1\le i\le j\le n),\\ X_{i,j}^-&=E_{\bar{\imath},j}+E_{\bar{\jmath},i} \quad (1\le i\le j\le n),\end{aligned} \tag{2.1}$$

where $E_{i,j}$ denotes the matrix unit of size $2n\times 2n$, that is, its (i,j)th entry is 1 and all other entries are 0. Note that they also form a basis for $\mathfrak{g}=\mathfrak{sp}_n$.

2.1.

Let $W=\mathbb{R}^{2n}$, which is equipped with the canonical symplectic form ω given by

$$\omega(v,w)={}^tvJ_nw \quad (v,w\in W). \tag{2.2}$$

Obviously, the natural left action of G on W defined by $v\mapsto gv$ (matrix multiplication) for $v\in W$ and $g\in G$ is symplectic, that is, $g^*\omega=\omega$ for all $g\in G$.

If we identify the canonical base vectors $e_i := {}^t(0,\ldots,0,\overset{i\text{th}}{1},0,\ldots,0)$ with ∂_{x_i} for $i=1,2,\ldots,n$ and with $\partial_{y_{i-n}}$ for $i=\bar{1},\bar{2},\ldots,\bar{n}$, then it is written as

$$\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \tag{2.3}$$

at $v = {}^t(x_1,\ldots,x_n,y_1,\ldots,y_n) \in W$.

LEMMA 2.1

The vector fields on W generated by the basis (2.1) for $\mathfrak{g}_0 = \mathfrak{sp}(n,\mathbb{R})$ in the sense of (1.4) are given by

$$\begin{aligned}
(X^0_{i,j})_W &= -x_j\partial_{x_i} + y_i\partial_{y_j} \quad (1 \le i,j \le n),\\
(X^+_{i,j})_W &= -(y_j\partial_{x_i} + y_i\partial_{x_j}) \quad (1 \le i \le j \le n),\\
(X^-_{i,j})_W &= -(x_j\partial_{y_i} + x_i\partial_{y_j}) \quad (1 \le i \le j \le n).
\end{aligned} \tag{2.4}$$

Proof

It is an easy exercise to show these formulae. □

Note that the orthogonal group $\mathrm{O}(1) = \{\pm 1\}$ also acts on W symplectically on the right.

PROPOSITION 2.2

Let (W,ω) be as above and $G = \mathrm{Sp}(n,\mathbb{R})$. Then the moment map $\mu : W \to \mathfrak{g}_0^* \simeq \mathfrak{g}_0$ is given by

$$\mu(v) = v\,{}^t v J_n = \begin{bmatrix} -x\,{}^t y & x\,{}^t x \\ -y\,{}^t y & y\,{}^t x \end{bmatrix} \tag{2.5}$$

for $v = {}^t(x_1,\ldots,x_n,y_1,\ldots,y_n) \in W$ with $x = {}^t(x_1,\ldots,x_n)$ and $y = {}^t(y_1,\ldots,y_n)$. In particular, μ is G-equivariant and is $\mathrm{O}(1)$-invariant.

Proof

To make this article self-contained, we include the proof (see, however, e.g., [2, Proposition 1.4.6]). It follows from Lemma 2.1 that

$$\begin{aligned}
\mathrm{d}\langle \mu, X^0_{i,j}\rangle &= \iota\big((X^0_{i,j})_W\big)\omega\\
&= \iota(-x_j\partial_{x_i} + y_i\partial_{y_j}) \sum_{k=1}^{n} \mathrm{d}x_k \wedge \mathrm{d}y_k\\
&= -x_j\,\mathrm{d}y_i - y_i\,\mathrm{d}x_j = -\mathrm{d}(y_i x_j).
\end{aligned}$$

Hence, one obtains that

$$\langle \mu, X^0_{i,j}\rangle = -y_i x_j.$$

Similar calculations yield

$$\langle \mu, X_{i,j}^{+} \rangle = -y_i y_j \qquad \text{and} \qquad \langle \mu, X_{i,j}^{-} \rangle = x_i x_j.$$

Therefore,

$$\begin{aligned}
\mu(v) &= \sum_{i,j} \langle \mu, X_{i,j}^{0} \rangle (X_{i,j}^{0})^{\vee} + \sum_{i \le j} \langle \mu, X_{i,j}^{+} \rangle (X_{i,j}^{+})^{\vee} + \sum_{i \le j} \langle \mu, X_{i,j}^{-} \rangle (X_{i,j}^{-})^{\vee} \\
&= \sum_{i,j} (-y_i x_j)(E_{j,i} - E_{\bar{\imath},\bar{\jmath}}) + \sum_{i \le j} (-y_i y_j) 2^{-\delta_{ij}} (E_{i,\bar{\jmath}} + E_{j,\bar{\imath}}) \\
&\quad + \sum_{i \le j} x_i x_j 2^{-\delta_{ij}} (E_{\bar{\imath},j} + E_{\bar{\jmath},i}) \\
&= \sum_{i,j} (-x_i y_j E_{i,j} + x_i x_j E_{i,\bar{\jmath}} - y_i y_j E_{\bar{\imath},j} + y_i x_j E_{\bar{\imath},\bar{\jmath}}) \\
&= \begin{bmatrix} -x^t y & x^t x \\ -y^t y & y^t x \end{bmatrix} = v^t v J_n
\end{aligned}$$

for $v = {}^t(x_1, \dots, y_n)$ with $x = {}^t(x_1, \dots, x_n)$ and $y = {}^t(y_1, \dots, y_n)$.

Now the $\mathrm{O}(1)$-invariance of μ is trivial, and the G-equivariance can be verified as

$$\mu(gv) = gv^t(gv)J_n = gv^t v^t g J_n = gv^t v J_n g^{-1} = \mathrm{Ad}(g)\mu(v),$$

since ${}^t g J_n = J_n g^{-1}$ for $g \in G$. This completes the proof. $\square$

It follows from the definitions of the Poisson bracket (1.1) and the symplectic form (2.3) that

$$\{x_i, y_j\} = -\delta_{i,j}, \qquad \{x_i, x_j\} = \{y_i, y_j\} = 0, \tag{2.6}$$

for $i, j = 1, \dots, n$. In view of (2.6), we quantize the classical observables by assigning

$$\widehat{x}_i = \text{multiplication by } x_i, \qquad \widehat{y}_i = -\mathrm{i}\partial_{x_i}, \tag{2.7}$$

so that $[\widehat{x}_i, \widehat{y}_j] = \mathrm{i}\delta_{i,j}$, as required. In what follows, we simply denote the multiplication operator by a function f by the same letter f if there is no risk of confusion.

Note that the quantization (2.7) corresponds to taking a Lagrangian subspace of W spanned by $e_1, \dots, e_n$. However, to obtain a representation of the complex Lie algebra $\mathfrak{g} = \mathfrak{sp}_n$, we will take a *complex* Lagrangian subspace of the complexification $W_{\mathbb{C}}$ defined by

$$V := \langle e_1, \dots, e_n \rangle_{\mathbb{C}}. \tag{2.8}$$

Therefore, the classical observables x_j, $j = 1, \dots, n$, are now the complex coordinates on V with respect to this basis.

Now, we quantize the moment map μ according to (2.7) and define the quantized moment map by $\widehat{\mu}$ as

$$\begin{aligned}\widehat{\mu} := \begin{bmatrix}\widehat{x}_1\\ \vdots\\ \widehat{y}_n\end{bmatrix}(\widehat{x}_1,\ldots,\widehat{y}_n)J_n &= \begin{bmatrix}x\\ -\mathrm{i}\partial_x\end{bmatrix}({}^t x, -\mathrm{i}{}^t\partial_x)J_n\\ &= \begin{bmatrix}\mathrm{i}x{}^t\partial_x & x{}^t x\\ \partial_x{}^t\partial_x & -\mathrm{i}\partial_x{}^t x\end{bmatrix}\end{aligned} \tag{2.9}$$

with $x = {}^t(x_1,\ldots,x_n)$ and $\partial_x = {}^t(\partial_{x_1},\ldots,\partial_{x_n})$.

Let $\mathcal{P}(V)$ denote the space of complex coefficient polynomial functions on V; that is, $\mathcal{P}(V) = \mathbb{C}[x_1,\ldots,x_n]$, and let $\mathcal{PD}(V)$ denote the ring of polynomial coefficient differential operators on V. Thus, each entry of $\widehat{\mu}$ is an element of $\mathcal{PD}(V)$.

THEOREM 2.3

For $X \in \mathfrak{g} = \mathfrak{sp}_n$, set $\pi(X) = \mathrm{i}\langle\widehat{\mu}, X\rangle$. Then the map

$$\pi : \mathfrak{g} \to \mathcal{PD}(V)$$

is a Lie algebra homomorphism. In terms of the basis (2.1), it is given by

$$\pi(X) = \begin{cases} -\frac{1}{2}(x_i\partial_{x_j} + \partial_{x_j}x_i) & \text{if } X = X^0_{i,j},\\ \mathrm{i}\partial_{x_i}\partial_{x_j} & \text{if } X = X^+_{i,j},\\ \mathrm{i}x_ix_j & \text{if } X = X^-_{i,j}.\end{cases} \tag{2.10}$$

Proof

Of course, one can verify that the commutation relations among the explicit form (2.10), which can be easily deduced from (2.9), coincide with those of the basis $\{X^\star_{i,j}\}$ for $\mathfrak{g}$. However, we will give another proof in the following.

The moment map μ induces a Lie algebra homomorphism from $\mathfrak{g}_0$ to $C^\infty(W)$; that is, if we write $H_X := \langle\mu, X\rangle$ for $X \in \mathfrak{g}_0$, then we have

$$\{H_X, H_Y\} = H_{[X,Y]} \quad (X, Y \in \mathfrak{g}_0). \tag{2.11}$$

Taking account of the fact that both the Poisson bracket and commutator are derivations, one sees that the relation (2.11) implies that

$$[\widehat{H}_X, \widehat{H}_Y] = -\mathrm{i}\widehat{H}_{[X,Y]}$$

as required in (1.2), since each function H_X is quadratic in the coordinate functions x_i, y_j for any $X \in \mathfrak{g}_0$ (see [2]) and the commutators among $\widehat{x}_i$ and $\widehat{y}_j$ are in the center of $\mathcal{PD}(V)$ for $i, j = 1, \ldots, n$. Hence, it follows from $\pi(X) = \mathrm{i}\widehat{H}_X$ that

$$\big[\pi(X), \pi(Y)\big] = \pi\big([X,Y]\big) \quad (X, Y \in \mathfrak{g}_0).$$

Now, extend the result to the complexification by linearity. $\square$

REMARK 2.4

By (2.9), one can rewrite $\pi(X) = \mathrm{i}\langle\widehat{\mu}, X\rangle$, $X \in \mathfrak{g}$, as

$$\pi(X) = \frac{\mathrm{i}}{2}\operatorname{tr}(\widehat{\mu}X) = \frac{\mathrm{i}}{2}\operatorname{tr}\left(\begin{bmatrix} x \\ -\mathrm{i}\partial_x \end{bmatrix}({}^t x, -\mathrm{i}{}^t\partial_x)J_n X\right)$$
$$= \frac{\mathrm{i}}{2}(\mathrm{i}{}^t\partial_x, {}^t x)X\begin{bmatrix} x \\ -\mathrm{i}\partial_x \end{bmatrix},$$

where the last equality follows from the fact that X is a member of $\mathfrak{g}$. Namely, our quantized moment map $\widehat{\mu}$ is essentially identical to the homomorphism $\varphi : U(\mathfrak{g}) \to \mathcal{A}$ given by Knapp–Vogan [14, Chapter I, Section 6, p. 98, Example], where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{A}$ denotes the Weyl algebra corresponding to our $\mathcal{PD}(V)$ with $n = 1$. This observation was the original motivation for the present work.

It is well known that the irreducible decomposition of the representation $(\pi, \mathcal{P}(V))$ of $\mathfrak{g}$ is given by $\mathcal{P}(V) = \mathcal{P}(V)_+ \oplus \mathcal{P}(V)_-$, where $\mathcal{P}(V)_+$ and $\mathcal{P}(V)_-$ are the subspaces consisting of even polynomials $f(x)$ satisfying $f(-x) = f(x)$ and of odd polynomials $f(x)$ satisfying $f(-x) = -f(x)$, respectively. It is also well known that this phenomena can be explained by the type of representations of $\mathrm{O}(1)$ which acts on V on the right.

2.2.

Let us consider the vector space $W^k := W \oplus \cdots \oplus W$, the direct sum of k copies of $W = \mathbb{R}^{2n}$, which can be identified with $\mathrm{Mat}_{2n\times k}(\mathbb{R})$. It is a symplectic vector space equipped with symplectic form ω_k given by

$$\omega_k(v, w) = \operatorname{tr}({}^t v J_n w) \quad (v, w \in W^k).$$

Let $e_{i,a}$ denote the matrix unit of size $2n \times k$ for $i = 1, \ldots, n$ and $a = 1, \ldots, k$. Under the identification $e_{i,a} \leftrightarrow \partial_{x_{i,a}}$ and $e_{\bar{\imath},a} \leftrightarrow \partial_{y_{i,a}}$, we write $v = {}^t[x_1, \ldots, x_n, y_1, \ldots, y_n] \in W^k$ with $x_i = (x_{i,1}, \ldots, x_{i,k})$ and $y_i = (y_{i,1}, \ldots, y_{i,k})$ being row vectors[1] of size k for $i = 1, \ldots, n$. Then ω_k is given by

$$\omega_k = \sum_{1\le i\le n, 1\le a\le k} \mathrm{d}x_{i,a} \wedge \mathrm{d}y_{i,a} \tag{2.12}$$

at $v = {}^t[x_1, \ldots, y_n]$. Note that $G = \mathrm{Sp}(n, \mathbb{R})$ acts on $W^k = \mathrm{Mat}_{2n\times k}(\mathbb{R})$ on the left, while the real orthogonal group $\mathrm{O}(k)$ acts on the right. Both actions are symplectic.

For brevity, let us write $x_\star \cdot y_\star = \sum_{a=1}^k x_{\star,a} y_{\star,a}$, the standard inner product between two row vectors $x_\star = (x_{\star,1}, \ldots, x_{\star,k})$ and $y_\star = (y_{\star,1}, \ldots, y_{\star,k})$ of size k in what follows.

[1] More precisely, one should write an element $v \in W^k = \mathrm{Mat}_{2n\times k}(\mathbb{R})$ as $v = {}^t[{}^t x_1, \ldots, {}^t x_n, {}^t y_1, \ldots, {}^t y_n]$; however, we will adopt this abbreviated notation in what follows.

PROPOSITION 2.5

Let (W^k,ω_k) be the symplectic G-vector space. Then the moment map $\mu : W^k \to \mathfrak{g}_0^* \simeq \mathfrak{g}_0$ is given by

(2.13a) $\mu(v) = v^t v J_n$

$$
\text{(2.13b)}\qquad = \left[\begin{array}{cccc|cccc}
-x_1\cdot y_1 & -x_1\cdot y_2 & \cdots & -x_1\cdot y_n & x_1\cdot x_1 & x_1\cdot x_2 & \cdots & x_1\cdot x_n\\
-x_2\cdot y_1 & -x_2\cdot y_2 & \cdots & -x_2\cdot y_n & x_2\cdot x_1 & x_2\cdot x_2 & \cdots & x_2\cdot x_n\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-x_n\cdot y_1 & -x_n\cdot y_2 & \cdots & -x_n\cdot y_n & x_n\cdot x_1 & x_n\cdot x_2 & \cdots & x_n\cdot x_n\\
\hline
-y_1\cdot y_1 & -y_1\cdot y_2 & \cdots & -y_1\cdot y_n & y_1\cdot x_1 & y_1\cdot x_2 & \cdots & y_1\cdot x_n\\
-y_2\cdot y_1 & -y_2\cdot y_2 & \cdots & -y_2\cdot y_n & y_2\cdot x_1 & y_2\cdot x_2 & \cdots & y_2\cdot x_n\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-y_n\cdot y_1 & -y_n\cdot y_2 & \cdots & -y_n\cdot y_n & y_n\cdot x_1 & y_n\cdot x_2 & \cdots & y_n\cdot x_n
\end{array}\right]
$$

for $v = {}^t[x_1,\dots,x_n,y_1,\dots,y_n] \in W^k$. In particular, μ is G-equivariant and is $\mathrm{O}(k)$*-invariant.*

Proof

When x_i, y_i, ∂_{x_i}, and ∂_{y_i} denote row vectors and the products stand for the inner product of row vectors, a simple calculation shows that the vector fields on W^k generated by the basis (2.1) for $\mathfrak{g}_0 = \mathfrak{sp}(n,\mathbb{R})$ are given by the same formulae as in Lemma 2.1, and thus, the same argument given in the proof of Proposition 2.2 produces the result. □

It follows from (2.12) that the Poisson brackets among the coordinate functions $x_{i,a}, y_{i,a}$, $i=1,\dots,n$, $a=1,\dots,k$, are given by

$$\{x_{i,a}, y_{j,b}\} = -\delta_{i,j}\delta_{a,b},$$

and all other brackets vanish. Therefore, we quantize them by assigning

$$\widehat{x}_{i,a} = x_{i,a} \qquad \text{and} \qquad \widehat{y}_{i,a} = -\mathrm{i}\partial_{x_{i,a}}$$

for $i=1,\dots,n$ and $a=1,\dots,k$.

Let V^k denote the direct sum $V\oplus\cdots\oplus V$ (k copies) with V given in (2.8). Since V^k can be identified with $\mathrm{Mat}_{n\times k}(\mathbb{C})$, the upper half of $W_{\mathbb{C}}^k = \mathrm{Mat}_{2n\times k}(\mathbb{C})$, we write an element of V^k as $x = (x_{i,a})_{i=1,\dots,n;a=1,\dots,k}$. Let $\mathcal{P}(V^k) = \mathbb{C}[x_{i,a};\ i=1,\dots,n, a=1,\dots,k]$ be the algebra of complex polynomial functions on V^k, and let $\mathcal{PD}(V^k)$ be the ring of polynomial coefficient differential operators on V^k. Note that the $x_{i,a}$'s are now complex variables and that the complex general linear group GL_k acts on V^k by matrix multiplication on the right and, thus, on $\mathcal{P}(V^k)$ by right translation:

$$\text{(2.14)}\qquad \rho(g)f(x) := f(xg) \quad \big(g\in\mathrm{GL}_k, f\in\mathcal{P}(V^k)\big).$$

The right action of GL_k on V^k is the restriction of the one on $W_{\mathbb{C}}^k$.

The quantized moment map $\widehat{\mu}$ in this case is also given by the same formula as (2.9):

$$\widehat{\mu} = \begin{bmatrix} \mathrm{i}x^t\partial_x & x^t x \\ \partial_x{}^t\partial_x & -\mathrm{i}\partial_x{}^t x \end{bmatrix}.$$

In this case, however, x and ∂_x are $(n \times k)$-matrices whose (i,a)th entries are the multiplication operator $x_{i,a}$ and the differential operator $\partial_{x_{i,a}}$ for $i = 1, \dots, n$ and $a = 1, \dots, k$, respectively.

LEMMA 2.6
For $x = (x_{i,a})_{i=1,\dots,n;a=1,\dots,k} \in V^k$ and $g \in \mathrm{GL}_k$, the following relations hold in $\mathrm{End}(\mathcal{P}(V^k))$:

$$\rho(g)^{-1}\partial_{x_{i,a}}\rho(g) = \sum_b g_{ab}\partial_{x_{i,b}}, \tag{2.15}$$

$$\rho(g)^{-1}x_{i,a}\rho(g) = \sum_b g^{ba}x_{i,b}, \tag{2.16}$$

where $g = (g_{ab})$ and $g^{-1} = (g^{ab})$.

Proof
Since $\partial_{x_{i,a}}$ is identified with $e_{i,a} \in \mathrm{Mat}_{n\times k}(\mathbb{C})$, one sees that

$$\big(\partial_{i,a}\big(\rho(g)f\big)\big)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(xg + te_{i,a}g) = \sum_{b=0}^{k} g_{ab}\frac{\partial f}{\partial x_{i,b}}(xg),$$

and hence,

$$\big(\rho(g)^{-1}\partial_{x_{i,a}}\rho(g)\big)f = \sum_{b=1}^{k} g_{ab}\frac{\partial f}{\partial x_{i,b}}$$

for $f \in \mathcal{P}(V^k)$. Thus, one obtains (2.15).

On the other hand, since

$$\big(\rho(g)^{-1}(x_{i,a}f)\big)(x) = \Big(\sum_{b=1}^{k} x_{i,b}g^{ba}\Big)f(xg^{-1}),$$

one has

$$\big(\rho(g)^{-1}x_{i,a}\rho(g)\big)f = \Big(\sum_{b=1}^{k} g^{ba}x_{i,b}\Big)f$$

and (2.16). □

Let us abbreviate as $\rho(g)a\rho(g)^{-1} =: \mathrm{Ad}_{\rho(g)}\, a$ for $a \in \mathcal{PD}(V^k)$ and $g \in \mathrm{GL}_k$. Moreover, for a given matrix $A = (a_{ij})$ with $a_{ij} \in \mathcal{PD}(V^k)$, let us denote by $\mathbf{Ad}_{\rho(g)}A = (\mathrm{Ad}_{\rho(g)}\, a_{ij})$ the matrix whose (i,j)th entries are equal to $\mathrm{Ad}_{\rho(g)}\, a_{ij}$.

COROLLARY 2.7

For $X \in \mathfrak{g} = \mathfrak{sp}_n$, *set* $\pi(X) = \mathrm{i}\langle \widehat{\mu}, X\rangle$. *Then the map*

$$\pi : \mathfrak{g} \to \mathcal{PD}(V^k)$$

is a Lie algebra homomorphism. In terms of the basis (2.1), *it is given by*

$$\pi(X) = \begin{cases} -\frac{1}{2}\sum_{a=1}^{k}(x_{i,a}\partial_{x_{j,a}} + \partial_{x_{j,a}}x_{i,a}) & \text{if } X = X^0_{i,j}, \\ \mathrm{i}\sum_{a=1}^{k}\partial_{x_{i,a}}\partial_{x_{j,a}} & \text{if } X = X^+_{i,j}, \\ \mathrm{i}\sum_{a=1}^{k} x_{i,a}x_{j,a} & \text{if } X = X^-_{i,j}. \end{cases} \tag{2.17}$$

Moreover, $\pi(X)$ *commutes with the action of the complex orthogonal group*[2] O_k*; that is,* $\pi(X) \in \mathcal{PD}(V^k)^{\mathrm{O}_k}$ *for all* $X \in \mathfrak{sp}_n$.

Proof

The same argument as in the proof of Theorem 2.3 shows that $\pi : \mathfrak{g} \to \mathcal{PD}(V^k)$ is a Lie algebra homomorphism and that (2.17) holds.

For the last statement, it follows from Lemma 2.6 that

$$\mathbf{Ad}_{\rho(g)^{-1}}x_i = x_i g^{-1} \qquad \text{and} \qquad \mathbf{Ad}_{\rho(g)^{-1}}\partial_{x_i} = \partial_{x_i}{}^t g$$

with $x_i = (x_{i,1}, \ldots, x_{i,k})$ and $\partial_{x_i} = (\partial_{x_{i,1}}, \ldots, \partial_{x_{i,k}})$ for $g \in \mathrm{GL}_k$. Hence, if $g \in \mathrm{O}_k$, then one has

$$\begin{bmatrix} \mathbf{Ad}_{\rho(g)^{-1}}x \\ -\mathrm{i}\mathbf{Ad}_{\rho(g)^{-1}}\partial \end{bmatrix} = \begin{bmatrix} x \\ -\mathrm{i}\partial \end{bmatrix} {}^t g$$

with $x = {}^t[x_1, \ldots, x_n]$ and $\partial = {}^t[\partial_{x_1}, \ldots, \partial_{x_n}]$, since ${}^t g = g^{-1}$. Therefore,

$$\begin{aligned} \mathbf{Ad}_{\rho(g)^{-1}}\widehat{\mu} &= \begin{bmatrix} \mathrm{i}\mathbf{Ad}_{\rho(g)^{-1}}(x\,{}^t\partial_x) & \mathbf{Ad}_{\rho(g)^{-1}}(x\,{}^t x) \\ \mathbf{Ad}_{\rho(g)^{-1}}(\partial_x{}^t\partial_x) & -\mathrm{i}\mathbf{Ad}_{\rho(g)^{-1}}(\partial_x{}^t x) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{i}\mathbf{Ad}_{\rho(g)^{-1}}x\,{}^t(\mathbf{Ad}_{\rho(g)^{-1}}\partial_x) & \mathbf{Ad}_{\rho(g)^{-1}}x\,{}^t(\mathbf{Ad}_{\rho(g)^{-1}}x) \\ \mathbf{Ad}_{\rho(g)^{-1}}\partial_x{}^t(\mathbf{Ad}_{\rho(g)^{-1}}\partial_x) & -\mathrm{i}\mathbf{Ad}_{\rho(g)^{-1}}\partial_x{}^t(\mathbf{Ad}_{\rho(g)^{-1}}x) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Ad}_{\rho(g)^{-1}}x \\ -\mathrm{i}\mathbf{Ad}_{\rho(g)^{-1}}\partial_x \end{bmatrix} \left[{}^t(\mathbf{Ad}_{\rho(g)^{-1}}x), -\mathrm{i}\,{}^t(\mathbf{Ad}_{\rho(g)^{-1}}\partial_x)\right] J_n \\ &= \begin{bmatrix} x \\ -\mathrm{i}\partial \end{bmatrix} {}^t g g \left[{}^t x, -\mathrm{i}\,{}^t\partial\right] J_n = \widehat{\mu}. \end{aligned}$$

This completes the proof. □

It is well known (see [13]) that the irreducible decomposition of $\mathcal{P}(V^k)$ under the joint action of $(\mathfrak{sp}_n, \mathrm{O}_k)$ is given by

$$\mathcal{P}(V^k) \simeq \sum_{\sigma \in \widehat{\mathrm{O}}_k, L(\sigma) \neq \{0\}} L(\sigma) \otimes V_\sigma, \tag{2.18}$$

[2] We realize the complex orthogonal group as $\mathrm{O}_k = \{g \in \mathrm{GL}_k; {}^t g g = 1_k\}$ in this section.

where V_σ is a representative of the class $\sigma \in \widehat{\mathrm{O}}_k$, the set of all equivalence classes of the finite-dimensional irreducible representation of O_k, and $L(\sigma) := \mathrm{Hom}_{\mathrm{O}_k}(V_\sigma, \mathcal{P}(V^k))$, which is an infinite-dimensional irreducible representation of $\mathfrak{sp}_n$. Moreover, it is also known that the action π restricted to $\mathfrak{k}$ lifts to the double cover $\tilde{K}_{\mathbb{C}}$ of the complexification $K_{\mathbb{C}}$ of the maximal compact subgroup K of $G = \mathrm{Sp}(n,\mathbb{R})$, which implies that $L(\sigma)$ is an irreducible $(\mathfrak{g}, \tilde{K}_{\mathbb{C}})$-module.

Note that our realization of the representation π in this section is the Schrödinger model of the oscillator representation of $\mathfrak{g} = \mathfrak{sp}_n$. We will need another realization of the representation in Section 5, that is, the *Fock model.*

3. Reductive dual pair $(\mathfrak{u}(p,q), \mathrm{GL}_k)$

Let G denote the indefinite unitary group defined by

$$\mathrm{U}(p,q) = \left\{ g \in \mathrm{GL}_n(\mathbb{C}); g^* I_{p,q} g = I_{p,q} \right\}$$

with $I_{p,q} = \left[\begin{smallmatrix} 1_p & \\ & -1_q \end{smallmatrix}\right]$, and put $n = p+q$ only in this section for brevity. Set $\mathfrak{g}_0 = \mathfrak{u}(p,q)$, the Lie algebra of G, and take a basis for $\mathfrak{g}_0$ as

$$\begin{aligned} X^c_{i,j} &= E_{i,j} - E_{j,i} \quad (1 \le i < j \le p \text{ or } p+1 \le i < j \le n), \\ Y^c_{i,j} &= \mathrm{i}(E_{i,j} + E_{j,i}) \quad (1 \le i \le j \le p \text{ or } p+1 \le i \le j \le n), \\ X^n_{i,j} &= E_{i,\bar{j}} + E_{\bar{j},i} \quad (1 \le i \le p, 1 \le j \le q), \\ Y^n_{i,j} &= \mathrm{i}(E_{i,\bar{j}} - E_{\bar{j},i}) \quad (1 \le i \le p, 1 \le j \le q), \end{aligned} \tag{3.1}$$

where $E_{i,j}$ denotes the matrix unit of size $n \times n$. Note that the $E_{i,j}$, $i,j = 1,\dots,n$, form a basis for $\mathfrak{g} = \mathfrak{gl}_n$, the complexified Lie algebra of $\mathfrak{g}_0 = \mathfrak{u}(p,q)$.

3.1.

Let $W = (\mathbb{C}^n)_{\mathbb{R}}$, the underlying real vector space of the complex vector space $\mathbb{C}^n$, and let $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the indefinite Hermitian form given by

$$H(z,w) := z^* I_{p,q} w \quad (z, w \in \mathbb{C}^n).$$

We regard W as a symplectic manifold with symplectic form $\omega = \mathrm{Im}\, H$, where $\mathrm{Im}\, H$ stands for the imaginary part of H. Under the identification $e_j \leftrightarrow \partial_{x_j}$ and $\mathrm{i}e_j \leftrightarrow \partial_{y_j}$ for $j = 1,\dots,n$, it is explicitly given by

$$\omega = \sum_{j=1}^{n} \epsilon_j \, \mathrm{d}x_j \wedge \mathrm{d}y_j \tag{3.2}$$

at $z = x + \mathrm{i}y \in W$ with $x = {}^t(x_1,\dots,x_n), y = {}^t(y_1,\dots,y_n) \in \mathbb{R}^n$, where

$$\epsilon_j := \begin{cases} 1 & (j = 1,\dots,p), \\ -1 & (j = p+1,\dots,n). \end{cases} \tag{3.3}$$

Then (W,ω) is a symplectic G-manifold since the natural action of G on $\mathbb{C}^n$ preserves the Hermitian form H.

LEMMA 3.1

The vector fields on W generated by the basis (3.1) for $\mathfrak{g}_0 = \mathfrak{u}(p,q)$ in the sense of (1.4) are given by

$$(3.4)\qquad \begin{aligned} (X^c_{i,j})_W &= -x_j\partial_{x_i} - y_j\partial_{y_i} + x_i\partial_{x_j} + y_i\partial_{y_j},\\ (Y^c_{i,j})_W &= y_j\partial_{x_i} - x_j\partial_{y_i} + y_i\partial_{x_j} - x_i\partial_{y_j},\\ (X^n_{i,j})_W &= -x_{\bar{j}}\partial_{x_i} - x_i\partial_{x_{\bar{j}}} - y_{\bar{j}}\partial_{y_i} - y_i\partial_{y_{\bar{j}}},\\ (Y^n_{i,j})_W &= y_{\bar{j}}\partial_{x_i} - x_{\bar{j}}\partial_{y_i} - y_i\partial_{x_{\bar{j}}} + x_i\partial_{y_{\bar{j}}}. \end{aligned}$$

Note that the unitary group $\mathrm{U}(1)$ also acts on W symplectically on the right.

PROPOSITION 3.2

Let (W,ω) be as above, and let $G = \mathrm{U}(p,q)$. Then the moment map $\mu : W \to \mathfrak{g}_0^* \simeq \mathfrak{g}_0$ is given by

$$(3.5)\qquad \mu(z) = -\frac{\mathrm{i}}{2} z z^* I_{p,q}$$

for $z = x + \mathrm{i}y \in W$ with $x = {}^t(x_1,\dots,x_n), y = {}^t(y_1,\dots,y_n) \in \mathbb{R}^n$. In particular, μ is G-equivariant and $\mathrm{U}(1)$-invariant.

Proof

It follows from Lemma 3.1 that

$$(3.6)\qquad \langle \mu, X\rangle = \begin{cases} \epsilon_i(x_iy_j - x_jy_i) & \text{if } X = X^c_{i,j},\\ \epsilon_i(x_ix_j + y_iy_j) & \text{if } X = Y^c_{i,j},\\ x_iy_{\bar{j}} - x_{\bar{j}}y_i & \text{if } X = X^n_{i,j},\\ x_ix_{\bar{j}} + y_iy_{\bar{j}} & \text{if } X = Y^n_{i,j}, \end{cases}$$

which can be rewritten in terms of the complex coordinates defined by $z_j = x_j + \mathrm{i}y_j$ $(j = 1,\dots,n)$ and their complex conjugates as

$$(3.7)\qquad \langle \mu, X\rangle = \begin{cases} \frac{\mathrm{i}}{2}\epsilon_i(z_i\bar{z}_j - z_j\bar{z}_i) & \text{if } X = X^c_{i,j},\\ \frac{1}{2}\epsilon_i(z_i\bar{z}_j + z_j\bar{z}_i) & \text{if } X = Y^c_{i,j},\\ \frac{\mathrm{i}}{2}(z_i\bar{z}_{\bar{j}} - z_{\bar{j}}\bar{z}_i) & \text{if } X = X^n_{i,j},\\ \frac{1}{2}(z_i\bar{z}_{\bar{j}} + z_{\bar{j}}\bar{z}_i) & \text{if } X = Y^n_{i,j}. \end{cases}$$

Hence,

$$\begin{aligned} \mu(z) &= \sum_{i<j}\langle \mu, X^c_{i,j}\rangle (X^c_{i,j})^\vee + \sum_{i\le j}\langle \mu, Y^c_{i,j}\rangle (Y^c_{i,j})^\vee\\ &\quad + \sum_{i,j}\langle \mu, X^n_{i,j}\rangle (X^n_{i,j})^\vee + \sum_{i,j}\langle \mu, Y^n_{i,j}\rangle (Y^n_{i,j})^\vee\\ &= -\frac{\mathrm{i}}{2}\sum_{1\le i,j\le p} z_i\bar{z}_j E_{i,j} + \frac{\mathrm{i}}{2}\sum_{1\le i,j\le q} z_{\bar{i}}\bar{z}_{\bar{j}} E_{\bar{i},\bar{j}} \end{aligned}$$

$$+\frac{\mathrm{i}}{2}\sum_{1\le i\le p,1\le j\le q} z_i\bar{z}_{\bar{j}}E_{i,\bar{j}}-\frac{\mathrm{i}}{2}\sum_{1\le i\le q,1\le j\le p} z_{\bar{\imath}}\bar{z}_jE_{\bar{\imath},j}$$
$$=-\frac{\mathrm{i}}{2}zz^*I_{p,q},$$

with $z={}^t(z_1,\dots,z_n)$.

The U(1)-invariance of μ is obvious, and the G-equivariance can be verified as

$$\mu(gz)=-\frac{\mathrm{i}}{2}(gz)(gz)^*I_{p,q}=-\frac{\mathrm{i}}{2}gzz^*g^*I_{p,q}=\mathrm{Ad}(g)\mu(z),$$

since $g^*I_{p,q}=I_{p,q}g^{-1}$ for $g\in\mathrm{U}(p,q)$. □

It follows from (3.2) that the Poisson brackets among the real coordinate functions x_i,y_i, $i=1,\dots,n$, are given by

$$\{x_i,y_j\}=-\epsilon_i\delta_{i,j}\quad(i,j=1,2,\dots,n), \tag{3.8}$$

and all other brackets vanish. In terms of the complex coordinates $z_j=x_j+\mathrm{i}y_j$, $j=1,2,\dots,n$, and their conjugates, it follows from (3.8) that the Poisson brackets among z_j and $\bar{z}_j$ are given by

$$\{z_i,\bar{z}_j\}=2\mathrm{i}\epsilon_i\delta_{i,j},\qquad\{z_i,z_j\}=\{\bar{z}_i,\bar{z}_j\}=0, \tag{3.9}$$

for $i,j=1,2,\dots,n$. In view of (3.9) we quantize z_i and $\bar{z}_i$ by assigning

$$\widehat{z}_i=z_i,\qquad\widehat{\bar{z}}_i=-2\epsilon_i\partial_{z_i}, \tag{3.10}$$

so that they satisfy

$$[\widehat{z}_i,\widehat{\bar{z}}_j]=2\epsilon_i\delta_{i,j},\qquad[\widehat{z}_i,\widehat{z}_j]=[\widehat{\bar{z}}_i,\widehat{\bar{z}}_j]=0, \tag{3.11}$$

for $i,j=1,2,\dots,n$. Therefore, we quantize the moment map μ and define the quantized moment map by $\widehat{\mu}$ as

$$\widehat{\mu}:=-\frac{\mathrm{i}}{2}\begin{bmatrix}\widehat{z}_1\\ \vdots\\ \widehat{z}_n\end{bmatrix}(\widehat{\bar{z}}_1,\dots,\widehat{\bar{z}}_n)I_{p,q}=\mathrm{i}\begin{bmatrix}z_1\\ \vdots\\ z_n\end{bmatrix}(\partial_{z_1},\dots,\partial_{z_n})=\mathrm{i}z{}^t\partial_z \tag{3.12}$$

with $z={}^t(z_1,\dots,z_n)$ and $\partial_z={}^t(\partial_{z_1},\dots,\partial_{z_n})$. Note that the quantization (3.10) corresponds to taking a complex Lagrangian subspace V' given by

$$V':=\left\langle\frac{1}{2}(e_1-\mathrm{i}Ie_1),\dots,\frac{1}{2}(e_n-\mathrm{i}Ie_n)\right\rangle_{\mathbb{C}}\subset W_{\mathbb{C}}, \tag{3.13}$$

where I denotes the complex structure on W defined by $e_j\mapsto\mathrm{i}e_j$, $\mathrm{i}e_j\mapsto-e_j$ for $j=1,\dots,n$. The classical observables $z_j=x_j+\mathrm{i}y_j$ can be regarded as the coordinates on V' with respect to this basis under the identification $e_j\leftrightarrow\partial_{x_j}$ and $\mathrm{i}e_j\leftrightarrow\partial_{y_j}$, $j=1,\dots,n$, and V' is naturally identified with $\mathbb{C}^n$. Let $\mathcal{P}(V')$ denote the algebra of complex coefficient polynomial functions on V, that is, $\mathcal{P}(V')=\mathbb{C}[z_1,\dots,z_n]$, and let $\mathcal{PD}(V')$ denote the ring of polynomial coefficient differential operators on V'.

THEOREM 3.3

For $X \in \mathfrak{g} = \mathfrak{gl}_n$, set $\pi(X) = \mathrm{i}\langle\widehat{\mu}, X\rangle$. Then the map

$$\pi : \mathfrak{g} \to \mathcal{PD}(V')$$

is a Lie algebra homomorphism. In terms of the basis $\{E_{i,j}\}$ for $\mathfrak{g}$, it is given by

$$\pi(E_{i,j}) = -z_j \partial_{z_i} \tag{3.14}$$

for $i, j = 1, \dots, n$.

Proof

The same argument as in Theorem 2.3 shows that π is a Lie algebra homomorphism, and (3.14) follows immediately from (3.12). □

It is clear from (3.14) that $\pi(X) \in \mathcal{PD}(V')^{\mathrm{GL}_1}$ for all $X \in \mathfrak{g}$, where GL_1 acts on V' on the right.

3.2.

Now let us consider W^k, the direct sum of k copies of $W = (\mathbb{C}^n)_{\mathbb{R}}$, which is identified with the underlying real vector space of $\mathrm{Mat}_{n\times k}(\mathbb{C})$. It is equipped with a symplectic form ω_k given by

$$\omega_k(z, w) = \operatorname{Im}\operatorname{tr}(z^* I_{p,q} w) \quad (z, w \in W^k)$$

and is still acted on by $G = \mathrm{U}(p, q)$ symplectically by matrix multiplication on the left. Under the identification of $e_{i,a} \leftrightarrow \partial_{x_{i,a}}$ and $\mathrm{i}e_{i,a} \leftrightarrow \partial_{y_{i,a}}$, we write an element of W^k as $z = {}^t[z_1, \dots, z_n]$, where $z_i = x_i + \mathrm{i}y_i$ are complex row vectors with $x_i = (x_{i,1}, \dots, x_{n,k})$ and $y_i = (y_{i,1}, \dots, y_{i,k})$ being real row vectors of size k for $i = 1, \dots, n$. Then ω_k is given by

$$\omega_k = \sum_{1\le i\le n, 1\le a\le k} \epsilon_i \, \mathrm{d}x_{i,a} \wedge \mathrm{d}y_{i,a} \tag{3.15}$$

at $z = {}^t[z_1, \dots, z_n] \in W^k$. Note that $\mathrm{U}(p, q)$ acts on W on the left, while $\mathrm{U}(k)$ acts on it on the right, and both actions are symplectic.

PROPOSITION 3.4

Let (W^k, ω_k) be the symplectic G-vector space as above. Then the moment map $\mu : W^k \to \mathfrak{g}_0^* \simeq \mathfrak{g}_0$ is given by the same formula as (3.5)

$$\mu = -\frac{\mathrm{i}}{2} z z^* I_{p,q}$$

with $z \in W^k = \mathrm{Mat}_{n\times k}(\mathbb{C})$. In particular, μ is G-equivariant and $\mathrm{U}(k)$-invariant.

Proof

As in the proof of Proposition 2.5, if we regard $x_i, y_i, \partial_{x_i}, \partial_{y_i}$ as row vectors and the products as the inner product on the space of row vectors, then a similar argument to that for Proposition 3.2 shows that the moment map $\mu : W^k \to \mathfrak{g}_0$ is given by (3.5), with the understanding that $z \in \mathrm{Mat}_{n\times k}(\mathbb{C})$. The $\mathrm{U}(k)$-invariance is obvious, and the G-equivariance is verified as in Proposition 3.2. $\square$

It follows from (3.15) that the Poisson brackets among the real coordinate functions $x_{i,a}, y_{i,a}$, $i = 1, \dots, n$, $a = 1, \dots, k$ are given by

$$\{x_{i,a}, y_{j,b}\} = -\epsilon_i \delta_{i,j} \delta_{a,b} \quad (i, j = 1, \dots, n, a, b = 1, \dots, k), \tag{3.16}$$

and all other brackets vanish. It follows from (3.16) that the Poisson brackets among the complex coordinates $z_{j,a} = x_{j,a} + \mathrm{i} y_{j,a}$ and their conjugates are given by

$$\{z_{i,a}, \bar{z}_{j,b}\} = 2\mathrm{i}\epsilon_i \delta_{i,j} \delta_{a,b} \tag{3.17}$$

for $i, j = 1, \dots, n$, $a, b = 1, \dots, k$, and all other brackets vanish. Therefore, we quantize $z_{i,a}$ and $\bar{z}_{i,a}$ by assigning

$$\widehat{z}_{i,a} = z_{i,a}, \qquad \widehat{\bar{z}}_{i,a} = -2\epsilon_i \partial_{z_{i,a}}, \tag{3.18}$$

so that the nontrivial commutators are given by

$$[\widehat{z}_{i,a}, \widehat{\bar{z}}_{j,b}] = 2\epsilon_i \delta_{i,j} \delta_{a,b}. \tag{3.19}$$

Let V'^k denote the direct sum of k copies of V', with V' as in (3.13). Since V'^k can be identified with $\mathrm{Mat}_{n\times k}(\mathbb{C})$, we write an element of V'^k as $z = (z_{i,a})_{i=1,\dots,n; a=1,\dots,k}$. Note then that GL_k acts on V'^k by matrix multiplication on the right and, hence, acts on $\mathcal{P}(V'^k)$ by right regular representation, which we denote also by ρ as in (2.14). Let $\mathcal{P}(V'^k) = \mathbb{C}[z_{i,a}; i = 1, \dots, n, a = 1, \dots, k]$ be the algebra of complex polynomial functions on V'^k, and let $\mathcal{PD}(V'^k)$ be the ring of polynomial coefficient differential operators on V'^k.

The quantized moment map $\widehat{\mu}$ is also given by the same formula as (3.12), namely,

$$\widehat{\mu} = \mathrm{i} z^t \partial_z.$$

In this case, however, z and ∂_z are $(n \times k)$-matrices whose (i, a)th entries are the multiplication operator $z_{i,a}$ and the differential operator $\partial_{z_{i,a}}$ for $i = 1, \dots, n$ and $a = 1, \dots, k$, respectively.

COROLLARY 3.5

For $X \in \mathfrak{g} = \mathfrak{gl}_n$, set $\pi(X) = \mathrm{i}\langle \widehat{\mu}, X\rangle$. Then the map

$$\pi : \mathfrak{g} \to \mathcal{PD}(V'^k)$$

is a Lie algebra homomorphism. In terms of the basis $\{E_{i,j}\}$ for $\mathfrak{g}$, it is given by

$$\pi(E_{i,j}) = -\sum_{a=1}^{k} z_{j,a} \partial_{z_{i,a}} \tag{3.20}$$

for $i, j = 1, \dots, n$. Moreover, $\pi(X)$ commutes with the action of the complex general linear group GL_k, that is, $\pi(X) \in \mathcal{PD}(V'^k)^{\mathrm{GL}_k}$ for all $X \in \mathfrak{g}$.

Proof
The first statement that π is a Lie algebra homomorphism can be shown as in the proof of Theorem 2.3. It remains to show that $\widehat{\mu}$ commutes with the action of GL_k, which can be done in the following way. By Lemma 2.6, one obtains that

$$\mathbf{Ad}_{\rho(g)^{-1}} z = zg^{-1} \qquad \text{and} \qquad \mathbf{Ad}_{\rho(g)^{-1}} \partial_z = \partial_z {}^t g,$$

from which it follows that

$$\mathbf{Ad}_{\rho(g)^{-1}}(z^t\partial_z) = (\mathbf{Ad}_{\rho(g)^{-1}} z)^t (\mathbf{Ad}_{\rho(g)^{-1}} \partial_z) = zg^{-1} g^t \partial_z = z^t \partial_z.$$

This completes the proof. $\square$

Similarly to the case of $\mathrm{Sp}(n,\mathbb{R})$, it is well known (see, e.g., [10], [5]) that the irreducible decomposition of $\mathcal{P}(V^k)$ under the joint action of $(\mathfrak{gl}_n, \mathrm{GL}_k)$ is given by

$$\mathcal{P}(V'^k) \simeq \sum_{\sigma \in \widehat{\mathrm{GL}}_k, L(\sigma) \neq \{0\}} L(\sigma) \otimes V_\sigma, \tag{3.21}$$

where V_σ is a representative of the class $\sigma \in \widehat{\mathrm{GL}}_k$, the set of all equivalence classes of the finite-dimensional irreducible representation of GL_k, and $L(\sigma) := \mathrm{Hom}_{\mathrm{GL}_k}(V_\sigma, \mathcal{P}(V'^k))$, which is a finite-dimensional irreducible representation of $\mathfrak{gl}_n$. It is also well known that the action π restricted to $\mathfrak{k}$ lifts to the complexification $K_\mathbb{C}$ of the maximal compact subgroup K of $G = \mathrm{U}(p,q)$, which implies that $L(\sigma)$ is an irreducible $(\mathfrak{g}, K_\mathbb{C})$-module.

4. Reductive dual pair $(\mathfrak{o}^*(2n), \mathrm{Sp}_k)$

In this section, let G denote the linear Lie group defined by

$$\begin{aligned} \mathrm{O}^*(2n) &= \left\{ g \in \mathrm{U}(n,n); {}^t g S g = S \right\} \\ &= \left\{ g \in \mathrm{GL}_{2n}(\mathbb{C}); g^* I_{n,n} g = I_{n,n}, {}^t g S g = S \right\}, \end{aligned} \tag{4.1}$$

where S denotes the nondegenerate symmetric matrix $\left[\begin{smallmatrix} & 1_n \\ 1_n & \end{smallmatrix}\right]$ of size $2n \times 2n$. Set $\mathfrak{g}_0 = \mathfrak{o}^*(2n)$, the Lie algebra of G, and take a basis for $\mathfrak{g}_0$ as

$$\begin{aligned} X^c_{i,j} &= E_{i,j} - E_{j,i} + E_{\bar{\imath},\bar{\jmath}} - E_{\bar{\jmath},\bar{\imath}} \quad (1 \le i < j \le n), \\ Y^c_{i,j} &= \mathrm{i}(E_{i,j} + E_{j,i} - E_{\bar{\imath},\bar{\jmath}} - E_{\bar{\jmath},\bar{\imath}}) \quad (1 \le i \le j \le n), \\ X^n_{i,j} &= E_{i,\bar{\jmath}} - E_{j,\bar{\imath}} - E_{\bar{\imath},j} + E_{\bar{\jmath},i} \quad (1 \le i < j \le n), \\ Y^n_{i,j} &= \mathrm{i}(E_{i,\bar{\jmath}} - E_{j,\bar{\imath}} + E_{\bar{\imath},j} - E_{\bar{\jmath},i}) \quad (1 \le i < j \le n), \end{aligned} \tag{4.2}$$

where $E_{i,j}$ denotes the matrix unit of size $2n \times 2n$. The complexified Lie algebra $\mathfrak{o}_{2n}$ of $\mathfrak{g}_0 = \mathfrak{o}^*(2n)$ is realized as

$$\mathfrak{o}_{2n} = \left\{ X \in \mathrm{Mat}_{2n}(\mathbb{C}); {}^t X S + S X = O \right\} \tag{4.3}$$

in this section, which we will denote by $\mathfrak{g}$ below. It has a basis

$$
\begin{aligned}
X^0_{i,j} &= E_{i,j} - E_{\bar{j},\bar{\imath}} \quad (1 \le i,j \le n),\\
X^+_{i,j} &= E_{i,\bar{j}} - E_{j,\bar{\imath}} \quad (1 \le i < j \le n),\\
X^-_{i,j} &= E_{\bar{j},i} - E_{\bar{\imath},j} \quad (1 \le i < j \le n).
\end{aligned} \tag{4.4}
$$

4.1.

Let $W = (\mathbb{C}^{2n})_{\mathbb{R}}$ and $\omega = \operatorname{Im} H$, where $H : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ is the Hermitian form given by

$$H(u,v) = u^* I_{n,n} v \quad (u, v \in \mathbb{C}^{2n}).$$

Namely, we consider the case we have discussed in Section 3 with $p = q = n$. Note in particular that ω can be written as

$$\omega = \sum_{j=1}^{n} (\mathrm{d}x_j \wedge \mathrm{d}y_j - \mathrm{d}x_{\bar{j}} \wedge \mathrm{d}y_{\bar{j}}) \tag{4.5}$$

at $v = x + \mathrm{i}y \in W$ with $x = {}^t(x_1, \ldots, x_{2n}), y = {}^t(y_1, \ldots, y_{2n}) \in \mathbb{R}^{2n}$. Then (W, ω) is a symplectic G-vector space, as above.

REMARKS 4.1

(i) There is another realization of the Lie group $\mathrm{O}^*(2n)$ as a group consisting of the complex orthogonal matrices; namely,

$$\mathrm{O}^*(2n) = \{g \in \mathrm{GL}_{2n}; {}^t g g = 1, {}^t g J_n g = J_n\}.$$

We temporarily denote this realization of $\mathrm{O}^*(2n)$ by G^γ, because the former realization G is isomorphic to G^γ by the correspondence $G \ni g \mapsto \gamma g \gamma^{-1} \in G^\gamma$ with $\gamma = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i}\end{smallmatrix}\right] \in \mathrm{U}(2n)$ (cf. [7]).

Let us consider the quaternionic vector space

$$\mathbb{H}^n := \left\{ v = {}^t(v_1, \ldots, v_n); v_i \in \mathbb{H} \ (i = 1, \ldots, n) \right\},$$

where $\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}; a, b, c, d \in \mathbb{R}\}$ denotes the skew field of quaternions. We regard $\mathbb{H}^n$ as a *right* $\mathbb{H}$-vector space. If we identify $\mathbf{i} \in \mathbb{H}$ with $\mathrm{i} \in \mathbb{C}$, then $\mathbb{H}^n$ is isomorphic to $\mathbb{C}^{2n}$ by the map

$$\phi_1 : \mathbb{H}^n \to \mathbb{C}^{2n}, \qquad v = v' + \mathbf{j}v'' \mapsto \begin{bmatrix} v' \\ v'' \end{bmatrix} \quad (v', v'' \in \mathbb{C}^n), \tag{4.6}$$

which is in fact a $\mathbb{C}$-isomorphism. Then G^γ is characterized as the group consisting of $\mathbb{H}$-linear transformations on $\mathbb{H}^n$ that preserve the quaternionic skew-Hermitian form C given by (see [5] for details)

$$C(u,v) := u^* \mathbf{j} v \quad (u, v \in \mathbb{H}^n). \tag{4.7}$$

(ii) There is another identification of $\mathbb{H}^n$ with a $\mathbb{C}$-vector space. Namely, there is an isomorphism of $\mathbb{H}^n$ onto $\mathrm{Mat}_{n\times 2}(\mathbb{C})$ given by

$$\phi_2 : \mathbb{H}^n \to \mathrm{Mat}_{n\times 2}(\mathbb{C}), \qquad v = v' + v''\mathbf{j} \mapsto [v', v'']. \tag{4.8}$$

In this case, however, $\mathbb{H}^n$ is regarded as a *left* $\mathbb{H}$-vector space, and the map ϕ_2 is a $\mathbb{C}$-isomorphism in this sense. Since $\mathbf{j}v'' = \bar{v}''\mathbf{j}$ for $v'' \in \mathbb{C}^n$, one sees that

$$(\phi_2 \circ \phi_1^{-1})\left(\begin{bmatrix} v' \\ v'' \end{bmatrix}\right) = [v', \bar{v}'']. \tag{4.9}$$

Note that $\phi_2 \circ \phi_1^{-1}$ is an $\mathbb{R}$-isomorphism from $\mathbb{C}^{2n}$ onto $\mathrm{Mat}_{n\times 2}(\mathbb{C})$.

More generally, let us consider $(\mathbb{H}^n)^k$, the direct sum of k copies of $\mathbb{H}^n$, which we regard as a left $\mathbb{H}$-vector space as above. Then the multiplication on $(\mathbb{H}^n)^k$ on the right by an element of $\mathrm{Mat}_k(\mathbb{H})$, say, $a + b\mathbf{j}$ with $a, b \in \mathrm{Mat}_k(\mathbb{C})$, corresponds to the multiplication on $\mathrm{Mat}_{n\times 2k}(\mathbb{C})$ on the right by the complex $(2k \times 2k)$-matrix $\left[\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right]$.

LEMMA 4.2

The vector fields on W generated by the basis (4.2) for $\mathfrak{g}_0 = \mathfrak{o}^*(2n)$ in the sense of (1.4) are given by

$$\begin{aligned}
(X^c_{i,j})_W &= -x_j\partial_{x_i} - y_j\partial_{y_i} + x_i\partial_{x_j} + y_i\partial_{y_j} - x_{\bar{j}}\partial_{x_{\bar{i}}} - y_{\bar{j}}\partial_{y_{\bar{i}}} + x_{\bar{i}}\partial_{x_{\bar{j}}} + y_{\bar{i}}\partial_{y_{\bar{j}}}, \\
(Y^c_{i,j})_W &= y_j\partial_{x_i} + y_i\partial_{x_j} - y_{\bar{j}}\partial_{x_{\bar{i}}} - y_{\bar{i}}\partial_{x_{\bar{j}}} - x_j\partial_{y_i} - x_i\partial_{y_j} + x_{\bar{j}}\partial_{y_{\bar{i}}} + x_{\bar{i}}\partial_{y_{\bar{j}}}, \\
(X^n_{i,j})_W &= -x_{\bar{j}}\partial_{x_i} + x_{\bar{i}}\partial_{x_j} + x_j\partial_{x_{\bar{i}}} - x_i\partial_{x_{\bar{j}}} - y_{\bar{j}}\partial_{y_i} + y_{\bar{i}}\partial_{y_j} + y_j\partial_{y_{\bar{i}}} - y_i\partial_{y_{\bar{j}}}, \\
(Y^n_{i,j})_W &= y_{\bar{j}}\partial_{x_i} - y_{\bar{i}}\partial_{x_j} + y_j\partial_{x_{\bar{i}}} - y_i\partial_{x_{\bar{j}}} - x_{\bar{j}}\partial_{y_i} + x_{\bar{i}}\partial_{y_j} - x_j\partial_{y_{\bar{i}}} + x_i\partial_{y_{\bar{j}}}.
\end{aligned} \tag{4.10}$$

For a given $v = \left[\begin{smallmatrix} v' \\ v'' \end{smallmatrix}\right] \in \mathbb{C}^{2n}$ with $v', v'' \in \mathbb{C}^n$, we set $v_+ := (\phi_2 \circ \phi_1^{-1})(v) = [v', \bar{v}''] \in \mathrm{Mat}_{n\times 2}(\mathbb{C})$ for brevity. By Remark 4.1(ii), $\mathrm{Sp}(1)$ acts on W on the right via the $\mathbb{R}$-isomorphism $\phi_2 \circ \phi_1^{-1}$.

PROPOSITION 4.3

Let (W, ω) be as above, and let $G = \mathrm{O}^*(2n)$ as in (4.1). Then the moment map $\mu : W \to \mathfrak{g}_0^* \simeq \mathfrak{g}_0$ is given by

$$\mu(v) = -\frac{\mathrm{i}}{2}\big(vv^* I_{n,n} - S^t(vv^* I_{n,n})S\big) \tag{4.11a}$$

$$= -\frac{\mathrm{i}}{2}\begin{bmatrix} v_+v_+^* & -v_+J_1{}^tv_+ \\ -\bar{v}_+J_1v_+^* & -\bar{v}_+{}^tv_+ \end{bmatrix} \tag{4.11b}$$

for $v = x + \mathrm{i}y \in W$ with $x = {}^t(x_1, \ldots, x_{2n}), y = {}^t(y_1, \ldots, y_{2n}) \in \mathbb{R}^{2n}$. In particular, μ is G-equivariant and $\mathrm{Sp}(1)$-invariant.

Proof

It follows from Lemma 4.2 that

$$\langle \mu, X\rangle = \begin{cases} x_iy_j - x_jy_i - x_{\bar{i}}y_{\bar{j}} + x_{\bar{j}}y_{\bar{i}} & \text{if } X = X^c_{i,j}, \\ x_ix_j + x_{\bar{i}}x_{\bar{j}} + y_iy_j + y_{\bar{i}}y_{\bar{j}} & \text{if } X = Y^c_{i,j}, \\ x_iy_{\bar{j}} - x_jy_{\bar{i}} + x_{\bar{i}}y_j - x_{\bar{j}}y_i & \text{if } X = X^n_{i,j}, \\ x_ix_{\bar{j}} - x_jx_{\bar{i}} - y_jy_{\bar{i}} + y_iy_{\bar{j}} & \text{if } X = Y^n_{i,j}, \end{cases} \tag{4.12}$$

which can be rewritten in terms of complex coordinates defined by $z_i := x_i + \mathrm{i}y_i$, $i = 1, \ldots, 2n$, and their complex conjugates as

$$\langle \mu, X \rangle = \begin{cases} -\frac{\mathrm{i}}{2}(\bar{z}_i z_j - \bar{z}_j z_i - \bar{z}_{\bar{\imath}} z_{\bar{\jmath}} + \bar{z}_{\bar{\jmath}} z_{\bar{\imath}}) & \text{if } X = X^c_{i,j}, \\ \frac{1}{2}(\bar{z}_i z_j + \bar{z}_j z_i + \bar{z}_{\bar{\imath}} z_{\bar{\jmath}} + \bar{z}_{\bar{\jmath}} z_{\bar{\imath}}) & \text{if } X = Y^c_{i,j}, \\ -\frac{\mathrm{i}}{2}(\bar{z}_i z_{\bar{\jmath}} - \bar{z}_j z_{\bar{\imath}} + \bar{z}_{\bar{\imath}} z_j - \bar{z}_{\bar{\jmath}} z_i) & \text{if } X = X^n_{i,j}, \\ \frac{1}{2}(\bar{z}_i z_{\bar{\jmath}} - \bar{z}_j z_{\bar{\imath}} - \bar{z}_{\bar{\imath}} z_j + \bar{z}_{\bar{\jmath}} z_i) & \text{if } X = Y^n_{i,j}. \end{cases} \tag{4.13}$$

Thus, setting $v' := {}^t(z_1, \ldots, z_n)$ and $v'' := {}^t(z_{\bar{1}}, \ldots, z_{\bar{n}})$, one obtains that

$$\begin{aligned} \mu(v) &= \sum_{i<j} \langle \mu, X^c_{i,j} \rangle (X^c_{i,j})^\vee + \sum_{i \le j} \langle \mu, Y^c_{i,j} \rangle (Y^c_{i,j})^\vee \\ &\quad + \sum_{i<j} \langle \mu, X^n_{i,j} \rangle (X^n_{i,j})^\vee + \sum_{i<j} \langle \mu, Y^n_{i,j} \rangle (Y^n_{i,j})^\vee \\ &= -\frac{\mathrm{i}}{2} \sum_{i,j=1}^{n} \big((z_i \bar{z}_j + \bar{z}_{\bar{\imath}} z_{\bar{\jmath}}) E_{i,j} - (\bar{z}_i z_j + z_{\bar{\imath}} \bar{z}_{\bar{\jmath}}) E_{\bar{\imath},\bar{\jmath}} \\ &\quad - (z_i \bar{z}_{\bar{\jmath}} - \bar{z}_{\bar{\imath}} z_j) E_{i,\bar{\jmath}} - (\bar{z}_i z_{\bar{\jmath}} - z_{\bar{\imath}} \bar{z}_j) E_{\bar{\imath},j} \big) \\ &= -\frac{\mathrm{i}}{2} \begin{bmatrix} v'\,{}^t\bar{v}' + \bar{v}''\,{}^t v'' & -v'\,{}^t\bar{v}'' + \bar{v}''\,{}^t v' \\ -\bar{v}'\,{}^t v'' + v''\,{}^t\bar{v}' & -\bar{v}'\,{}^t v' - v''\,{}^t\bar{v}'' \end{bmatrix} \\ &= -\frac{\mathrm{i}}{2} \left(\begin{bmatrix} v' \\ v'' \end{bmatrix} {}^t(\bar{v}', -\bar{v}'') + \begin{bmatrix} \bar{v}'' \\ -\bar{v}' \end{bmatrix} {}^t(v'', -v') \right) \\ &= -\frac{\mathrm{i}}{2} \big(vv^* I_{n,n} - S^t(vv^* I_{n,n}) S \big). \end{aligned}$$

Rewriting (4.11a), one obtains the second expression (4.11b).

The $\mathrm{Sp}(1)$-invariance of μ immediately follows from (4.11b), and the G-equivariance can be verified in the following way. If $g \in G$, then

$$\begin{aligned} \mu(gv) &= -\frac{\mathrm{i}}{2} \big(gvv^* g^* I_{n,n} - S^t(gvv^* g^* I_{n,n}) S \big) \\ &= -\frac{\mathrm{i}}{2} \big(gvv^* I_{n,n} g^{-1} - S^t(gvv^* I_{n,n} g^{-1}) S \big), \end{aligned}$$

since $g^* I_{n,n} = I_{n,n} g^{-1}$. The second term in the parentheses on the right-hand side equals

$$S^t g^{-1\,t}(vv^* I_{n,n})^t g S = g S^t(vv^* I_{n,n}) S g^{-1},$$

since ${}^t g S = S g^{-1}$. Thus,

$$\mu(gv) = -\frac{\mathrm{i}}{2} \big(gvv^* I_{n,n} g^{-1} - g S^t(vv^* I_{n,n}) S g^{-1} \big) = \mathrm{Ad}(g)\mu(v).$$

This completes the proof. $\square$

It follows from (4.5) that the Poisson brackets among x_i, y_i, $i=1,\dots,2n$, are given by

$$\{x_i, y_j\} = -\delta_{i,j}, \qquad \{x_{\bar{\imath}}, y_{\bar{\jmath}}\} = \delta_{i,j}, \tag{4.14}$$

for $i,j=1,\dots,n$, and all other brackets vanish. In terms of complex coordinates $z_j = x_j + \mathrm{i}y_j$ for $j=1,\dots,2n$ and their conjugates, it follows from (4.14) that the Poisson brackets among them are given by

$$\{z_i, \bar{z}_j\} = \{\bar{z}_{\bar{\imath}}, z_{\bar{\jmath}}\} = 2\mathrm{i}\delta_{i,j} \tag{4.15}$$

for $i,j=1,\dots,n$, and all other brackets vanish, as in (3.9). In view of (4.15), we quantize them by assigning

$$\begin{aligned} \widehat{z}_i &= z_i, & \widehat{\bar{z}}_i &= -2\partial_{z_i}, \\ \widehat{\bar{z}}_{\bar{\imath}} &= \bar{z}_{\bar{\imath}}, & \widehat{z}_{\bar{\imath}} &= -2\partial_{\bar{z}_{\bar{\imath}}}, \end{aligned} \tag{4.16}$$

for $i=1,\dots,n$, so that the nontrivial commutators among the quantized operators are given by

$$[\widehat{z}_i, \widehat{\bar{z}}_j] = [\widehat{\bar{z}}_{\bar{\imath}}, \widehat{z}_{\bar{\jmath}}] = 2\delta_{i,j} \tag{4.17}$$

for $i,j=1,\dots,n$.

Let I denote a complex structure on W defined by $e_j \mapsto \mathrm{i}e_j$ and $\mathrm{i}e_j \mapsto -e_j$ for $j=1,\dots,2n$. Under the identification $e_j \leftrightarrow \partial_{x_j}$ and $\mathrm{i}e_j \leftrightarrow \partial_{y_j}$, the classical observables z_j and $\bar{z}_j$ introduced above can be regarded as the coordinate functions on $W_{\mathbb{C}}$ with respect to the basis $\frac{1}{2}(e_j - \mathrm{i}Ie_j)$ and $\frac{1}{2}(e_j + \mathrm{i}Ie_j)$, respectively, for $j=1,\dots,2n$. Note that $\bar{z}_j$ is no longer the complex conjugate of z_j, since x_i and y_i are now complex functions. Then the quantization (4.16) corresponds to taking a complex Lagrangian subspace V given by

$$V = \Big\langle \frac{1}{2}(e_j - \mathrm{i}Ie_j), \frac{1}{2}(e_{\bar{\jmath}} + \mathrm{i}Ie_{\bar{\jmath}}); j=1,\dots,n \Big\rangle_{\mathbb{C}}. \tag{4.18}$$

For simplicity, we set $w_j := \bar{z}_{\bar{\jmath}}$, $j=1,\dots,n$, and we write an element of $V = \mathrm{Mat}_{n\times 2}(\mathbb{C})$ as $[z,w]$ with $z = {}^t(z_1,\dots,z_n)$ and $w = {}^t(w_1,\dots,w_n)$ in what follows.

Now, we quantize the moment map μ according to (4.16) by using its first expression (4.11a) and define the quantized moment map by $\widehat{\mu}$ as

$$\widehat{\mu} := -\frac{\mathrm{i}}{2}\left(\begin{bmatrix} \widehat{z}_1 \\ \vdots \\ \widehat{z}_{\bar{n}} \end{bmatrix} (\widehat{\bar{z}}_1, \dots, \widehat{\bar{z}}_{\bar{n}}) I_{n,n} - SI_{n,n} \begin{bmatrix} \widehat{\bar{z}}_1 \\ \vdots \\ \widehat{\bar{z}}_{\bar{n}} \end{bmatrix} (\widehat{z}_1, \dots, \widehat{z}_{\bar{n}}) S \right) \tag{4.19a}$$

$$= -\frac{\mathrm{i}}{2}\left(\begin{bmatrix} z \\ -2\partial_w \end{bmatrix} (-2{}^t\partial_z, {}^tw) I_{n,n} - SI_{n,n} \begin{bmatrix} -2\partial_z \\ w \end{bmatrix} ({}^tz, -2{}^t\partial_w) S \right)$$

$$= \mathrm{i} \begin{bmatrix} z\,{}^t\partial_z + w\,{}^t\partial_w & \frac{1}{2}(z\,{}^tw - w\,{}^tz) \\ 2(\partial_z{}^t\partial_w - \partial_w{}^t\partial_z) & -(\partial_w{}^tw + \partial_z{}^tz) \end{bmatrix}, \tag{4.19b}$$

where $z={}^t(z_1,\dots,z_n)$, $w={}^t(w_1,\dots,w_n)$, $\partial_z={}^t(\partial_{z_1},\dots,\partial_{z_n})$, and $\partial_w={}^t(\partial_{w_1},\dots,\partial_{w_n})$. The quantization of the second expression (4.11b), that is,

$$\widehat{\mu}=-\frac{\mathrm{i}}{2}\begin{bmatrix}\widehat{v}_+{}^t\widehat{\overline{v}}_+ & -\widehat{v}_+J_1{}^t\widehat{v}_+\\ -\widehat{\overline{v}}_+J_1{}^t\widehat{\overline{v}}_+ & -\widehat{\overline{v}}_+{}^t\widehat{v}_+\end{bmatrix}, \tag{4.19c}$$

produces the same result as (4.19b), where $\widehat{v}_+=[z,w]$ and $\widehat{\overline{v}}_+=[-2\partial_z,-2\partial_w]$.

Let $\mathcal{P}(V)$ denote the algebra of complex coefficient polynomials on V, that is, $\mathcal{P}(V)=\mathbb{C}[z_1,\dots,z_n,w_1,\dots,w_n]$, and let $\mathcal{PD}(V)$ denote the ring of polynomial coefficient differential operators on V. Note that the complex symplectic group of rank one

$$\mathrm{Sp}_1=\{g\in\mathrm{GL}_2;{}^tgJ_1g=J_1\}$$

acts on V by matrix multiplication on the right and, hence, on $\mathcal{P}(V)$ by right regular representation, which we denote by ρ, as in (2.14). The right action of Sp_1 on V coincides with the one on $\mathrm{Mat}_{n\times 2}(\mathbb{C})$ mentioned in Remark 4.1(ii).

THEOREM 4.4

For $X\in\mathfrak{g}=\mathfrak{o}_{2n}$, set $\pi(X)=\mathrm{i}\langle\widehat{\mu},X\rangle$. Then the map

$$\pi:\mathfrak{g}\to\mathcal{PD}(V)$$

is a Lie algebra homomorphism. In terms of the basis (4.4) for $\mathfrak{g}$, it is given by

$$\pi(X)=\begin{cases}-(z_j\partial_{z_i}+w_j\partial_{w_i}+\delta_{i,j}) & \text{if } X=X^0_{i,j},\\ 2(\partial_{z_i}\partial_{w_j}-\partial_{w_i}\partial_{z_j}) & \text{if } X=X^+_{i,j},\\ \frac{1}{2}(z_jw_i-w_jz_i) & \text{if } X=X^-_{i,j}.\end{cases} \tag{4.20}$$

Moreover, $\pi(X)$ commutes with the action of Sp_1; that is, $\pi(X)\in\mathcal{PD}(V)^{\mathrm{Sp}_1}$ for all $X\in\mathfrak{g}$.

Proof

It suffices to prove that $\pi(X)$ commutes with the right action of Sp_1. For this, we use the second expression (4.19c) of $\widehat{\mu}$. It follows from Lemma 2.6 that

$$\mathbf{Ad}_{\rho(g)^{-1}}\widehat{v}_+=\widehat{v}_+g^{-1}\qquad\text{and}\qquad\mathbf{Ad}_{\rho(g)^{-1}}\widehat{\overline{v}}_+=\widehat{\overline{v}}_+{}^tg$$

for $g\in\mathrm{GL}_2$. Therefore, if $g\in\mathrm{Sp}_1$, then one obtains

$$\begin{aligned}\mathbf{Ad}_{\rho(g)^{-1}}\widehat{\mu}&=-\frac{\mathrm{i}}{2}\begin{bmatrix}\widehat{v}_+g^{-1t}(\widehat{\overline{v}}_+{}^tg) & -\widehat{v}_+g^{-1}J_1{}^t(\widehat{v}_+g^{-1})\\ -\widehat{\overline{v}}_+{}^tgJ_1{}^t(\widehat{\overline{v}}_+{}^tg) & -\widehat{\overline{v}}_+{}^tg^t(\widehat{v}_+g^{-1})\end{bmatrix}\\ &=-\frac{\mathrm{i}}{2}\begin{bmatrix}\widehat{v}_+{}^t\widehat{\overline{v}}_+ & -\widehat{v}_+J_1{}^t\widehat{v}_+\\ -\widehat{\overline{v}}_+J_1{}^t\widehat{\overline{v}}_+ & -\widehat{\overline{v}}_+{}^t\widehat{v}_+\end{bmatrix}=\widehat{\mu},\end{aligned}$$

since ${}^tgJ_1g=J_1$. This completes the proof. □

4.2.
Now let us consider W^k, the direct sum of k copies of $W=(\mathbb{C}^{2n})_{\mathbb{R}}$, which we identify with $\mathrm{Mat}_{2n\times k}(\mathbb{C})$. It is equipped with a symplectic form given by

$$\omega_k(u,v)=\operatorname{Im}\operatorname{tr}(u^* I_{n,n} v)\quad (u,v\in W^k)$$

and is still acted on by $G=\mathrm{O}^*(2n)$ symplectically by matrix multiplication on the left. Under the identification of $e_{i,a}\leftrightarrow \partial_{x_{i,a}}$ and $\mathrm{i}e_{i,a}\leftrightarrow \partial_{y_{i,a}}$, we write an element of W^k as $v={}^t[v_1,\dots,v_{2n}]$, where $v_i=x_i+\mathrm{i}y_i$ are complex row vectors with $x_i=(x_{i,1},\dots,x_{i,k})$ and $y_i=(y_{i,1},\dots,y_{i,k})$ being real row vectors of size k for $i=1,\dots,2n$. Then ω_k is given by

$$\omega_k=\sum_{1\le i\le n,1\le a\le k}(\mathrm{d}x_{i,a}\wedge \mathrm{d}y_{i,a}-\mathrm{d}x_{\bar{\imath},a}\wedge \mathrm{d}y_{\bar{\imath},a}) \tag{4.21}$$

at $v={}^t[v_1,\dots,v_{2n}]\in \mathrm{Mat}_{2n\times k}(\mathbb{C})$. Moreover, the isomorphisms ϕ_1 and ϕ_2 defined by (4.6) and (4.8), respectively, naturally extend to the one between $(\mathbb{H}^n)^k$ and $\mathrm{Mat}_{2n\times k}(\mathbb{C})$ and the one between $(\mathbb{H}^n)^k$ and $\mathrm{Mat}_{n\times 2k}(\mathbb{C})$, respectively, which we denote by the same symbols. Then $\mathrm{Sp}(k)$ acts on W^k on the right via the $\mathbb{R}$-isomorphism $\phi_2\circ\phi_1^{-1}$, as above.

PROPOSITION 4.5
Let (W^k,ω_k) be the symplectic G-vector space as above. Then the moment map $\mu: W^k\to \mathfrak{g}_0^*\simeq\mathfrak{g}_0$ is given by the same formulae as (4.11). Namely, for $v={}^t[v',v'']\in W^k$ with $v',v''\in\mathrm{Mat}_{n\times k}(\mathbb{C})$,

$$\begin{aligned}\mu(v)&=-\frac{\mathrm{i}}{2}\big(vv^*I_{n,n}-S^t(vv^*I_{n,n})S\big)\\ &=-\frac{\mathrm{i}}{2}\begin{bmatrix} v_+v_+^* & -v_+J_k{}^tv_+\\ -\bar{v}_+J_kv_+^* & -\bar{v}_+{}^tv_+\end{bmatrix},\end{aligned} \tag{4.22}$$

where $v_+=(\phi_2\circ\phi_1^{-1})(v)\in\mathrm{Mat}_{n\times 2k}(\mathbb{C})$. In particular, μ is G-equivariant and $\mathrm{Sp}(k)$-invariant.

Proof
The vector fields on W^k generated by the basis for $\mathfrak{g}_0$ are given by the same formulae as (4.10) in Lemma 4.2, with the understanding that $x_i,y_i,\partial_{x_i},\partial_{y_i}$ are row vectors and the products stand for the inner product of row vectors. Now, exactly the same argument as in Proposition 4.3 implies the proposition. □

It follows from (4.21) that the Poisson brackets among the real coordinate functions $x_{i,a},y_{i,a}$ are given by

$$\{x_{i,a},y_{j,b}\}=-\delta_{i,j}\delta_{a,b},\qquad \{x_{\bar{\imath},a},y_{\bar{\jmath},b}\}=\delta_{i,j}\delta_{a,b} \tag{4.23}$$

and all other brackets vanish. Hence, the nontrivial ones among the complex coordinate functions are given by

$$\{z_{i,a},\bar{z}_{j,b}\}=\{\bar{z}_{\bar{\imath},a},z_{\bar{\jmath},b}\}=2\mathrm{i}\delta_{i,j}\delta_{a,b} \tag{4.24}$$

for $i, j = 1, \ldots, n$ and $a, b = 1, \ldots, k$. Therefore, we quantize $z_{i,a}$ and $\bar{z}_{i,a}$ by assigning

$$\begin{aligned} \widehat{z}_{i,a} &= z_{i,a}, & \widehat{\bar{z}}_{i,a} &= -2\partial_{z_{i,a}}, \\ \widehat{\bar{z}}_{\bar{\imath},a} &= \bar{z}_{\bar{\imath},a}, & \widehat{z}_{\bar{\imath},a} &= -2\partial_{\bar{z}_{\bar{\imath},a}} \end{aligned} \tag{4.25}$$

so that the nontrivial commutators among the quantized operators are given by

$$[\widehat{z}_{i,a}, \widehat{\bar{z}}_{j,b}] = [\widehat{\bar{z}}_{\bar{\imath},a}, \widehat{z}_{\bar{\jmath},b}] = 2\delta_{i,j}\delta_{a,b} \tag{4.26}$$

for $i, j = 1, \ldots, n$ and $a, b = 1, \ldots, k$.

Let V^k denote the direct sum of k copies of V, with V as in (4.18). Since V^k can be identified with $\mathrm{Mat}_{n\times 2k}(\mathbb{C})$, we write an element of V^k as $[z, w]$, where $z = (z_{i,a})$ and $w = (w_{i,a})$ are elements of $\mathrm{Mat}_{n\times k}(\mathbb{C})$, and we set $w_{i,a} = \bar{z}_{\bar{\imath},a}$ for $i = 1, \ldots, n$ and $a = 1, \ldots, k$ for simplicity, as above. Let $\mathcal{P}(V^k) = \mathbb{C}[z_{i,a}, w_{i,a}; i = 1, \ldots, n, a = 1, \ldots, k]$ be the algebra of complex polynomial functions on V^k, and let $\mathcal{PD}(V^k)$ be the ring of polynomial coefficient differential operators on V^k. Then the complex symplectic group Sp_k acts on V^k by matrix multiplication on the right and, hence, on $\mathcal{P}(V^k)$ by right regular representation, which we denote by ρ, as usual.

The quantized moment map $\widehat{\mu}$ is given by the same formula as (4.19b):

$$\widehat{\mu} = \mathrm{i}\begin{bmatrix} z^t\partial_z + w^t\partial_w & \frac{1}{2}(z^t w - w^t z) \\ 2(\partial_z{}^t\partial_w - \partial_w{}^t\partial_z) & -(\partial_w{}^t w + \partial_z{}^t z)\end{bmatrix}.$$

Here, z (resp., w) and ∂_z (resp., ∂_w) now denote $(n\times k)$-matrices whose (i,a)th entries are the multiplication operator $z_{i,a}$ (resp., $w_{i,a}$) and the differential operator $\partial_{z_{i,a}}$ (resp., $\partial_{w_{i,a}}$) for $i = 1, \ldots, n$ and $a = 1, \ldots, k$.

COROLLARY 4.6

For $X \in \mathfrak{g} = \mathfrak{o}_{2n}$, set $\pi(X) = \mathrm{i}\langle\widehat{\mu}, X\rangle$. Then the map

$$\pi : \mathfrak{g} \to \mathcal{PD}(V^k)$$

is a Lie algebra homomorphism. In terms of the basis (4.4) for $\mathfrak{g}$, it is given by

$$\pi(X) = \begin{cases} -\sum_{a=1}^k (z_{j,a}\partial_{z_{i,a}} + w_{j,a}\partial_{w_{i,a}} + k\delta_{i,j}) & \text{if } X = X^0_{i,j}, \\ 2\sum_{a=1}^k (\partial_{z_{i,a}}\partial_{w_{j,a}} - \partial_{w_{i,a}}\partial_{z_{j,a}}) & \text{if } X = X^+_{i,j}, \\ \frac{1}{2}\sum_{a=1}^k (z_{j,a}w_{i,a} - w_{j,a}z_{i,a}) & \text{if } X = X^-_{i,j}. \end{cases} \tag{4.27}$$

Moreover, $\pi(X)$ commutes with the action of the complex symplectic group Sp_k, that is, $\pi(X) \in \mathcal{PD}(V^k)^{\mathrm{Sp}_k}$ for all $X \in \mathfrak{g}$.

Proof

The proof is essentially the same as that of Theorem 4.4. □

Similarly to the cases discussed above, it is well known (see, e.g., [10], [5]) that the irreducible decomposition of $\mathcal{P}(V^k)$ under the joint action of $(\mathfrak{o}_{2n}, \mathrm{Sp}_k)$ is given by

$$\mathcal{P}(V^k) \simeq \sum_{\sigma \in \widehat{\mathrm{Sp}}_k, L(\sigma) \neq \{0\}} L(\sigma) \otimes V_\sigma, \tag{4.28}$$

where V_σ is a representative of the class $\sigma \in \widehat{\mathrm{Sp}}_k$, the set of all equivalence classes of the finite-dimensional irreducible representation of Sp_k, and $L(\sigma) := \mathrm{Hom}_{\mathrm{Sp}_k}(V_\sigma, \mathcal{P}(V^k))$, which is an infinite-dimensional irreducible representation of $\mathfrak{o}_{2n}$. It is also well known that the action π restricted to $\mathfrak{k}$ lifts to the complexification $K_\mathbb{C}$ of the maximal compact subgroup K of $G = \mathrm{O}^*(2n)$, which implies that $L(\sigma)$ is an irreducible $(\mathfrak{g}, K_\mathbb{C})$-module.

5. Lagrangian subspace

In this section, we take complex Lagrangian subspaces of $W_\mathbb{C}$ different from the ones considered in the previous sections in the cases where $G = \mathrm{O}^*(2n)$ and $\mathrm{U}(p,q)$, and we quantize the moment map to obtain finite-dimensional representations of $\mathfrak{o}_{2n}$ and the oscillator representation of $\mathfrak{u}(p,q)$. Finally, we make an observation that the image of the Lagrangian subspace coincides with the associated variety of the corresponding irreducible $(\mathfrak{g}, K_\mathbb{C})$ (or $(\mathfrak{g}, \tilde{K}_\mathbb{C})$)-modules occurring in the irreducible decomposition of the space consisting of polynomial functions on the Lagrangian subspace under the joint action of $(\mathfrak{g}, G')$.

5.1.

Let $G = \mathrm{O}^*(2n)$, and let (W, ω) be the symplectic G-vector space we discussed in Section 4, that is, $W = (\mathbb{C}^{2n})_\mathbb{R}$ and ω is given by (4.5). Let us now consider another complex Lagrangian subspace $V' \subset W_\mathbb{C}$ defined by

$$V' := \left\langle \frac{1}{2}(e_1 - \mathrm{i}Ie_1), \dots, \frac{1}{2}(e_{2n} - \mathrm{i}Ie_{2n}) \right\rangle_\mathbb{C} \tag{5.1}$$

and the corresponding quantization

$$\widehat{z}_i = z_i, \qquad \widehat{\bar{z}}_i = -2\epsilon_i \partial_{z_i}, \tag{5.2}$$

for $i = 1, \dots, 2n$ as in Section 3, which also satisfy (4.26). Here I denotes the complex structure on W mentioned in Section 4. Then the quantized moment map, which we denote by the same symbol $\widehat{\mu}$, is given by

$$\begin{aligned} \widehat{\mu} &= -\frac{\mathrm{i}}{2}\left(\begin{bmatrix} \widehat{z}_1 \\ \vdots \\ \widehat{z}_{\bar{n}} \end{bmatrix} (\widehat{\bar{z}}_1, \dots, \widehat{\bar{z}}_{\bar{n}}) I_{n,n} - S I_{n,n} \begin{bmatrix} \widehat{\bar{z}}_1 \\ \vdots \\ \widehat{\bar{z}}_{\bar{n}} \end{bmatrix} (\widehat{z}_1, \dots, \widehat{z}_{\bar{n}}) S \right) \\ &= -\mathrm{i} \begin{bmatrix} -z'{}^t\partial_{z'} + \partial_{z''}{}^t z'' & -z'{}^t\partial_{z''} + \partial_{z''}{}^t z' \\ -z''{}^t\partial_{z'} + \partial_{z'}{}^t z'' & -z''{}^t\partial_{z''} + \partial_{z'}{}^t z' \end{bmatrix} \end{aligned} \tag{5.3}$$

with $z' = {}^t(z_1, \dots, z_n)$, $z'' = {}^t(z_{\bar{1}}, \dots, z_{\bar{n}})$, $\partial_{z'} = {}^t(\partial_{z_1}, \dots, \partial_{z_n})$, and $\partial_{z''} = {}^t(\partial_{z_{\bar{1}}}, \dots, \partial_{z_{\bar{n}}})$. Therefore, in terms of the basis (4.4) for $\mathfrak{g} = \mathfrak{o}_{2n}$, $\pi(X) := \mathrm{i}\langle \widehat{\mu}, X \rangle$

is given by

$$\pi(X)=\begin{cases} z_j\partial_{z_i}-z_{\bar\imath}\partial_{z_{\bar\jmath}} & \text{if } X=X^0_{i,j},\\ z_{\bar\jmath}\partial_{z_i}-z_{\bar\imath}\partial_{z_j} & \text{if } X=X^+_{i,j},\\ z_i\partial_{z_{\bar\jmath}}-z_j\partial_{z_{\bar\imath}} & \text{if } X=X^-_{i,j}.\end{cases}\tag{5.4}$$

Since each $\pi(X)$ preserves the degree of a homogeneous polynomial $f\in\mathcal{P}(V')=\mathbb{C}[z_1,\dots,z_{2n}]$ for $X\in\mathfrak{g}$, any irreducible representation occurring in the irreducible decomposition of $\mathcal{P}(V')$ is finite-dimensional.

5.2.

On the contrary, we will apply the quantization procedure introduced in Section 4 to the case discussed in Section 3. Namely, let $G=\mathrm{U}(p,q)$, and let (W,ω) be the symplectic G-vector space, that is, $W=(\mathbb{C}^{p+q})_{\mathbb{R}}$ and ω is given by (3.2). Now we quantize the complex coordinate functions $z_j=x_j+\mathrm{i}y_j$ and $\bar z_j=x_j-\mathrm{i}y_j$ in the following way (cf. (4.16)):

$$\begin{aligned} \widehat{z}_i&=z_i, & \widehat{\bar z}_i&=-2\partial_{z_i} & (i&=1,\dots,p),\\ \widehat{\bar z}_{\bar\jmath}&=\bar z_{\bar\jmath}, & \widehat{z}_{\bar\jmath}&=-2\partial_{\bar z_{\bar\jmath}} & (j&=1,\dots,q),\end{aligned}\tag{5.5}$$

which also satisfy (3.11). This quantization corresponds to taking a complex Lagrangian subspace $V\subset W_{\mathbb{C}}$ defined by

$$V=\Big\langle \frac12(e_i-\mathrm{i}Ie_i),\frac12(e_{\bar\jmath}+\mathrm{i}Ie_{\bar\jmath}); i=1,\dots,p, j=1,\dots,q\Big\rangle_{\mathbb{C}},\tag{5.6}$$

where I denotes the complex structure on W mentioned in Section 3. For simplicity, we will write $w_j:=\bar z_{\bar\jmath}$, $j=1,\dots,q$, as in the previous section, and write an element of V as $\left[\begin{smallmatrix} z\\ w\end{smallmatrix}\right]$ with $z\in\mathbb{C}^p$ and $w\in\mathbb{C}^q$. Then the quantized moment map, which we denote by the same symbol $\widehat{\mu}$, is given by

$$\begin{aligned}\widehat{\mu}&=-\frac{\mathrm{i}}{2}\begin{bmatrix}\widehat{z}_1\\ \vdots\\ \widehat{z}_n\end{bmatrix}(\widehat{\bar z}_1,\dots,\widehat{\bar z}_n)I_{p,q}=-\frac{\mathrm{i}}{2}\begin{bmatrix} z\\ -2\partial_w\end{bmatrix}(-2\,{}^t\partial_z,-{}^tw)\\ &=\mathrm{i}\begin{bmatrix} z\,{}^t\partial_z & \frac12 z\,{}^tw\\ -2\partial_w{}^t\partial_z & -\partial_w{}^tw\end{bmatrix}\end{aligned}\tag{5.7}$$

with $z={}^t(z_1,\dots,z_p)$, $\partial_z={}^t(\partial_{z_1},\dots,\partial_{z_p})$, $w={}^t(w_1,\dots,w_q)$, and $\partial_w={}^t(\partial_{w_1},\dots,\partial_{w_q})$. In terms of the basis $\{E_{i,j}\}$ for $\mathfrak{g}=\mathfrak{gl}_n$, $\pi(X):=\mathrm{i}\langle\widehat{\mu},X\rangle$ is given by

$$\pi(X)=\begin{cases} -z_j\partial_{z_i} & \text{if } X=E_{i,j}\ (i,j=1,\dots,p),\\ 2\partial_{z_i}\partial_{w_j} & \text{if } X=E_{i,\bar\jmath}\ (i=1,\dots,p; j=1,\dots,q),\\ -\frac12 z_jw_i & \text{if } X=E_{\bar\imath,j}\ (i=1,\dots,q; j=1,\dots,p),\\ \partial_{w_j}w_i & \text{if } X=E_{\bar\imath,\bar\jmath}\ (i,j=1,\dots,q).\end{cases}\tag{5.8}$$

Let us now consider the k direct sum $W^k_{\mathbb{C}}$ and its subspace V^k with V given in (5.6), which is identified with $\mathrm{Mat}_{n\times k}(\mathbb{C})$. Then GL_k acts on V^k on the right by

$$\begin{bmatrix} z \\ w \end{bmatrix} \mapsto \begin{bmatrix} zg \\ w^t g^{-1} \end{bmatrix} \tag{5.9}$$

for $g \in \mathrm{GL}_k$, with $z = (z_{i,a}) \in \mathrm{Mat}_{p\times k}(\mathbb{C})$ and $w = (w_{j,a}) \in \mathrm{Mat}_{q\times k}(\mathbb{C})$ and, hence, on $\mathcal{P}(V^k)$ by right regular representation, which we denote by ρ, as usual. Note that (5.9) is the holomorphic extension of the standard right action of $\mathrm{U}(k)$ on $\mathrm{Mat}_{n\times k}(\mathbb{C})$ given by $Z \mapsto Zg$ for $Z \in \mathrm{Mat}_{n\times k}(\mathbb{C})$ and $g \in \mathrm{U}(k)$. Then, understanding that z and ∂_z (resp., w and ∂_w) in (5.7) stand for $(p \times k)$-matrices $(z_{i,a})$ and $(\partial_{z_{i,a}})$ (resp., $(q\times k)$-matrices $(w_{j,a})$ and $(\partial_{w_{j,a}})$) as in the previous sections, one obtains the following.

THEOREM 5.1

For $X \in \mathfrak{g} = \mathfrak{gl}_n$, set $\pi(X) := \mathrm{i}\langle\widehat{\mu}, X\rangle$. Then the map

$$\pi : \mathfrak{g} \to \mathcal{PD}(V^k)$$

is a Lie algebra homomorphism. Moreover, $\pi(X)$ commutes with the action of GL_k on V^k, that is, $\pi(X) \in \mathcal{PD}(V^k)^{\mathrm{GL}_k}$ for all $X \in \mathfrak{g}$.

Proof

We only show that $\pi(X) \in \mathcal{PD}(V^k)^{\mathrm{GL}_k}$ for $X \in \mathfrak{g}$. It follows from Lemma 2.6 that

$$\begin{aligned} \mathbf{Ad}_{\rho(g)^{-1}} z &= zg^{-1}, \qquad & \mathbf{Ad}_{\rho(g)^{-1}} w &= w^t g, \\ \mathbf{Ad}_{\rho(g)^{-1}} \partial_z &= \partial_z{}^t g, \qquad & \mathbf{Ad}_{\rho(g)^{-1}} \partial_w &= \partial_w g^{-1} \end{aligned}$$

for $g \in \mathrm{GL}_k$. Hence, one obtains that

$$\begin{aligned} \mathbf{Ad}_{\rho(g)^{-1}} \widehat{\mu} &= -\frac{\mathrm{i}}{2} \begin{bmatrix} \mathbf{Ad}_{\rho(g)^{-1}} z \\ -2\mathbf{Ad}_{\rho(g)^{-1}} \partial_w \end{bmatrix} \left[-2^t(\mathbf{Ad}_{\rho(g)^{-1}} \partial_z), {}^t(\mathbf{Ad}_{\rho(g)^{-1}} w)\right] \\ &= -\frac{\mathrm{i}}{2} \begin{bmatrix} zg^{-1} \\ -2\partial_w g^{-1} \end{bmatrix} [-2g^t \partial_z, g^t w] \\ &= -\frac{\mathrm{i}}{2} \begin{bmatrix} z \\ -2\partial_w \end{bmatrix} g^{-1} g[-2^t\partial_z, {}^t w] = \widehat{\mu}. \end{aligned}$$

This completes the proof. □

Therefore, the irreducible decomposition of $\mathcal{P}(V^k)$ is given by

$$\mathcal{P}(V^k) \simeq \sum_{\sigma \in \widehat{\mathrm{GL}}_k, L(\sigma)\neq\{0\}} L(\sigma) \otimes V_\sigma, \tag{5.10}$$

where V_σ is a representative of the class $\sigma \in \widehat{\mathrm{GL}}_k$, the set of all equivalence classes of the finite-dimensional irreducible representation of GL_k, and $L(\sigma) := \mathrm{Hom}_{\mathrm{GL}_k}(V_\sigma, \mathcal{P}(V^k))$ (see [13]). It is also known that $L(\sigma)$ is an irreducible $(\mathfrak{g}, K_\mathbb{C})$-module of infinite dimension for any $\sigma \in \widehat{\mathrm{GL}}_k$ such that $L(\sigma) \neq \{0\}$, where

$\mathfrak{g}=\mathfrak{gl}_{p+q}$ and $K_{\mathbb{C}}$ is the complexification of the maximal compact subgroup K of $G=\mathrm{U}(p,q)$.

5.3.

One can uniquely extend the moment map $\mu : W \to \mathfrak{g}_0$ to the map from $W_{\mathbb{C}}$ into $\mathfrak{g}$, which we denote by $\mu_{\mathbb{C}}$. Then the images by $\mu_{\mathbb{C}}$ of the complex Lagrangian subspaces that have been considered in this and previous sections are all equal to the associated varieties of the corresponding representations, which we will see below case by case.

5.3.1.

First we consider the cases where $G=\mathrm{U}(p,q)$ and $\mathrm{O}^*(2n)$. Let $K_{\mathbb{C}}$ be the complexification of the maximal compact group K of G. Then it is well known that $K_{\mathbb{C}}$ acts on $\mathfrak{p}$ with the irreducible decomposition $\mathfrak{p}=\mathfrak{p}^+\oplus\mathfrak{p}^-$ and that the orbit space decomposition of $\mathfrak{p}^+$ under $K_{\mathbb{C}}$ is given by

$$\mathfrak{p}^+ = \bigsqcup_{j=0}^{r} \mathcal{O}_j^{K_{\mathbb{C}}} \quad (r := \mathbb{R}\text{-}\operatorname{rank} G), \tag{5.11}$$

where

$$\mathcal{O}_j^{K_{\mathbb{C}}} = \left\{ \begin{bmatrix} O & C \\ O & O \end{bmatrix} ; \begin{array}{l} C \in \mathrm{Mat}_{p\times q}(\mathbb{C}), \\ \operatorname{rank} C = j \end{array} \right\} \quad \text{for } G=\mathrm{U}(p,q), \tag{5.12}$$

$$\mathcal{O}_j^{K_{\mathbb{C}}} = \left\{ \begin{bmatrix} O & C \\ O & O \end{bmatrix} ; \begin{array}{l} C \in \mathrm{Mat}_{n\times n}(\mathbb{C}), {}^tC + C = O, \\ \operatorname{rank} C = 2j \end{array} \right\} \quad \text{for } G=\mathrm{O}^*(2n). \tag{5.13}$$

Moreover, if we denote the closure of an orbit $\mathcal{O}$ by $\overline{\mathcal{O}}$, then (see [15])

$$\overline{\mathcal{O}_j^{K_{\mathbb{C}}}} = \bigsqcup_{i\le j} \mathcal{O}_i^{K_{\mathbb{C}}}. \tag{5.14}$$

Therefore, in view of the explicit formulae (3.5) and (4.11), one finds that

$$\mu_{\mathbb{C}}(V'^k) = \overline{\mathcal{O}_0^{K_{\mathbb{C}}}} = \{0\} \qquad \text{and} \qquad \mu_{\mathbb{C}}(V^k) = \overline{\mathcal{O}_m^{K_{\mathbb{C}}}} \tag{5.15}$$

with $m=\min(k,r)$. Since the associated varieties of the finite-dimensional representations and those of the irreducible representations $L(\sigma)$ occurring in (5.10) and (4.28) are equal to $\{0\}$ and $\overline{\mathcal{O}_m^{K_{\mathbb{C}}}}$, respectively (cf. [3]), one concludes that the image of the complex Lagrangian subspace V^k or V'^k by $\mu_{\mathbb{C}}$ coincides with the associated variety corresponding to the irreducible representations occurring in $\mathcal{P}(V^k)$ or in $\mathcal{P}(V'^k)$.

5.3.2.

To see that this is the case for $G=\mathrm{Sp}(n,\mathbb{R})$, we realize the symplectic group over $\mathbb{R}$ as its Cayley transform:

$$G^\gamma := \{\gamma g \gamma^{-1} ; g \in G\} = \mathrm{Sp}_n \cap \mathrm{U}(n,n),$$

with $\gamma = \frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{smallmatrix}\right]$. In the rest of this section, however, let us denote G^γ just by G, and use the same symbols to denote the Cayley transforms of subgroups, Lie algebras, and so on as those of the corresponding objects by abuse of notation if there is no risk of confusion.

One can also obtain the so-called *Fock model* of the oscillator representation by the canonical quantization of the moment map in the following way: let us denote by I the complex structure on $W = \mathbb{R}^{2n}$ defined by $e_i \mapsto e_{\bar{\imath}}$ and $e_{\bar{\imath}} \mapsto -e_i$ for $i = 1, \dots, n$, and introduce complex coordinates $z_i := x_i + \mathrm{i}y_i$ and their conjugates $\bar{z}_i := x_i - \mathrm{i}y_i$, $i = 1, \dots, n$. Namely, we regard $W = \mathbb{R}^{2n}$ as $(\mathbb{C}^n)_{\mathbb{R}}$; more precisely, let us define an $\mathbb{R}$-vector space W_a by $W_a = \{[\begin{smallmatrix} z \\ \bar{z} \end{smallmatrix}]; z \in \mathbb{C}^n\}$ and an $\mathbb{R}$-isomorphism from W_a onto W by

$$\varphi_\gamma : W_a \to W, \qquad \begin{bmatrix} z \\ \bar{z} \end{bmatrix} \mapsto \gamma \begin{bmatrix} z \\ \bar{z} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} z + \bar{z} \\ -\mathrm{i}(z - \bar{z}) \end{bmatrix},$$

for $z = {}^t(z_1, \dots, z_n)$ and $\bar{z} = {}^t(\bar{z}_1, \dots, \bar{z}_n) \in \mathbb{C}^n$. The Cayley transform G acts on W_a by $v \mapsto gv$ (matrix multiplication) for $g \in G$ and $v \in W_a$, with respect to which φ_γ is equivariant. Moreover, one sees that $\varphi_\gamma^* \omega = \frac{\mathrm{i}}{2} \sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$ and that the moment map $\mu : W_a \to \mathfrak{g}_0$ is given by

$$\mu(v) = \frac{\mathrm{i}}{2} v\, {}^t v J_n = \frac{\mathrm{i}}{2} \begin{bmatrix} -z\,{}^t\bar{z} & z\,{}^t z \\ -\bar{z}\,{}^t\bar{z} & \bar{z}\,{}^t z \end{bmatrix} \tag{5.16}$$

for $v = {}^t(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n) \in W_a$.

REMARK 5.2

If we temporarily distinguish the Cayley transform $\mathfrak{g}_0^\gamma$ from $\mathfrak{g}_0$ only in this remark, it is easily verified that the following diagram is commutative:

$$\begin{array}{ccc} W_a & \xrightarrow{\ \mu\ } & \mathfrak{g}_0^\gamma \\ {\scriptstyle \varphi_\gamma}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{Ad}(\gamma)} \\ W & \xrightarrow[\ \mu\ ]{} & \mathfrak{g}_0 \end{array} \tag{5.17}$$

where the upper horizontal map denotes the moment map given by (5.16), while the lower horizontal one is given by (2.5).

The Poisson brackets among z_i and $\bar{z}_i$ are given by (3.9) with all $\epsilon_i = 1$. Therefore, we quantize z_i and $\bar{z}_i$ by assigning

$$\widehat{z}_i = z_i, \qquad \widehat{\bar{z}}_i = -2\partial_{z_i}, \tag{5.18}$$

so that they satisfy (3.11) with all $\epsilon_i = 1$. This quantization corresponds to the choice of the complex Lagrangian subspace V of $W_{\mathbb{C}} = \mathbb{C}^{2n}$ given by

$$V = \Big\langle \frac{1}{2}(e_1 - \mathrm{i}e_{\bar{1}}), \dots, \frac{1}{2}(e_n - \mathrm{i}e_{\bar{n}}) \Big\rangle_{\mathbb{C}}. \tag{5.19}$$

Then the quantized moment map, which we denote by $\widehat{\mu}$ as always, is given by

$$
\begin{aligned}
\widehat{\mu} &= \frac{\mathrm{i}}{2}\begin{bmatrix}\widehat{z}_1\\ \vdots \\ \widehat{\overline{z}}_n\end{bmatrix}(\widehat{z}_1,\dots,\widehat{\overline{z}}_n)J_n = \frac{\mathrm{i}}{2}\begin{bmatrix} z \\ -2\partial_z\end{bmatrix}({}^t z, -2{}^t\partial_z)J_n \\
&= \mathrm{i}\begin{bmatrix} z\,{}^t\partial_z & \frac{1}{2}z\,{}^t z \\ -2\partial_z\,{}^t\partial_z & -\partial_z\,{}^t z\end{bmatrix}
\end{aligned}
\tag{5.20}
$$

with $z = {}^t(z_1,\dots,z_n)$, $\partial_z = {}^t(\partial_{z_1},\dots,\partial_{z_n})$. In terms of the basis $\{X^{\star}_{i,j}\}$ for $\mathfrak{g} = \mathfrak{sp}_n$, $\pi(X) := \mathrm{i}\langle\widehat{\mu}, X\rangle$ is given by

$$
\pi(X) = \begin{cases} -\frac{1}{2}(z_j\partial_{z_i} + \partial_{z_i} z_j) & \text{if } X = X^0_{i,j}, \\ 2\partial_{z_i}\partial_{z_j} & \text{if } X = X^+_{i,j}, \\ -\frac{1}{2}z_j z_i & \text{if } X = X^-_{i,j}. \end{cases}
\tag{5.21}
$$

Now let us take the k direct sum $W^k_{\mathbb{C}}$ and its subspace V^k, with V as in (5.19). When V^k is identified with $\mathrm{Mat}_{n\times k}(\mathbb{C})$, GL_k acts on V^k on the right and, hence, on $\mathcal{P}(V^k)$ by right regular representation. Then, if one understands that z and ∂_z in (5.20) stand for $(n\times k)$-matrices $(z_{i,a})$ and $(\partial_{z_{i,a}})$, respectively, and sets $\pi(X) = \mathrm{i}\langle\widehat{\mu}, X\rangle$ for $X \in \mathfrak{g} = \mathfrak{sp}_n$, then one can show that the map $\pi : \mathfrak{g} \to \mathcal{PD}(V^k)$ is a Lie algebra homomorphism and that $\pi(X) \in \mathcal{PD}(V^k)^{\mathrm{O}_k}$ for all $X \in \mathfrak{g}$. The irreducible decomposition of $\mathcal{P}(V^k)$ under the joint action of $(\mathfrak{sp}_n, \mathrm{O}_k)$ is of course the same as (2.18).

It is known that the orbit space decomposition under $K_{\mathbb{C}}$ of $\mathfrak{p}^+$ is given by the same formula as (5.11) with

$$
\mathcal{O}^{K_{\mathbb{C}}}_j = \left\{ \begin{bmatrix} O & C \\ O & O\end{bmatrix};\ \begin{matrix} C \in \mathrm{Mat}_{n\times n}(\mathbb{C}), {}^tC = C, \\ \operatorname{rank} C = j \end{matrix} \right\},
$$

and its closure $\overline{\mathcal{O}^{K_{\mathbb{C}}}_j}$ is given by the same formula as (5.14) (see [15]). Therefore, in view of (5.16), one finds that

$$
\mu_{\mathbb{C}}(V^k) = \overline{\mathcal{O}^{K_{\mathbb{C}}}_m}
\tag{5.22}
$$

with $m = \min(k, r)$. Hence, the image of the complex Lagrangian subspace V^k by $\mu_{\mathbb{C}}$ coincides with the associated variety corresponding to the irreducible representations occurring in $\mathcal{P}(V^k)$, as in the previous cases.

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