Seminatural bundles of rank two, degree one, and $c_2 = 10$ on a quintic surface

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To the memory of Professor Masaki Maruyama

Abstract In this paper we continue our study of the moduli space of stable bundles of rank two and degree 1 on a very general quintic surface. The goal in this paper is to understand the irreducible components of the moduli space in the first case in the "good" range, which is $c_2 = 10$. We show that there is a single irreducible component of bundles which have seminatural cohomology, and we conjecture that this is the only component for all stable bundles.

0. Introduction

This paper is the next in a series, starting with [13], in which we study the moduli spaces of rank two bundles of odd degree on a very general quintic hypersurface $X \subset \mathbb{P}^3$. This series is dedicated to Professor Maruyama, who brought us together in the study of moduli spaces, a subject in which he was one of the first pioneers.

In the first paper, we showed that the moduli space $M_X(2, 1, c_2)$, of stable bundles of rank 2, degree 1, and given c_2, is empty for $c_2 \leq 3$, irreducible for $4 \leq c_2 \leq 9$, and good (i.e., generically smooth of the expected dimension $4c_2 - 20$) for $c_2 \geq 10$. On the other hand, Nijsse [15] has shown that the moduli space is irreducible for $c_2 \geq 16$ using the techniques of O'Grady [16] and [17]. This leaves open the question of irreducibility for $10 \leq c_2 \leq 15$.

CONJECTURE 0.1

The moduli space $M_X(2, 1, 10)$ is irreducible.

We have not yet formulated an opinion about the cases $11 \leq c_2 \leq 15$.

In the present paper, due to lack of time and for length reasons, we treat a special case of the conjecture: the case of bundles with *seminatural cohomology*, meaning that only at most one of $h^0(E(n))$, $h^1(E(n))$, or $h^2(E(n))$ can be nonzero for each n. Let $M_X^{sn}(2, 1, 10)$ denote the open subvariety of the moduli space consisting of bundles with seminatural cohomology. In Section 3 we show that

the seminatural condition is a consequence of assuming just $h^0(E(1)) = 5$. The main result of this paper is as follows.

THEOREM 0.2

The moduli space $M_X^{sn}(2, 1, 10)$ is irreducible of dimension 20.

Recall from [13] that our inspiration to look at this question came from the recent results of Yoshioka, for the case of Calabi–Yau surfaces originating in [14]. Yoshioka [20] and [21] shows that the moduli spaces are irreducible for all positive values of c_2, when X is an abelian or K3 surface. His results apply, for example, when X is a general quartic hypersurface. We thought it was a natural question to look at the case of a quintic hypersurface, which is one of the first cases where X has general type, with $K_X = \mathcal{O}_X(1)$ being as small as possible.

Remark on the difficulty of this project. We were somewhat surprised by the diversity of techniques needed to treat this question. Much of the difficulty stems from the possibilities of overdetermined intersections which need to be ruled out at various places in the argument. This question is inherently very delicate, because there is not, to our knowledge, any general principle which would say whether the moduli space is *supposed to be* irreducible or not. On the one hand, the present case is close to the abelian or K3 case, so it is not too surprising if the moduli space remains irreducible; however, on the other hand, at some point new irreducible components will be appearing, as has been shown by the first author in [12]. So, we are led to analyze a number of cases for various aspects of the argument. If any case is mistakenly ignored, it might hide a new irreducible component which would then be missed.

A natural question to wonder about is whether "derived algebraic geometry" could help here, but it would seem that those techniques need to be further developed in order to apply to some basic geometric situations such as we see here. Furthermore, each place in the argument where some case is ruled out constitutes a possible reason for there to be additional irreducible components in more complicated situations (such as on a sextic hypersurface). So, in addition to the theorem itself, which only goes a little way into the range that remains to be treated, much of the interest lies in the geometric situations which are encountered along the way.

1. Notation and outline

Throughout the paper, $X \subset \mathbb{P}^3$ denotes a very general quintic hypersurface, and E is a stable rank two vector bundle of degree one* with determinant $\bigwedge^2 E \cong$

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*This represents a change in notation from [13], where we considered bundles of degree -1. For the present considerations, bundles of degree 1 are more practical in terms of Hilbert polynomials. We apologize for this inconvenience, but luckily the indexation by second Chern class stays the same. Indeed, if E has degree 1, then $c_2(E) = c_2(E(-1))$ as can be seen, for

$\mathcal{O}_X(1)$ such that $c_2(E) = 10$. The moduli space of stable bundles in general has been the subject of much work (see [3]–[6], [9]–[11], [14], [16], [17], [20], [21]), but the special case $M_X(2, 1, 10)$ considered here goes into somewhat new and uncharted territory.

Note that $\operatorname{Pic}(X) = \mathbb{Z}$ is generated by $\mathcal{O}_X(1)$. The canonical bundle is $K_X = \mathcal{O}_X(1)$. For any n we have $H^1(\mathcal{O}_X(n)) = 0$. For $n \leq 4$ the map $H^0(\mathcal{O}_{\mathbb{P}^3}(n)) \to H^0(\mathcal{O}_X(n))$ is an isomorphism.

The Hilbert polynomial of E is $\chi(E(n)) = 5n^2$. In particular, $\chi(E) = 0$. We will be assuming that E is general in some irreducible component of the moduli space. From the previous paper [13] using some techniques for bounding the singular locus which had also been introduced in [8] and [22], it follows that E is unobstructed, so if $\operatorname{End}^0(E)$ denotes the trace-free part of $\operatorname{End}(E)$, then $H^2(\operatorname{End}^0(E)) = 0$. Note however that $H^2(\mathcal{O}_X) = \mathbb{C}^4$; indeed, it is dual to $H^0(\mathcal{O}_X(1)) = H^0(\mathcal{O}_{\mathbb{P}^3}(1))$. Thus

$$H^2(E \otimes E^*) \cong \mathbb{C}^4.$$

The dual bundle is given by $E^* = E(-1)$, so duality says that $H^i(E(n)) \cong H^{2-i}(E(-n))$.

The dimension of any irreducible component of the moduli space is the expected one, 20. The subspace of bundles E with $H^0(E) \neq 0$ has dimension < 20 (see our previous paper [13]), so a general E has $H^0(E) = 0$. It follows from duality that $H^2(E) = 0$, and by $\chi(E) = 0$ we get $H^1(E) = 0$. Throughout the paper (except at one place in Section 10), we consider only bundles with $H^0(E) = 0$.

Duality says that $H^2(E(1))$ is dual to $H^0(E(-1)) = 0$. Since $\chi(E(1)) = 5$, if we set $f := h^1(E(1))$, then $h^0(E(1)) = 5 + f$. In particular there are at least 5 linearly independent sections of $E(1)$ which may be viewed as maps $s : \mathcal{O}_X(-1) \to E$ or, equivalently, $s : \mathcal{O}_X \to E(1)$. Note that the zero set of s has to be of codimension 2, as any codimension-one component would be a divisor integer multiple of the hyperplane class, but $h^0(E) = 0$, so this cannot happen. If we choose one such map s, then we get the *standard exact sequence*

$$(1.1) \qquad\qquad 0 \to \mathcal{O}_X(-1) \to E \to J_{P/X}(2) \to 0$$

and its twisted versions such as

$$(1.2) \qquad\qquad 0 \to \mathcal{O}_X \to E(1) \to J_{P/X}(3) \to 0.$$

Here $J_{P/X}$ denotes the ideal of $P \subset X$.

One of our main tools is to consider the subscheme of zeros $P \subset X$ as a subscheme $P \subset \mathbb{P}^3$ with ideal sheaf $J_{P/\mathbb{P}^3}$. These two ideal sheaves are related by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(n-5) \to J_{P/\mathbb{P}^3}(n) \to J_{P/X}(n) \to 0.$$

example, on the bundle $E = \mathcal{O}_X \oplus \mathcal{O}_X(1)$ with $c_2(E) = c_2(E(-1)) = 0$. Thus, the moduli space of stable bundles $M_X(2, 1, c_2)$ we look at here is isomorphic to $M_X(2, -1, c_2)$ considered in [13].

It follows that for any $n < 5$, $h^0(J_{P/X}(n)) = h^0(J_{P/\mathbb{P}^3}(n))$, so these are inter-changeable in this case. Similarly, for any n we have $h^1(J_{P/X}(n)) = h^1(J_{P/\mathbb{P}^3}(n))$.

Calculation of the Chern class $c_2(E) = 10$ shows that $P \subset X$ is a subscheme of length 20. It is a union of possibly nonreduced points, which are locally complete intersections, that is, defined by two equations. Furthermore, as is classically well known (see [2], [7]), P satisfies the Cayley–Bacharach condition for $\mathcal{O}_X(4)$ which we denote by CB(4), saying that any subscheme $P' \subset P$ of colength 1 imposes the same number of conditions as P on sections of $\mathcal{O}_X(4)$. The extension class is governed by an element of $H^1(J_{P/X}(4))^*$, and we have the exact sequence

$$0 \to H^0\big(J_{P/X}(4)\big) \to H^0\big(\mathcal{O}_X(4)\big) \to \mathcal{O}_P(4) \to H^1\big(J_{P/X}(4)\big) \to 0.$$

To get a locally free E, the extension class should be nonzero on each vector coming from a point in P, the existence of such a class being exactly the condition CB(4). Note that $h^0(\mathcal{O}_X(4)) = 35$, and define $e := h^1(J_{P/X}(4)) - 1$. Then $e \geq 0$ (the extension cannot be split; indeed this is part of the CB(4)-condition), and $h^0(J_{P/X}(4)) = 16 + e$.

The "well-determined" case is when $e = 0$. Then the extension class is well defined up to a scalar multiple which does not affect the isomorphism class of E, and the existence of the nonzero class in $H^1(J_{P/X}(4))$ is expected to impose 16 conditions on the 20 points, giving 24 the expected dimension of the Hilbert scheme of such subschemes $P \subset X$. When these are viewed as subschemes $P \subset \mathbb{P}^3$ with $h^1(J_{P/\mathbb{P}^3}(4)) = 1$, the expected dimension is 44. In Section 3, we will show $f = 0 \Rightarrow e = 0$, and in that case the bundle E has seminatural cohomology.

Outline of the proof

Here now is the overall structure of the proof of Theorem 0.2. The next section discusses some preliminary techniques and results. In Section 3, we consider the condition of seminaturality for E. This means that for any n, at most one cohomology group of $E(n)$ is nonzero. For our case, it means more precisely that $h^0(E(n)) = h^1(E(n)) = 0$ for $n \leq 0$; and $h^1(E(n)) = h^2(E(n)) = 0$ for $n \geq 0$. In particular, $h^i(E) = 0$ for $i = 0, 1, 2$. The cohomology dimensions are obtained from the Euler characteristic, for example, $h^0(E(1)) = 5$. Our main object of study will be a nonzero section $s \in H^0(E(1))$, with zero scheme denoted $P \subset X \subset \mathbb{P}^3$. As we have said above, the bundle is then expressed (see (1.1)) as an extension of $J_{P/X}(2)$ by $\mathcal{O}_X(-1)$, and the subscheme $P \subset X$, which has length 20, is a local complete intersection (lci), and has to satisfy CB(4). The seminaturality condition implies $e = 0$, so the extension class governing this extension is unique up to a scalar. Furthermore, seminaturality of E is equivalent to the conditions $h^0(J_{P/X}(3)) = 4$ and $h^0(J_{P/X}(2)) = 0$.

We will be looking at several moduli schemes, Hilbert schemes, and incidence schemes. The moduli space of interest is, for fixed general X, the moduli space of seminatural bundles E (which are, as always, semistable of degree 1 with $c_2 = 10$). Over this moduli space, the $H^0(E(1))$ form a bundle of rank 5, so we may look at the associated $\mathbb{P}^4$-bundle which is the moduli of pairs (E, s) where

$s \in H^0(E(1))$ is a nonzero section. Since it is a projective space bundle over the moduli space, it suffices to prove that it is irreducible. In view of the unicity of the extension class, this is isomorphic to the Hilbert scheme of finite local complete intersection subschemes $P \subset X$ of length 20 which satisfy CB(4) and the seminaturality conditions $h^0(J_{P/X}(3)) = 4$ and $h^0(J_{P/X}(2)) = 0$, denoted by $\mathbf{H}_X^{sn}$ in Section 4.3. We would like to show that $\mathbf{H}_X^{sn}$ is irreducible.

In the middle parts of the proof, we will disregard X and consider the Hilbert scheme $\mathbf{H}_{\mathbb{P}^3}^{sn}$ of subschemes $P \subset \mathbb{P}^3$ which are contained in at least one smooth quintic hypersurface X, which are lci, satisfy CB(4), and satisfy the seminaturality conditions which may be written as $h^0(J_{P/\mathbb{P}^3}(3)) = 4$ and $h^0(J_{P/\mathbb{P}^3}(2)) = 0$ so they do not depend on X. We will prove that $\mathbf{H}_{\mathbb{P}^3}^{sn}$ is irreducible. An argument is needed to go from this statement to irreducibility of $\mathbf{H}_X^{sn}$. This will be done in Section 10, and we refer the reader there for an explanation of how this is done using the very nice idea given to us by Hirschowitz for the previous paper.

Before getting to the middle of the proof, we first look more closely at our seminatural bundle E over X in Section 4. The goal is to get as much information as possible on the structure of the zero scheme P of a general section s. Ideally, we would like to show that P is a set of 20 distinct points. However, it turns out that we could only show that P is a union of the form $P' \cup P''$ where P', located at a point independent of s, is either empty, a single point, or a point of multiplicity 2; and P'' consists of 20, 19, or 18 distinct points which move around in X and are permuted doubly transitively by the Galois action as the section s varies. The basic idea for showing this property is to look at lines in $\mathbb{P}^3$ which cut the surface in 5 points and to analyze the conditions which these points might impose on s.

In Section 4.3 we discuss the Hilbert schemes of $P \subset X$ and $P \subset \mathbb{P}^3$ and give their dimensions. We prove an important technical result, Proposition 4.5, saying that the locus of points where $E(1)$ is not generated by global sections has dimension zero. This allows us to continue in Section 5 with the analysis of the structure of the zero scheme $P = P' \cup P''$, completing the proof of the property described in the previous paragraph.

Section 6 begins what might be called the middle phase of the proof, investigating the Hilbert scheme $\mathbf{H}_{\mathbb{P}^3}^{sn}$ of subschemes $P \subset \mathbb{P}^3$ which can occur as zero schemes of sections s. From the first part of the proof, we know that a general element P will decompose as $P = P' \cup P''$ with the properties described before. We introduce something new: choose a general subspace $U \subset H^0(J_{P/\mathbb{P}^3}(3))$ of dimension 2, which defines a subscheme $Z \subset \mathbb{P}^3$. In view of what was said before, we can show that Z has dimension 1, that is, that it is a complete intersection of two cubics. In particular, it is a curve of degree 9. The next technical difficulty that needs to be discussed is the fact that the general Z could well be reducible, breaking up into a union of curves with total degree 9. Using the structure results on $P = P' \cup P''$, we are able to analyze how P' and P'' might meet the different components of Z. The main techniques here are various kinds of

dimension counts, aiming to show that different special cases have to correspond to subvarieties of smaller dimension in the total space of (P, U).

The first step, in Section 6, is to understand what the possible decompositions of Z can look like, and what are the dimensions of the strata. This seems to be an interesting and difficult question in general, because there might be strata parameterizing curves Z which are complete intersections of two cubics, containing nonreduced components. We do not discuss the most complicated of these strata, hence the subscript $(\ )_{\mathrm{rci}}$ on our parameter spaces indicating *reduced complete intersections*. The structure results for $P = P' \cup P''$ allow us to argue in most cases that (P, U) leads to a reduced Z, with a dimension estimate for the possibly nonreduced case covered by Lemma 6.5.

In Section 7 we consider the "common curve case," where for a fixed subscheme P, the family of complete intersection curves Z defined by $U \subset H^0(J_{P/\mathbb{P}^3}(3))$ contains a component Q_1 which is fixed, in the sense that it does not depend on U, and the big collection of "almost all" points $P'' \subset P$ is contained in Q_1. Of course, the potential extra point P' could be contained in the variable part. By a dimension count, we show that this case corresponds to a strict subvariety of the Hilbert scheme.

We therefore conclude in Theorem 7.4 that the general point must correspond to the "variable curve case," meaning that the component of Z which contains the big moving collection of at least 18 points P'' has to be movable as a function of the choice of $U \subset H^0(J_{P/\mathbb{P}^3}(3))$. In Section 8, we consider this case, with the further hypothesis that the general Z has two or more irreducible components. The points of P'' have to lie in the same component Z'' of Z by double transitivity of the Galois action, and it turns out that Z'' has degree ≥ 6. A dimension count then allows us to conclude that the space of possibilities here has too small a dimension, so again this cannot contribute a general point.

These reductions show that a general P in any irreducible component must be contained in a complete intersection Z of two cubics, such that Z is reduced and irreducible of degree 9. This then is the general case, treated in Section 9. The reader might wish to start by consulting the beginning of that section, since it contains a description of the nonempty irreducible open set of the Hilbert scheme consisting of $P \subset Z$ where Z is a smooth complete intersection of two cubic hypersurfaces. The proof of Theorem 9.1, saying that this is the only irreducible component of $\mathbf{H}^{sn}_{\mathbb{P}^3}$, consists of an analysis of what happens when Z acquires singularities: more dimension counting shows that the singular case contributes a subvariety of strictly smaller dimension.

As was said before, Section 10 is devoted to a monodromy argument for going from irreducibility of $\mathbf{H}^{sn}_{\mathbb{P}^3}$ to irreducibility of $\mathbf{H}^{sn}_X$ and hence of the moduli space of seminatural bundles on a fixed general quintic X, to complete the proof of Theorem 0.2. The basic idea is to isolate one preferred irreducible component of $\mathbf{H}^{sn}_X$ which will be preserved by the Galois action as X moves around in its parameter space. Irreducibility of the total space $\mathbf{H}^{sn}_{\mathbb{P}^3}$ then implies that the preferred component must be the only one.

In Section 11, we give a heuristic discussion of some ideas for treating the moduli space of bundles which are no longer necessarily seminatural, to show that no new irreducible components appear (see Conjecture 0.1). We hope to treat this case in detail in a future paper.

2. Preliminaries

2.1. Cayley–Bacharach

A subscheme $P \subset X$ satisfies the *Cayley–Bacharach condition* CB(n) if, for any subscheme $P' \subset P$ of colength 1 (i.e., the kernel of $\mathcal{O}_P \to \mathcal{O}_{P'}$ has length 1), the map

$$H^0\big(J_{P/X}(n)\big) \to H^0\big(J_{P'/X}(n)\big)$$

is an isomorphism. In other words, P imposes the same conditions on degree n forms as any colength 1 subscheme. We have the same terminology for $P \subset \mathbb{P}^3$, and if $P \subset X \subset \mathbb{P}^3$, then CB($n$) for $P \subset X$ is equivalent to CB(n) for $P \subset \mathbb{P}^3$, so we do not distinguish the notations.

LEMMA 2.1

If a zero-dimensional subscheme $P \subset X$ satisfies CB(n)*, then it satisfies* CB(m) *for any $m \leq n$.*

Proof

Suppose $P' \subset P$ has colength 1. Choose a section $g \in H^0(\mathcal{O}_X(n-m))$ nonvanishing at all points of P; then if $f \in H^0(J_{P'/X}(m))$, we have $fg \in H^0(J_{P'/X}(n))$. By CB(n), fg vanishes on P, but g is a unit near any point of P so f vanishes on P, proving CB(m). $\qquad\square$

As was noted previously, the Cayley–Bacharach condition corresponds exactly to existence of a locally free extension of the ideal sheaf of P; refer to the previous section for the notation.

2.2. Subschemes of locally planar curves

The results of [1] allow us to estimate the dimension of the Hilbert scheme of zero-dimensional subschemes of a curve, as was used in some detail in [13]. Mainly, as soon as the curve is locally planar or even locally embeddable in a smooth surface, the space of subschemes of length ℓ has dimension $\leq \ell$.

2.3. Residual subschemes

If $W \subset X$ is a divisor and P is a zero-dimensional subscheme, we obtain the *residual subscheme* $P^\perp$ of P with respect to W, such that $\ell(P^\perp) + \ell(P \cap W) = \ell(P)$. It is characterized by the property that sections of $\mathcal{O}_X(n)(-W)$ which vanish on $P^\perp$, map to sections of $\mathcal{O}_X(n)$ vanishing on P. If P is reduced, then $P^\perp$ is just the union of those points of P which are not in W; if P contains some nonreduced schematic points, then the structure of $P^\perp$ may be more complicated.

LEMMA 2.2

If P satisfies CB(3) *and $P'' \subset P$ is a subscheme of colength 2, suppose P'' is contained in a quadric. Then P is contained in the same quadric.*

Proof

The residual subscheme $P^\perp$ for the quadric has length ≤ 2. If it is nonempty, we can choose a linear form containing a subscheme of colength 1 of $P^\perp$, corresponding to a subscheme $P^1 \subset P$ of colength 1. Applying CB(3) to the product of the quadric and the linear form is a contradiction, so $P^\perp = \emptyset$ and we are done. $\square$

2.4. Restriction to a plane section

Suppose $H \subset \mathbb{P}^3$ is a hyperplane, and let $Y := H \cap X$. By the genericity assumption on X, in particular $\mathrm{Pic}(X)$ generated by $\mathcal{O}_X(1)$, we get that Y has to be reduced and irreducible. Its canonical sheaf is $\mathcal{O}_Y(2)$. When Y is smooth, then, it is a plane curve of degree 5 and genus 6. We have an exact sequence

$$0 \to E \to E(1) \to E_Y(1) \to 0.$$

From the vanishing of $H^i(E)$ it follows that $H^2(E(1)) = 0$ (but this is also clear from duality), and

$$H^0\big(E(1)\big) \xrightarrow{\cong} H^0\big(E_Y(1)\big),$$

$$H^1\big(E(1)\big) \xrightarrow{\cong} H^1\big(E_Y(1)\big).$$

Suppose $L \subset \mathbb{P}^3$ is a line. A generic X does not contain any lines, so $L \cap X$ is a finite subscheme of length $\ell(L \cap X) = 5$. We claim that for a general plane H containing L, the intersection $Y = H \cap X$ is smooth. This holds by Bertini's theorem away from the base locus of the linear system of planes passing through L, so we just have to see that it also holds at a point $x \in L \cap X$. Note that $T_x L \subset T_x X$ is a one-dimensional subspace. A general H will have tangent space which is a general plane in $T_x \mathbb{P}^3$ containing $T_x L$. Thus, a general plane H containing L has tangent space $T_x H$ which does not contain $T_x X$; in particular the intersection $T_x H \cap T_x X = T_x L$ is transverse. This implies that $H \cap X$ is smooth at x. This works for all the finitely many points $x \in L \cap X$, so the general section $Y = H \cap X$ is smooth. It is therefore a smooth plane curve of degree 5 and genus 6. Notice that $L \subset H$, so $L \cap X \subset Y$.

Pick Y as in the previous paragraph, suppose that $Q \subset L \cap X$ is a finite subscheme of length 4, and suppose that $x \in H^0(E(1))$ is a section vanishing on Q. Then $s|_Y$ is a section of $H^0(E(1))$ vanishing on $Q \subset Y$. As Y is smooth, the finite subscheme Q is a Cartier divisor. The section $s|_Y$ corresponds to a map $\mathcal{O}_Y \to E(1)$ which, since it vanishes on Q, gives a map

$$\mathcal{O}_Y(Q) \to E(1).$$

Let $Q' \subset Y$ be the divisor of zeros of s, in particular $Q \subset Q'$, and s extends to a strict map, that is, an inclusion of a subvector bundle

$$\mathcal{O}_Y(Q') \hookrightarrow E(1).$$

The quotient line bundle is $\mathcal{O}_Y(3-Q')$ where the notation here combines $\mathcal{O}_Y(3)$, which is three times the hyperplane divisor (which has degree 5 on Y), with the divisor Q'. In particular, $\mathcal{O}_Y(3-Q')$ is a line bundle of degree $15-\ell(Q')$. We obtain an exact sequence

$$0 \to \mathcal{O}_Y(Q') \to E_Y(1) \to \mathcal{O}_Y(3-Q') \to 0$$

leading to the long exact sequence of cohomology. This construction will be used many times in Section 4.

Another useful construction is the following. Write $L \cap Y = x+y+u+v+w$, possibly with some of the points being the same. Take a linear form containing w as an isolated zero, and divide by the equation of L. This gives a meromorphic function whose polar divisor is $x+y+u+v$. Equivalently, $\mathcal{O}_Y(x+y+u+v)$ has a section nonvanishing at the points x,y,u,v. This will be used often without too much further notice below.

3. The seminatural condition

HYPOTHESIS 3.1
Assume that $h^0(E(1)) = 5$, and assume that $H^i(E) = 0$ for $i = 0,1,2$.

Recall that the second part is true for any E general in its irreducible component as discussed above.

The goal of this section is to show that Hypothesis 3.1 implies that $f = 0$ and E has seminatural cohomology, which in this case means $H^0(E(n)) = 0$ for $n \le 0$, $H^2(E(n)) = 0$ for $n \ge 0$, and $H^1(E(n)) = 0$ for all n. Our main theorem (Theorem 0.2) is the statement that there is only a single irreducible component corresponding to such bundles, so Hypothesis 3.1 will be in effect throughout the rest of the paper.

LEMMA 3.2
If $h^0(E(1)) = 5$, then $f = 0$; in other words, $H^1(E(1)) = 0$. If $s : \mathcal{O}(-1) \to E$ has a scheme of zeros P, then saying $h^0(E(1)) = 5$ is equivalent to requiring that $h^0(J_{P/X}(3)) = 4$, and saying that all $h^i(E) = 0$ is equivalent to requiring that $h^0(J_{P/X}(2)) = 0$. These conditions are also the same as saying $h^0(J_{P/\mathbb{P}^3}(3)) = 4$ and $h^0(J_{P/\mathbb{P}^3}(2)) = 0$.

Proof
As discussed above, $h^2(E(1)) = 0$, so the fact that $\chi(E(1)) = 5$ gives the first statement. For the second statement, use the fact that $H^1(\mathcal{O}_X(n)) = 0$ for all n, and use the long exact sequences of cohomology for the extension $E(1)$ of $J_{P/X}(3)$ by $\mathcal{O}_X$ and similarly E of $J_{P/X}(2)$ by $\mathcal{O}_X(-1)$. For the last phrase recall that $h^0(J_{P/X}(n)) = h^0(J_{P/\mathbb{P}^3}(n))$ for $n < 5$ because $h^0(\mathcal{O}_X(n)) = h^0(\mathcal{O}_{\mathbb{P}^3}(n))$. $\qquad\square$

The first part of the seminatural condition is easy to see.

LEMMA 3.3

Under our Hypothesis 3.1, $H^0(E(n)) = 0$ for $n \leq 0$, and $H^2(E(n)) = 0$ for $n \geq 0$.

Proof
Since $H^0(E) = 0$, it follows that $H^0(E(n)) = 0$ for all $n \leq 0$, and for $n \geq 0$, $H^2(E(n))$ is dual to $H^0(E(-n)) = 0$. $\qquad\square$

The main step towards the seminatural condition is the next twist.

PROPOSITION 3.4

We also have $H^1(E(2)) = 0$.

Proof
If $Y = H \cap X$ is a smooth plane section, we claim $H^0(E_Y(-1)) = 0$. If not, then we would get an inclusion $\mathcal{O}_Y(1) \hookrightarrow E_Y$, hence $\mathcal{O}_Y(2) \hookrightarrow E_Y(1)$. However, Y is a curve of genus 6 and $K_Y = \mathcal{O}_Y(2)$, so $H^0(\mathcal{O}_Y(2))$ has dimension 6. This gives $h^0(E_Y(1)) \geq 6$. Consider the exact sequence

$$0 \to E \to E(1) \to E_Y(1) \to 0.$$

The fact that $H^1(E) = 0$ implies that $H^0(E(1))$ surjects onto $H^0(E_Y(1))$, so $h^0(E(1)) \geq 6$. This is a contradiction to $h^0(E(1)) = 5$, showing that $H^0(E_Y(-1)) = 0$.

To show that $H^1(E(2)) = 0$, it suffices by duality to show that $H^1(E(-2)) = 0$. Consider the exact sequence

$$0 \to E(-2) \to E(-1) \to E_Y(-1) \to 0.$$

Again by duality from Lemma 3.2, $H^1(E(-1)) = 0$, so the long exact sequence gives an isomorphism between $H^0(E_Y(-1))$ and $H^1(E(-2))$. From the previous paragraph we obtain $H^1(E(-2)) = 0$. This proves the proposition. $\qquad\square$

COROLLARY 3.5

Under Hypothesis 3.1, E has seminatural cohomology: $H^1(E(n)) = 0$ for all n.

Proof
By duality it suffices to consider $n \geq 0$, and we have already done $n = 0, 1, 2$. Consider the case $n = 3$. This could be done by continuing as in Proposition 3.4, but here is another argument. Choose an inclusion $s : \mathcal{O}(-1) \to E$, and let P be the subscheme of zeros of s. Choose a general plane section $Y = H \cap X$ such that H passes through one point $z \in P$ in a general direction. Then $s|_Y$ has a zero at z, of multiplicity m with $1 \leq m \leq 5$. Indeed, P cannot contain a 6-fold fat point whose length is 21, because P has length 20. Thus the multiplicity of a general plane section of P at any point z is ≤ 5. The section s restricted to Y therefore induces a strict inclusion of vector bundles from $\mathcal{O}_Y(m \cdot z)$ to $E(1)$, hence an exact sequence, of the form

$$0 \to \mathcal{O}_Y(2 + m \cdot z) \to E(3) \to \mathcal{O}_Y(5 - m \cdot z) \to 0.$$

Note that the subline bundle has degree $10 + m$ and the quotient line bundle has degree $25 - m$, so both of these have vanishing H^1 by duality. It follows that $H^1(E(3)) = 0$. For any $n \geq 4$ a similar argument (but Y does not even need to pass through a point of P) shows that $H^1(E(n)) = 0$. $\qquad\square$

COROLLARY 3.6

It follows that $e = 0$, which is to say that for any inclusion $s : \mathcal{O}(-1) \to E$, if P is the subscheme of zeros of s, then $h^0(J_{P/X}(4)) = 16$.

Proof
Choose an inclusion s, and consider the exact sequence

$$0 \to \mathcal{O}_X(1) \to E(2) \to J_{P/X}(4) \to 0.$$

Notice that $H^2(\mathcal{O}(1)) = H^2(K_X) = \mathbb{C}$. The long exact sequence of cohomology then reads

$$0 \to H^1(E(2)) \to H^1(J_{P/X}(4)) \to \mathbb{C} \to 0,$$

since $H^2(E(2)) = 0$ and $H^1(\mathcal{O}_X(n)) = 0$ for all n. The previous conclusion says the term on the left $H^1(E(2))$ vanishes, so $H^1(J_{P/X}(4)) = \mathbb{C}$. It is generated by the nonzero extension class governing the exact sequence corresponding to s. On the other hand we have

$$0 \to J_{P/X}(4) \to \mathcal{O}_X(4) \to \mathcal{O}_P(4) \to 0,$$

so the map $H^0(\mathcal{O}_X(4)) = \mathbb{C}^{35} \to \mathcal{O}_P(4) = \mathbb{C}^{20}$ has cokernel $H^1(J_{P/X}(4))$ of dimension 1. It follows that the kernel $H^0(J_{P/X}(4))$ has dimension 16. $\qquad\square$

COROLLARY 3.7

Pick a section $s \in H^0(E(1))$, and let P be its subscheme of zeros. The extension class defining E as an extension of $J_{P/X}(2)$ by $\mathcal{O}_X(-1)$ is unique up to a scalar.

Proof
Recall from where e was defined that the space of extensions, $H^1(J_{P/X}(4))^*$, has dimension $e + 1$. Thus, the condition $e = 0$ means that this is a line: the extension is unique up to scalars, and for a given subscheme P there is a unique bundle extension E up to isomorphism. $\qquad\square$

COROLLARY 3.8

If $Y = H \cap X$ is a plane section, then $H^0(E_Y) = 0$. Also, $H^0(E(1)) \overset{\cong}{\to} H^0(E_Y(1))$ and $H^1(E_Y(1)) = 0$.

Proof
Consider the exact sequence

$$0 \to E(-1) \to E \to E_Y \to 0.$$

From $H^0(E) = 0$ and $H^1(E(-1)) = 0$ we get $H^0(E_Y) = 0$. Similarly, the exact sequence

$$0 \to E \to E(1) \to E_Y(1) \to 0$$

together with $H^i(E) = 0$ gives $H^i(E(1)) \overset{\cong}{\to} H^i(E_Y(1))$. $\qquad\qquad$ □

4. The structure of the base loci

Let $B_2 \subset X$ be the subset of points where all sections of $H^0(E(1))$ vanish. Let $B_1 \subset X$ be the subset of points x such that the image of $H^0(E(1)) \to E(1)_x$ has dimension ≤ 1 (in particular, $B_2 \subset B_1$). These are the *base loci* of sections of $E(1)$. In this section, we obtain some information about these base loci, which will allow us to to deduce, in Section 5, that the zero scheme of a general section s has some fairly strong general position properties.

4.1. There is at most one point in B_2

PROPOSITION 4.1

The subset B_2 has at most one point, and if it exists, then the sections of $H^0(E(1))$ define this reduced point as a subscheme.

Proof

Suppose $p \neq q$ are two points of B_2. Then all sections of $E(1)$ vanish at p and q. Consider a plane section $Y = H \cap X$ such that $p, q \in Y$ but Y is general for this property; in particular Y is smooth (see Section 2.4). The map

$$E(1)_p \oplus E(1)_q \to H^1\big(E_Y(1)(-p-q)\big)$$

is injective. Furthermore, it is surjective since $H^1(E_Y(1)) = 0$.

Let L denote the line through p and q. It intersects Y in a divisor denoted $p + q + u + v + w$. Some of the points u, v, w may be equal or equal to p or q.

We have an exact sequence

$$0 \to E_Y \to E_Y(1) \to E_{L \cap Y}(1) \to 0,$$

and on the other hand, the exact sequence

$$0 \to E(-1) \to E \to E_Y \to 0$$

gives $H^1(E_Y) \overset{\cong}{\to} H^2(E(-1)) \cong H^0(E(1))^* \cong \mathbb{C}^5$. Hence the image of $H^0(E_Y(1)) \to E_{L \cap Y}(1)$ has codimension 5, and since $L \cap Y$ is a finite subscheme of length 5, $E_{L \cap Y}(1) \cong \mathbb{C}^{10}$, so the image has dimension 5 too.

We may impose the condition of vanishing at two points u, v and obtain a nonzero section $s \in H^0(E_Y(1)(-p-q-u-v))$. This has the required meaning when some of the points coincide, using the previous paragraph. However, the section s then does not vanish at the third point w; otherwise we would get a section in $H^0(E_Y(1)(-L \cap Y)) = H^0(E_Y)$, contradicting Corollary 3.8.

This section generates a subline bundle $M \subset E_Y(1)$, with $M = \mathcal{O}_Y(p + q + u + v + D)$ for an effective divisor D not passing through w. Note that $\mathcal{O}_Y(p + q +$

$u + v$) has a nonzero section, corresponding to the quotient of a linear form (on the plane H) vanishing at w but not along L, divided by a linear form vanishing along L. If D does not contain both p and q, then we would get a section of $E(1)$ nonvanishing at one of those points, contradicting our assumption $p, q \in B_2$. Therefore $D \geq p + q$. It follows that $w \neq p, q$. The same reasoning works for u and v too, so u, v, w are three points distinct from p or q.

Our section s comes from a section in $H^0(E(1))$ corresponding to $\mathcal{O}(-1) \to E$, and the subscheme of zeros P contains p, q, u, v. These are four points on the line L, so any cubic form vanishing at P has to vanish along L. In particular, elements of $H^0(J_{P/X}(3))$ vanish at w. This implies that elements of $H^0(E(1))$ evaluate at w to elements in the line $M_w \subset E(1)_w$. Thus $w \in B_1$.

This reasoning holds even if w coincides, say, with v; it means that all elements of $J^0(J_{P/X}(3))$ have to vanish in the tangent direction corresponding to the additional point w glued onto v, which still gives a rank one condition on the values of sections of $E(1)$ at the point w.

The same reasoning holds for u and v. If at least two of the points u, v, w are distinct, then we obtain this way at least two points of B_1 along the line L. Then, vanishing at these two points consists of two conditions, so we can impose further vanishing at the third point (even if it is a tangential point at one of the other two) and obtain a nonzero section which vanishes at all five points. As before this yields a nonzero section of $H^0(E_Y)$ contradicting Corollary 3.8.

It remains to consider the case when all three points are the same, that is to say, $L \cap Y = p + q + 3u$ with $u \neq p, q$, and choosing a section vanishing at p, q and two times at u generates a subbundle $M = \mathcal{O}_Y(ap + bq + 2u + D) \hookrightarrow E(1)$ with D an effective divisor distinct from p, q, u, and $a, b \geq 2$. Recall that if either $a = 1$ or $b = 1$, then this would give a section of $E(1)$ nonvanishing at p or q contradicting our assumption $p, q \in B_2$.

As above, we have $u \in B_1$. Therefore, choosing a section in $H^0(E_Y(1)(-2u))$ represents only 3 conditions rather than 4; hence there are two linearly independent such sections s_1, s_2. We claim that the values of these two sections, in $E_Y(1)(-2u)_u$, are linearly independent. Indeed, otherwise a combination of the two would vanish again at u, and this would give a section of $E_Y(1)(-3u)$ which also vanishes at $p, q \in B_2$. This would give a nonzero element of $H^0(E_Y)$, which can't happen.

Let a and b be the smallest possible orders of vanishing of s_i at p and q, respectively, and by linear combinations we can assume that both of them vanish to those orders. They give maps

$$M_1 = \mathcal{O}_Y(ap + bq + 2u + D_1) \overset{s_1}{\to} E_Y(1),$$

$$M_2 = \mathcal{O}_Y(ap + bq + 2u + D_2) \overset{s_2}{\to} E_Y(1),$$

and the resulting map

$$M_1 \oplus M_2 \to E_Y(1)$$

has image of rank 2 at the point u by the previous paragraph. Therefore it is injective. It follows that $\deg(M_1 \oplus M_2) \leq \deg(E_Y(1)) = 15$. Suppose $\deg(D_1)$ is the smaller of the two; then we get $a + b + 2 + \deg(D_1) \leq 7$. We may also by symmetry assume $a \geq b$. Write $M = M_1$ and $D = D_1$. There are three possibilities:

$$M = \mathcal{O}_Y(2p + 2q + 2u + d), \qquad D = (d), \qquad \deg(M) = 7,$$

$$M = \mathcal{O}_Y(2p + 2q + 2u), \qquad D = 0, \qquad \deg(M) = 6,$$

or

$$M = \mathcal{O}_Y(3p + 2q + 2u), \qquad D = 0, \qquad \deg(M) = 7.$$

In each case, let $N := E_Y(1)/M = \mathcal{O}_Y(3) \otimes M^{-1}$ be the quotient bundle. Recall that $\mathcal{O}_Y(3) = \mathcal{O}_Y(3p + 3q + 9u)$ and $K_Y = \mathcal{O}_Y(2) = \mathcal{O}_Y(2p + 2u + 6v)$. We have an exact sequence

$$H^0(N) \to N_p \oplus N_q \to H^1\big(N(-p-q)\big) \to H^1(N).$$

The rightmost map is dual to

$$H^0(K_Y \otimes N^{-1}) \to H^0\big(K_Y \otimes N^{-1}(p+q)\big).$$

Notice however that $N^{-1} = \mathcal{O}_Y(-3) \otimes M$, so $K_Y \otimes N^{-1} = M(-1) = M(-p - q - 3u)$. Hence our rightmost map is dual to

$$H^0\big(M(-p-q-3u)\big) \to H^0\big(M(-3u)\big).$$

This map is surjective; indeed, the condition $p, q \in B_2$ means that all sections of M must vanish at p and q, and sections of $M(-3u)$ are in particular sections of M, so every element of $H^0(M(-3u))$ must come from an element of $H^0(M(-p-q-3u))$. This surjectivity translates by duality to the statement that the rightmost map in the above exact sequence is injective. It follows that $H^0(N) \to N_p \oplus N_q$ is surjective.

In other words, the values of global sections of N at p and q span a two-dimensional space. Since on the other hand the values of sections of $E_Y(1)$ must vanish at p and q, this implies from the exact sequence

$$H^0\big(E_Y(1)\big) \to H^0(N) \to H^1(M)$$

that we have $h^1(M) \geq 2$.

Consider now the three cases, the first case being $M = \mathcal{O}_Y(2p + 2q + 2u + d)$, with $\chi(M) = 2$, so $h^1(M) \geq 2$ implies that $h^0(M) \geq 4$. Vanishing at $2u$ imposes two conditions, which leaves $h^0(M(-2u)) = h^0(\mathcal{O}_Y(2p + 2q + d)) \geq 2$. These sections must vanish at p and q, so we get $h^0(\mathcal{O}_Y(p + q + d)) \geq 2$. Now, our two independent sections of $\mathcal{O}_Y(p + q + d)$ cannot vanish at both p and q because Y is not $\mathbb{P}^1$, so there are no functions with a single nontrivial pole at d. We get a section of $\mathcal{O}_Y(p + q + d)$ whose value at one of p or q is nonzero. Multiplying this by the section of $\mathcal{O}_Y(p + q + 2u)$ nonvanishing at p and q gives a section of M nonvanishing at p or q, a contradiction which treats the first case.

In the next case, $M = \mathcal{O}_Y(2p + 2q + 2u)$ with $\chi(M) = 1$, so $h^1(M) \geq 2$ gives $h^0(M) \geq 3$. As usual, sections of M have to vanish at p and q, so $h^0(M(-p-q)) = h^0(\mathcal{O}_Y(p + q + 2u)) \geq 3$. But notice that $\mathcal{O}_Y(p + q + 2u) = \mathcal{O}_Y(1)(-u)$. The map $\mathbb{C}^3 = H^0(\mathcal{O}_H(1)) \to H^0(\mathcal{O}_Y(1))$ is an isomorphism; and $\mathcal{O}_Y(1)$ is generated by its global sections. Hence vanishing of a section at u imposes a nontrivial condition, giving $h^0(\mathcal{O}_Y(1)(-u)) = 2$. This contradicts the previous estimation of ≥ 3. This contradiction completes this case.

In the last case, $M = \mathcal{O}_Y(3p + 2q + 2u)$ with $\chi(M) = 2$, so $h^1(M) \geq 2$ gives $h^0(M) \geq 4$. This is similar to the first case. Vanishing at $2u$ imposes two conditions, and then the sections must further vanish at p and q, which leaves $h^0(M(-2u)) = h^0(M(-p - q - 2u)) = h^0(\mathcal{O}_Y(2p + q + d)) \geq 2$. If we have a section here which is nonzero at either p or q, then multiplying it by the section of $\mathcal{O}_Y(p + q + 2u)$ nonvanishing at p gives a section of M nonvanishing at p or q, a contradiction. Therefore, both sections in $H^0(\mathcal{O}_Y(2p + q + d))$ have to vanish further at p and q. This would give $h^0(\mathcal{O}_Y(p + d)) \geq 2$. That can happen only if Y is a hyperelliptic curve.

But a smooth plane curve of degree 5, having a very ample canonical linear system cut out by conics, is never hyperelliptic.

We have now finished showing that it is impossible to have two distinct points $p, q \in B_2$. The same proof works equally well if q is infinitesimally near p; this double point defines a tangent direction, and L should be chosen as the tangent line in this direction. The main case as before is when $L \cap Y = 2p + 3u$, and as before we get three cases: either $M = \mathcal{O}_Y(4p + 2u + d)$, $M = \mathcal{O}_Y(4p + 2u)$, or $M = \mathcal{O}_Y(5p + 2u)$. The main principle here is that sections of M have to vanish on both p and the nearby point q, that is to say, they have to vanish to order 2 at p. With this, the same proofs as above hold, so this shows that if B_2 is nonempty, then it is a single reduced point. This completes the proof of the proposition.

$\square$

4.2. Local structure of B_1 at a point of B_2

For the next discussion, we assume that there is a point $p' \in B_2$, unique by the above. Consider the schematic structure of B_1 around this point p'. An argument similar to the one above, allows us to show that B_1 cannot contain the third infinitesimal neighborhood of p'; however, we have not been able to rule out the possibility that it might contain the second neighborhood. We will formulate this statement precisely in the form of the following lemma, even though we have not really defined the schematic structure of B_1. Recall that $H^0(E(1)) \overset{\cong}{\to} H^0(E_Y(1))$.

LEMMA 4.2

Suppose $Y \subset X$ is a general plane section passing through p'. Choose $s \in H^0(E_Y(1))$, vanishing to order 1 at p'. Let $M \subset E_Y(1))$ be the subline bundle generated by s. Then there exists a section $t \in H^0(E_Y(1))$ such that the projection of t as a section of $N := E_Y(1)/M$ vanishes to order at most 2 at p'.

Proof

Suppose on the contrary that all sections vanish to order ≥ 3 in N. As this is true on a general Y, we may also specialize Y and it remains true. In particular, choose a tangent line L to X at p' such that the second fundamental form vanishes. Choose a smooth plane section Y corresponding to a plane containing L. Then $Y \cap L$ is a divisor of class $\mathcal{O}_Y(1)$, and we can write

$$Y \cap L = 3p' + u + v,$$

where, as far as we know for now, u and v might be the same, and one or both might be equal to p'.

Choose a nonzero section $s \in H^0(E_Y(1))$ which has only a simple zero at p'. Recall that this is possible by the result that B_2 is reduced in Proposition 4.1. Let $M \subset E_Y(1)$ be the subline bundle generated by s, and let $N := E_Y(1)/M$ be the quotient. The contrary hypothesis says that all sections of $E_Y(1)$ vanish to order 3 at p', when projected into N. This means that the condition of a section vanishing to order 3 at p' imposes only two additional conditions. Indeed, étale-locally we can choose a basis for $E_Y(1)$ compatible with the subbundle M and impose two conditions stating that the first coordinate (corresponding to M) vanishes to order 3. (It automatically vanishes to order 1 already.) This implies that the section vanishes, since the second coordinate vanishes to order 3 by hypothesis.

Now since $h^0(E_Y(1)) = 5$, we can impose two further conditions and obtain a section t vanishing at u. The divisor of vanishing of t is therefore $ap' + u + D$ where $a \geq 3$. If $a = 3$, then we would get a morphism

$$\mathcal{O}_Y(3p' + u) \to E_Y(1)$$

nonzero at p', but the line bundle $\mathcal{O}_Y(3p' + u)$ has a section nonvanishing at p', and this would contradict $p' \in B_2$. Therefore we can conclude that $a \geq 4$.

We now note that u and v must be distinct from p'. For example, if $Y \cap L = 4p' + u$, choose a section s vanishing at u, and as described above, we can assume vanishing to order 3 at p', which imposes two additional conditions. If s vanishes to order ≥ 4 at p' this would give a section in $H^0(E_Y)$, which cannot happen, so we can assume that $M = \mathcal{O}_Y(3p' + u)$, and again this has a section nonvanishing at p', contradicting $p' \in B_2$. So, this case cannot happen.

Similarly if $Y \cap L = 5p'$, vanishing to order 3 imposes two conditions, and vanishing to order 4 imposes two more conditions. So again there is a section s which generates $M = \mathcal{O}_Y(4p') \subset E_Y(1)$, but this M has a section nonvanishing at p', contradicting $p' \in B_2$. From these arguments we conclude that u, v are different from p'.

Next, use the fact that a cubic polynomial on L vanishing at 4 points in $L \cap X$ must also vanish on the fifth point. Suppose first that $u \neq v$. Our section t viewed as a section of $E(1)$ defines a zero scheme P, which contains its zeros on Y. In particular, P contains the scheme $3p'$ on L as well as the scheme u. Note on the other hand that $v \notin P$; otherwise we would get a section in $H^0(E_Y)$. We conclude that any element of $H^0(J_{P/X}(3))$ has to vanish at v. It follows

that $v \in B_1$. By symmetry, we get also $u \in B_1$. Now, vanishing of sections at u and v imposes 2 conditions, and vanishing at $3p' \subset Y$ imposes 2 conditions as discussed above. This gives a section in $H^0(E_Y(1))$ vanishing at all of $Y \cap L$, hence a nonzero section of $H^0(E_Y)$. We get a contradiction in this case.

To finish the proof of the lemma, we have to treat the case where $u = v$, that is, $Y \cap L = 3p' + 2u$. As described previously, $u \neq p'$. Basically the same argument as before gives $u \in B_1$. Indeed, we can consider a section t which vanishes at $3p' + u$. It cannot have a zero of order 2 at u; otherwise we would get $H^0(E_Y) \neq 0$. Let $M \subset E_Y(1)$ be the subline bundle generated by t, and let N be the quotient. Write $M = \mathcal{O}_Y(ap' + u + D)$ with $a \geq 4$ and D disjoint from p', u. We may also consider t as a section defined over X, inducing a quotient morphism $E(1) \to \mathcal{O}_X(3)$. When restricted to Y this provides a morphism $E_Y(1) \to \mathcal{O}_Y(3)$ which is the same as the map to N generically. Hence it must factor through $E_Y(1) \to N \to \mathcal{O}_Y(3)$. This is more precisely given by $N = \mathcal{O}_Y(3)(-ap' - u - D)$. Sections of $E(1)$ map to sections of $\mathcal{O}_X(3)$ vanishing on the zero locus P of t, which contains $3p' + u \subset L$ (a subscheme of length 4). These sections must vanish on all of L; hence they vanish on $2u$. Thus the image of any section in N has to be a section of $\mathcal{O}_Y(3)(-ap' - 2u - D) = N(-u)$. This means that the sections of $E(1)$, evaluated at u, must lie in M_u. In other words, $u \in B_1$ as claimed.

Vanishing of a section at u therefore imposes a single condition. So there are two linearly independent sections t_1, t_2 which vanish at $3p' + u$. No nonzero linear combination of these can have a zero of order 2 at u. It follows that the derivatives of t_1 and t_2 at u are linearly independent. Let M_1 and M_2 denote the subline bundles of $E_Y(1)$ generated by the t_i. We have

$$M_i = \mathcal{O}_Y(a_i p' + u + D_i)$$

with $a_i \geq 4$ and $D_i \cap u = \emptyset$. But the line bundle $\mathcal{O}_Y(3p' + u)$ has a section non-vanishing at u, so M_i has a section nonvanishing at u. But the $M_i(u) \subset E(1)_u$ are generated by the derivatives of t_i, which are linearly independent. Thus the $M_i(u)$ generate $E(1)_u$. But as there are sections of M_i nonvanishing at u, this contradicts $u \in B_1$. This completes the proof of the lemma. □

COROLLARY 4.3
Suppose $p' \in B_2$. Then for a general section $s \in H^0(E(1))$, the scheme of zeros of s locally at p' is either the reduced point p', or a length 2 subscheme (infinitesimal tangent vector) at p'.

Proof
From the proposition before, the sections of $E(1)$ define p' as a reduced subscheme. This means that for any tangent direction, there is at least one section whose derivative in that direction does not vanish. So, if $Y \subset X$ is a generic curve through p', then the zero scheme P of a general section s has $P \cap Y = \{p'\}$ being a reduced subscheme locally at p'. It follows that P is curvilinear at p'.

Assume that the general P has length ≥ 3 locally at p'. Consider the two sections s, t given by Lemma 4.2. Their zero sets are therefore curvilinear subschemes of length ≥ 3 at p'. Given s, we may choose t general; then the zero set of t is transverse to that of s at p'. For otherwise this would mean that the tangent directions of the zero sets are always the same, but that would give an infinitesimal tangent vector in B_2 contradicting Proposition 4.1. So these curvilinear subschemes are transversal. We may choose local coordinates at p' so that they go along the coordinate axes, up to order 3 at least. If x, y are these coordinates with $p' = (0,0)$, we may write

$$s = xa, \qquad t = yb \quad \text{modulo terms of order 3}$$

where a and b are sections of $E(1)$ nonvanishing at p'. Furthermore, Y is transverse to $(x = 0)$. We may assume that the subline bundle of $E_Y(1)$ generated by s is generated by $a|_Y$.

Notice that if $b(0,0)$ is linearly independent from $a(0,0)$, then $s + t = xa + yb$ would be a section whose zero scheme is the reduced point p', so we would be done. Therefore we may assume, after possibly multiplying by a scalar, that $b(0,0) = a(0,0)$.

The conclusion of the lemma says that t is not a section of this subline bundle to order 3, which means that $b|_Y$ is not a multiple of a to order 2, that is, modulo quadratic terms. We may therefore write

$$b = a + xb_x + yb_y + \cdots$$

with one of b_x, b_y nonzero modulo $a(0,0)$. Look at the section $s + \lambda t$ for variable $\lambda \in \mathbb{C}$; its leading term is $(x + \lambda y)a(0,0)$. By our contrary hypothesis, we suppose that the zero set of this section is curvilinear to order 3 for all λ, which means that there is a factorization of $s + \lambda t$ as a multiple of a single section of $E(1)$, up to terms of order 3. The first term has to be $(x + \lambda y)a(0,0)$, so we can write

$$s + \lambda t = \big(x + \lambda y + q(x,y)\big)(a + xf_x + yf_y).$$

This expands to

$$xa + \lambda y(a + xb_x + yb_y) = \big(x + \lambda y + q(x,y)\big)(a + xf_x + yf_y)$$

or, simplifying (always modulo terms of order 3),

$$\lambda y(xb_x + yb_y) = q(x,y)a + (x + \lambda y)(xf_x + yf_y).$$

Now compare terms modulo the section generated by a; we get

$$f_x = 0, \qquad f_y = b_y \quad \text{modulo } a(0,0),$$

and from the xy term we get $\lambda b_x = f_y$, again modulo $a(0,0)$. Putting these together gives $\lambda b_x = b_y$ modulo $a(0,0)$, for all λ. This is possible only if b_x and b_y are multiples of $a(0,0)$; but the conclusion of Lemma 4.2 said that this was not the case. This contradiction completes the proof of the corollary. $\square$

The above discussion may seem somewhat complicated: let us explain the geometric picture, in terms of a schematic notion of the base locus B_1. The problem

is that B_1 could have some "layers" surrounding the point $p' \in B_2$. Locally, this would mean that the subsheaf of $E(1)$ generated by global sections looks like a rank 1 subsheaf over B_1, the layers of which would give a certain infinitesimal neighborhood of p'. In the lemma, we say that if we cut by a general plane section Y going through p', then the intersection with B_1 has length at most 2. Intuitively this means that while B_1 might have a single layer around p', it cannot have two layers. Notice that in some directions B_1 might be bigger, but in a general direction it has length 2. Then, in the corollary, we say that if the general section has a curvilinear zero set of length ≥ 3, that would mean that B_1 had to have at least two layers around p'.

4.3. Dimension of the CB-Hilbert scheme

Let $\mathbf{H}_X$ denote the Hilbert scheme of subschemes $P \subset X$ which satisfy CB(4). Let $\mathbf{H}_{\mathbb{P}^3}$ denote the Hilbert scheme of subschemes $P \subset \mathbb{P}^3$ which satisfy CB(4) and which are contained in at least one smooth quintic. We call these the *CB-Hilbert schemes*.

Let $\mathbf{H}_X^{sn}$ and $\mathbf{H}_{\mathbb{P}^3}^{sn}$ denote the subschemes parameterizing P such that $h^0(J_P(3)) = 4$ and $h^0(J_P(2)) = 0$, and (in the second case) such that P is contained in at least one smooth quintic surface. In that case, as we have seen in Lemma 3.2, any bundle E extending $J_{P/X}(2)$ by $\mathcal{O}_X(-1)$ has seminatural cohomology, so we call them the *seminatural CB-Hilbert schemes*. Furthermore, as in Corollary 3.7, the isomorphism class of E is uniquely determined by P. Since E is stable, it does not have any nontrivial automorphisms.

PROPOSITION 4.4

The seminatural CB-Hilbert scheme $\mathbf{H}_X^{sn}$ has pure dimension 24; the seminatural CB-Hilbert scheme $\mathbf{H}_{\mathbb{P}^3}^{sn}$ has pure dimension 44. Denote by $\mathbf{H}_X^{sn}[2]$ and $\mathbf{H}_{\mathbb{P}^3}^{sn}[2]$ the fiber bundles over these, parametrizing pairs (P, U) where P is a seminatural CB-Hilbert point and $U \subset H^0(J_{P/\mathbb{P}^3}(3))$ is a 2-dimensional subspace. These have pure dimensions 28 and 48, respectively.

Proof

A point in $\mathbf{H}_X^{sn}$ corresponds to a choice of bundle E in $M_X^{sn}(2, 1, 10)$ plus a section $s \in H^0(E(1))$ up to scalar. As the moduli space has dimension 20 and, for the seminatural case, $\mathbb{P}H^0(E(1))$ has dimension 4, the total dimension of $\mathbf{H}_X^{sn}$ is 24. The Hilbert scheme of pairs (P, X) with $P \in \mathbf{H}_X^{sn}$ fibers over the 55-dimensional space of quintics X (note that $h^0(\mathcal{O}_{\mathbb{P}^3}(5)) = 56$) with 24-dimensional fibers, so it has dimension 79. On the other hand, for a fixed $P \in \mathbf{H}_{\mathbb{P}^3}^{sn}$, the space of quintics X containing P is $\mathbb{P}H^0(J_{P/\mathbb{P}^3}(5))$. Notice that if P is contained in at least one X, then the discussion of Section 3 implies that $h^1(J_{P/\mathbb{P}^3}(5)) = 0$, so $h^0(J_{P/\mathbb{P}^3}(5)) = 36$ and the space of quintics containing P is an open subset of $\mathbb{P}^{35}$. So, the dimension of the Hilbert scheme $\mathbf{H}_{\mathbb{P}^3}^{sn}$ is $79 - 35 = 44$. The fiber bundles parameterizing choices of $U \subset H^0(J_{P/\mathbb{P}^3}(3))$ are bundles of Grassmanians of dimension 4, so they have dimensions 28 and 48, respectively. All irreducible

components have the same dimension because the same discussion works for all of them. □

4.4. The base locus B_1 has dimension zero

PROPOSITION 4.5

Suppose that E is a general point of its irreducible component. The subset B_1 of points at which $E(1)$ is not generated by global sections has dimension zero. Equivalently, if s is a general section of $E(1)$ and P its subscheme of zeros, then the base locus in X of the linear system of cubics $H^0(J_{P/X}(3))$ has dimension zero. (It remains possible that the base locus in $\mathbb{P}^3$ could have dimension 1; indeed, that will be a major case treated in Section 7 below.)

Proof

The proof takes up the rest of this section, using three further lemmas. Note first the equivalence of the two formulations. The section s generates a rank one subsheaf of $E(1)$ at all points outside P. Thus, if B_1 had positive dimension, this would mean that all sections restrict to multiples of s on B_1, so all sections of $H^0(J_{P/X}(3))$ would factor as a function vanishing on B_1 times some other function. So the second statement implies the first. In the other direction, suppose all elements of the linear system factored as fg where g is a fixed form, either linear or quadratic. Then the zero set of g would provide a positive-dimensional component of B_1.

To be proven is that the elements of the linear system $H^0(J_{P/X}(3))$ cannot all share a common factor g. Suppose to the contrary that they did (a hypothesis which will be in effect until the end of the proof of the proposition), and let $W \subset X$ be the zero set of g. It is a divisor in either the linear system $\mathcal{O}_X(1)$ or $\mathcal{O}_X(2)$, which is to say that it is either a plane section or a quadric section of X.

Let $P^\perp \subset X$ be the residual subscheme of P along W (i.e., roughly speaking, $P - P \cap W$). Recall that we are assuming that P is not contained in a quadric section, so $P \not\subset W$ and $P^\perp$ is nonempty. The statement that elements of $H^0(J_{P/X}(3))$ vanish along W means that the map

$$H^0\big(J_{P^\perp/X}(3)(-W)\big) \to H^0\big(J_{P/X}(3)\big)$$

is an isomorphism. Recall also that the right-hand side has dimension 4 in our situation, so we get $h^0(J_{P^\perp/X}(3)(-W)) = 4$ too.

It is now easy to rule out the case where W is a quadric section. Indeed, in that case we would have $h^0(J_{P^\perp/X}(1)) = 4$, but $h^0(\mathcal{O}_X(1)) = 4$ and the space of sections generates $\mathcal{O}_X(1)$ everywhere, so there are at most 3 sections vanishing on a nonempty subscheme $P^\perp$, giving a contradiction.

Therefore, we may now say that W is a plane section of X. From above, $h^0(J_{P^\perp/X}(2)) = 4$.

Next, we claim that $P^\perp$ satisfies CB(3), that is, the Cayley–Bacharach property for $\mathcal{O}_X(3)$, which is the same as $\mathcal{O}_X(4)(-W)$. Indeed, if f is a section of $\mathcal{O}_X(3)$ and g is the equation of W, then fg is a section of $\mathcal{O}_X(4)$. Suppose

$P^3 \subset P^\perp$ is a colength 1 subscheme (defined by an ideal of length 1). Then it induces a colength 1 subscheme $P' \subset P$ such that P^3 is the residual of P': notice that as a module $\mathcal{O}_{P^\perp}$ may be viewed as isomorphic to the ideal (g) inside $\mathcal{O}_P$, so an ideal of length 1 in $\mathcal{O}_{P^\perp}$ gives an ideal of length 1 in $\mathcal{O}_P$. Now if f vanishes on P^3, then fg vanishes on P', so by CB(4) for P we get that fg vanishes on P, which in turn says that f vanishes on $P^\perp$. This proves that $P^\perp$ satisfies CB(3) as claimed. It follows that $P^\perp$ also satisfies CB(2).

The next remark is that $P^\perp$ is not contained in a plane, for if it were, then the union of this plane with the one defining W would be a quadric containing P, contrary to our situation.

We have the following lemma, which is a preliminary version of the structural result of Proposition 5.1 below. Notice that here we have not yet shown that B_1 has dimension zero, so we use the specific current situation in the proof instead.

LEMMA 4.6

In the situation of the proof of Proposition 4.5, consider a general section $t \in H^0(E(1))$, and let $P \subset X$ be its subscheme of zeros. Then P decomposes as a disjoint union $P = P' \sqcup P''$ such that P'' is reduced, and P' is either empty or consists of a point p', or an infinitesimal tangent vector at p'; in the latter two cases p' is the unique point of B_2.

Proof

Choose first any section $s \in H^0(E(1))$ with zero scheme P, corresponding to a subsheaf $\mathcal{O}_X \subset E(1)$. Let r be another section linearly independent from s, and let $F \subset E(1)$ be the subsheaf generated by r and s. Let $K := E(1)/F$ be the quotient. Let $\tilde{r}$ be the image of r considered as a section of $J_{P/X}(3)$. Within the situation of the proof of the proposition, the elements of the linear system $H^0(J_{P/X}(3))$ are supposed to have a common factor defining the divisor W, which we have seen is a plane section. Thus, the zero scheme $Z(\tilde{r})$ decomposes as $W \cup D$ where D is a quadric section. For r sufficiently general, D does not contain W. (Otherwise the quadric section $2W$ would be a common zero of the linear system, and we have ruled that out.) Thus, $Z(\tilde{r})$ is smooth on the complement of a finite set. Now, we can choose $t \in H^0(E(1))$ so that it is nonvanishing at all isolated points of B_1 (except maybe $p' \in B_2$) and at the finite set of singularities. Thus, if the zero scheme $P(t)$ of t meets W, it meets it at a point where $Z(\tilde{r})$ is reduced (which we think of heuristically as points where B_1 is reduced even though we have not given a scheme structure to B_1). Furthermore, since $P(t)$ moves (except maybe at $p' \in B_2$), its intersection with W is reduced. On the other hand, at points located on W, $P(t)$ has to be locally contained in W; otherwise we could add a small multiple of r to split off the part of the subscheme sticking out of W. These together imply that the points of $P(t)$ contained in W are reduced (except possibly at B_2). The points outside of W are reduced because they are located at places where $E(1)$ is generated by global sections, again with the possible exception of $p' \in B_2$. From Corollary 4.3, the local structure of $P(t)$

near the possible single point $p' \in B_2$ is at most an infinitesimal tangent vector
of length 2. This gives the claim of the lemma.   □

LEMMA 4.7
The length of $P^\perp$ is ≤ 10.

Proof
Consider first the case where B_2 is empty, or P is reduced at $p' \in B_2$, or else $p' \notin W$. In this case, $P = P^W \cup P^\perp$ with $P^W = P \cap W$ a reduced subscheme. For fixed
W, the dimension of the space of choices of P^W is $\leq \ell(P^W)$. On the other hand,
$P^\perp$ is located at the intersection $C_1 \cap C_2 \cap X$ where C_1 and C_2 are quadrics whose
intersection has dimension 1. Furthermore, no component of $C_1 \cap C_2$ is contained
in X; indeed, the former has degree 4 while curves in X have degree ≥ 5 because
of the condition $\text{Pic}(X) = \langle \mathcal{O}_X(1) \rangle$. Therefore, $C_1 \cap C_2 \cap X$ is a finite set. Since
$P^\perp$ is reduced except for a possible tangent vector at the unique point $p' \in B_2$,
we get that the dimension of the set of choices of $P^\perp$ for a given C_1, C_2 is ≤ 1.
On the other hand, suppose C_1, C_2, C_3 are three general quadrics through $P^\perp$. If
their intersection is finite, it contains at most 8 points, but with $\ell(P^\perp) \geq 11$ this
cannot happen and we must have a nontrivial curve in the intersection; this means
that a double intersection $C_1 \cap C_2$ has to split into two pieces. The dimension of
the space of such double intersections is the dimension of the Grassmannian of
2-planes in $H^0(\mathcal{O}(2)) = \mathbb{C}^{10}$. This Grassmannian has dimension 16. However, as
may be seen by a calculation of the possible cases of splitting, the subvariety of
the Grassmannian corresponding to double intersections which split into at least
two components, is ≤ 14. Together with the possible one-dimensional choice of
tangent vector at p', we get altogether that the space of choices for $P^\perp$, together
with the two-dimensional subspace spanned by C_1, C_2, is ≤ 15. Putting in P^W, we
get that the dimension of the space of choices of P plus a 2-dimensional subspace
of $H^0(J_{P/X}(3))$ is less than $15 + 3 + \ell(P^W)$. The 3 is for the space of choices of
plane section W. Now if $\ell(P^\perp) \geq 11$, then $\ell(P^W) \leq 9$ and this dimension is ≤ 27.
The dimension of the corresponding bundle over the seminatural CB-Hilbert
scheme $\mathbf{H}_X^{sn}[2]$ is 28 (see Proposition 4.4) so such a bundle E cannot be general
in its irreducible component.

We are left to treat the case where the unique point $p' \in B_2$ lies on W, and
P includes a tangent vector here. Let P^1 denote the subscheme of P located
set-theoretically along W, and let P^2 be the complement. Then the dimension
of the space of choices of P^1 is still $\ell(P^1)$, and the same argument as above gives
that the dimension of the space of choices of P^2 plus a two-dimensional subspace
of quadrics is ≤ 15. We get as before $\ell(P^2) \leq 10$. On the other hand, the tangent
vector at p' might go outside of W and contribute to $P^\perp$. If the tangent vector
stays inside W, then $P^\perp = P^2$ and we are done. If the tangent vector goes outside
of W, then the estimate from above says only that $\ell(P^\perp) \leq 11$; however, we get an
additional condition saying that the quadrics have to vanish at this point $p' \in W$,

and this condition (which may be seen, e.g., as a condition on the choice of P^1 once P^2 and the quadrics are fixed) gets us back to the estimate $\ell(P^\perp) \leq 10$. $\quad\square$

LEMMA 4.8

There is a plane section $V \subset X$ such that $V \cap P^\perp$ has length ≥ 5.

Proof

Suppose not; that is to say, suppose that any plane section meets $P^\perp$ in a subscheme of length ≤ 4. In order to obtain a contradiction, we show that under this hypothesis, $\ell(P^\perp) \geq 11$.

Choose a plane section meeting $P^\perp$ in a subscheme of length ≥ 3, call this intersection $P^\perp_+$, and let $P^\perp_-$ denote the residual subscheme. Then the condition CB(3) for $P^\perp$ implies CB(2) for $P^\perp_-$. The results of our previous paper [13] therefore apply:

(a) $\ell(P^\perp_-) \geq 4$;
(b) if $\ell(P^\perp_-) = 4$ or 5, then $P^\perp_-$ is contained in a line;
(c) if $\ell(P^\perp_-) = 6$ or 7, then $P^\perp_-$ is contained in a plane.

However, our hypothesis for the proof of the lemma says that no plane contains a subscheme of $P^\perp$ of length ≥ 5, so the cases $\ell(P^\perp_-) = 5$, 6, 7 cannot happen. If $\ell(P^\perp_-) = 4$, then $P^\perp_-$ is contained in a line, and we can choose a plane which meets, furthermore, a point of $P^\perp_+$, again giving a plane with more than 5 points. This shows that we must have $\ell(P^\perp_-) \geq 8$, and since $\ell(P^\perp_+) \geq 3$ we get $\ell(P^\perp) \geq 11$ as claimed (under the hypothesis contrary to the lemma). This contradicts the estimate of Lemma 4.7, which completes the proof of the present lemma. $\quad\square$

Now choose a plane section V such that $V \cap P^\perp$ has maximal length. Write $P^\perp_+ = P^\perp \cap V$, and let $P^\perp_-$ be the residual subscheme with respect to V. Then $P^\perp_-$ satisfies CB(2). If $\ell(P^\perp_+) = 5$, then $\ell(P^\perp_-) \leq 5$ and by [13], $P^\perp_-$ must consist of 4 or 5 points on a line. Choose a new plane section passing through this line but not meeting $P^\perp_+$; we conclude that $P^\perp_+$ must also consist of 5 points on a line, but then in fact we could choose a plane section meeting $P^\perp$ in 6 points. Thus the case $\ell(P^\perp_+) = 5$ does not happen. If $\ell(P^\perp_+) \geq 7$, then $P^\perp_-$ would consist of ≤ 3 points, but there are no such subschemes satisfying CB(2), so this cannot happen either. We conclude that $\ell(P^\perp_+) = 6$; hence $P^\perp_-$ must be 4 points on a line. If y is any point of $P^\perp_+$, then there is a plane containing $P^\perp_-$ and y, so the remaining points of $P^\perp_+$ are either on this same plane or else contained in a line. If the two lines meet at a point, this would give a plane section containing too many points. Hence, we conclude that there are 2 skew lines containing at least 8 of the 10 points in $P^\perp$. Because of CB(3) for $P^\perp$, in fact all of the points must be on the two skew lines.

Count now the dimension of the space of such configurations: there are 8 parameters for the two skew lines. Once this configuration is fixed, the subscheme

$P^\perp$ is specified up to a finite set of choices. The choice of W counts for 3, and the choice of 10 points in W counts for 10. The full dimension of this space of choices is therefore ≤ 21. As in the proof of Lemma 4.7, the case where our double point at $p' \in B_2$ lies on W but the tangent direction extends out of W does not add an extra dimension because we get a point participating in $P^\perp$ which constrains the choice of points on W. In view of the fact that $\dim(\mathbf{H}_X^{n,s}) = 24$, this situation cannot happen for a general E.

This contradiction completes the proof of the proposition, for a general E in its irreducible component; the base locus B_1 has dimension zero. $\square$

5. The structure of a general zero scheme P

The previous results were as close as we could get to saying that $E(1)$ is generated by global sections, with the techniques we could find. Choose a general section $s \in H^0(E(1))$, and let P denote its scheme of zeros. If $y \in B_1$ (but not in B_2), then a general section will not vanish at y. Furthermore, if there is a point p' in B_2, then the structure of P near p' is at most an infinitesimal tangent vector.

PROPOSITION 5.1

Suppose $s \in H^0(E(1))$ is a general element, and let P be the subscheme of zeros. We can write $P = P' \cup P''$ where P' consists of the possible point of P located at B_2, and P'' is all the rest. With this notation, P'' consists of 18, 19, or 20 isolated points, and P' is, respectively, an infinitesimal tangent vector at $p' \in B_2$; or the isolated point p'; or empty. At any point $y \in P''$, the map

$$H^0\big(J_{P/X}(3)\big) \to J_y/J_y^2(3)$$

is surjective, meaning that y is locally the complete intersection of two general sections of $H^0(J_{P/X}(3))$.

Proof

By Proposition 4.5, the base locus B_1 has dimension zero. For any point $z \in B_1$ with $z \notin B_2$, a sufficiently general section $s \in H^0(E(1))$ is nonzero at z. Therefore, for s general the scheme of zeros P does not meet B_1 except possibly at B_2. Divide P into two pieces, P' at B_2 and P'' which does not meet either B_1 or B_2. By Proposition 4.1, the base locus B_2 consists of at most one point which we shall denote by p' if it exists. Therefore, P' is either empty or has a single point. By Corollary 4.3, the zero scheme of a general s at p' has length at most two. So, if P' has a point, then it is scheme-theoretically either this reduced point, or an infinitesimal tangent vector there.

At a point $y \in P''$, since y is not in the base locus B_1, it means that $E(1)$ is generated by global sections at y. From the standard exact sequence (1.2) for s we see that $E(1)_y = J_y/J_y^2(3)$, so the generation of $E(1)_y$ by global sections is exactly the surjectivity of the last claim in the proposition. $\square$

The points of P'' are "interchangeable." This can be phrased using Galois theory. Write $H^0(E(1)) = \mathbb{A}^5_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t_1, \ldots, t_5]$. Put $K = \mathbb{C}(t_1, \ldots, t_5)$, and let $\mathbf{s} \in \mathbb{A}^5_K$ be the tautological point. Think of $\mathbf{s} \in H^0(X_K, E(1))$. Let $P \subset X_K$ be the subscheme of zeros. The decomposition $P = P' \cup P''$ is canonical, hence defined over K. On the other hand, P'' consists of 18 to 20 points, but the points are only, distinguishable over $\overline{K}$, which is to say, $P''_{\overline{K}} \subset X(\overline{K})$ is a set with 18, 19, or 20 points. The Galois group $\operatorname{Gal}(\overline{K}/K)$ acts.

PROPOSITION 5.2

The action of $\operatorname{Gal}(\overline{K}/K)$ on the set $P''_{\overline{K}}$ is doubly transitive: it means that any pair of points can be mapped to any other pair.

Proof

For general s, the part P'' is contained in the open subset X^g where $E(1)$ is generated by global sections. Suppose x_0, y_0 and x_1, y_1 are two pairs of points in P''. Consider a continuous path of pairs $(x(t), y(t))$ contained in $X^g \times X^g$ defined for $t \in [0,1] \subset \mathbb{R}$, with $(x(0), y(0)) = (x_0, y_0)$ and $(x(t), y(t)) = (x_0, y_0)$. Vanishing of a section at $x(t)$ and $y(t)$ imposes 4 conditions on elements of the 5-dimensional space $H^0(E(1))$, so we get a family of sections $s(t)$ leading to a family of subschemes $P''(t)$. For a general choice of path, the $P''(t)$ will all be reduced with 18, 19, or 20 points. At $t = 0$ and $t = 1$, the section is the same as s up to a scalar since it is uniquely determined by the vanishing conditions. We obtain an element of the fundamental group of an open subset of the parameter space of sections s, whose action on the covering determined by the points in P'', sends (x_0, y_0) to (x_1, y_1). This shows that the action is doubly transitive, and it is the same as the Galois action after applying the Grothendieck correspondence between Galois theory and covering spaces. $\square$

COROLLARY 5.3

Suppose P is the scheme of zeros of a general section $s \in H^0(E(1))$, written $P = P' \cup P''$ as above. Let $Z \subset \mathbb{P}^3$ be the intersection of two cubic hypersurfaces corresponding to general elements of $H^0(J_{P/\mathbb{P}^3}(3))$. This Z is a complete intersection: $\dim(Z) = 1$. There is a single irreducible component Z'' of Z such that P'' is contained in the smooth locus of Z''. At points of P'', Z'' is transverse to X. The only irreducible components of Z which can be nonreduced are those, other than Z'', which are fixed as the cubic hypersurfaces vary.

Proof

The points of P'' lie in the subset X^g where sections generate $E(1)$. In the standard exact sequence (1.2), sections of $E(1)$ map to sections of $J_{P/X}(3)$, and the fiber $E(1)_x$ maps to $J_x/J_x^2(3)$ for $x \in P''$. As sections generate the fiber, it implies that sections of $J_{P/X}(3)$ generate the ideal J_x (which is the maximal ideal at x). Under the isomorphism $H^0(J_{P/\mathbb{P}^3}(3)) \cong H^0(J_{P/X}(3))$, two general sections of $J_{P/\mathbb{P}^3}(3)$ therefore have linearly independent derivatives at $x \in P''$

when restricted to X, so the same is true of the cubics in $\mathbb{P}^3$, which means that Z is a transverse complete intersection at any $x \in P''$. The doubly transitive action from Proposition 5.2 implies that for general sections and general choice of Z, the points of P'' must all lie in the same irreducible component Z'' of Z. Note, on the other hand, that any other component of Z must also have dimension 1; otherwise we would get a 1-dimensional base locus B_1 of sections of $E(1)$ on X, and this possibility has been ruled out in Proposition 4.5.

Note that Z'' is reduced since its smooth locus is nonempty. If Z_i is a nonreduced component, then by Sard's theorem it has to be a fixed part of the family of complete intersections of the form Z. □

6. Complete intersections of two cubics

We need to know something about what curves Z can arise as the complete intersection of two cubics in $\mathbb{P}^3$. The degree of Z is 9. If Z is smooth, then $K_Z = \mathcal{O}_Z(2)$ is a line bundle of degree 18, so the genus of Z is 10.

The choice of Z corresponds to a choice of two-dimensional subspace $U \subset H^0(\mathcal{O}_{\mathbb{P}^3}(3)) = \mathbb{C}^{20}$; furthermore $U = H^0(J_Z(3))$ and $h^0(\mathcal{O}_Z(3)) = 18$ (as may be seen from the exact sequences of restriction to one of the cubics C and then from C to Z). The dimension of the Grassmannian of 2-dimensional subspaces of $\mathbb{C}^{20}$ is $2 \cdot (20 - 2) = 36$. Denote this Grassmannian by $\mathbf{G}$; we have a universal family

$$\mathbf{Z} \subset \mathbf{G} \times \mathbb{P}^3.$$

Let $\mathbf{G}_{\mathrm{rci}}$ denote the subset of U defining a reduced complete intersection, that is, such that the fiber Z_U of $\mathbf{Z}$ over U has dimension 1 and is reduced.

Suppose $Z = Z' \cup Z''$ is a decomposition with $d' := \deg(Z')$, $d'' := \deg(Z'')$; so $d' + d'' = 9$ and we may assume $d' \le d''$. We are not saying necessarily that the pieces Z' and Z'' are irreducible, though. This gives $d' \le 4$.

LEMMA 6.1

Suppose Z is a complete intersection of two cubic hypersurfaces in $\mathbb{P}^3$. If Z_i is a reduced irreducible component of Z of degree ≤ 5, then either Z_i is contained in a quadric, or the normalization of Z_i has genus $g = 0$ or 1 and the space of such curves has dimension ≤ 20.

Proof

If $\deg(Z_i) \le 4$, then it is contained in a quadric, so we may assume the degree is 5. Let $Y \to Z_i$ be the normalization, and let g denote the genus of Y. Projecting from a point on Z_i gives a presentation of Y as the normalization of a plane curve of degree 4, so it has genus $g \le 3$. The line bundle $\mathcal{O}_Y(2)$ has degree 10, which is therefore in the range $\ge 2g - 1$, so $h^0(\mathcal{O}_Y(2)) = 11 - g$. If $g \ge 2$, then this is ≤ 9 and the map $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_Y(2))$ is not injective, giving a quadric containing Z_i.

If $g = 0$, $Y \cong \mathbb{P}^1$, and the embedding to $\mathbb{P}^3$ corresponds to a map $\mathbb{C}^4 \to H^0(\mathcal{O}_Y(1)) \cong \mathbb{C}^6$. This yields 24 parameters, minus 1 for scalars, minus 3 for $\mathrm{Aut}(\mathbb{P}^1)$, so there are 20 parameters.

If $g = 1$, the moduli space of elliptic curves provided with a line bundle $\mathcal{O}_Y(1)$ of degree 5 has dimension 1. (The line bundles are all equivalent via translations.) Here $h^0(\mathcal{O}_Y(1)) = \mathbb{C}^5$, so the space of parameters for the embedding has dimension 19. This gives a 20-dimensional space altogether.   □

LEMMA 6.2
Let $\mathbf{G}_{\mathrm{rci}}(6,3)$ denote the locally closed subset of $\mathbf{G}_{\mathrm{rci}}$ parameterizing reduced complete intersections Z_U such that $Z_U = Z' \cup Z''$ with Z'' irreducible of degree 6. The degree 3 piece Z' is allowed to have other irreducible components. Then $\mathbf{G}_{\mathrm{rci}}(6,3)$ is the union of four irreducible components parameterizing

(a) *the case where Z' is a rational normal space cubic;*
(b) *the case where Z' is a plane cubic;*
(c) *the case where Z' is a disjoint union of a plane conic and a line; and*
(d) *the case where Z' is a disjoint union of three lines.*

These have dimensions 28, 30, 26, and 24, respectively. For (a) there is an open set on which $Z_U = Z' \cup Z''$ with Z' and Z'' being smooth and meeting transversally in 8 points.

Proof
We divide into cases corresponding to the piece Z' obtained by removing the degree 6 irreducible component Z''. In the first case (a), we include all degree 3 curves Z' which are connected chains of rational curves with no loops or self-intersections, which then have to span $\mathbb{P}^3$. In these cases, $H^0(\mathcal{O}_{Z'}(1))$ is always 4-dimensional, and Z' deforms to a smooth rational normal space cubic. For a given Z', the space of choices of U is the Grassmannian of 2-planes in $H^0(J_{Z'/\mathbb{P}^3}(3))$ and $H^0(\mathcal{O}_{Z'}(3))$ has dimension 10; one can check (case by case) that the restriction map from $H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ is surjective, so $h^0(J_{Z'/\mathbb{P}^3}(3)) = 10$ and the Grassmannian has dimension 16. The space of choices of rational normal space cubic is irreducible, equal to the space of choices of basis for the 4-dimensional space $H^0(\mathcal{O}_{Z'}(1))$ ($16d$), modulo scalars ($1d$), and the automorphisms of the rational curve Z' ($3d$). In the case of a chain the dimension of the automorphism group goes up, so those pieces are of smaller dimension in the closure of the open set where Z' is smooth. The dimension of this component is therefore

$$\dim \mathbf{G}_{\mathrm{rci}}(6,3)^{(a)} = 16 + 16 - 1 - 3 = 28.$$

A general point corresponds to a smooth Z' with general choice of U yielding a smooth curve Z'' of degree 6 meeting Z' at 8 points.

The remaining possibilities are (b), (c), and (d), which are irreducible, and one counts the dimensions as

(b) a plane $3d$ plus a cubic $9d$ plus a subspace of $H^0(J_{Z'/\mathbb{P}^3}(3)) = \mathbb{C}^{11}$, $18d$ for a total of 30;

(c) a plane $3d$ plus a conic $5d$ plus a disjoint line $4d$ plus a subspace of $H^0(J_{Z'/\mathbb{P}^3}(3)) = \mathbb{C}^9$, $14d$ for a total of 26;

(d) three lines $12d$ plus a subspace of $H^0(J_{Z'/\mathbb{P}^3}(3)) = \mathbb{C}^8$, $12d$ for a total of 24. $\qquad\square$

LEMMA 6.3

Let $\mathbf{G}_{\mathrm{rci}}(7,2)$ denote the locally closed subset of $\mathbf{G}_{\mathrm{rci}}$ parameterizing reduced complete intersections Z_U such that $Z_U = Z' \cup Z''$ with Z'' irreducible of degree 7. The degree 2 piece Z' is allowed to have other irreducible components. Then $\mathbf{G}_{\mathrm{rci}}(7,2)$ is the union of two irreducible components parameterizing

(a) *the case where Z' is a plane conic; and*
(b) *the case where Z' is a disjoint union of two lines.*

These have dimensions 30 and 28, respectively. For (a) there is an open set on which $Z_U = Z' \cup Z''$ with Z' and Z'' being smooth and meeting transversally in 6 points.

Proof
The complementary curve Z' has degree 2. If irreducible, it has to be a plane conic. If reducible, it is the union of two lines. If the lines meet, this still corresponds to (a); if they are disjoint, it is case (b). Both cases have irreducible spaces of parameters.

To count the dimensions, in case (a) the choice of plane $H \cong \mathbb{P}^2 \subset \mathbb{P}^3$ is $3d$, the choice of conic in the plane is $5d$, and $h^0(\mathcal{O}_{Z'}(3)) = 7$. One can check that the restriction map is surjective (since Z' is reduced there are only two cases, an irreducible conic or two crossing lines), so $h^0(J_{Z'/\mathbb{P}^3}(3)) = 13$ and the Grassmannian of 2-planes in here has dimension 22. The total dimension is therefore $3 + 5 + 22 = 30$.

In case (b) the choice of two disjoint lines is 8-dimensional, and $h^0(J_{Z'/\mathbb{P}^3}(3)) = 12$; the Grassmannian of 2-planes has dimension 20, so the total dimension here is 28. $\qquad\square$

LEMMA 6.4

The subvariety $\mathbf{G}_{\mathrm{rci}}(8,1)$ parameterizing $Z_U = Z' \cup Z''$ with Z'' irreducible of degree 8, is irreducible of dimension 32. It has an open set on which Z'' is smooth and meets the line Z' transversally in 4 points.

Proof
Note that Z' has to be a line. The space of lines has dimension 4 and $h^0(J_{Z'/\mathbb{P}^3}(3)) = 16$. The Grassmannian of 2-planes U in $H^0(J_{Z'/\mathbb{P}^3}(3))$ has dimension 28, so the total dimension is 32. The general element is contained in a smooth cubic surface, on which the relevant linear system defining Z'' has no base points

so a general Z'' is smooth; the intersection $Z' \cap Z''$ has 4 points by the adjunction formula. □

In the nonreduced case, we can give a bound for the dimension. It would be interesting to have a better understanding of the strata.

LEMMA 6.5

The variety parameterizing $Z_U = Z' \cup Z''$ such that Z'' is reduced and irreducible of degree 6 or 7 has dimension ≤ 30.

Proof

From Lemmas 6.2 and 6.3, the dimension is ≤ 30 for the case when all of Z is reduced. If Z is nonreduced, then noting that it has no embedded points since it is a complete intersection, and that the nonreduced part has degree ≤ 3, it must contain a uniquely determined line with multiplicity ≥ 2. The dimension count is similar to the case of Lemma 6.4, but we choose two extra points to measure the nonreduced structure.

Consider the space of choices (Z, a, b) where a and b are distinct points on the double (or triple) line of Z, provided with normal directions to the line that are contained in Z. Because Z has at least a double structure along the line, the space of choices of (Z, a, b) has dimension at least two more than that of the space of choices of Z. Thus, we need to show that the space of (Z, a, b) has dimension ≤ 32.

For a given line L with two distinct points a, b together with normal directions to L at a and b, the space of cubics vanishing along L and vanishing in the given normal directions has dimension 14. Therefore, the Grassmannian of 2-planes in here, parameterizing complete intersections Z containing L plus the normal directions at a and b, has dimension 24. Adding to this the 4-dimensional space of choices of L, plus 2 for the choices of points, plus 2 for the choices of normal directions gives 32. This proves the lemma. □

7. The common curve case

Consider a general bundle E in its irreducible component, a general section $s \in H^0(E(1))$, and a general two-dimensional subspace $U \subset W := H^0(J_{P/\mathbb{P}^3}(3)) \cong H^0(J_{P/X}(3))$. Let $Z \subset \mathbb{P}^3$ be the intersection defined by U, which is a complete intersection by Corollary 5.3. Write $P = P' \cup P''$ as usual, and let $Z'' \subset Z$ be the irreducible component containing P''.

Let $Q \subset \mathbb{P}^3$ be the intersection of the four independent cubics spanning $W = H^0(J_{P/\mathbb{P}^3}(3))$. It is contained in Z; indeed, it is the intersection of Z with the other two cubics spanning the complement of $U \subset W$. Hence $\dim(Q) \leq 1$, and also of course $P \subset Q$. Write $Q = Q_0 \cup Q_1$ where Q_1 is the union of 1-dimensional pieces of Q and Q_0 is the remaining zero-dimensional part. Notice that $Q_1 \cap P$ and $Q_0 \cap P$ correspond to Galois invariant pieces in the situation where s is

a generic geometric point, so by Proposition 5.2 it follows that if $P'' \cap Q_1$ is nonempty, then $P'' \subset Q_1$ and similarly for Q_0.

Our situation therefore breaks down into two distinct cases:

– the *common curve case* when the 1-dimensional part Q_1 contains the big variable part P''; or

– the *variable curve case* when $P'' \subset Q_0$.

In this section, we would like to rule out the first possibility; reasoning by contradiction, suppose on the contrary that we are in the common curve case. Since $Q_1 \subset Z$, and by Corollary 5.3 there is a single irreducible component Z'' of Z containing P'', it follows that $Z'' \subset Q_1$. The common curve case is therefore equivalent to the following hypothesis, which will be in effect throughout the section until it is ruled out.

HYPOTHESIS 7.1

All of the sections in $W = H^0(J_{P/\mathbb{P}^3}(3))$ vanish along Z''.

LEMMA 7.2

Suppose that $\deg(Z'') \neq 6$. Then Z'' is contained in a quadric, from which it follows that P is contained in a quadric.

Proof

If we can show that Z'' is contained in a quadric, then it follows that P is contained in the same quadric by Lemma 2.2. Suppose $\deg(Z'') \leq 5$. Then by Lemma 6.1, either Z'' is contained in a quadric or it runs in a space of dimension ≤ 20. In the latter case, for each choice of Z'' we have a space of possible choices of P of dimension 20, $19 + 3 = 22$, or $18 + 5 = 23$ depending on whether P' is empty, a single point, or an infinitesimal tangent vector. In all cases, this results in a space of possible subschemes P of dimension $\leq 43 < 44$, so by Proposition 4.4 it cannot contribute to a general point in the irreducible component.

Suppose $\deg(Z'') = d \geq 7$ (and of course $Z'' \subset Z$, so $d \leq 9$). Choose a hyperplane $H \cong \mathbb{P}^2 \subset \mathbb{P}^3$ not passing through a point of P'', and let $A := H \cap Z'' \subset \mathbb{P}^2$ be the intersection. It is finite of length d. Using Hypothesis 7.1, we get a 4-dimensional space W of sections of $H^0(J_{Z''/\mathbb{P}^3}(3))$. (One can note that, by the same argument as in Lemma 2.2, sections of $\mathcal{O}(3)$ vanishing on Z'' vanish also on P so $H^0(J_{Z''/\mathbb{P}^3}(3)) = H^0(J_{P/\mathbb{P}^3}(3))$.) Consider the restriction map

$$r : W \to H^0(J_{A/\mathbb{P}^2}(3)).$$

If w lies in the kernel, it means that w factors as the linear form defining H times a quadric. Then this quadric contains Z'', so it contains P by Lemma 2.2.

We now show that r is not injective.

LEMMA 7.3

Suppose $A \subset \mathbb{P}^2$ is a subscheme of length 7. If A does not contain 5 points on

a line (i.e., the intersection with any line has length ≤ 4), then A imposes 7 independent conditions on $H^0(\mathcal{O}_{\mathbb{P}^2}(3))$.

Proof

Choose a line L with maximal value of the length ℓ of $L \cap A$. Then $2 \leq \ell \leq 4$. On the 10-dimensional space of cubics we can try to impose 3 further conditions and should prove that this makes sections vanish. Imposing 0, 1, or 2 additional conditions on cubics restricted to L makes them vanish. The residual subscheme A' of A with respect to L has length $7 - \ell$, and we have to show that it imposes this number of conditions on conics. Again choosing a line L' with maximal contact (of order $2 \leq \ell' \leq 3$) with A', imposing 0 or 1 additional conditions we get vanishing of the conics on L'; we are left with a further residual subscheme A'' of length $7 - \ell - \ell'$, which is between 0 and 3; however if A'' consisted of 3 colinear points, that would imply $\ell \geq 3$, so A'' would have length ≤ 2 and this is ruled out. Hence A'' imposes independent conditions on linear sections. We conclude that A imposed 7 independent conditions on cubics.   □

To finish the proof of Lemma 7.2, consider the subscheme A from that proof. It has length 7, 8, or 9. Since H was general, no 5 points of A lie on a line. Applying Lemma 7.3 to a subscheme of length 7, we see that A has to impose at least 7 conditions on cubics. But $r(W)$ is a subspace of the 10-dimensional $H^0(\mathcal{O}_{\mathbb{P}^2}(3))$, vanishing on A. Thus $\dim(r(W)) \leq 3$ showing that r cannot be injective. This completes the proof.   □

To finish this section, we just have to consider the case when $\deg(Z'') = 6$. Consider first the case where a general Z is reduced, and apply Lemma 6.2. Notice that for each choice of Z a reduced complete intersection $Z = Z_U$, $U \in \mathbf{G}_{\mathrm{rci}}(6,3)$, the space of possible choices of P has dimension ≤ 20. From $P \subset Z$ this is clear when P is reduced. The other possibility is that P' is an infinitesimal tangent vector. In that case, P'' is to be chosen in the smooth subset of Z'', giving an 18-dimensional space of choices. When P' is in the smooth part of Z' it really only corresponds to a 1-dimensional space of choices, giving 19 in all; when P' is in the singular set of Z, the choice of p' is zero-dimensional and the choice of tangent vector ≤ 2-dimensional, so we get a space of choices of dimension ≤ 20 in all.

From Lemma 6.2, the space of choices of pairs (P, Z) in case (a) has dimension $\leq 28 + 20 = 48$. This is the same as the dimension of the component of the Hilbert scheme we are looking at. However, a general pair (Z, P) with $Z = Z_U$ in the 28-dimensional piece $\mathbf{G}_{\mathrm{rci}}(6,3)^{(a)}$ and $P \subset Z$ general does not occur. Indeed, the degree 6 piece Z'' is a smooth curve of genus 3, so there are 22 sections of $\mathcal{O}(4)$ on Z'', and imposing up to 20 conditions cannot make the sections vanish there; on the other hand, we could start by imposing up to 2 independent conditions from P' on the degree 3 piece Z'. Thus, a general choice of $P \subset Z$ imposes 20 conditions on the 27-dimensional space $H^0(\mathcal{O}_Z(4))$, leaving only 7

sections to add to $h^0(J_Z(4)) = 8$ giving 15. So, for a general choice of $P \subset Z$ we have $h^0(J_{P/\mathbb{P}^3}(4)) = 15$ and P cannot satisfy CB(4). Hence the space of (P, Z) such that Z decomposes with a degree 6 piece Z'' is a proper subspace of our irreducible component, so for general bundles E this case does not occur.

In case (b) of Lemma 6.2, consider the plane H containing Z'. The subspace U of cubics vanishing on Z has dimension 2, whereas $H^0(J_{Z'/H}(3))$ has dimension 1. (The plane cubic Z' determines its equation uniquely up to a scalar.) Therefore the restriction map from U to $H^0(\mathcal{O}_H(3))$ is not injective, but an element $u \in U$ mapping to zero on H must be a product of a quadric and the linear equation of H. This gives a quadric containing Z'' and hence P. So, for bundles with $h^0(E) = 0$, this case does not occur.

In cases (c) and (d) of Lemma 6.2, the total dimension is $\leq 26 + 20$ which is too small, so these do not contribute for general bundles E.

This completes the analysis of the case where a general Z is reduced. If a general Z is nonreduced, the nonreduced components Z_i must be fixed, but different from Z''. As Z'' is also fixed (when we vary the two-dimensional subspace U of cubics), there must be at least one variable component Z_j. The degree of the complementary piece to Z'' is 3, so the only possibility is a fixed line of multiplicity two and a variable line of multiplicity 1. But then we have a 4-dimensional space of cubics passing through the degree 8 curve $Z'' \cup Z_i$, so as in the proof of Lemma 7.2, this would give a quartic containing $Z'' \cup Z_i$.

We have finished the proof of the following theorem ruling out the common curve case.

THEOREM 7.4

Hypothesis 7.1 leads to a contradiction. Therefore, for a general seminatural bundle E in its irreducible component and a general section $s \in H^0(E(1))$ defining a scheme of zeros P, if the intersection of the four cubics passing through P has a 1-dimensional piece Q_1, then the big interchangeable collection of points $P'' \subset P$ does not meet Q_1. In other words, we are in the variable curve case.

8. The reducible variable curve case

The common curve case is ruled out by the previous section. Hence we are in the variable curve case, when $P'' \subset Q_0$. It means that the choice of W, which determines Q, then determines P'' and hence almost P'. (Note however that P' could still be in a 1-dimensional piece of Q_1.) Let $U \subset W$ be a general 2-dimensional subspace determining a complete intersection $Z = Z_U$. In this section, we consider the case when Z is not irreducible, a possibility which we would like to rule out.

As was argued before, the points of P'' are indistinguishable under the Galois group; the subspace U may be chosen over the same field as P, so P'' must be contained in the smooth points of a single irreducible component Z'' of Z. Write $Z = Z' \cup Z''$ where the remaining piece Z' is allowed to be reducible.

Applying Lemmas 6.1 (as in the first paragraph of the proof of Lemma 7.2) and 2.2 as well as the hypothesis $h^0(E) = 0$ so P is not contained in a quadric gives $\deg(Z'') \geq 6$.

The idea is to use a dimension count. The dimensions of the cases go all the way up to $\dim \mathbf{G}_{\mathrm{rci}}(8,1) = 32$. However, the subspace W determines P'', and in turn W is determined by a smaller subset of points than P, so the dimension count can still work.

Choose a subscheme $P_{16} \subset P$ of length 16 as follows. Start with P_2 of length 2 containing P'. Note that P_2 imposes 2 independent conditions on $H^0(\mathcal{O}_{\mathbb{P}^3}(3))$. Then for $3 \leq i \leq 16$ let $P_i := P_{i-1} \cup \{p_i\}$ with p_i chosen in P'' such that it imposes a nontrivial condition on $H^0(J_{P_{i-1}/\mathbb{P}^3}(3))$. This exists because

$$h^0\big(J_{P/\mathbb{P}^3}(3)\big) = 4 < 20 - (i-1) = h^0\big(J_{P_{i-1}/\mathbb{P}^3}(3)\big).$$

For $i = 16$ we get P_{16} imposing 16 independent conditions, and $P' \subset P_{16}$. It follows that

$$W = H^0\big(J_{P/\mathbb{P}^3}(3)\big) = H^0\big(J_{P_{16}/\mathbb{P}^3}(3)\big).$$

In particular, W is determined by P_{16}. However, the remaining four points of $P - P_{16}$ are all in P''; in particular they are reduced points. Because of the "variable curve case" Theorem 7.4, the intersection Q of the cubics in W has dimension zero at the points of P''; therefore, the locations of the remaining four points are determined (up to a finite choice) by W. We get that P is determined by P_{16}.

We may now count the dimension of the space of choices of pair (P, Z) where $Z = Z_U$ for a general subspace $U \subset W$. The space of choices of Z containing a degree 6 or degree 7 piece is ≤ 30 by Lemma 6.5. The dimension of the space of choices of P_{16} inside Z is ≤ 16 if we assume Z reduced, or ≤ 17 in any case, so the total dimension there is ≤ 47, which is too small. For the case of Z containing a piece Z'' of degree 8, we get a dimension of $32 + 16 = 48$, so this looks possible. However, the general element Z of the parameter space corresponds to the union of a smooth degree 8 curve Z'' meeting a line Z' in 4 points. In order to get to dimension 48, we must have P general; in particular P'' is a general collection of 18 points in Z''. Now Z'' has genus 7. The line bundle $\mathcal{O}_{Z'}(3)(-P'')$ is a general one of degree 6, which on a curve of genus 7 will not have any sections. Hence, all cubics containing P must vanish on Z'', which would put us back into the "constant curve case." So, this case does not occur.

We have finished ruling out the possibility that Z would be reducible, resulting in the following theorem.

THEOREM 8.1
For a general seminatural bundle E in its irreducible component and a general section $s \in H^0(E(1))$ defining a scheme of zeros P, choose a general 2-dimensional subspace $U \subset W = H^0(J_{P/\mathbb{P}^3}(3))$ defining a complete intersection Z_U. Then Z_U is irreducible.

9. Subschemes of an irreducible degree 9 curve

In this section we complete the proof that the Hilbert scheme $\mathbf{H}_{\mathbb{P}^3}^{sn}$ is irreducible, by treating the case $P \subset Z_U$ where Z_U is an irreducible complete intersection of degree 9.

We first indicate how to construct an open set of the Hilbert scheme. Consider a smooth complete intersection curve Z_U for a general 2-dimensional subspace $U \subset H^0(\mathcal{O}_{\mathbb{P}^3}(3))$. The Grassmannian of choices of U has dimension 36, and there is a dense open set where $Z = Z_U$ is smooth of genus 10.

Now $P \subset Z$ will be a subscheme of length 20, which is a Cartier divisor since Z is smooth. By varying any collection of 10 points, we obtain a family which surjects to the Jacobian $\mathrm{Jac}^{20}(Z)$. The line bundle $L = \mathcal{O}_Z(4)(-P)$ has degree $36 - 20 = 16$. Note that the map

$$H^0\big(\mathcal{O}_{\mathbb{P}^3}(4)\big) \to H^0\big(\mathcal{O}_Z(4)\big)$$

is surjective, with kernel of dimension 8. Hence, in order to obtain $h^0(J_{P/\mathbb{P}^3}(4)) = 16$ one should ask for $h^0(\mathcal{O}_Z(4)(-P)) = 8$, that is, $h^0(L) = 8$. As $g = 10$ we get $\chi(L) = 16 + 1 - 10 = 7$. The condition $h^0(L) = 8$ is therefore equivalent to $h^1(L) = 1$ or by duality, $h^0(K_Z \otimes L^{-1}) = 1$. Now, $K_Z = \mathcal{O}_Z(2)$ has degree 18, so $M := K_Z \otimes L^{-1}$ is a line bundle of degree 2. Asking for it to have a section is equivalent to asking that $M \cong \mathcal{O}_Z(x+y)$ for a degree 2 effective divisor $(x)+(y) \in Z^{(2)} \subset \mathrm{Jac}^2(X)$. The dimension of choices of M is 2, and the space of choices is nonempty and irreducible. For each choice of M, we have $L := K_Z \otimes M^{-1}$, and the space of choices of divisor P such that $\mathcal{O}_Z(4)(-P) = L$ is a projective space of dimension

$$\#(P) - \dim\big(\mathrm{Jac}(Z)\big) = 10.$$

Putting these together, we get a nonempty irreducible 12-dimensional space of choices of $P \subset Z$ such that $h^0(J_{P/\mathbb{P}^3}(4)) = 16$. Including the variation of Z in a 36-dimensional space, these fit together to form a nonempty irreducible 48-dimensional variety.

If we replace P by a subscheme $P_1 \subset P$ of colength 1 in the above argument, then M changes to $M_1 = M(z) = \mathcal{O}_Z(x + y + z)$ where $(z) = P - P_1$. As this is a general point of Z, we still have $h^0(M_1) = 1$ giving the Cayley–Bacharach condition $\mathrm{CB}(4)$ for P. Hence, there is a dense open subset of the 48-dimensional variety parameterizing pairs (Z, P) where P satisfies $\mathrm{CB}(4)$. This is our irreducible component of $\mathbf{H}_{\mathbb{P}^3}^{sn}[2]$. Abstracting out the choice of Z gives an irreducible 44-dimensional component of the Hilbert scheme $\mathbf{H}_{\mathbb{P}^3}^{sn}$.

THEOREM 9.1

The irreducible component constructed above is the only one in $\mathbf{H}_{\mathbb{P}^3}^{sn}$.

Proof

The argument above shows the basic idea. However, we need to do some more work to treat the case when Z is singular and especially the possibility of a

point or infinitesimal tangent vector in P'. The first step is to rule out this last possibility.

LEMMA 9.2

A general P in its irreducible component is reduced.

Proof

Given P we can choose a quintic surface X containing it and write $P = P' \cup P''$. We have P'' reduced, and if P' is nonreduced, it consists of a single infinitesimal tangent vector. Furthermore we may assume that P is at a smooth point of its Hilbert scheme. Choose a local smoothing infinitesimal deformation of P'; we would like to extend that to a deformation of P preserving the CB(4) condition. As the Cayley–Bacharach property is open, it is equivalent to preserving the property $h^0(J_{P/\mathbb{P}^3}(4)) = 16$. One can check that the obstruction to finding a deformation of P'' which, when added to the given deformation of P', preserves $h^0(J_{P/\mathbb{P}^3}(4))$, would be the existence of a section $t \in H^0(J_{P/\mathbb{P}^3}(4))$ such that t vanishes to order 2 at all the points of P'' in $\mathbb{P}^3$.

Consider a complete intersection of cubics Z containing P, and we may assume that P'' lies on the smooth locus of Z. From the results of the previous sections, we may assume that Z is an irreducible curve of degree 9. Hence $\mathcal{O}_Z(4)$ is a line bundle of degree 36. Our section t vanishes at $2P'' \subset Z$ but also at the points of P'. Together these are at least 38 points, so it follows that t vanishes on Z. Let $C \subset \mathbb{P}^3$ be one of the cubics defining Z. The residual of the scheme $2P''$ of multiplicity 2 at P'', intersected with C, consists of all the points of P''. The restriction $t|_C$, divided by the other equation of Z, corresponds to a linear section vanishing at these points; but the points of P'' are not all contained in a plane (indeed they are not even contained in a quadric), so $t|_C = 0$. Then t divided by the equation of C is a linear form again vanishing on P'', so it is zero. Thus, $t = 0$.

This proves that the obstruction to lifting our smoothing deformation of P' to a deformation of P vanishes. Therefore, for a general point P the piece P' has to consist of at most a single reduced point. This proves the lemma. $\square$

Suppose next that P is a Cartier divisor on Z. This will always be the case at points of P'' which are smooth points of Z, but it remains a possibility that P' is a nonmovable point at a singularity of Z. We will deal with this problem below, but for now in the interest of better explaining the argument, assume that $L := \mathcal{O}_Z(4)(-P)$ is a line bundle which we may think of as being a restriction from a small analytic neighborhood of Z.

Now Z is a complete intersection, so duality still applies. This can be seen, for example, by using Serre duality on $\mathbb{P}^3$ and the equations for Z which provide resolutions for $\mathcal{O}_Z$; the local Ext sheaves may be tensored with L, which exists on a neighborhood of Z. We get

$$H^i(Z, L|_Z) \cong H^{1-i}\big(Z, L^{-1} \otimes \mathcal{O}_Z(2)\big)^*.$$

Applying this to $L = \mathcal{O}_Z(4)(-P)$, we get

$$h^1(L|_Z) = h^0\big(\mathcal{O}_Z(-2)(P)\big).$$

On the other hand, $\chi(L) = 7$ and, as before, $h^0(J_{P/\mathbb{P}^3}(4)) = h^0(L|_Z) + 8$, so the condition $h^0(J_{P/\mathbb{P}^3}(4)) = 16$ is equivalent to $h^1(L|_Z) = 1$, which in turn is equivalent to asking that the degree 2 line bundle $\mathcal{O}_Z(-2)(P)$ be effective.

The Picard scheme $\mathrm{Pic}^0(Z)$ is still a group scheme, hence smooth; and its tangent space at the origin is $H^1(\mathcal{O}_Z)$. The exact sequences for $Z \subset C \subset \mathbb{P}^3$ (where C is one of the cubics cutting out Z) give $H^1(\mathcal{O}_Z) \cong H^3(\mathcal{O}_{\mathbb{P}^3}(-6)) = H^0(\mathcal{O}_{\mathbb{P}^3}(2))^*$ which is 10-dimensional. So the group scheme as well as its torsors $\mathrm{Pic}^d(Z)$ are 10-dimensional. An infinitesimal argument with exact sequences also shows that for 10 general points in Z, the map from the product of their tangent spaces to the Picard scheme is surjective. As P consists of 20 points, and the Picard scheme has dimension 10, at least 10 points can move generally, keeping the same divisor P up to linear equivalence.

The effective divisors form a two-dimensional subscheme of $\mathrm{Pic}^2(Z)$. Thus, at a general $P \subset Z$ satisfying $h^0(J_{P/\mathbb{P}^3}(4)) = 16$, the Hilbert scheme of such P has dimension 12. The locus of singular Z has dimension ≤ 35, so the pairs (Z, P) with Z singular lie on a subscheme of dimension ≤ 47 and cannot therefore correspond to a general bundle E in its irreducible component. This finishes the proof of Theorem 9.1 in the case where P corresponds to a Cartier divisor.

Some further argument is needed for the general case. The reader may calculate directly that the dimension of the space of (Z, P), such that Z is a nodal curve and P contains a point p' located at the node, is <48 and does not contribute. This indicates that we do not get a new irreducible component in this way.

To give a more complete argument, consider (Z, P) with Z singular (but still reduced and irreducible) and P including a point $p' \in P'$ located at a singular point of Z. Consider general hyperplanes $H \subset \mathbb{P}^3$ passing through p', let $K := (H \cap Z)_{p'}$ (meaning the local piece of $H \cap Z$ at p'), and let $P^+ = P'' \cup K$. This is now a Cartier divisor on Z, so the previous considerations apply. Let ℓ denote the length of K. The condition that Z is not contained in a plane means that the general intersection $H \cap Z$ cannot be concentrated at a single point; on the other hand p' is singular in Z, so $2 \leq \ell \leq 8$. The exact sequences for complete intersections imply that K imposes ℓ independent conditions on cubics.

Our point p' is in the base locus B_2 for the bundle E, meaning that sections in $H^0(J_{P/X}(3))$ vanish to order ≥ 2 at p' in X. This is true for any general quintic X passing through P, so sections of $H^0(J_{P/\mathbb{P}^3}(3))$ vanish to order ≥ 2 at p' in $\mathbb{P}^3$. In particular, Z contains the multiplicity two fat point at p'. In turn, this implies that K contains the multiplicity two fat point at p' in H.

We have $h^0(J_{P^+/\mathbb{P}^3}(3)) \geq 17 - \ell$, which translates, using duality and calculating the Euler characteristic, into $h^0(\mathcal{O}_Z(-2)(P^+)) \geq 1$. That is to say, $\mathcal{O}_X(-2)(P^+)$ is an effective line bundle of degree $\ell + 1$. (The case $\ell = 1$ would correspond to the case treated previously.) The dimension of the space of choices

of P^+ satisfying this effectivity condition, at general P'' in its linear system, is $\leq 11 + \ell$. Note that since P is reduced, P^+ determines P.

For a given $K \subset H$, the space of choices of Z passing through K is the Grassmannian of 2-planes in $\mathbb{C}^{20-\ell}$, so it has dimension $2(18 - \ell) = 36 - 2\ell$. We consider the space of choices of (p', H, Z, P). The choices of $p' \in H$ form a 5-dimensional space. Let k denote the dimension of the space of choices of $K \subset H$ located at a given point p'. Then altogether, the space of choices of (p', H, Z, P) has dimension

$$\leq 5 + (11 + \ell) + (36 - 2\ell) + k = 52 + k - \ell.$$

This should be compared with the dimension of the Hilbert scheme, plus the number of choices of H (2-dimensional) for each P, which is to say 50.

The dimension count is now taken care of by noting that $K \subset H$ contains the fat point of multiplicity 2 at p' and this part is fixed without parameters. The remaining parameters for the choice of K therefore correspond to the length of the remaining subscheme, which is to say, $k \leq \ell - 3$. This gives a count of ≤ 49 for the space of (p', H, Z, P) corresponding to the singular situation, which is < 50 so it does not contribute to the general points of the Hilbert scheme of (Z, P). This completes the proof of Theorem 9.1. $\qquad\square$

10. Bundles on the quintic

To complete the proof of Theorem 0.2, we should go back from the Hilbert scheme of CB(4)-subschemes in $\mathbb{P}^3$, to the Hilbert scheme of CB(4)-subschemes of a general quintic X. Note first of all that we have looked above at the Hilbert scheme $\{(Z, P)\} := \mathbf{H}^{sn}_{\mathbb{P}^3}[2]$ of pairs (Z, P). However, for a given P the space of choices of Z is just a Grassmannian of 2-planes $U \subset W \cong \mathbb{C}^4$. So, irreducibility of the 48-dimensional Hilbert scheme $\{(Z, P)\}$ implies irreducibility of the 44-dimensional Hilbert scheme $\{P\} := \mathbf{H}^{sn}_{\mathbb{P}^3}$. The incidence variety of pairs (P, X) such that X is a smooth quintic hypersurface containing P will be denoted by $\{(P, X)\}$. The map $\{(P, X)\} \to \{P\}$ is a fibration in projective spaces of dimension 35; indeed by the seminatural condition P imposes 20 conditions on the 56-dimensional space $H^0(\mathcal{O}_{\mathbb{P}^3}(5))$, and we should also divide out by scalars. Thus, the incidence variety $\{(P, X)\}$ is irreducible of dimension 79. The space of quintics denoted $\{X\}$ is an open subset of $\mathbb{P}^{55}$, and the Hilbert scheme of choices of P for a given general X is the 24-dimensional fiber of the map

$$\{(P, X)\} \to \{X\}.$$

Up to now, we have shown that the source of this map is irreducible. An additional argument is needed to show that the fibers are irreducible. We will use the same argument as was used in [13], which was pointed out to us by A. Hirschowitz.

The idea is to say that there is a specially determined irreducible component of each fiber; then this component is invariant under the Galois action of the Galois group of the function field of the base, on the collection of irreducible components of the fiber. On the other hand, irreducibility of the total space means

that the Galois group acts transitively on the set of irreducible components of the fiber, and together these imply that the fiber is irreducible.

In order to isolate a special irreducible component, notice that the singular locus of the moduli space of bundles was identified in [13]. It has some explicit irreducible components corresponding to the choice of CB(2)-subschemes of length 10 in X, yielding the case of bundles with $H^0(E) \neq 0$. (This is the case we have been explicitly avoiding throughout the bulk of the argument above.) We consider the 19-dimensional component of the singular locus whose general point is a bundle E fitting into an exact sequence

$$0 \to \mathcal{O}_X \to E \to J_R(1) \to 0$$

where $R \subset Y$ is a general collection of 10 points on $Y = X \cap C$ for a quadric C.

For a general such bundle E, there is a unique co-obstruction, which is to say a unique exact sequence as above, and the Zariski normal space to the singular locus may naturally be identified with $H^1(E)$, which has dimension 2. The second-order obstruction map is the same as the quadratic form associated to the symmetric bilinear form obtained from duality $H^1(E) \cong H^1(E^*(1))^* = H^1(E^*)$. This quadratic form defines a pair of lines inside $H^1(E)$. These are the two actual normal directions of the moduli space of bundles along the singular locus at E. In order to show that this component of the singular locus meets a canonically defined irreducible component of the moduli space, it suffices to show that these two lines are interchanged as R moves about in the Hilbert scheme of 10-tuples of points in Y.

The 2-dimensional space $H^1(E)$, together with its quadratic form, depends only on the arrangement $R \subset \mathbb{P}^3$ of 10 points on a quadric $C \cong \mathbb{P}^1 \times \mathbb{P}^1$, in a way we now explain. The homogeneous coordinates of the 10 points give a map $\mathbb{C}^4 \to \mathbb{C}^{10}$. We get a map $\mathrm{Sym}^2(\mathbb{C}^4) \to \mathbb{C}^{10}$, and the equation of the quadric C is an element of the kernel; as $\mathrm{Sym}^2(\mathbb{C}^4)$ has dimension 10 itself, there is an element $\xi = (\xi_1, \dots, \xi_{10})$ in the cokernel, unique up to scalars. The CB(2)-condition, which holds for general R, corresponds to asking that $\xi_i \neq 0$ for all $1 \leq i \leq 10$. Therefore ξ defines a nondegenerate symmetric bilinear form on $\mathbb{C}^{10}$ denoted

$$\langle X, Y \rangle := X \Delta(\xi) Y^t = \sum_{i=1}^{10} \xi_i x_i y_i.$$

The condition that ξ vanish on the image of $\mathrm{Sym}^2(\mathbb{C}^4)$ says that $\mathbb{C}^4 \subset \mathbb{C}^{10}$ is an isotropic subspace. In other words, it is contained in its orthogonal subspace $\mathbb{C}^4 \subset (\mathbb{C}^4)^\perp \cong \mathbb{C}^6$. The quotient $\mathbb{C}^6 / \mathbb{C}^4$ is our two-dimensional space $H^1(E)$, and $\Delta(\xi)$ induces a quadratic form here. We are interested in the two isotropic lines. Fix 9 of the points in a general way; then our two-dimensional subspace with quadratic form depends on a single choice of $r_{10} \in C$. A calculation shows that the discriminant divisor of the quadratic form contains reduced components in C. So if one has a curve of points $r_{10} \in Y$ which intersect this divisor transversally, the two lines are interchanged when we go around the intersection point on the curve. Now, one can choose X to pass through the given $r_1, \dots, r_9$ as well as

transversally through a general reduced point on the discriminant divisor. For such X, the tangent directions are interchanged as R moves around in $Y = X \cap C$, so the same is also true for any general X.

This completes the construction of a specified irreducible component of the moduli space of bundles. Notice that for the singular points E constructed above, we still have $H^1(E(1)) = 0$, so as soon as we move off the singular locus to get $H^1(E) = 0$, this gives a bundle with seminatural cohomology. Thus, our specified irreducible component corresponds to bundles with seminatural cohomology. Now, Hirschowitz's argument plus Theorem 9.1 saying that the Hilbert scheme of choices $\{P\}$ is irreducible combine to show that there is only one irreducible component in the moduli space of stable bundles on X of degree 1 and $c_2 = 10$ having seminatural cohomology. This completes the proof of Theorem 0.2.

11. Some ideas for the nonseminatural case

We indicate here how one should be able to treat Conjecture 0.1. Notice that we made the hypothesis that $H^1(E(1)) = 0$, and this implied seminatural cohomology. So, in the nonseminatural case we have $h^1(E(1)) \geq 1$ and $h^0(E(1)) \geq 6$. If $s : \mathcal{O} \to E(1)$ with subscheme of zeros P, then $h^0(J_{P/X}(3)) \geq 5$.

The first step will be to show that sections of $E(1)$ have a base locus consisting of at most one point p' and that a general P has to be reduced at P', with 19 points making up P'' with doubly transitive Galois action. This should be similar to our arguments of Sections 4 and 5.

One can also point out, right away, that this allows us to rule out the "common curve case" as in Section 7; indeed even in the case when Q_1 has degree 6, the same argument as we used for degrees 7 and 8 works to show that Q_1 would have to be contained in a quadric.

So, we are in the variable curve case. If Z is a complete intersection of two cubics passing through P, then $Z = Z' \cup Z''$ with Z'' irreducible, containing P'' in its smooth locus. Part of the argument consisted of ruling out $\deg(Z'') < 9$ by a dimension count. Here we cannot just transpose the arguments; indeed the dimension of the Hilbert scheme of possible collections P might be strictly smaller than 44, because each P can contribute a positive-dimensional space of extension classes.

So we should divide the argument into two cases. If $h^1(J_{P/X}(4)) = 1$, that is, $e = 0$ in the notation of Section 1, then the dimension of $\mathrm{Ext}^1(J_{P/X}(2), \mathcal{O}_X(-1))$ is 1 and the extension class is unique up to scalars. In this case, the dimension of the Hilbert scheme $\{P\}$ remains 44 (and including the complete intersection curve Z gives $\{(Z, P)\}$ of dimension 48). The dimension count may then proceed as we have done, and this should allow us to treat this case.

In the case when $h^1(J_{P/X}(4)) \geq 2$, each choice of P corresponds to a positive-dimensional space of choices of extension class up to scalars. However, in this case we can degenerate the extension class to one which no longer satisfies the Cayley–Bachrach condition—meaning that, viewed as a dual element to $\mathcal{O}_P(4)$, it vanishes on one or more points.

The doubly transitive Galois action on P implies that the images of the points P in the projective space of extension classes cannot generically bunch up in groups of more than one. Therefore, it is possible to degenerate the extension class towards one which vanishes at exactly one point of P. This means an extension which corresponds to a torsion-free sheaf E with a singularity at a single point. It therefore corresponds to a point in the boundary of the moduli space, at the boundary component coming from $M_X(2,1,9)$. This boundary piece has codimension 1, and we should be able to analyze the nearby bundles and conclude that we remain in the principal irreducible component. (Indeed it suffices to say that nearby bundles have seminatural cohomology.)

The technique of localizing our picture on the boundary of the moduli space is obviously a necessary and important one which needs to be further developed in order to treat this type of question. This will be left for a future work.

Another interesting direction will be to look at Reider's theory of nonabelian Jacobians [18] and [19] for bundles on a general quintic surface. The structures we have encountered in an ad hoc way in the course of our proof are actually pieces of Reider's theory.

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