On G/N-Hilb of N-Hilb

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Abstract In this paper we consider the iterated G-equivariant Hilbert scheme G/N-Hilb(N-Hilb) and prove that G/N-Hilb(N-Hilb($\mathbb{C}^3$)) is a crepant resolution of $\mathbb{C}^3/G$ isomorphic to the moduli space $\mathcal{M}_\theta(Q)$ of θ-stable representations of the McKay quiver Q for certain stability condition θ. We provide several explicit examples to illustrate this construction. We also consider the problem of when G/N-Hilb(N-Hilb) is isomorphic to G-Hilb showing the fact that these spaces are most of the times different.

1. Introduction

Let X be a nonsingular quasi-projective complex 3-fold, and let $G \subset \operatorname{Aut} X$ be a finite subgroup such that the stabilizer subgroup of any point $x \in X$ acts on the tangent space T_xX as a subgroup of $\mathrm{SL}(T_xX)$. Let G-Hilb(X) be the fine moduli space of G-clusters, and let $\mathcal{Z}$ be the universal subscheme. We have the following celebrated theorem of Bridgeland, King, and Reid [BKR, Theorem 1.2].

THEOREM 1.1 ([BKR])

We have that $Y = G\text{-Hilb}(X)$ is irreducible and $f : Y \to X/G$ is a crepant resolution. Furthermore, $\Phi : \mathrm{D}^b(\operatorname{Coh} Y) \longrightarrow \mathrm{D}^b(\operatorname{Coh}^G X)$ is an equivalence of derived categories where Φ is the Fourier–Mukai transform with kernel $\mathcal{O}_{\mathcal{Z}}$.

Our framework is the following: let $G \subset \mathrm{SL}(3,\mathbb{C})$ be finite, and let $N \trianglelefteq G$ be a normal subgroup. First consider the action of N on $\mathbb{C}^3$, and take the crepant resolution $Y := N\text{-Hilb}(\mathbb{C}^3)$. Next act with G/N on Y to obtain $G/N\text{-Hilb}(Y)$:

$$\begin{tikzcd}
G/N\text{-Hilb}(Y) \arrow[dr, "\tau_2"'] & & Y = N\text{-Hilb}(\mathbb{C}^3) \arrow[dl, "\pi_2"] \arrow[dr, "\tau_1"'] & & \mathbb{C}^3 \arrow[dl, "\pi_1"] \\
& Y/(G/N) \arrow[dr, "\phi_2"'] & & \mathbb{C}^3/N \arrow[dl, "\phi_1"] & \\
& & \mathbb{C}^3/G & &
\end{tikzcd}$$

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As an immediate consequence of Theorem 1.1 we have the following corollary.

COROLLARY 1.2
G/N-Hilb(N-Hilb($\mathbb{C}^3$)) is a crepant resolution of $\mathbb{C}^3/G$.

A similar construction was considered before by the second author in [Ito1] and [Ito2] in the case of trihedrals subgroups in $\mathrm{SL}(3,\mathbb{C})$. The trihedral group is a non-Abelian finite subgroup generated by diagonal matrices and the matrix $T := \left(\begin{smallmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{smallmatrix}\right)$. In this case, $G = N \rtimes T$ is a semidirect product. Then in Ito's construction, we require that $T = G/N$ act on the crepant resolution Y of $\mathbb{C}^3/N$ symmetrically on the exceptional locus. Therefore, this construction gives the G/N-Hilb(N-Hilb($\mathbb{C}^3$)) when $Y = N$-Hilb($\mathbb{C}^3$).

This construction can be extended in a natural way to obtain crepant resolutions of $\mathbb{C}^3/G$ for any finite nonsimple group $G \subset \mathrm{SL}(3,\mathbb{C})$ (see [YY] for a classification of such groups). In general, if we consider the sequence of normal subgroups N_i of the form $N_0 := G$, $N_1 := N \lhd G$, $N_2 \lhd N_0/N_1, \ldots, N_i \lhd \big(\cdots((G/N_1)/N_2)\cdots\big)/N_{i-1}$ for $i \geq 1$ and N_i is normal in G, the iterated equivariant Hilbert scheme

$$N_i\text{-Hilb}\big(N_{i-1}\text{-Hilb}(\cdots(N_1\text{-Hilb}(\mathbb{C}^3))\cdots\big)$$

described in this paper is crepant. In particular it is always possible to find such a crepant resolution with N_i Abelian for all i.

Denote by $\mathrm{Irr}(G)$ the set of irreducible representations of G, and let (Q,R) be the McKay quiver of G with relations R. For $\mathbf{d} = (\dim(\rho))_{\rho\in\mathrm{Irr}(G)}$ and any generic θ in the space of stability conditions Θ we can define $\mathcal{M}_{\theta,\mathbf{d}}(Q,R)$ to be the moduli space of θ-stable representations of Q satisfying the relations R. Moreover, there exists a chamber decomposition of Θ such that the geometric invariant theory (GIT) quotient $\mathcal{M}_{\theta,\mathbf{d}}(Q,R)$ is constant for all θ in any given (open) chamber C. Thus, we also denote this moduli space simply by $\mathcal{M}_C$. Thus we may also denote this moduli space simply by $\mathcal{M}_C$. The methods in [BKR] can be applied to prove that $\tau : \mathcal{M}_C \longrightarrow \mathbb{C}^3/G$ is a crepant resolution and $\Phi_C : D(\mathcal{M}_C) \longrightarrow \mathrm{D}^G(\mathbb{C}^3)$ is an equivalence of categories (see [CI, Section 2]).

For these moduli spaces, the problem of whether every (projective) crepant resolution of $\mathbb{C}^3/G$ is a moduli of representations of the McKay quiver was treated by Craw and Ishii [CI, Theorem 1.1] in the case of Abelian group actions.

THEOREM 1.3 ([CI])
For a finite Abelian subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$ let $Y \to \mathbb{C}^3/A$ be a projective crepant resolution. Then $Y \cong \mathcal{M}_C$ for some chamber $C \subset \Theta$.

Then Craw and Ishii proposed the following conjecture.

CONJECTURE 1.4

For a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$ let $Y \to \mathbb{C}^3/G$ be a projective crepant resolution. Then $Y \cong \mathcal{M}_C$ for some chamber $C \subset \Theta$.

In this paper we show that the projective crepant resolution G/N-Hilb(N-Hilb($\mathbb{C}^3$)) is a fine moduli space for a particular chamber $C \subset \Theta$ as follows.

THEOREM 1.5 (= 2.7)

Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be finite, and let $N \trianglelefteq G$ be a normal subgroup. The crepant resolution G/N-Hilb(N-Hilb($\mathbb{C}^3$)) is isomorphic to a moduli space of G-constellations $\mathcal{M}_C$ for some chamber $C \subset \Theta$.

Thus our main result shows that the conjecture holds for the family G/N-Hilb(N-Hilb($\mathbb{C}^3$)) of crepant resolutions for general $G \subset \mathrm{SL}(3,\mathbb{C})$.

As we see in this paper, the varieties underlying the fine moduli spaces G-Hilb($\mathbb{C}^3$) and G/N-Hilb(N-Hilb($\mathbb{C}^3$)) are in general nonisomorphic quasi-projective varieties. Even when they coincide, as moduli spaces of representations of the McKay quiver almost always they belong to different chambers in the space of stability conditions Θ, or in other words, the corresponding tautological vector bundles are not the same.

It is therefore natural to ask when the iterated Hilbert scheme G/N-Hilb(N-Hilb($\mathbb{C}^3$)) is isomorphic to G-Hilb($\mathbb{C}^3$). In this paper we give a complete answer for this problem when they are considered as moduli spaces, that is, both the underlying variety and the tautological vector bundle coincide. For the problem of when they are isomorphic as algebraic varieties, we present the list of such cases when the group G is Abelian. For non-Abelian cases, we prove that G/N-Hilb(N-Hilb($\mathbb{C}^3$)) and G-Hilb($\mathbb{C}^3$) are nonisomorphic varieties when G is a non-Abelian small subgroup of $\mathrm{GL}(2,\mathbb{C})$ embedded in $\mathrm{SL}(3,\mathbb{C})$ and $N = G \cap \mathrm{SL}(3,\mathbb{C})$, and for some polyhedral groups G in $\mathrm{SO}(3)$. These results suggest that the moduli spaces $\mathcal{M}_C$ are actually varying. We summarize the results in this direction in the following theorems.

THEOREM 1.6 (THEOREM 7.3, COROLLARY 7.4, PROPOSITION 7.5)

Let $G \subset \mathrm{GL}(2,\mathbb{C})$ be a finite small subgroup, and let $N \neq G, \{1\}$ be a normal subgroup. Let $Y := G/N$-Hilb(N-Hilb($\mathbb{C}^2$)).

(i) *If $G \subset \mathrm{GL}(2,\mathbb{C})$, then $Y \cong G$-Hilb($\mathbb{C}^2$) as moduli spaces if and only if $G \cong (1/rs)(1,1)$ and $N \cong \frac{1}{s}(1,1)$ for some $r, s \geq 2$.*

(ii) *If $G \subset \mathrm{SL}(2,\mathbb{C})$, then $Y \cong G$-Hilb($\mathbb{C}^2$) as algebraic varieties.*

(iii) *If $G \not\subset \mathrm{SL}(2,\mathbb{C})$ is non-Abelian and $N = G \cap \mathrm{SL}(2,\mathbb{C})$, then Y and G-Hilb($\mathbb{C}^2$) are nonisomorphic as algebraic varieties.*

THEOREM 1.7 (THEOREMS 7.3, 7.7, COROLLARY 7.14)
Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a finite small subgroup, and let $N \neq G, \{1\}$ be a normal subgroup. Let $Y := G/N$-Hilb(N-Hilb($\mathbb{C}^3$)).

(i) *$Y \cong G$-Hilb($\mathbb{C}^3$) as moduli spaces if and only if $G \cong (1/2r)(1,1,2r-2)$ and $N \cong (1/2)(1,1,0)$.*

(ii) *If G is Abelian, then $Y \cong G$-Hilb($\mathbb{C}^3$) as algebraic varieties if and only if we are in one of the following situations:*

(1) *$G/N \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for some $m > 1$;*

(2) *$G \cong (1/r)(1,1,r-2)$ or $G \cong \frac{1}{r}(1,r-1,0)$, that is, $\mathbb{C}^3/G$ has a unique crepant resolution;*

(3) *$G \cong (1/2r)(1,a,-a-1)$ with $(2r,a)=1$, $a^2 \equiv 1 \pmod{4r}$, and $N \cong (1/2)(1,1,0)$;*

(4) *there is a subgroup $G' \subset G$ containing N such that (G',N) fits into either (2) or (3) and $G/G' \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for some $m > 1$.*

(iii) *If $G \subset \mathrm{SO}(3)$ is of type D_{2n} or G_{12} (defined in Sections 6.1, 6.3, respectively) and N is the maximal Abelian subgroup, or if G is isomorphic to a non-Abelian finite small subgroup of $\mathrm{GL}(2,\mathbb{C})$ and $N = G \cap \mathrm{SL}(2,\mathbb{C})$, then Y and G-Hilb($\mathbb{C}^3$) are nonisomorphic as algebraic varieties.*

This paper is organized as follows. In Section 2, we introduce moduli spaces of G-constellations and find the stability for G/N-Hilb(N-Hilb($\mathbb{C}^3$)) to prove Theorem 1.5. In Section 3, we recall representations of semidirect products and fix some notation used in examples. In Section 4, we show some examples of G/N-Hilb(N-Hilb($\mathbb{C}^3$)) when G is Abelian. The construction of crepant resolutions for Abelian quotient singularities are well known as toric resolutions where one of them is G-Hilb($\mathbb{C}^3$). However, if we have several choices for normal subgroups N of G, then we have several G/N-Hilb(N-Hilb($\mathbb{C}^3$)) which can be obtained by a finite sequence of flops from G-Hilb($\mathbb{C}^3$). Moreover, the actions of G/N on N-Hilb($\mathbb{C}^3$) are also interesting. In Section 5, we introduce the notion of *skeletons* to compute local coordinates in examples. In Section 6, we give several examples of G/N-Hilb(N-Hilb($\mathbb{C}^3$)) when G is non-Abelian. We also describe the G-constellations and McKay quiver with relations. In Section 7, we investigate when G/N-Hilb(N-Hilb($\mathbb{C}^3$)) is isomorphic to G-Hilb($\mathbb{C}^3$) as moduli spaces and as algebraic varieties.

2. The moduli space of G-constellations and G/N-Hilb(N-Hilb)

Recall that G-Hilb($\mathbb{C}^n$) is the moduli space of G-clusters, where a *G-cluster $\mathcal{Z}$* is a G-invariant subscheme $\mathcal{Z} \subset \mathbb{C}^n$ such that $H^0(\mathcal{O}_Z) \cong R_G$, the regular representation of G, as $\mathbb{C}[G]$-modules. Thus a point $y \in G/N$-Hilb(N-Hilb($\mathbb{C}^n$)) is a G/N-cluster of N-clusters.

2.1. A family of G-constellations

The first observation that appears is that y may not be a G-cluster. Therefore, in order to construct the moduli space of such objects we need a generalized notion of G-cluster called *G-constellation* (see [CI] and [Cra2, Chapter 5]).

DEFINITION 2.1

A *G-constellation* $\mathcal{F}$ on X is a G-equivariant coherent sheaf on X such that $H^0(\mathcal{F}) \cong R_G$ as $\mathbb{C}[G]$-modules.

Notice that a G-cluster is a G-equivariant $\mathbb{C}[x_1, \dots, x_n]$-module $\mathcal{O}_{\mathcal{Z}}$ generated from 1 mod $I_{\mathcal{Z}}$, which is precisely a G-constellation generated from the trivial representation ρ_0.

Equivalently, when $X = \mathbb{C}^n$ a G-equivariant coherent sheaf $\mathcal{F}$ on X is a representation of the McKay quiver Q satisfying the relations R. This identification was first stated in [IN, Section 3] and rewritten in the language of G-constellations in [CI, Section 2.1]. For an explicit description of the relations R see [BSW]. Recall that the McKay quiver is the quiver with $\operatorname{Irr} G$ as its vertex set, and $\dim_{\mathbb{C}} \operatorname{Hom}(\rho, V \otimes \rho')$ arrows from ρ to ρ'.

We fix the following notation in this paper. Denote by $Y_1 := N\text{-Hilb}(\mathbb{C}^n)$, and denote by $Y_2 := G/N\text{-Hilb}(Y_1)$. Then we have the diagram

$$\begin{tikzcd}
 & \mathcal{Z}_2 \arrow[dl, "p_2"'] \arrow[dr, "q_2"] & & \mathcal{Z}_1 \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & \\
Y_2 \arrow[dr, "\tau_2"'] & & Y_1 \arrow[dl, "\pi_2"] \arrow[dr, "\tau_1"'] & & \mathbb{C}^n \arrow[dl, "\pi_1"] \\
 & Y_1/(G/N) & & \mathbb{C}^n/N &
\end{tikzcd}$$

where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are the universal families for Y_1 and Y_2, respectively.

LEMMA 2.2

Every point in the connected component of G/N-Hilb(N-Hilb($\mathbb{C}^n$)) dominating $\mathbb{C}^3/G$ is a G-constellation on $\mathbb{C}^n$. More precisely, there is a canonically defined flat family of G-constellations parameterized by this connected component of G/N-Hilb(N-Hilb($\mathbb{C}^n$)).

Proof

Consider the fiber product $\mathcal{Z}_2 \times_{Y_1} \mathcal{Z}_1 \subset Y_2 \times Y_1 \times \mathbb{C}^n$, and consider the projection $p_{20} : Y_2 \times Y_1 \times \mathbb{C}^n \longrightarrow Y_2 \times \mathbb{C}^n$ onto the first and third factors. Then $p_{20*}(\mathcal{O}_{\mathcal{Z}_2 \times_{Y_1} \mathcal{Z}_1})$ is a G-equivariant coherent sheaf on $Y_2 \times \mathbb{C}^n$, flat over Y_2. For a closed point $y \in Y_2$, let $\mathcal{O}_W = q_{2*} p_2^* \mathcal{O}_y \subset Y_1$ be the corresponding G/N-cluster. Then the fiber of $p_{20*}(\mathcal{O}_{\mathcal{Z}_2 \times_{Y_1} \mathcal{Z}_1})$ over y is $q_{1*} p_1^* \mathcal{O}_W$. Especially, if y lies over a free orbit in $\mathbb{C}^n/G$, then $q_{1*} p_1^* \mathcal{O}_W$ is the G-cluster supported by the free orbit and it is the regular representation as a G-module. Since $p_{20*}(\mathcal{O}_{\mathcal{Z}_2 \times_{Y_1} \mathcal{Z}_1})$ is flat over Y_2, it is a flat family of G-constellations in the connected component containing free orbits. $\square$

2.2. Stability for G/N-Hilb(N-Hilb($\mathbb{C}^3$))

Let $\Theta = \Theta^G := \{\theta \in \mathrm{Hom}_{\mathbb{Z}}(R[G], \mathbb{Q}) | \theta(R_G) = 0\}$ with $R[G]$ the representation ring. The notion of stability by [Kin] translates into the language of G-constellations as follows. For $\theta \in \Theta$, a G-constellation $\mathcal{F}$ is θ-stable (or θ-semistable) if $\theta(\mathcal{E}) = \theta(H^0(\mathcal{E})) > 0 = \theta(\mathcal{F})$ (or $\theta(\mathcal{E}) \geq 0$) for $0 \subsetneq \mathcal{E} \subsetneq \mathcal{F}$. With a quiver-theoretic point of view, if M is a representation of Q of dimension vector $\mathbf{d} = (d_i)_{i \in Q_0}$, then the notion of stability for M is given as follows: let $\theta \in \mathbb{Q}^{Q_0}$, and define $\theta(M) := \sum \theta_i d_i$. Then M is θ-stable (or θ-semistable) if $\theta(M') > 0 = \theta(M)$ (or $\theta(M') \geq 0$) for $0 \subsetneq M' \subsetneq M$. More generally, θ-stability and semistability are defined for a G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^3$ with finite support such that $\theta(H^0(\mathcal{F})) = 0$ in the same way.

It is known from [IN] that G-Hilb($\mathbb{C}^3$) can be considered as a moduli $\mathcal{M}_\theta$ of θ-stable representations of the McKay quiver of G satisfying the relations, for any stability condition $\theta \in \Theta$ satisfying $\theta(\rho) > 0$ for every nontrivial irreducible representation ρ of G (and hence $\theta(\rho_0) < 0$ for the trivial representation ρ_0). We call such θ a *0-generated* stability.

REMARK 2.3

In general, the chamber of stability parameters defining G-Hilb is larger than the cone defined by the inequalities above.

Let $\theta^N \in \Theta^N$ and $\theta^{G/N} \in \Theta^{G/N}$ be ***0**-generated* stabilities for N and G/N, respectively. (In the following argument, θ^N and $\theta^{G/N}$ can be arbitrary parameters in the chambers of N-Hilb and G/N-Hilb, respectively. However, we assume they are *0*-generated to simplify the proof of Lemma 2.5 below.)

DEFINITION 2.4

Let $\rho \in \mathrm{Irr}(G)$ and $\theta \in \Theta$. We define

$$\theta(\rho) := \begin{cases} \theta^N(\rho|_N) + \varepsilon \cdot \theta^{G/N}(\rho) & \text{if } \rho \in \mathrm{Irr}(G/N), \\ \theta^N(\rho|_N) & \text{if } \rho \notin \mathrm{Irr}(G/N), \end{cases}$$

where $0 < \varepsilon \ll 1$.

Note that $\theta^N \in \Theta^N$ can be regarded as an element of Θ by composing the map $\theta^N : R[N] \to \mathbb{Q}$ with the restriction map $R[G] \to R[N]$. The condition $\rho \in \mathrm{Irr}(G/N)$ as a representation of G means that ρ is trivial for every element in N. It is straightforward to check that $\theta(R_G) = 0$ for the regular representation R_G as required.

LEMMA 2.5

The parameter θ defined in Lemma 2.4 is generic.

Proof

For any nonzero subrepresentation $S \subsetneq R_G$, $\theta(S) \neq 0$ by the choice of θ. This implies that θ is generic. $\square$

From now on we restrict ourselves to the case $G \subset \mathrm{SL}(3,\mathbb{C})$. Consider the functor

$$\Phi : D^b(\mathrm{Coh}^{G/N} Y_1) \to D^b(\mathrm{Coh}^G \mathbb{C}^3)$$

defined by $\Phi(-) = \mathbb{R}q_{1*}p_1^*(-)$. Then it is an equivalence of triangulated categories by [BKR] (see also [IU, Theorem 3.1]). Let $\mathrm{Coh}_0^{G/N}(Y_1)$ denote the Abelian category of G/N-equivariant coherent sheaves on Y_1 with zero-dimensional supports. Then Φ sends objects of $\mathrm{Coh}_0^G(Y_1)$ to G-equivariant sheaves with zero-dimensional supports and Φ is exact on $\mathrm{Coh}_0^G(Y_1)$.

LEMMA 2.6

Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a finite subgroup, and let N be a normal subgroup of G. Let θ^N be a 0-generated stability parameter for N. Then for an object $E \in \mathrm{Coh}_0^{G/N} Y_1$, $\Phi(E) \in \mathrm{Coh}^G \mathbb{C}^3$ is θ^N-semistable. Moreover, if $\mathcal{F} \subseteq \Phi(E)$ is a G-equivariant subsheaf of $\Phi(E)$ with $\theta^N(\mathcal{F}) = 0$, then there is a G/N-equivariant subsheaf F of E such that $\mathcal{F} = \Phi(F)$.

Proof

Let $F_0 : D^b(\mathrm{Coh}^G \mathbb{C}^3) \to D^b(\mathrm{Coh}^N \mathbb{C}^3)$ and $F_1 : D^b(\mathrm{Coh}^{G/N} Y_1) \to D^b(\mathrm{Coh}\, Y_1)$ be the forgetful functors. We have a commutative diagram

$$\begin{tikzcd}
D^b(\mathrm{Coh}^{G/N} Y_1) \arrow[r, "\Phi", "\sim"'] \arrow[d, "F_1"'] & D^b(\mathrm{Coh}^G \mathbb{C}^3) \arrow[d, "F_0"] \\
D^b(\mathrm{Coh}\, Y_1) \arrow[r, "\sim", "\Phi^N"'] & D^b(\mathrm{Coh}^N \mathbb{C}^3)
\end{tikzcd}$$

where Φ^N is the functor which is defined in the same way as Φ and is an equivalence by [BKR]. Since $F_1(E)$ has a filtration in $\mathrm{Coh}\, Y_1$ whose factors are skyscraper sheaves, $\Phi^N(F_1(E))$ has a filtration in $\mathrm{Coh}^N \mathbb{C}^3$ whose factors are N-clusters. Since N-clusters are θ^N-stable, $F_0(\Phi(E)) \cong \Phi^N(F_1(E))$ is θ^N-semistable. Now for any G-invariant subsheaf $\mathcal{F}$ of $\Phi(E)$, we have $\theta^N(\mathcal{F}) = \theta^N(F_0(\mathcal{F})) \geq 0$ by the semistability of $F_0(\Phi(E))$, which shows that $\Phi(E)$ is θ^N-semistable.

Suppose $\mathcal{F} \subseteq \Phi(E)$ is a G-invariant subsheaf with $\theta^N(\mathcal{F}) = 0$. Then $\mathcal{F}$ is also θ^N-semistable by the definition of semistability for a G-equivariant coherent sheaf. Moreover, $F_0(\Phi(E))$ is also θ^N-semistable as an N-equivariant coherent sheaf as in the previous paragraph, and hence so is $F_0(\mathcal{F})$. Consider the Jordan–Hölder filtrations on the θ^N-semistable N-equivariant coherent sheaves $F_0(\Phi(E))$, $F_0(\mathcal{F})$, and $F_0(\Phi(E)/F)$, respectively, whose factors are θ^N-stable. Since the Jordan–Hölder factors of $F_0(\Phi(E))$ are N-clusters, those of $F_0(\mathcal{F})$ $\big($and $F_0(\Phi(E)/F))\big)$ are also N-clusters by the Jordan–Hölder theorem for semistable sheaves. Note that N-clusters are of the form $\Phi^N(\mathcal{O}_y)$ for $y \in Y_1$. Therefore, there is a filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_l = F_0(\mathcal{F})$$

such that $\mathcal{G}_i/\mathcal{G}_{i-1} \cong \Phi(\mathcal{O}_{y_i})$ for some $y_i \in Y_1$. Since the equivalence Φ^N induces an isomorphism

$$\operatorname{Hom}\bigl(\mathcal{O}_y, F_1(E)/G\bigr) \cong \operatorname{Hom}\bigl(\Phi^N(\mathcal{O}_y), F_0(\Phi(E))/\Phi^N(G)\bigr)$$

for any closed point $y \in Y_1$ and any subsheaf $G \subset F_1(E)$, induction on i shows that there is a unique subsheaf $G_i \subset F_1(E)$ such that $\Phi^N(G_i) = \mathcal{G}_i$ for each i. Especially, $G := G_l$ is a unique subsheaf of $F_1(E)$ such that $F_0(\mathcal{F}) = \Phi^N(G)$. G must be preserved by the action of G/N by its uniqueness, which shows that G is of the form $F_1(F)$ for a G/N-invariant subsheaf $F \subseteq E$. $\square$

THEOREM 2.7

Let $\theta \in \Theta$ be as in Definition 2.4. Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a finite subgroup, and let N be a normal subgroup of G. Then

$$G/N\text{-Hilb}\bigl(N\text{-Hilb}(\mathbb{C}^3)\bigr) \cong \mathcal{M}_C$$

for the chamber $C \subset \Theta$ which contains θ.

Proof

G/N-Hilb(N-Hilb($\mathbb{C}^3$)) parameterizes a family of G-constellations of the form $\Phi(\mathcal{O}_W)$ where $W \subset Y_1$ is a G/N-cluster. Take a G-invariant subsheaf $\mathcal{F}$ of $\Phi(\mathcal{O}_W)$. If $\theta^N(\mathcal{F}) > 0$, then we have $\theta(\mathcal{F}) > 0$ by the assumption $\varepsilon \ll 1$ and we may assume $\theta^N(\mathcal{F}) = 0$. In this case, there is a G/N-invariant subsheaf $F \subseteq \mathcal{O}_W$ such that $\mathcal{F} = \Phi(F)$ by Lemma 2.6. Note that we have $\tau_{1*}F \cong (\pi_{1*}\mathcal{F})^N$ by the definition of Φ, which implies that the number of copies of an irreducible representation of G/N appearing in $H^0(F)$ is the same as that in $H^0(\mathcal{F})$, proving $\theta(\mathcal{F}) = \varepsilon\theta^{G/N}(F) > 0$. Thus we obtain the θ-stability of $\Phi(\mathcal{O}_W)$, and hence Φ induces a morphism $f : G/N\text{-Hilb}(N\text{-Hilb}(\mathbb{C}^3)) \to \mathcal{M}_C$. Now since $G \subset \mathrm{SL}(3,\mathbb{C})$, both are crepant resolutions of $\mathbb{C}^3/G$, so f is an isomorphism. $\square$

REMARK 2.8

In the above proof, we can replace θ^N by an arbitrary G/N-invariant generic stability parameter for N-constellations. Especially, we have similar results for iterated constuctions such as “Hilb of Hilb of Hilb.”

Note that to let $\theta^{G/N}$ be general, we have to construct the moduli space of θ-stable G-constellations on a quasi-projective variety with G-action, which could be done by patching local constructions.

3. Representations of semidirect products

We recall representations of semidirect products from [Ser, Section 8.2] to compute several examples in Section 6. We also use the same notation for an Abelian group G with a subgroup N in Section 4. Let G be a finite group obtained as the semidirect product $N \rtimes H$ of subgroups N and H. We assume that the normal subgroup N is Abelian. All the examples in Section 6 are of this form. Let $\mathrm{Irr}(N) = \{\sigma_0, \dots, \sigma_{p-1}\}$ be the set of irreducible representations of N

Table 1. Irreducible representations of G from the action of G/N into $\mathrm{Irr}(N)$

σ_0	$\mathrm{Orb}(\sigma_1)$	$\dots$	$\mathrm{Orb}(\sigma_k)$
ρ_0^0	ρ_1^0		ρ_k^0
ρ_0^1			
$\vdots$	$\vdots$	$\dots$	$\vdots$
	$\rho_1^{h_1}$		$\rho_k^{h_k}$
$\rho_0^{h_0}$			

where $\dim(\sigma_i) = 1$, and denote by $I = \{0, \dots, p-1\}$ the set of subindices. Let us also denote by $\mathrm{Irr}(G/N) = \{\tau_0, \dots, \tau_{h_0}\}$ the set of irreducible representations of $G/N = H$ with $d_j^{G/N} := \dim(\tau_j)$.

The group $H = G/N$ acts on $\mathrm{Irr}(N)$ as follows: H acts on N by conjugation and thus on $\mathrm{Irr}(N)$ by $h \cdot \sigma(n) = \sigma(h^{-1}nh)$, for $h \in H$, $\sigma \in \mathrm{Irr}(N)$, and $n \in N$. Choose a set of representatives of the classes in $\mathrm{Irr}\, N$ under the action of G/N, and denote by $\tilde{I} = \{0, \dots, k\} \subseteq I$ the corresponding subset of subindices. For any $i \in \tilde{I}$ consider the orbit $\mathrm{Orb}(\sigma_i)$ of σ_i under G/N of length n_i. Let G_i be the stabilizer, and let $\mathrm{Irr}(G_i) = \{\tau_i^0, \dots, \tau_i^{h_i}\}$ be the set of irreducible representations of G_i. Recall that if σ_i and σ_j are in the same orbit, then G_i and G_j are conjugate, in particular, isomorphic. The trivial representation $\sigma_0 \in \mathrm{Irr}(N)$ is always fixed so that $G_0 = G/N$ and $\tau_0^j = \tau_j$ for all j.

The irreducible representations of G are obtained as follows: for every $i \in \tilde{I}$ the representations in the orbit $\mathrm{Orb}(\sigma_i)$ combine to give $h_i + 1$ irreducible representations ρ_i^j for $j = 0, \dots, h_i$ with $\dim(\rho_i^j) = n_i \dim(\tau_i^j)$ (see Table 1). In other words, they are induced by the representations of $N \rtimes G_i$ obtained as the tensor product of the extensions of σ_i and τ_i^j to $N \rtimes G$. In particular, if σ_i is fixed by G/N, then it give rise to $h_0 + 1$ irreducible representations, each corresponding to an irreducible representation of G/N. Note that ρ_0^0 is the trivial representation of G. Then ρ_i^j are all the irreducible representations of G by [Ser, Section 8.2].

REMARK 3.1

The action of G/N on $\mathrm{Irr}(N)$ to produce $\mathrm{Irr}(G)$ can be translated into the McKay quiver N, where every vertex corresponds to an irreducible representation of N. Then G/N acts on the set of vertices and on the set of arrows of Q, as well as on the path algebra $\Bbbk Q$ permuting the set of primitive idempotents $\{e_i | i \in I\}$. We thus can construct the McKay quiver of G as the (G/N)-orbifold quiver of the McKay quiver of N (see [Dem] for the general formulation and [NdC] for the case of binary dihedral groups in $\mathrm{GL}(2, \mathbb{C})$).

Let us describe the stability parameter defined in Definition 2.4 which is shown in Table 2. We are going to use the **0**-generated stabilities for the groups N and

Table 2. Stability condition θ in terms of θ^N and $\theta^{G/N}$

σ_0	$\mathrm{Orb}(\sigma_1)$	$\dots$	$\mathrm{Orb}(\sigma_k)$
$\theta_0^N + \varepsilon\theta_0^{G/N}$	$\dim(\tau_1^0)\cdot\sum_{\sigma_i\in\mathrm{Orb}(\sigma_1)}\theta_i^N$		$\dim(\tau_k^0)\cdot\sum_{\sigma_i\in\mathrm{Orb}(\sigma_k)}\theta_i^N$
$d_1^{G/N}\theta_0^N + \varepsilon\theta_1^{G/N}$			
$\vdots$	$\vdots$	$\dots$	$\vdots$
$d_{h_0}^{G/N}\theta_0^N + \varepsilon\theta_{h_0}^{G/N}$	$\dim(\tau_1^{h_1})\cdot\sum_{\sigma_i\in\mathrm{Orb}(\sigma_1)}\theta_i^N$		$\dim(\tau_k^{h_k})\cdot\sum_{\sigma_i\in\mathrm{Orb}(\sigma_k)}\theta_i^N$

G/N separately, so let us denote them as follows:

$$\theta^N \in \mathbb{Q}^p \text{ such that } \theta_i^N := \theta^N(\sigma_i) > 0 \text{ for } i \neq 0 \text{ and } \sigma_i \in \mathrm{Irr}(N),$$

$$\theta^{G/N} \in \mathbb{Q}^{h_0+1} \text{ such that } \theta_j^{G/N} := \theta^{G/N}(\tau_j) > 0 \text{ for } j \neq 0 \text{ and } \tau_j \in \mathrm{Irr}(G/N).$$

In particular we have $\sum_{i=0}^{p-1} d_i^N\theta_i^N = 0$ and $\sum_{i=0}^{h_0} d_j^{G/N}\theta_j^{G/N} = 0$, so that $\theta_0^N = -\sum_{i=1}^{p-1} d_i^N\theta_i^N$ and $\theta_0^{G/N} = -\sum_{j=1}^{h_0} d_j^{G/N}\theta_j^{G/N}$.

4. The case G Abelian

Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a finite Abelian subgroup, and let $A \lhd G$ be a normal subgroup of G with $|A| = p$ and $|G/A| = q$. After introducing the toric notation that is needed, we describe how to calculate the triangulation of the junior simplex Δ corresponding to $Y := G/A\text{-Hilb}(A\text{-Hilb}(\mathbb{C}^3))$, and we construct explicitly every G-constellation in Y from the A-clusters. Then we describe a method to calculate the local coordinates of a moduli space of G-constellations using the McKay quiver and finish the section describing the stability condition in the Abelian case.

4.1. How to calculate $A/N\text{-Hilb}(N\text{-Hilb}(\mathbb{C}^3))$

Every element of G can be written of the form $g = \mathrm{diag}(\varepsilon^{a_1}, \varepsilon^{a_2}, \varepsilon^{a_3})$ where ε is an rth primitive root of unity and $0 \leq a_i < r$. Let $L \supset \mathbb{Z}^3$ be the lattice generated by the elements of G written in the form $\frac{1}{r}(a_1, a_2, a_3)$, and let $M := L^\vee$ be the dual lattice of Laurent monomials. The *junior simplex* is the triangle $\Delta \subset L_\mathbb{R} := L\otimes_\mathbb{Z}\mathbb{R}$ with vertices the standard basis e_1,e_2,e_3. We denote by $\mathbb{R}^2_\Delta$ the affine plane spanned by Δ and $\mathbb{Z}^2_\Delta := L \cap \mathbb{R}^2_\Delta$. Recall that Δ contains all lattice points with $a_1 + a_2 + a_3 = r, a_i \geq 0$, and triangulations of Δ are in one-to-one correspondence with crepant resolutions of $\mathbb{C}^3/G$.

First consider the action of A on $\mathbb{C}^3$. In [CR] Craw and Reid give a method to triangulate Δ into p regular triangles Δ_i which produces the crepant resolution $A\text{-Hilb}(\mathbb{C}^3)$. This triangulation shows that $A\text{-Hilb}(\mathbb{C}^3) \cong \bigcup_{i=1}^p Y_i$ where $Y_i := \sigma(\Delta_i) \cong \mathbb{C}^3_{\varepsilon_i,\eta_i,\zeta_i}$ is the affine toric variety associated to the triangle Δ_i, and ε_i, η_i, and ζ_i are Laurent monomials in x, y, and z.

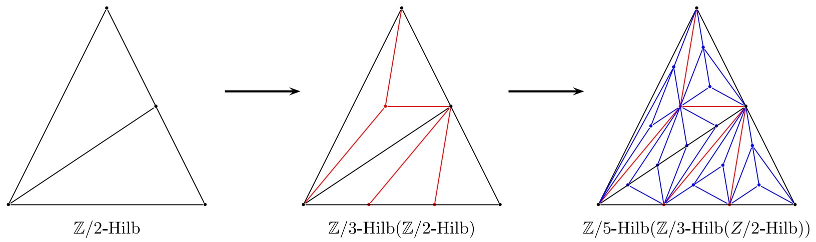


Figure 1. Successive triangulations of Δ for a group G of order $r = 30 = 2 \cdot 3 \cdot 5$.

The action of G/A on A-Hilb$(\mathbb{C}^3)$ is again Abelian, so it is given by diagonal matrices; thus it acts on every Y_i separately. For every triangle Δ_i with $i = 1, \dots, p$, we form the the toric singular quotient $Y_i/(G/A)$ and take G/A-Hilb(Y_i) as crepant resolution. Therefore,

$$G/A\text{-Hilb}\big(A\text{-Hilb}(\mathbb{C}^3)\big) = \bigcup_{i=1}^{p} G/A\text{-Hilb}(Y_i).$$

In other words, the triangulation of Δ which gives G/A-Hilb$(A$-Hilb$(\mathbb{C}^3))$ is produced in two steps. First, calculate A-Hilb$(\mathbb{C}^3)$ according to [CR] to obtain $\Delta = \bigcup_{i=1}^{p} \Delta_i$. Second, triangulate every Δ_i into q regular triangles with the same method according to the $\mathbb{Z}/q$-action of G/A into Y_i to produce $\Delta = \bigcup \Delta_{ij}$ for $i = 1, \dots, p$, $j = 1, \dots, q$. Obviously, the same process of successive triangulations can be done as many times as nontrivial normal subgroups we have in a filtration of G (see Figure 1).

We now look at how G/A-Hilb$(A$-Hilb$(\mathbb{C}^3))$ can be constructed explicitly as moduli of G-constellations. As an A-constellation, a point $\mathcal{F} \in A$-Hilb$(\mathbb{C}^3)$ is an A-equivariant coherent sheaf on $\mathbb{C}^3$ such that $H^0(\mathcal{F}) \cong \mathbb{C}[A] \cong \bigoplus_{\sigma \in \mathrm{Irr}(A)} \sigma$, and $\mathcal{F} \cong \mathcal{O}_Z$ for some A-cluster Z. Therefore, locally at $U \subset A$-Hilb$(\mathbb{C}^3)$ we can take a basis $\Gamma := \{\lambda_\sigma \mid \sigma \in \mathrm{Irr}(A), \lambda_\sigma \text{ is } \sigma\text{-semi-invariant}\}$ of $H^0(\mathcal{F})$ to be an A-graph. That is, λ_σ is a monomial in $\mathbb{C}[x,y,z]$, and if $x^i y^j z^k \in \Gamma$, then $x^{i'} y^{j'} z^{k'} \in \Gamma$ for any $i' \le i$, $j' \le j$ and $k' \le k$ (see [Nak]). We call Γ the *building block* for U.

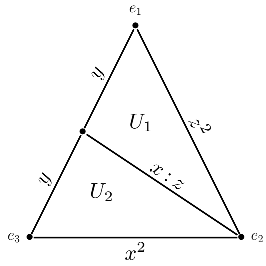
It is also known from [Nak] that $U = \sigma(\Delta_i) \cong \mathbb{C}^3_{a,b,c}$ where $a = \frac{f_\sigma}{\lambda_\sigma}$, $b = \frac{f_{\sigma'}}{\lambda_{\sigma'}}$, and $c = \frac{f_{\sigma''}}{\lambda_{\sigma''}}$ are Laurent monomials in x, y, and z, where f_σ, $f_{\sigma'}$, and $f_{\sigma''}$ are σ-, σ'-, and σ''-semi-invariants, respectively. Then, an open set $V = \sigma(\Delta_{ij}) \subset G/N$-Hilb$(U)$ is determined by a (G/N)-graph $\Omega := \{\omega_\tau \mid \tau \in \mathrm{Irr}(G/N)\} \cong \mathbb{C}[G/N]$ where ω_τ are now monomials in $\mathbb{C}[a,b,c]$. Thus, a point $\mathcal{Z} \in V$ as a G-equivariant module can be written in the form

$$\mathcal{Z} = \big\{\omega_\tau \Gamma \mid \tau \in \mathrm{Irr}(G/N)\big\} = \big\{\omega_\tau \lambda_\sigma \mid \tau \in \mathrm{Irr}(G/N), \sigma \in \mathrm{Irr}(N)\big\} \cong \mathbb{C}[G].$$

In other words, the resulting G-constellations arising from the open set U are obtained by multiplying the building block Γ by the q different (G/N)-graphs Ω.

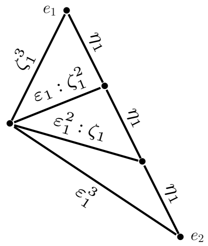
EXAMPLE 4.1

Let $G = \frac{1}{6}(1,2,3) = \frac{1}{2}(1,0,1) \times \frac{1}{3}(1,2,0) \cong \mathbb{Z}/6\mathbb{Z}$. Take the normal subgroup in

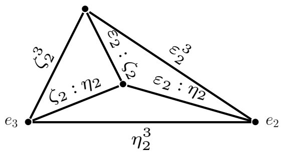


U_i	$\varepsilon_i, \eta_i, \zeta_i$	A-graph	G/A-action type on U_i
U_1	$\frac{x}{z}, z^2, y$	$\boxed{1\ z}$	$\frac{1}{3}(1,0,2)$
U_2	$\frac{z}{x}, x^2, y$	$\boxed{1\ x}$	$\frac{1}{3}(1,1,1)$

Figure 2. Toric fan and coordinates for $(1,0,1)$-Hilb$(\mathbb{C}^3)$.



U_{ij}	Coordinates	G/A-graph
U_{11}	$\frac{\varepsilon_1}{\zeta_1^2}, \zeta_1^3, \eta_1$	$\boxed{1\ \zeta_1\ \zeta_1^2}$
U_{12}	$\frac{\zeta_1^2}{\varepsilon_1}, \frac{\varepsilon_1^2}{\zeta_1}, \eta_1$	$\boxed{1\ \zeta_1\ \varepsilon_1}$
U_{13}	$\varepsilon_1^3, \frac{\zeta_1}{\varepsilon_1^2}, \eta_1$	$\boxed{1\ \varepsilon_1^2\ \varepsilon_1}$



U_{ij}	Coordinates	G/A-graph
U_{21}	$\frac{\varepsilon_2}{\zeta_2}, \frac{\eta_2}{\zeta_2}, \zeta_2^3$	$\boxed{1\ \zeta_2\ \zeta_2^2}$
U_{22}	$\frac{\eta_2}{\varepsilon_2}, \frac{\zeta_2}{\varepsilon_2}, \varepsilon_2^3$	$\boxed{1\ \varepsilon_2\ \varepsilon_2^2}$
U_{23}	$\frac{\varepsilon_2}{\eta_2}, \frac{\zeta_2}{\eta_2}, \eta_2^3$	$\boxed{1\ \eta_2\ \eta_2^2}$

Figure 3. $\frac{1}{3}(1,0,2)$-Hilb(U_1) and $\frac{1}{3}(1,1,1)$-Hilb(U_2).

G to be $A = \frac{1}{2}(1,0,1)$. The triangulation of the junior simplex $\Delta = \Delta_1 \cup \Delta_2$ corresponding to A-Hilb$(\mathbb{C}^3) = U_1 \cup U_2$ and the toric coordinates are given in Figure 2.

The action of $G/A \cong \frac{1}{3}(1,2,0)$ on A-Hilb$(\mathbb{C}^3)$ leaves invariant the open sets U_1 and U_2, sending $(\varepsilon_1, \eta_1, \zeta_1) \mapsto (\omega\varepsilon_1, \eta_1, \omega^2\zeta_1)$ and $(\varepsilon_2, \eta_2, \zeta_2) \mapsto (\omega\varepsilon_2, \omega\eta_2, \omega\zeta_2)$, respectively, where ω is a primitive cubic root of unity. Therefore, each of the quotient open sets $U_i/(G/A)$ contains the singularities $\frac{1}{3}(1,0,2)$ and $\frac{1}{3}(1,1,1)$, respectively, which we resolve with the crepant resolutions (G/A)-Hilb(U_i) for $i = 1,2$. The triangulations of Δ_1 and Δ_2 are shown in Figure 3.

In constructing the resolution $\mathbb{Z}/2\mathbb{Z}$-Hilb$(\mathbb{C}^3)$ we added only one new lattice point to Δ, namely, $\frac{1}{2}(1,0,1)$, producing the subdivision $\Delta = \Delta_1 \cup \Delta_2$. To construct $\mathbb{Z}/3\mathbb{Z}$-Hilb$(\mathbb{Z}/2\mathbb{Z}$-Hilb$(\mathbb{C}^3))$ we now introduce the remaining lattice points, namely, $\frac{1}{6}(1,2,3)$, $\frac{1}{6}(2,4,0)$, and $\frac{1}{6}(4,2,0)$, and triangulate Δ_1 and Δ_2 according to the algorithm in [CR]. Changing back to the coordinates x, y, and z we obtain the fan shown in Figure 4.

For $j = 1,2,3$ the basis for the G-constellations of the open set U_{1j} are given by multiplying every basis element in the (G/A)-graphs $\Omega_1 = \{1, \zeta_1, \zeta_1^2\}$, $\Omega_2 = \{1, \zeta_1, \varepsilon_1\}$, and $\Omega_3 = \{1, \varepsilon_1, \varepsilon_1^2\}$ by the building block $\Gamma = \{1, z\}$ coming from the open set U_1. Similarly for the open sets $U_{2j} \subset \mathbb{Z}/3\mathbb{Z}$-Hilb$(\mathbb{Z}/2\mathbb{Z}$-Hilb$(\mathbb{C}^3))$.

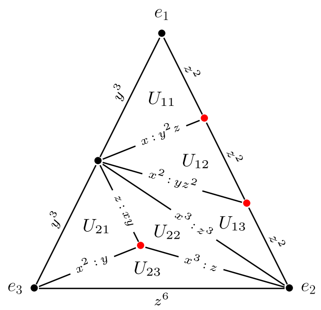
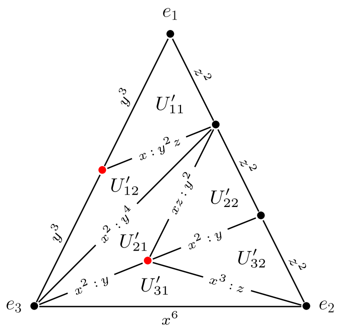


Figure 4. Toric fan for $\mathbb{Z}/3\mathbb{Z}$-Hilb($\mathbb{Z}/2\mathbb{Z}$-Hilb($\mathbb{C}^3$)).



	G-constellations
M'_{11}	$\{1,y,y^2,z,yz,y^2z\}$
M'_{12}	$\{1,y,y^2,\frac{x}{y^2},\frac{x}{y},x\}$
M'_{21}	$\{1,x,y,\frac{y^2}{x},y^2,\frac{y^3}{x}\}$
M'_{22}	$\{1,x,y,z,xz,yz\}$
M'_{31}	$\{1,x,x^2,x^3,x^4,x^5\}$
M'_{32}	$\{1,x,x^2,z,xz,x^2z\}$

Figure 5. Toric fan $\mathbb{Z}/2\mathbb{Z}$-Hilb($\mathbb{Z}/3\mathbb{Z}$-Hilb($\mathbb{C}^3$)) and the corresponding G-constellations.

More precisely, the G-constellations are

$$M_{11} = \{1\cdot\boxed{1\ z}, \zeta_1\cdot\boxed{1\ z}, \zeta_1^2\cdot\boxed{1\ z}\} = \{1,z,y,yz,y^2,y^2z\},$$

$$M_{12} = \{1\cdot\boxed{1\ z}, \zeta_1\cdot\boxed{1\ z}, \varepsilon_1\cdot\boxed{1\ z}\} = \left\{1,z,y,yz,\frac{x}{z},x\right\},$$

$$M_{13} = \{1\cdot\boxed{1\ z}, \varepsilon_1^2\cdot\boxed{1\ z}, \varepsilon_1\cdot\boxed{1\ z}\} = \left\{1,z,\frac{x^2}{z^2},\frac{x^2}{z},\frac{x}{z},x\right\},$$

$$M_{21} = \{1\cdot\boxed{1\ x}, \zeta_2\cdot\boxed{1\ x}, \zeta_2^2\cdot\boxed{1\ x}\} = \{1,x,y,xy,y^2,xy^2\},$$

$$M_{22} = \{1\cdot\boxed{1\ x}, \varepsilon_2\cdot\boxed{1\ x}, \varepsilon_2^2\cdot\boxed{1\ x}\} = \left\{1,x,\frac{z}{x},z,\frac{z^2}{x^2},\frac{z^2}{x}\right\},$$

$$M_{23} = \{1\cdot\boxed{1\ x}, \eta_2\cdot\boxed{1\ x}, \eta_2^2\cdot\boxed{1\ x}\} = \{1,x,x^2,x^3,x^4,x^5\}.$$

Let now $A=\frac{1}{3}(1,2,0)$ be the normal subgroup. Then A-Hilb$(\mathbb{C}^3)\cong V_1\cup V_2\cup V_3$ where $V_i\cong\mathbb{C}^3$. The quotient group $G/A\cong\frac{1}{2}(1,0,1)$ produces a $\mathbb{Z}/2\mathbb{Z}$-action on every V_i for $i=1,2,3$. The resolution of these singularities is translated into the junior simplex $\Delta=\Delta_1\cup\Delta_2\cup\Delta_3$ by adding the points $\frac{1}{6}(3,0,3)$ and $\frac{1}{6}(1,2,3)$, triangulating in the only possible way as in Figure 5. All crepant resolutions of $\mathbb{C}^3/\frac{1}{6}(1,2,3)$ are shown in Figure 6.

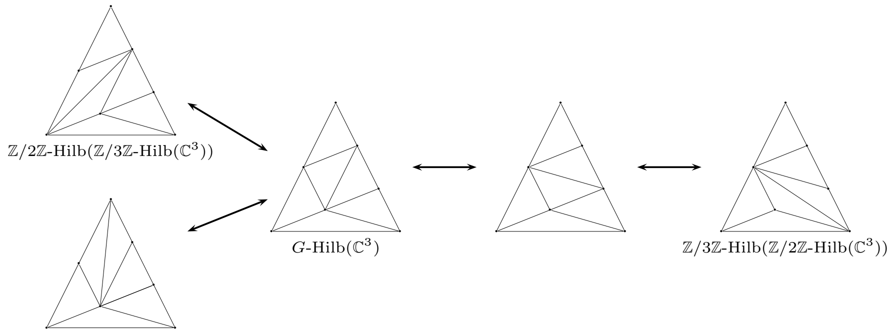


Figure 6. Crepant resolutions of $\mathbb{C}^3/G$ with $G=\frac{1}{6}(1,2,3)$.

Table 3. $\mathrm{Irr}(G)$ and the stability condition θ in terms of θ_N and $\theta_{G/N}$

$\mathrm{Irr}(G) =$

σ_0	σ_1	$\cdots$	σ_{p-1}
ρ_0^0	ρ_1^0	$\cdots$	ρ_{p-1}^0
ρ_0^1	ρ_1^1	$\cdots$	ρ_{p-1}^1
$\vdots$	$\vdots$	$\ddots$	$\vdots$
ρ_0^{q-1}	ρ_1^{q-1}	$\cdots$	ρ_{p-1}^{q-1}

$\theta =$

σ_0	σ_1	$\cdots$	σ_{p-1}
$-\sum_{i=1}^{p-1}\theta_i^N - \varepsilon\sum_{j=1}^{q-1}\theta_j^{G/N}$	θ_1^N	$\cdots$	θ_{p-1}^N
$-\sum_{i=1}^{p-1}\theta_i^N + \varepsilon\theta_1^{G/N}$	θ_1^N	$\cdots$	θ_{p-1}^N
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$-\sum_{i=1}^{p-1}\theta_i^N + \varepsilon\theta_{q-1}^{G/N}$	θ_1^N	$\cdots$	θ_{p-1}^N

4.2. θ-stability in the Abelian case

Every irreducible representation $\sigma_i \in \mathrm{Irr}(N)$ is fixed, so all stabilizers G_i are isomorphic to G/N. The irreducible representations of G are therefore distributed as in Table 3. Since $n_i = d_i^N = d_j^{G/N} = 1$ for all $i = 0,\dots,p-1$, $j = 0,\dots,q-1$, we have $\dim(\rho) = 1$ for $\rho \in \mathrm{Irr}(G)$. The stability condition $\theta \in \Theta$ for which the crepant resolution G/A-Hilb(A-Hilb($\mathbb{C}^3$)) $\cong \mathcal{M}_\theta$ is given also in Table 3. Notice that in this case $\theta_0^N = -\sum_{i=1}^{p-1}\theta_i^N$ and $\theta_0^{G/N} = -\sum_{j=1}^{q-1}\theta_i^{G/N}$, so the fact that $0 < \varepsilon \ll 1$ implies that $\theta(\rho_0^k) < 0$ for all k.

EXAMPLE 4.2

Let us take the group $G = \frac{1}{6}(1,2,3)$ and consider $\mathbb{Z}/3\mathbb{Z}$-Hilb($\mathbb{Z}/2\mathbb{Z}$-Hilb($\mathbb{C}^3$)). The distribution of $\mathrm{Irr}(G)$ and the stability condition θ are shown in Figure 7 where $a, b_1, b_2 \in \mathbb{Q}$ are positive numbers and $0 < \varepsilon \ll 1$.

The stability condition is in this case given clockwise around the McKay quiver. By checking the subrepresentations in every affine piece we can see that chamber $C \subset \Theta$ is given by

$$\theta_2, \theta_4 < 0, \qquad \theta_2 + \theta_5 > 0, \qquad \theta_1 + \theta_4 > 0,$$

$$\mathrm{Irr}(G) = \begin{array}{|c|c|}\hline \rho_0^0 & \rho_1^0 \\ \hline \rho_0^1 & \rho_1^1 \\ \hline \rho_0^2 & \rho_1^2 \\ \hline \end{array} \qquad \theta := \begin{array}{|c|c|}\hline \theta_0 & \theta_1 \\ \hline \theta_2 & \theta_3 \\ \hline \theta_4 & \theta_5 \\ \hline \end{array} = \begin{array}{|c|c|}\hline -a-\varepsilon(b_1+b_2) & a \\ \hline -a+\varepsilon b_1 & a \\ \hline -a+\varepsilon b_2 & a \\ \hline \end{array}$$

Figure 7. $\mathrm{Irr}(G)$ and stability condition for $\mathbb{Z}/3$-Hilb($\mathbb{Z}/2$-Hilb($\mathbb{C}^3$)).

$$\theta_3 > 0, \qquad \theta_4 + \theta_5 > 0, \qquad \theta_0 + \theta_1 + \theta_3 > 0.$$

Thus if we take $0 < \varepsilon < a/\max\{b_1, b_2\}$, every inequality is satisfied by θ.

5. Local coordinates

In this section, we introduce some notation and terminology to illustrate non-Abelian examples in the next section. Let (Q,R) be a quiver with relations, where R is a two-sided ideal in the path algebra $\mathbb{C}Q$ of Q. Let Q' be a connected quiver, and let $\phi : Q' \to Q$ be a morphism of quivers, that is, a pair of morphisms $\phi_0 : Q'_0 \to Q_0$ and $\phi_1 : Q'_1 \to Q_1$ between the respective vertex and arrows sets. For a vertex v of Q, let d_v be the number of vertices in the preimage $H_v := \{\phi^{-1}(v)\} \subseteq Q'_0$, and let $\mathbb{C}^{H_v} \cong \mathbb{C}^{d_v}$ be the vector space with a distinguished basis $\{e_w \mid w \in H_v\}$. Notice that for an arrow $a \in Q_1$ with the head $h(a)$ and the tail $t(a)$, a linear map $\mathbb{C}^{H_{t(a)}} \to \mathbb{C}^{H_{h(a)}}$ is given by a matrix in $\mathrm{Mat}_{d_{t(a)} \times d_{h(a)}}$.

To the pair (Q', ϕ) we construct a representation $S_{Q'}$ of dimension vector (d_v) such that for an arrow $a \in Q_1$, the associated matrix in $\mathrm{Mat}_{d_{t(a)} \times d_{h(a)}}$ is given by writing $k_\alpha \in \mathbb{C}$, $k_\alpha \neq 0$ at the $(t(\alpha), h(\alpha))$-entry for every $\alpha \in \phi^{-1}(a) \subseteq Q'_1$ and 0 everywhere else. Note that $S_{Q'}$ can be regarded as the direct image of a representation of Q' with dimension vector $(1, \ldots, 1)$ whose linear maps are nonzero by the morphism ϕ.

DEFINITION 5.1

We say that (Q', ϕ) is a *skeleton* if the representation $S_{Q'}$ verifies the relations R for a suitable choice of $(k_\alpha)_{\alpha \in Q'_1} \in (\mathbb{C}^*)^{Q'_1}$ and the isomorphism class of $S_{Q'}$ does not depend on such a choice. By abuse of notation we also call $S_{Q'}$ a skeleton.

When the quiver Q is the McKay quiver, it happens often (for instance, in every family of examples treated in this paper) that a suitable subset $\mathcal{U}$ of skeletons determines an open cover of the moduli space $\mathcal{M}_C$ for a given $C \subset \Theta$. In other words, the conditions $k_a \neq 0$ for all $a \in Q'$ determine an open set $U_{Q'} \subset \mathcal{M}_C$, and the union of such open sets for skeletons in $\mathcal{U}$ form an open cover. Here, the skeleton $S_{Q'}$ is the representation corresponding to the origin in the affine open set $U_{Q'} \subseteq \mathbb{C}^m$ for some m (cf. [NdCS, Section 7]).

For Abelian groups in $\mathrm{SL}(3, \mathbb{C})$, the set $\mathcal{U}$ is determined by the torus fixed points (see [Ish, Section 3]). As will be shown in the examples of the following sections, in the case of the iterated Hilbert scheme G/N-Hilb(N-Hilb($\mathbb{C}^3$)) the subset $\mathcal{U}$ is induced by the skeletons defining the open cover of N-Hilb($\mathbb{C}^3$).

The local coordinates of an open set $U_{Q'}$ associated with a skeleton (Q', ϕ) can be obtained explicitly as follows. Fix $(k_\alpha) \in (\mathbb{C}^*)^{Q'_1}$ in the definition of skeletons. Consider representations of Q which associate to each arrow $a \in Q_1$ a matrix in $\mathrm{Mat}_{d_{t(a)} \times d_{h(a)}}$ whose $(h(\alpha), t(\alpha))$-entry is k_α for $\alpha \in \phi^{-1}(a)$. These representations form an affine space whose coordinates are the remaining entries of the matrices. If we consider only representations which satisfy the relations R, we obtain an affine scheme $U_{Q'} \subseteq \mathbb{C}^m$ for some m which contains $S_{Q'}$ as the representation corresponding to the origin. In good cases including all the examples in this paper, $U_{Q'}$ becomes an affine open neighbourhood of $S_{Q'}$ in the moduli space of representations of (Q, R), and we can specify the entries of the matrices which form the local coordinates around $S_{Q'}$.

Let $S := \mathbb{C}[x, y, z]$, and for every $\rho \in \mathrm{Irr}(G)$ consider the Cohen–Macaulay S^G-module $S_\rho := (S \otimes \rho^*)^G$. We have the tautological bundle $\mathcal{R}_\rho$ on the moduli space of G-constellations whose global sections form the module S_ρ. On the open set $U_{Q'}$ corresponding to a skeleton Q', the vertices of Q' correspond to sections of $\mathcal{R}_\rho$'s over $U_{Q'}$, where we always assume the vertex over the trivial representation ρ_0 corresponds to 1. These sections can be regarded as rational sections of S_ρ over $\mathbb{C}^3/G$. If we take a basis $u_1, \ldots, u_d$ of the representation space ρ, a rational section of S_ρ over $\mathbb{C}^3/G$ is of the form $\sum_{i=1}^d f_i u_i^*$, where f_i are rational functions in the ρ-part of $\mathbb{C}(x, y, z)$ and $u_1^*, \ldots, u_d^*$ form the dual basis of ρ^*. Then such a rational section is given by a d-tuple $(f_1, \ldots, f_d)$ of rational functions which spans the representation ρ in $\mathbb{C}(x, y, z)$.

EXAMPLE 5.2

Let $M_{22} = \left\{1, x, \frac{z}{x}, z, \frac{z^2}{x^2}, \frac{z^2}{x}\right\}$ be the G-constellation defining the open $U_{22} \in \mathbb{Z}/3\mathbb{Z}\text{-Hilb}(Y)$ where $Y := \mathbb{Z}/2\mathbb{Z}\text{-Hilb}(\mathbb{C}^3)$. Let Q be the McKay quiver of G with the usual commutativity relations deriving from $xy = yx$, $xz = zx$, and $yz = zy$ (see Figure 8), and consider M_{22} as a representation of Q. Then by choosing the basis element at every vector space $\mathbb{C}_\rho$ to be given by the unique element $\lambda_\rho \in M$, we have $x \cdot x = a \cdot \frac{z}{x}$, $y \cdot 1 = b \cdot \frac{z}{x}$, and $z \cdot \frac{z^2}{x^2} = c \cdot x$ for some $a, b, c \in \mathbb{C}$. This implies that $a = \frac{x^3}{z}$, $b = \frac{xy}{z}$, and $c = \frac{z^3}{x^3}$, which are precisely the local coordinates of $\sigma(\Delta_{22})$. Since after change of basis any nonzero map can be chosen to be 1, the skeleton for U_{22} in this case is formed by the linear maps equal to 1.

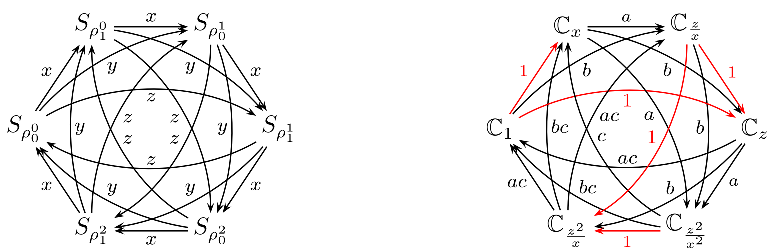


Figure 8. McKay quiver for $G = \frac{1}{6}(1,2,3)$ and the open set $U_{22} \cong \mathbb{C}^3_{a,b,c}$.

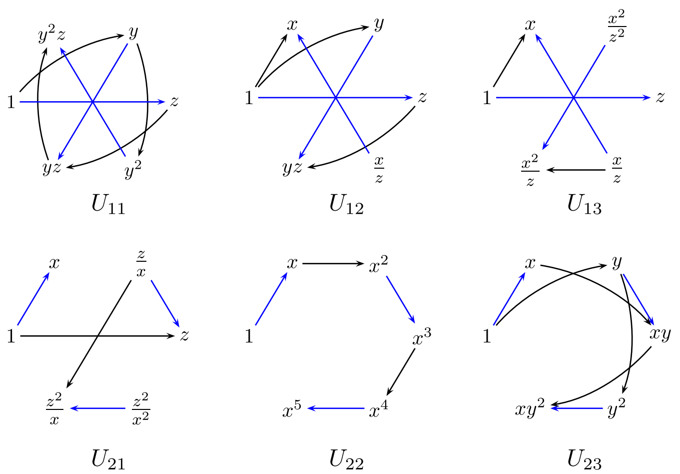


Figure 9. Skeletons for $\mathbb{Z}/3$-Hilb($\mathbb{Z}/2$-Hilb($\mathbb{C}^3$)) with the corresponding G-constellations.

Table 4. $\mathrm{Irr}(D_{2n})$ for (a) $n = 2m$ even, and (b) $n = 2m+1$ odd

σ_0	$\{\sigma_1, \sigma_{2m-1}\}$		$\{\sigma_{m-1}, \sigma_{m+1}\}$	σ_m
ρ_0^0	ρ_1^0	$\cdots$	ρ_{m-1}^0	ρ_m^0
ρ_0^1				ρ_n^1

(a)

σ_0	$\{\sigma_1, \sigma_{2m}\}$		$\{\sigma_m, \sigma_{m+1}\}$
ρ_0^0	ρ_1^0	$\cdots$	ρ_m^0
ρ_0^1			

(b)

The skeletons in every open set in $\mathbb{Z}/3\mathbb{Z}$-Hilb(Y) are shown in Figure 9. Notice that in the skeleton for U_{ij} it is possible to find the skeletons $U_1, U_2 \subset Y$, repeated $|\mathbb{Z}/3\mathbb{Z}| = 3$ times.

6. Non-Abelian examples

6.1. Dihedral groups $D_{2n} \subset \mathrm{SO}(3)$.

These groups are generated by

$$D_{2n} := \left\langle \alpha = \frac{1}{n}(1,-1,0), \beta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle,$$

and they have order $2n$. The normal subgroup $N := \langle \alpha \rangle$ has n 1-dimensional irreducible representations $\sigma_i(\alpha) = \varepsilon^i$ where ε is an nth root of unity and $i = 0, \ldots, n-1$. The action of $G/N = \langle \overline{\beta} \rangle \cong \mathbb{Z}/2\mathbb{Z}$ gives the irreducible representations of G as in Table 4 depending on the parity of n.

The dimension vectors are ${}^1_1 2 \ldots 2 {}^1_1$ and ${}^1_1 2 \ldots 2$, respectively. In this case, as well as for any subgroup $G \subset \mathrm{SO}(3)$, the fiber over the origin $f^{-1}(0)$ of any crepant resolution $f : Y \to \mathbb{C}^3/G$ has dimension 1. The description of $f^{-1}(0)$ in the case of $Y = G\text{-Hilb}(\mathbb{C}^3)$ was first given by [GNS] (see [NdCS] for the rest of the crepant resolutions of $\mathbb{C}^3/G$).

The open cover of N-Hilb$(\mathbb{C}^3)$ is given by n open sets $\bigcup_{j=1}^{n} U_j$, covering $n-1$ rational curves E_i for $i = 1, \ldots, n-1$. The action G/N on N-Hilb$(\mathbb{C}^3)$ identifies U_i and U_{n-i+1} for $i = 1, \ldots, n$.

If $n = 2m$ is even, then E_m is fixed by G/N having two fixed lines L_+ and L_- crossing transversally the $\mathbb{P}^1$ covered by U_m and U_{m+1}. They give rise to E_+ and E_-, respectively. If $n = 2m+1$ is odd, then the open set U_{m+1} is fixed by G/N and there is just one fixed line L, producing the new rational curve E.

Diagonalizing the action of G/N we see that in both cases these singularities are of type $\frac{1}{2}(1,1,0)$. By blowing up these singular lines it follows that the dual graph of the fiber over the origin of the singularity is shown in Figure 10.

The fiber over the origin in G-Hilb$(\mathbb{C}^3)$ with the degrees of the normal bundles in each of the rational curves is shown in Figure 11 (see [GNS] and [NdCS] for details). We can therefore see the difference between G/N-Hilb$(N$-Hilb$(\mathbb{C}^3))$ and G-Hilb$(\mathbb{C}^3)$: in the case n even the graph is different, whereas in the case n odd the difference resides in the degrees of the normal bundles. This concludes the proof of the D_{2n}-case in Theorem 1.7(iii).

We illustrate the construction of G/N-Hilb$(N$-Hilb$(\mathbb{C}^3))$ in this case by an example. The general case is analogous.

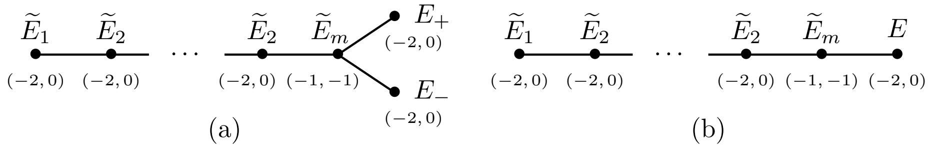


Figure 10. Dual graph of $f^{-1}(0)$ for G/N-Hilb$(N$-Hilb$(\mathbb{C}^3))$ for (a) $n = 2m$ even, and (b) $n = 2m+1$ odd. The curve $\widetilde{E}_i$ denote the strict transform of E_i and the numbers denote the degree of the normal bundle at every curve.

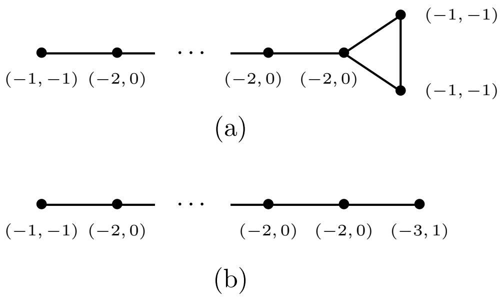
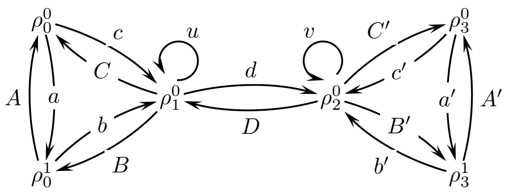


Figure 11. Dual graph of $f^{-1}(0)$ for G-Hilb$(\mathbb{C}^3)$ for (a) $n = 2m$ even, and (b) $n = 2m+1$ odd.



$$bC=0, b'C'=0, cB=0, c'B'=0,$$
$$Ca=uB, C'a'=vB', Ac=bu, A'c'=b'v,$$
$$BA=uC, B'B'=vC', ab=cu, a'b'=c'v$$
$$Bb+Cc=dD, B'b'+C'c'=Dd$$
$$Du=vD, ud=dv$$

Figure 12. The McKay quiver of D_{12} with relations.

Table 5. Stability condition for $D_{12}/N\text{-Hilb}(N\text{-Hilb}(\mathbb{C}^3))$ with $N=\frac{1}{6}(1,-1,0)$

	σ_0	$\{\sigma_1,\sigma_5\}$	$\{\sigma_2,\sigma_4\}$	σ_3
$\theta :=$	$-\sum_{i=1}^5 a_i-\varepsilon b$	a_1+a_5	a_2+a_4	a_3
	$-\sum_{i=1}^5 a_i+\varepsilon b$			a_3

EXAMPLE 6.1

Let $G=D_{12}$. In this case the McKay quiver with relations (Q,R) is given in Figure 12 where the relations R provide the Morita equivalence between $\mathbb{C}Q/\langle R\rangle$ and $S*G$, and are obtained following [BSW].

In this case the stability condition given in Definition 2.4 is shown in Table 5, where $a_i, b>0$ for $i=1,\dots,5$ and $0<\varepsilon\ll 1$.

We start by considering the gluing of U_1 and U_2 with U_5 and U_6 by the action of G/N. Since U_3 and U_4 contain the fixed part, we will treat them separately. We need to consider the N-constellations at the origin of each open chart, so after choosing a basis for every $H^0(gZ)$ for $g\in G/N$ consisting of N-graphs, we obtain the G-constellations shown in Figure 13.

In the skeleton of the open sets the dots stacked vertically in the two middle vertices of the McKay quiver denote the two linearly independent vectors e_1 and e_2 in the corresponding vector space at that vertex.

With a similar calculation as in Section 5 we can calculate the local coordinates to obtain

$$V_1\cong \operatorname{Spec}\mathbb{C}\Big[\frac{z(x^3+y^3)}{x^3-y^3}, \frac{xy}{(x^3+y^3)^2}, (x^3+y^3)^2\Big],$$

$$V_2\cong \operatorname{Spec}\mathbb{C}\Big[\frac{z(x^3+y^3)}{x^3-y^3}, \frac{x^2y^2}{(x^3+y^3)^2}, \frac{(x^3+y^3)^2}{xy}\Big],$$

and $\alpha=\frac{x^3-y^3}{x^3+y^3}$.

Let us now consider the G-constellations arising from the blowup of the fixed lines L_+ and L_-. In the open set $U_3\cong\mathbb{C}^3_{a,b,c}$ with $a=x^4/y^2$, $b=y^3/x^3$, and $c=z$, every N-cluster is given by the ideal $I_{a,b,c}=(x^4-ay^2, y^3-bx^3, xy-ab, z-c)$. Then the lines $L_\pm$ are defined by $b=\pm1$, $c=0$, which means that the ideals defining the lines are $I_{L_\pm}=(x^4-ay^2, y^3\mp x^3, xy\mp a, z)$. Therefore we can choose as basis for the N-constellations at these lines the N-graphs $\Gamma_\pm=\{1,x,y,x^2,y^2,x^3\pm y^3\}$, which are invariant under the action of G/N.

N-constellation

$Z_1 = \{1, x, x^2, x^3, x^4, x^5\}$,
$gZ_1 = \{1, y, y^2, y^3, y^4, y^5\}$

G-constellation

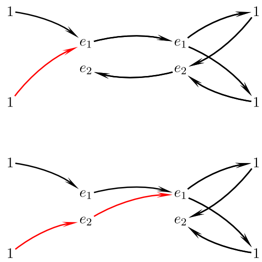
1	(x, y)	(x^2, y^2)	$x^3 + y^3$
α	(x^5, y^5)	(x^4, y^4)	$x^3 - y^3$

$Z_2 = \{1, y, x, x^2, x^3, x^4\}$,
$gZ_2 = \{1, x, y, y^2, y^3, y^4\}$

1	(x, y)	(x^2, y^2)	$x^3 + y^3$
α	$(\alpha x, \alpha y)$	(x^4, y^4)	$x^3 - y^3$

Skeleton

Open set



V_1

V_2

Figure 13. G-constellations at the open sets V_1 and V_2.

For the line L_+ we have $b = 1$. By the change of coordinate $b^+ = 1 - b = \frac{x^3 - y^3}{x^3}$, we have that the action of G/N as $\frac{1}{2}(1,1,0)$ is defined on $\mathbb{C}^3_{A,B,C}$ where $A = c$, $B = \frac{2-b^+}{b^+}$, $C = a(b^+)^2$. This implies that the rational curve E_+ is covered by the open sets $V_3 \cong \mathbb{C}^3_{B^2,A/B,C}$, $V_6 \cong \mathbb{C}^3_{A^2,B/A,C}$, and E^+ is given by the ratio $(A : B) = (z(x^3 - y^3) : x^3 + y^3)$. In terms of G-constellations, the calculation is the same as in the Abelian case, where V_3 and V_6 are the open cover of $\frac{1}{2}(1,1,0)$-Hilb$(\mathbb{C}^3_{A,B,C})$ (see Figure 14). In the case of V_3, for instance, we have the nonzero maps coming from the N-constellations Γ_+ and $\Gamma_+ \cdot B$ (generated from ρ_0^0 and ρ_0^1, resp.), and the extra arrow comes from the fact that $(x^3 + y^3)B = (x^3 - y^3) + \frac{4x^3y^3}{x^3 - y^3}$, which is induced by the G/N-equivariant map $\Gamma_+ \to B\Gamma_+$.

Similarly, for the line L_- we have $b = -1$. By the change of coordinate $b^- = 1 - b = \frac{x^3 + y^3}{x^3}$, the action of G/N as $\frac{1}{2}(1,1,0)$ is defined on $\mathbb{C}^3_{A',B',C'}$ where $A' = c$, $B' = \frac{b^- - 2}{b^-}$, $C' = a(b^-)^2$. This implies that the rational curve E_- is covered by the open sets $V_4 \cong \mathbb{C}^3_{B'^2,A'/B',C'}$, $V_5 \cong \mathbb{C}^3_{A'^2,B'/A',C'}$, and E^- is given by the ratio $(A' : B') = (z(x^3 + y^3) : x^3 - y^3)$. In terms of G-constellations, the calculation is similar but now taking Γ_- instead of Γ_+ (see Figure 14).

The skeletons provide the choices of nonzero variables in the representation space rep(Q). For example, in the case V_3 we can choose $c = (1,0)$, $b = (0,1)$, $d = \mathrm{Id}$, $C_1 = 1$, $B = \binom{1}{1}$ and use the relations to obtain the representation space of $V_3 \subset \mathcal{M}_\theta$ shown in Figure 15, where $K = c'(C' - 1)$. Thus $V_3 \cong \mathbb{C}^3_{a,c'C'}$ where

$$a = \frac{z(x^3 + y^3)}{x^3 - y^3}, \qquad c' = \frac{(x^3 + y^3)^2}{x^2y^2}, \qquad \text{and} \qquad C' = \frac{(x^3 - y^3)^2}{(x^3 + y^3)^2}.$$

G-constellation

$\Gamma_+ \cdot \{1, B\}$:

1	(x,y)	(x^2,y^2)	x^3+y^3
B	(xB,yB)	(x^2B,y^2B)	$(x^3+y^3)B$

$\Gamma_+ \cdot \{1, z\}$:

1	(x,y)	(x^2,y^2)	x^3+y^3
z	(xz,yz)	(x^2z,y^2z)	$(x^3+y^3)z$

$\Gamma_- \cdot \{1, B'\}$:

1	(x,y)	(x^2,y^2)	$(x^3-y^3)B'$
B'	(xB',yB')	(x^2B',y^2B')	x^3-y^3

$\Gamma_+ \cdot \{1, z\}$:

1	(x,y)	(x^2,y^2)	$z(x^3-y^3)$
z	(xz,yz)	(x^2z,y^2z)	x^3-y^3

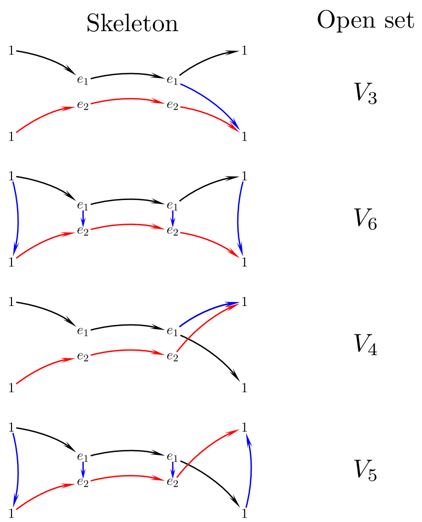


Figure 14. G-constellations at the open sets covering the exceptional curves E^+ and E^-, where $B = \frac{x^3+y^3}{x^3-y^3}$ and $B' = \frac{x^3-y^3}{x^3+y^3}$.

6.2. D_8-Hilb($\frac{1}{3}(1,2,0)$-Hilb($\mathbb{C}^3$))

Let G be the group $N \rtimes G/N \subset \mathrm{SL}(3,\mathbb{C})$ where N is generated by $g = \frac{1}{3}(1,2,0)$ and $G/N \cong D_8 \subset \mathrm{SO}(3)$ is generated by $\alpha = \frac{1}{4}(1,3,0)$ and $\beta = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. Let $\mathrm{Irr}(N) = \{\sigma_0, \sigma_1, \sigma_2\}$ where $\sigma_i(g) = \omega^i$ for $i = 0,1,2$, and let ω be a primitive cube root of unity. In this case we have $\mathrm{Orb}(\sigma_0) = \sigma_0$ and $\mathrm{Orb}(\sigma_1) = \{\sigma_1, \sigma_2\}$ with stabilizers $G_0 = G/N \cong D_8$ and $G_1 = \langle\alpha\rangle \cong \mathbb{Z}/4\mathbb{Z}$. Therefore $\mathrm{Irr}\, G_0 = \{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4\}$ where $\dim(\tau_i) = 1$ for $i = 0, \ldots, 3$ and $\dim(\tau_4) = 2$. The irreducible representations of G are shown in Table 6.

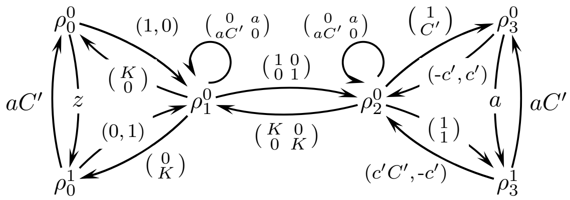


Figure 15. Open set $V_3 \subset \mathcal{M}_\theta$.

Table 6. Irreducible representations of G with their dimensions and the stability condition for $G/N\text{-Hilb}(N\text{-Hilb}(\mathbb{C}^3))$ with $a,b,c>0$ and $0<\varepsilon\ll 1$

σ_0	$\{\sigma_1,\sigma_2\}$
ρ_0^0	ρ_1^0
ρ_0^1	ρ_1^1
ρ_0^2	ρ_1^2
ρ_0^3	ρ_1^3
ρ_0^4	

, $\mathbf{d} =$

σ_0	$\{\sigma_1,\sigma_2\}$
1	2
1	2
1	2
1	2
2	

, $\theta =$

σ_0	$\{\sigma_1,\sigma_2\}$
$-a-b-5\varepsilon c$	$a+b$
$-a-b+\varepsilon c$	$a+b$
$-a-b+\varepsilon c$	$a+b$
$-a-b+\varepsilon c$	$a+b$
$-2a-2b+\varepsilon c$	

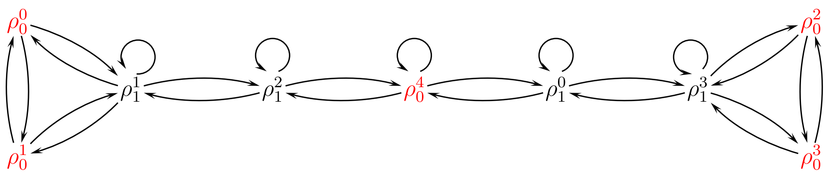


Figure 16. The McKay quiver of G.

The McKay quiver shown in Figure 16 coincides with the McKay quiver of $D_{24} \subset \mathrm{SO}(3)$ as in Section 6.1 since $G \cong D_{24} = \langle \frac{1}{12}(1,11,0), \beta \rangle$.

The scheme N-Hilb$(\mathbb{C}^3)$ can be covered by 3 open sets U_i for $i=1,2,3$ with the corresponding distinguished N-constellations $Z_1=\{1,x,x^2\}$, $Z_2=\{1,x,y\}$, and $Z_3=\{1,y,y^2\}$. By the action of $G/N\cong D_8$ the open set U_2 is fixed while U_1 and U_3 are identified.

The fixed open set U_2 has coordinates $a=\frac{x^2}{y}$, $b=\frac{y^2}{x}$, and $c=z$. If we denote $(+):=a^2+b^2$ and $(-):=a^2-b^2$, we have that G/N-Hilb$(\mathbb{C}^3_{a,b,c})$ is covered by 5 open sets given by the following G/N constellations:

$$\begin{aligned}
\Gamma_1 &= \big\{1,a,b,(+),(-),a(+),b(+),-(+)(-)\big\},\\
\Gamma_2 &= \big\{1,a,b,(+),(-),a(+),b(+),c\big\},\\
\Gamma_3 &= \big\{1,a,b,(+),(-),ac,-bc,c\big\},\\
\Gamma_4 &= \big\{1,a,b,c(-),(-),ac,-bc,c\big\},\\
\Gamma_5 &= \big\{1,a,b,(+),c(+),ac,-bc,c\big\}.
\end{aligned}$$

We obtain in this way the open sets V_i given by the G-constellations $\Gamma_i\cdot Z_2$, for $i=1,\dots,5$ shown in Figure 17.

In the case of the orbit $\{U_1,U_3\}$ we have $U_1\cong\mathbb{C}^3_{d,e,f}$ with coordinates $d=x^3$, $e=y/x^2$, $f=z$, and $U_3\cong\mathbb{C}^3_{d',e',f'}$ with coordinates $d'=y^3$, $e'=x/y^2$, $f'=z$. In each of the open sets there exists a fixed line with stabilizer subgroup $G_1=\langle\alpha\rangle\cong\mathbb{Z}/4\mathbb{Z}$. This implies that we have to consider the G_1-graphs

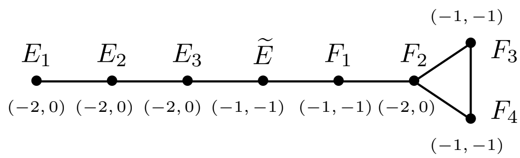
$$\begin{aligned}
\Omega_1 &= \{1,c,c^2,c^3\}, &\qquad \Omega_2 &= \{1,c,c^2,d\},\\
\Omega_3 &= \{1,c,d,d^2\}, &\qquad \Omega_4 &= \{1,d,d^2,d^3\}
\end{aligned}$$

in U_1, and

$$\begin{aligned}
\Omega_1' &= \{1,c',c'^2,c'^3\}, &\qquad \Omega_2' &= \{1,c',c'^2,d'\},\\
\Omega_3 &= \{1,c',d',d'^2\}, &\qquad \Omega_4 &= \{1,d',d'^2,d'^3\}
\end{aligned}$$

in U_3. The identification of U_1 and U_3 by β produces the open sets U_i given by the G-constellations $\Omega_i\cdot Z_1\cup\Omega_i'\cdot Z_3$ for $i=1,\dots,4$ (see Figure 17).

Let $Y:=\frac{1}{3}(1,2,0)$-Hilb$(\mathbb{C}^3)$. Then the exceptional fiber over the origin $\pi^{-1}(0)$ of the crepant resolution $\pi:Y\to\mathbb{C}^3/N$ consists of two $(-2,0)$-rational curves intersecting in one point. The action of G/N interchanges these two curves, producing in $Y/(G/N)$ a single rational curve E with singularities of types $\frac{1}{4}(1,3,0)$ and D_8 at 0 and ∞, respectively. The fiber $\phi^{-1}(0)$ of the crepant resolution $\phi: G/N$-Hilb$(Y)\to\mathbb{C}^3/G$ is therefore given by the following dual graph:



where E_i are covered by U_i for $i=1,\dots,3$, F_j are covered by V_j for $j=1,\dots,4$, and $\widetilde{E}$ is the strict transform of E.

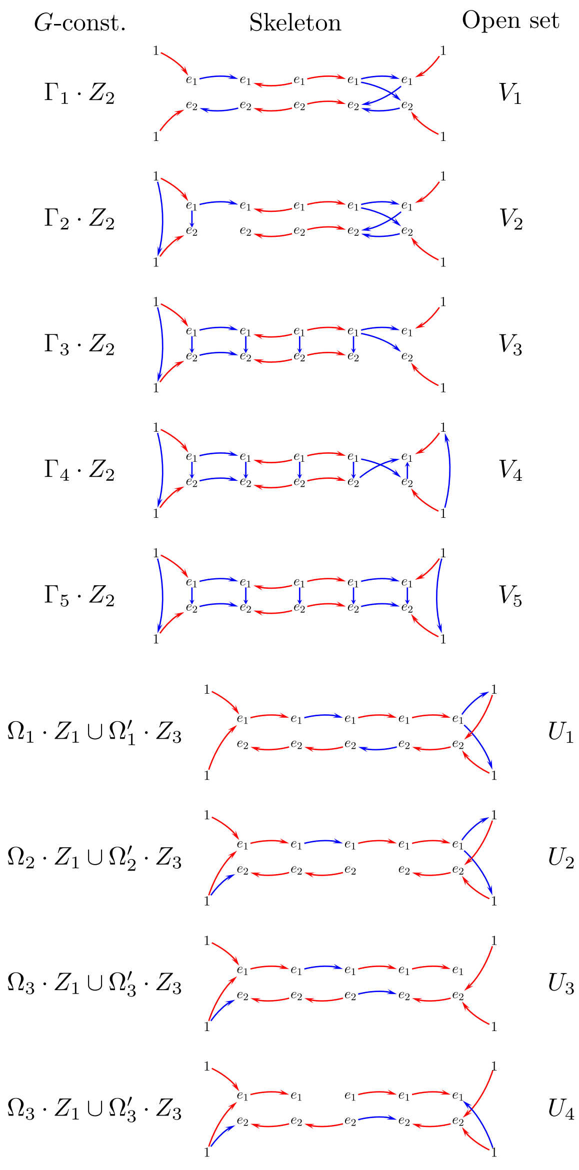


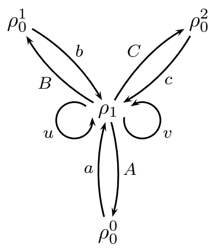
Figure 17. G-constellations for D_8-Hilb$(\frac{1}{3}(1,2,0)$-Hilb$(\mathbb{C}^3))$.

6.3. Trihedral group of order 12

Let $G := G_{12} \subset \mathrm{SO}(3)$ be the trihedral group of order 12 generated by $N := \langle \frac{1}{2}(1,1,0), \frac{1}{2}(1,0,1) \rangle$ and $T := \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. The Abelian normal subgroup $N \cong \mathbb{Z}/2 \times \mathbb{Z}/2\mathbb{Z}$ with $\mathrm{Irr}(N) = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ induces the irreducible representations of G as in Table 7. The McKay quiver with relations (Q, R) is given in Figure 18.

Table 7. Irreducible representations of G with their dimensions and the stability condition for G/N-Hilb(N-Hilb($\mathbb{C}^3$)) with $a_i, b > 0$ for $i = 1,2,3$ and $0 < \varepsilon \ll 1$

$$
\begin{array}{c|c|}
\sigma_0 & \{\sigma_1,\sigma_2,\sigma_3\} \\ \hline
\rho_0^0 & \\
\rho_0^1 & \rho_1 \\
\rho_0^2 & \\ \hline
\end{array}\ ,\quad
\mathbf{d} = \begin{array}{c|c|}
\sigma_0 & \{\sigma_1,\sigma_2,\sigma_3\} \\ \hline
1 & \\
1 & 3 \\
1 & \\ \hline
\end{array}\ ,\quad
\theta = \begin{array}{c|c|}
\sigma_0 & \{\sigma_1,\sigma_2,\sigma_3\} \\ \hline
-\sum a_i - 2\varepsilon b & \\
-\sum a_i + \varepsilon b & a_1 + a_2 + a_3 \\
-\sum a_i + \varepsilon b & \\ \hline
\end{array}
$$



$$
\begin{gathered}
uA = vA,\ au = av, \\
uB = \omega vB,\ bu = \omega bv, \\
uC = \omega^2 vC,\ cu = \omega^2 cv, \\
Aa + \omega Bb + \omega^2 Cc = u^2, \\
Aa + \omega^2 Bb + \omega Cc = v^2
\end{gathered}
$$

Figure 18. McKay quiver of G with relations.

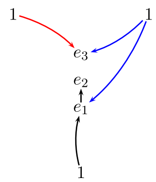
N-constellation	G-constellation	Skeleton	Open set
$Z_1 = \{1,x,y,xy\} \in U_1$, $T \cdot Z_1 = \{1,y,z,yz\} \in U_2$, $T^2 \cdot Z_1 = \{1,x,z,xz\} \in U_3$	$\begin{array}{\|c\|c\|} \hline 1 & (x,y,z) \\ \hline \lambda & (xy,xz,yz) \\ \hline \mu & (x\lambda, y\lambda, z\lambda) \\ \hline \end{array}$		V_1

Figure 19. G-constellation arising from the orbit $\{U_1, U_2, U_3\}$.

By acting first on $\mathbb{C}^3_{x,y,z}$ with N we have that N-Hilb($\mathbb{C}^3$) is given by 4 affine open sets U_i, $i = 1,\dots,4$. The N-constellations at each of these open sets are

$$
Z_1 = \{1,x,y,xy\}, \qquad Z_2 = \{1,y,z,yz\},
$$
$$
Z_3 = \{1,x,z,xz\}, \qquad Z_4 = \{1,x,y,z\}.
$$

The action of $G/N = \langle \bar{T} \rangle \cong \mathbb{Z}/3\mathbb{Z}$ identifies the open sets U_1, U_2 and U_3, and fixes U_4, inducing the corresponding action on the N-constellations. The orbit $\{U_1, U_2, U_3\}$ give rise to the open set $V_1 \subset G/N$-Hilb(N-Hilb($\mathbb{C}^3$)) shown in Figure 19.

It follows from the same method as Section 5 that $\lambda = \frac{R_2}{R_0}$ and $\mu = \frac{R_1}{R_0}$, where $R_0 := y^2z^2 + x^2z^2 + x^2y^2$, $R_1 := y^2z^2 + \omega x^2z^2 + \omega^2 x^2y^2$ and $R_2 := y^2z^2 + \omega^2 x^2z^2 + \omega x^2y^2$. The local coordinates of this open set are written at the end of the section.

The remaining case is the fixed N-constellation Z_4 in U_4. The open set $U_4 \cong \mathbb{C}^3_{a,b,c}$ has coordinates $a = \frac{yz}{x}$, $b = \frac{xz}{y}$, and $c = \frac{xy}{z}$, and we have $T(a) = b$,

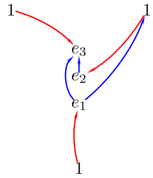
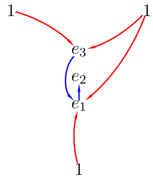
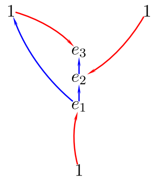
	G-constellation	Skeleton	Open set
$Z_4\cdot\{1,\alpha,\alpha^2\}$:	$\begin{array}{|c|c|}\hline 1 & (x,y,z)\\ \hline \alpha & (\alpha,\alpha x,\alpha y)\\ \hline \alpha^2 & (\alpha^2,\alpha^2 x,\alpha^2 y)\\ \hline\end{array}$		V_2
$Z_4\cdot\{1,\alpha,\beta\}$:	$\begin{array}{|c|c|}\hline 1 & (x,y,z)\\ \hline \alpha & (\alpha,\alpha x,\alpha y)\\ \hline \beta & (\beta,\beta x,\beta y)\\ \hline\end{array}$		V_3
$Z_4\cdot\{1,\beta^2,\beta\}$:	$\begin{array}{|c|c|}\hline 1 & (x,y,z)\\ \hline \beta^2 & (\beta^2,\beta^2 x,\beta^2 y)\\ \hline \beta & (\beta,\beta x,\beta y)\\ \hline\end{array}$		V_4

Figure 20. G-constellations arising from the nonisolated $\frac{1}{3}(1,2,0)$ line.

$T(b)=c$, and $T(c)=a$. On the other hand, diagonalizing the action of $T\cong\mathbb{Z}/3\mathbb{Z}$, we can consider it to be of type $\frac{1}{3}(1,2,0)$ on $\mathbb{C}^3_{\alpha,\beta,\gamma}$ with the new coordinates $\alpha=a+\omega^2 b+\omega c$, $\beta=a+\omega b+\omega^2 c$, and $\gamma=a+b+c$. That is,

$$\alpha=\frac{f_1}{f_3},\qquad \beta=\frac{f_2}{f_3},\qquad \gamma=\frac{f_0}{f_3}$$

where $f_0:=x^2+y^2+z^2$, $f_1:=x^2+\omega^2y^2+\omega z^2$, $f_2:=x^2+\omega y^2+\omega^2z^2$, $f_3:=xyz$, and ω is a primitive cube root of unity.

The situation is therefore identical to the Abelian case, so we need to consider the distinguished G/N-constellations in $\frac{1}{3}(1,2,0)$-Hilb$(\mathbb{C}_{\alpha,\beta,\gamma})$, namely, $\Gamma_1=\{1,\alpha,\alpha^2\}$, $\Gamma_2=\{1,\alpha,\beta\}$, and $\Gamma_3=\{1,\beta,\beta^2\}$. They give rise to the G-constellations $Z_4\cdot\Gamma_i$ shown in Figure 20.

It can be checked that the matrices giving the open sets $V_i\subset\mathcal{M}_\theta(Q,R)$ are the following:

V_1: $a=(1,0,0)$, $b=(0,0,1)$, $c=(1,c_2,1)$, $A=\begin{pmatrix} C_1+C_3\\ -c_2C_1A_1+c_2B_1C_3\\ C_1(c_2^2A_1-1)\end{pmatrix}$, $B=\begin{pmatrix}B_1\\0\\A_1\end{pmatrix}$, $C=\begin{pmatrix}C_1\\0\\B_1(c_2^2C_1-1)\end{pmatrix}$, $u=\begin{pmatrix}0&1&0\\ C_3-\omega C_1 & \omega^2c_2C_1 & \omega^2C_1+\omega B_1\\ \omega^2c_2C_3 & -\omega^2(c_2^2C_1-1) & -\omega^2c_2C_1\end{pmatrix}$, $v=\begin{pmatrix}0&1&0\\ -\omega^2C_1+C_3 & \omega c_2C_1 & \omega C_1+\omega^2B_1\\ \omega c_2C_3 & -\omega(c_2^2C_1-1) & -\omega c_2C_1\end{pmatrix}$;

V_2: $a=(1,0,0)$, $b=(B_1-b_3^2B_3,1,b_3)$, $c=(0,0,1)$, $A=\begin{pmatrix} b_1B_1+b_3B_3 \\ B_3-B_1A_1 \\ (b_1b_3+1)B_1B_3\end{pmatrix}$, $B=\begin{pmatrix} B_1 \\ 0 \\ B_3\end{pmatrix}$, $C=\begin{pmatrix} 1 \\ 0 \\ A_1\end{pmatrix}$, $u=\begin{pmatrix} 0 & 1 & 0 \\ -\omega^2b_1B_1+b_3B_3 & \omega B_1 & \omega^2+\omega b_3B_1 \\ \omega B_3 & -\omega b_3B_3 & -\omega B_1\end{pmatrix}$, $v=\begin{pmatrix} 0 & 1 & 0 \\ -\omega b_1B_1+b_3B_3 & \omega^2B_1 & \omega+\omega^2b_3B_1 \\ \omega^2B_3 & -\omega^2b_3B_3 & -\omega^2B_1\end{pmatrix}$;

V_3: $a=(1,0,0)$, $b=(0,0,1)$, $c=(c_1,1,1)$, $A=\begin{pmatrix} c_1C_1+C_3 \\ -C_1A_1+B_1C_3 \\ C_1(A_1-c_1^2)\end{pmatrix}$, $B=\begin{pmatrix} B_1 \\ 0 \\ A_1\end{pmatrix}$, $C=\begin{pmatrix} C_1 \\ 0 \\ B_1(C_1-c_1)\end{pmatrix}$, $u=\begin{pmatrix} 0 & 1 & 0 \\ C_3-\omega C_1c_1 & \omega^2C_1 & \omega^2C_1+\omega B_1 \\ \omega^2(A_1-C_1c_1) & -\omega^2(C_1-c_1) & -\omega^2C_1\end{pmatrix}$, $v=\begin{pmatrix} 0 & 1 & 0 \\ C_3-\omega^2C_1c_1 & \omega C_1 & \omega C_1+\omega^2B_1 \\ \omega(A_1-C_1c_1) & -\omega(C_1-c_1) & -\omega C_1\end{pmatrix}$;

V_4: $a=(1,0,0)$, $b=(0,0,1)$, $c=(C_1-c_3^2C_3,1,c_3)$, $A=\begin{pmatrix} c_1C_1+C_3 \\ C_3-C_1A_1 \\ (c_1c_3+1)C_1C_3\end{pmatrix}$, $B=\begin{pmatrix} 1 \\ 0 \\ A_1\end{pmatrix}$, $C=\begin{pmatrix} C_1 \\ 0 \\ C_3\end{pmatrix}$, $u=\begin{pmatrix} 0 & 1 & 0 \\ -\omega C_1^2+\omega c_3^2C_1C_3+c_3C_3 & \omega^2C_1 & \omega+\omega^2c_3C_1 \\ \omega^2C_3 & -\omega^2c_3C_3 & -\omega^2C_1\end{pmatrix}$, $v=\begin{pmatrix} 0 & 1 & 0 \\ -\omega^2C_1^2+\omega^2c_3^2C_1C_3+c_3C_3 & \omega C_1 & \omega^2+\omega c_3C_1 \\ \omega C_3 & -\omega c_3C_3 & -\omega C_1\end{pmatrix}$.

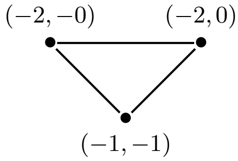
As in Section 5, by using the McKay quiver as the quiver between the Cohen–Macaulay modules S_ρ we can compute the local coordinates at every open set obtaining

$$
\begin{aligned}
V_1 &= \mathbb{C}^3_{B_1,c_2,C_1} = \mathbb{C}[-f_1R_0/R_2, -\sqrt{3}f_3/R_0, -f_2R_0/R_1],\\
V_2 &= \mathbb{C}^3_{b_3,B_1,B_3} = \mathbb{C}[-R_1/(\sqrt{3}f_2f_3), \sqrt{3}f_1f_3/R_2, \sqrt{3}f_2^2f_3/R_2],\\
V_3 &= \mathbb{C}^3_{B_1,c_1,C_1} = \mathbb{C}[-\sqrt{3}f_1f_3/R_2, -R_0/(\sqrt{3}f_3), \sqrt{3}f_2f_3/R_1],\\
V_4 &= \mathbb{C}^3_{c_3,C_1,C_3} = \mathbb{C}[R_2/(\sqrt{3}f_1f_3), \sqrt{3}f_2f_3/R_1, \sqrt{3}f_1^2f_3/R_1].
\end{aligned}
$$

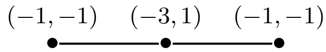
Moreover, the gluing between the different open sets is given as follows:

$$
\begin{aligned}
V_4 \ni (c_3,C_1,C_3) &\longleftrightarrow (-c_3^{-1}, C_1-c_3^2C_3, C_1) \in V_3,\\
V_1 \ni (B_1,c_2,C_1) &\longleftrightarrow (B_1c_2, c_2^{-1}, c_2C_1) \in V_3,\\
V_2 \ni (b_3,B_1,B_3) &\longleftrightarrow (-B_1, B_1-b_3^2B_3, -b_3^{-1}) \in V_3.
\end{aligned}
$$

Hence, the fibers of the origin of the quotient space are three rational curves meeting at a point. The dual graph with the appropriate degrees of the normal bundles is the following:



In this case the chamber $C \subset \Theta$, for which $G/N\text{-Hilb}(N\text{-Hilb}(\mathbb{C}^3)) \cong \mathcal{M}_C$ is given by the inequalities $\theta_1+\theta_3>0$, $\theta_2+\theta_3>0$, and $\theta_1+\theta_2+\theta_3<0$. On the other hand, the fiber over zero in G-Hilb$(\mathbb{C}^3)$ is given by the dual graph



and the chamber C' for G-Hilb$(\mathbb{C}^3) \cong \mathcal{M}_{C'}$ is given by $\theta_i > 0$ for $i \neq 0$ (see [NdCS] for details). This concludes the proof of the case G_{12} in Theorem 1.7(iii).

7. When does Hilb of Hilb coincide with Hilb?

In this section we study the relation between G-Hilb and G/N-Hilb(N-Hilb). Since both can be constructed as moduli spaces of representations of the McKay quiver, we may ask when they are isomorphic as moduli spaces (i.e., their tautological bundles coincide) and if not, when their underlying algebraic varieties are isomorphic. We answer these questions in many cases.

Considering them as moduli spaces we have G-Hilb $\cong \mathcal{M}_C$ and G/N-Hilb(N-Hilb) $\cong \mathcal{M}_{C'}$ for chambers $C, C' \subset \Theta$, where C is the chamber containing the zero-generated stability and C' is the chamber containing the parameter in Definition 2.4. Then the problem in this case is to determine which groups G admit a normal subgroup N such that $C = C'$. We give a complete answer to this question in the cases $G \subset \mathrm{GL}(2,\mathbb{C})$ and $G \subset \mathrm{SL}(3,\mathbb{C})$ in Theorem 7.3.

As algebraic varieties, in dimension 2 and for $G \subset \mathrm{SL}(2,\mathbb{C})$, there is nothing to prove since both of them are minimal resolutions of $\mathbb{C}^2/G$ and thus isomorphic. For non-Abelian subgroups in $\mathrm{GL}(2,\mathbb{C})$ we treat the case when $N = G \cap \mathrm{SL}(2,\mathbb{C})$ and conclude that they are nonisomorphic (see Proposition 7.5). For $G \subset \mathrm{SL}(3,\mathbb{C})$ we give a complete answer when the group G is Abelian by using the method of Craw and Reid [CR] to obtain the triangulation of the junior simplex Δ which corresponds to G-Hilb$(\mathbb{C}^3)$. As we saw in Section 4 the triangulation for G/N-Hilb(N-Hilb$(\mathbb{C}^3)$) is given by using the Craw–Reid method in two steps, first for N and then for G/N. Comparing both triangulations we are able to describe in Theorem 7.7 all possible configurations for G and N for which there is an isomorphism of varieties over $\mathbb{C}^3/G$.

We finish treating some non-Abelian small subgroups $G \subset \mathrm{SL}(3,\mathbb{C})$ for which G-Hilb$(\mathbb{C}^3)$ is not isomorphic to G/N-Hilb(N-Hilb$(\mathbb{C}^3)$), in particular, finite subgroups of SO(3) of types D_{2n} and G_{12} with N being the maximal normal subgroup, and non-Abelian intransitive subgroups with $N = G \cap \mathrm{SL}(2,\mathbb{C})$.

7.1. As moduli spaces

Let $G \subset \mathrm{GL}(n,\mathbb{C})$, assuming either $n = 3$ and $G \subset \mathrm{SL}(3,\mathbb{C})$ or $n = 2$ and $G \subset \mathrm{GL}(2,\mathbb{C})$ small. Let N be a normal subgroup in G. With the same notation as in Section 2, let $Y_1 := N$-Hilb$(\mathbb{C}^n)$ and $Y_2 := G/N$-Hilb(Y_1) with universal families $\mathcal{Z}_1$ and $\mathcal{Z}_2$, respectively, and denote by $\mathcal{U} := p_{20*}(\mathcal{O}_{\mathcal{Z}_1 \times_{Y_1} \mathcal{Z}_2})$ the flat family over Y_2 of G-constellations by the projection $p_{20} : Y_2 \times Y_1 \times \mathbb{C}^n \longrightarrow Y_2 \times \mathbb{C}^n$.

LEMMA 7.1

Put $X_1 = \mathbb{C}^n/N$ *and* $X_2 = Y_1/(G/N)$. *Then,* $\mathcal{Z}_1$ *is the reduced part of the fiber product* $Y_1 \times_{X_1} \mathbb{C}^n$ *and* $\mathcal{Z}_2$ *is the reduced part of* $Y_2 \times_{X_2} Y_1$.

Proof

It is sufficient to prove the latter statement. Here, $\mathcal{Z}_2$ is obviously a closed subscheme of $Y_2 \times_{X_2} Y_1$, and the morphism $\mathcal{Z}_2 \to Y_2 \times_{X_2} Y_1$ is an isomorphism over the generic point of Y_2. Now $\mathcal{O}_{\mathcal{Z}_2}$, regarded as an $\mathcal{O}_{Y_2}$-module via the finite morphism p_2, is locally free and therefore is the quotient of $\mathcal{O}_{Y_2 \times_{X_2} Y_1}$ by the torsion part, which must be the nilradical. $\square$

As we know, G-Hilb$(\mathbb{C}^n) \cong \mathcal{M}_C$ and $Y_2 \cong \mathcal{M}_{C'}$ are both resolutions of $\mathbb{C}^n/G$ isomorphic to a moduli space of G-constellations for some chambers $C, C' \subset \Theta$. Then,

$$\begin{aligned} C = C' &\iff \exists \text{ a closed subscheme } \mathcal{Z} \subset Y_2 \times \mathbb{C}^n \\ &\qquad \text{such that } \mathcal{U} \cong \mathcal{O}_{\mathcal{Z}} \text{ in } \mathrm{Coh}(Y_2 \times \mathbb{C}^n) \\ &\iff \phi : \mathcal{Z}_2 \times_{Y_1} \mathcal{Z}_1 \longrightarrow Y_2 \times \mathbb{C}^n \text{ is a closed immersion} \\ &\implies \phi \text{ is injective on the } \mathbb{C}\text{-valued points.} \end{aligned}$$

If we denote by $Y_1^{G/N}$ the fixed locus of the action of G/N into Y_1, we obtain the following sufficient condition for Y_2 and G-Hilb$(\mathbb{C}^n)$ being nonisomorphic as moduli spaces.

LEMMA 7.2

If $\tau_1^{-1}(0) \nsubseteq Y_1^{G/N}$, *then* $C \neq C'$.

Proof

Assume $y, g(y) \in \tau_1^{-1}(0) \subset Y_1$ are two distinct points for some $g \in G/N$. Then there exists a (G/N)-cluster $W \in Y_2$ such that $y, g(y) \in \mathrm{Supp}(W)$. Then (W, y), $(W, g(y)) \in \mathcal{Z}_2$ and $(y, 0), (g(y), 0) \in \mathcal{Z}_1$, which implies that $(W, y, 0), (W, g(y), 0) \in \mathcal{Z}_2 \times_{Y_1} \mathcal{Z}_1$ are two distinct points. But then $\phi(W, y, 0) = \phi(W, g(y), 0) = (W, 0)$, so ϕ is not injective. $\square$

As the following theorem shows, in dimensions 2 and 3 the cases when G/N-Hilb(N-Hilb) and G-Hilb coincide as moduli spaces are very few.

THEOREM 7.3

(i) *Let* $G \subset \mathrm{GL}(2, \mathbb{C})$ *be a finite small subgroup, and let* $N \neq G, \{1\}$ *be a normal subgroup in* G. *Then*

$$C = C' \iff G \cong \frac{1}{rs}(1,1) \qquad \text{and} \qquad N \cong \frac{1}{s}(1,1).$$

(ii) *Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be a finite subgroup, and let $N \neq G, \{1\}$ be a normal subgroup in G. Then,*

$$C = C' \iff G \cong \frac{1}{2r}(1,1,2r-2) \qquad \textit{and} \qquad N \cong \frac{1}{2}(1,1,0).$$

Proof
We begin by proving (ii). Recall that if $N \subsetneq G \subset \mathrm{SL}(3,\mathbb{C})$, then $\dim Y_1^{G/N} \leq 1$. Therefore if $C = C'$, then by Lemma 7.2 we must have $\dim(\tau_1^{-1}(0)) \leq 1$. Moreover, since the $\tau^{-1}(0)$ is connected it must consist of a single curve. Indeed, if we have more than one curve in $\tau^{-1}(0)$ fixed by G/N, then at any intersection point of two curves a 2-dimensional subspace of the tangent space is fixed; thus $\dim Y_1^{G/N} > 1$, a contradiction. Especially, the Grothendieck group of coherent sheaves on Y_1 whose supports are contained in $\tau^{-1}(0)$ is of rank two.

Therefore $N \cong \mathbb{Z}/2\mathbb{Z}$ as in [IN], and we can suppose it to be isomorphic to $\frac{1}{2}(1,1,0)$. Then N-Hilb$(\mathbb{C}^3) \cong U_1 \cup U_2$ where $U_1 \cong \mathbb{C}^3_{x^2,z,\frac{y}{x}}$, $U_2 \cong \mathbb{C}^3_{y^2,z,\frac{x}{y}}$, and $\tau^{-1}(0) = E \cong \mathbb{P}^1$ with coordinates $x : y$. After extending the action of G/N on $\mathbb{C}[x,y,z]^N$ naturally into $\mathbb{C}(x,y,z)^N$, we have that G/N fixes E if $g(\frac{x}{y}) = \frac{x}{y}$ for all $g \in G/N$. In other words, g as an element of G can be written in the form $\left(\begin{smallmatrix} \varepsilon & 0 & a \\ 0 & \varepsilon & b \\ 0 & 0 & \varepsilon^{n-2} \end{smallmatrix}\right)$ with $a, b \in \mathbb{C}$ and ε a primitive nth root of unity. But the group N is normal in G, so g must commute with any element in N. This implies that $a = b = 0$, and since G contains $\frac{1}{2}(1,1,0)$ as a subgroup, n has to be an even number. Thus $G \cong \frac{1}{2r}(1,1,2r-2)$ for some $r > 1$.

Conversely, if the group is of the form $G \cong \frac{1}{2r}(1,1,2r-2)$ and $N \cong \frac{1}{2}(1,1,0)$, then by the construction of G-constellations in the Abelian case of Section 4, we see that the elements ω_τ for $\tau \in \mathrm{Irr}(G/N)$ are not Laurent monomials. More precisely, for U_1 we have $\omega_{\tau_0} = 1$, $\omega_{\tau_1} = x^2$ and for U_2 we have $\omega_{\tau_0} = 1$, $\omega_{\tau_1} = y^2$, so there are no Laurent monomials in the G-constellations $\mathcal{Z}$ of G/N-Hilb(N-Hilb$(\mathbb{C}^3)$). This means that they are precisely the G-graphs of G-Hilb$(\mathbb{C}^3)$; thus the chambers are the same.

The proof of (i) follows the same argument. If $C = C'$, then by Lemma 7.2 we have $\tau^{-1}(0) \subset Y_1^{G/N}$. Since $\tau^{-1}(0)$ is a chain of rational curves, then $\dim(Y_1^{G/N}) = 1$, which in particular implies that G/N is not small. As in the proof of (ii) we have that $\tau^{-1}(0)$ must consists of a single rational curve; thus we may assume N to be isomorphic $\frac{1}{s}(1,1)$ for some $s \geq 2$. The exceptional divisor $E \cong \mathbb{P}^1$ in Y_1 has coordinates $(x : y)$, and it is invariant under G/N. As before, it follows that any $g \in G/N$ has to be of the form $\frac{1}{n}(1,1)$ for some $n \geq 2$, and since N is a subgroup, we have $n = rs$ for some $r \geq 2$.

Conversely if $G \cong \frac{1}{rs}(1,1)$ and $N \cong \frac{1}{s}(1,1)$, the action of G/N in the two affine pieces of Y_1 is of type $1/rs(s,0)$, which is not as small as expected. (In other words, $Y_1/(G/N) \cong Y_1$ is nonsingular.) In terms of G-constellations Y_1 has two building blocks $\Gamma_1 = \{1, \ldots, x^{s-1}\}$ and $\Gamma_2 = \{1, \ldots, y^{s-1}\}$, and after the action of G/N we obtain the G-constellations $\mathcal{Z}_1 = \{1, \ldots, x^{rs-1}\}$ and $\mathcal{Z}_2 = \{1, \ldots, y^{rs-1}\}$, so the chambers coincide. $\square$

As an immediate consequence, for any finite subgroup $G \subset \mathrm{SL}(2,\mathbb{C})$ they are never the same moduli space.

COROLLARY 7.4

Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup. Then $C \neq C'$.

Proof

If G is in $\mathrm{SL}(2,\mathbb{C})$, then $rs = 2$; thus either $r = 1$ or $s = 1$, which contradicts $N \neq G, \{1\}$, and we are done. $\square$

7.2. As varieties

In this section we treat the problem of when G-Hilb and G/N-Hilb(N-Hilb) are isomorphic as algebraic varieties. We start with the dimension 2 case.

7.2.1. G/N-Hilb(N-Hilb($\mathbb{C}^2$))

Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. It is well known that the minimal resolution of $\mathbb{C}^2/G$ is unique. Therefore, since both G-Hilb($\mathbb{C}^2$) and G/N-Hilb(N-Hilb($\mathbb{C}^2$)) are minimal they are isomorphic.

Now let G be a finite small subgroup of $\mathrm{GL}(2,\mathbb{C})$, and take $N = G \cap \mathrm{SL}(2,\mathbb{C})$. In this particular case we have the following result, which proves Theorem 1.6(iii).

PROPOSITION 7.5

Let $G \subset \mathrm{GL}(2,\mathbb{C})$ be a finite non-Abelian subgroup such that $G \not\subseteq \mathrm{SL}(2,\mathbb{C})$, and let $N = G \cap \mathrm{SL}(2,\mathbb{C})$. Then G/N-Hilb(N-Hilb($\mathbb{C}^2$)) is not a minimal resolution of $\mathbb{C}^2/G$.

Proof

In this proof, we use the notation $Y_1 = N$-Hilb($\mathbb{C}^2$), $X_2 = Y_1/(G/N)$, and $Y_2 = G/N$-Hilb(Y_1). Then $\tau_1 : Y_1 \to \mathbb{C}^2/N$ is a crepant resolution, and Y_2 is the minimal resolution of X_2. Let $\{E_i\} = \{E_0, E_1, \ldots\}$ be the exceptional curves on Y_1. Since G is non-Abelian, we may assume that E_0 intersects three other exceptional curves, E_1, E_2, and E_3. We denote by $\bar{E}_i$ the images of E_i in X_2 and by i' the G/N-orbit of i. Since Y_1 is a crepant resolution, we have $K_{Y_1} \equiv 0$. Therefore, if e_i denotes the ramification index of τ_1 along E_i, we have $K_{X_2} \equiv -\sum_{i'} \bigl(1 - \frac{1}{e_i}\bigr) \bar{E}_i$. Now since the action of G/N fixes every point on E_0, we see $e_0 = |G/N|$ and $e_1 = e_2 = e_3 = 1$. The canonical bundle of Y_2 is written

$$K_{Y_2} \equiv -\sum_{i'} \Bigl(1 - \frac{1}{e_i}\Bigr) \widetilde{E}_i + \sum_j a_j F_j$$

where $\widetilde{E}_i$ is the proper transform of $\bar{E}_i$ and $\{F_j\}$ are the exceptional curves of τ_2 with discrepancies a_j. Since $-1 < a_j \leq 0$, it follows that

$$K_{Y_2} \cdot \widetilde{E}_1 = -\Bigl(1 - \frac{1}{e_0}\Bigr) + \sum_{F_j \cap \widetilde{E}_1 \neq \emptyset} a_j < 0,$$

which shows that K_{Y_2} is not nef. $\square$

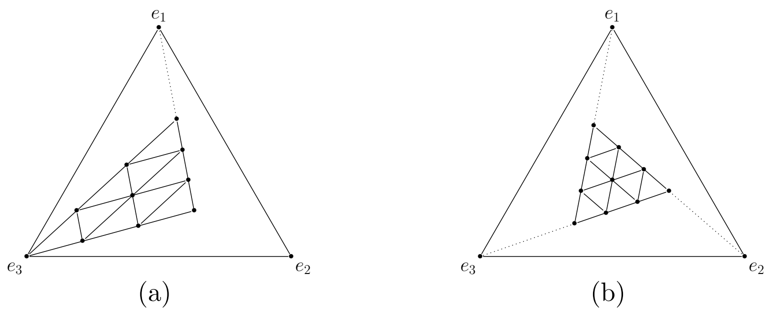


Figure 21. Types of regular triangles: (a) corner triangle and (b) meeting of champions

7.2.2. Finite Abelian subgroups in $\mathrm{SL}(3,\mathbb{C})$

In this section we assume that $G \subset \mathrm{SL}(3,\mathbb{C})$ is a finite Abelian subgroup. We use the same notation as in Section 4. We start by recalling the properties of the triangulation of the junior simplex Δ constructed by Craw and Reid in [CR] corresponding to G-$\mathrm{Hilb}(\mathbb{C}^3)$ that we need. By abusing the notation, in what follows we identify G-$\mathrm{Hilb}(\mathbb{C}^3)$ with its corresponding triangulation of Δ given by [CR].

A *regular triangle* of side r in Δ is a lattice triangle with $r+1$ points on each edge. In G-$\mathrm{Hilb}(\mathbb{C}^3)$ every regular triangle of side r is triangulated with the *regular tesselation*, which is done by drawing $r-1$ parallel lines to the sides of the regular triangle, obtaining r^2 regular triangles of side 1. There are only two types of regular triangles appearing in G-$\mathrm{Hilb}(\mathbb{C}^3)$, namely, the *corner triangle* and the *meeting of champions*, both shown in Figure 21.

In particular, the sides of a regular triangle always extend to one of the vertices e_i. From the construction we can deduce the following properties that we use repeatedly in the rest of the section.

PROPOSITION 7.6

Consider the triangulation of Δ *corresponding to* G-$\mathrm{Hilb}(\mathbb{C}^3)$*. Then, we have the following.*

(i) *Any line contained in* Δ *either passes through one of the vertices* e_i *for* $i=1,2,3$ *or is contained in a regular triangle.*

(ii) *The valency of a vertex* v *in* Δ *is either* 3*,* 4*,* 5*, or* 6*.*

Proof

Part (i) follows from the construction of the triangulation of Δ, and part (ii) forms [CR, Corollary 1.4]. □

THEOREM 7.7

Let G *be a finite nonsimple Abelian subgroup of* $\mathrm{SL}(3,\mathbb{C})$*, and let* N *be a normal subgroup of* G*, with* $N \neq G, \{1\}$*. Then*

$$G/N\text{-Hilb}\big(N\text{-Hilb}(\mathbb{C}^3)\big) \cong G\text{-Hilb}(\mathbb{C}^3) \tag{1}$$

as algebraic varieties if and only if we are in one of the following situations:

(1) *$G/N \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for some $m > 1$;*

(2) *$G \cong \frac{1}{r}(1,1,r-2)$ or $G \cong \frac{1}{r}(1,r-1,0)$; that is, $\mathbb{C}^3/G$ has a unique crepant resolution;*

(3) *$G \cong \frac{1}{2r}(1,a,-a-1)$ with $(2r,a) = 1$, $a^2 \equiv 1 \pmod{4r}$, and $N \cong \frac{1}{2}(1,1,0)$;*

(4) *there is a subgroup $G' \subset G$ containing N such that (G',N) fits into either (2) or (3) and $G/G' \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for some $m > 1$.*

The proof of the theorem is deduced from Lemmas 7.8–7.12. The first lemma shows the biggest family of Abelian groups for which we have an isomorphism of varieties and constitutes case (1) in Theorem 7.7. The rest of the cases are in some sense sporadic modulo case (4).

LEMMA 7.8
If $G/N \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, then $G/N\text{-Hilb}(N\text{-Hilb}(\mathbb{C}^3)) \cong G\text{-Hilb}(\mathbb{C}^3)$ as varieties.

Proof
Assume $G/N \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. Let $L \supset L' \supset \mathbb{Z}^3$ be the toric lattices for $\mathbb{C}^3/G$ and $\mathbb{C}^3/N$, respectively. Then the assumption implies $L/L' \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. Since L and L' are generated by elements on the junior simplex, we have decompositions $L = L_0 \oplus \mathbb{Z}e_1$ and $L' = L'_0 \oplus \mathbb{Z}e_1$, where $L_0 = L \cap \mathbb{R}^2_\Delta$ and $L'_0 = L' \cap \mathbb{R}^2_\Delta$.

Then we have $L_0/L'_0 \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for the two-dimensional lattices $L_0 \supset L'_0$, which implies $L_0 = (1/m)L'_0$. So the Newton polygon for G at e_1 is $1/m$ times that for G'; thus the triangulations are the same by [CR]. $\square$

The following lemma justifies case (4) in Theorem 7.7 and allows us to obtain isomorphism as varieties between Y_2 and G-Hilb$(\mathbb{C}^3)$ by combining cases (1), (2), and (3).

LEMMA 7.9
Suppose that there exists a surjection $\phi : G/N \twoheadrightarrow \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ for some $m > 1$, and let G' be the pullback of $\operatorname{Ker}(\phi)$ to G. Then equation (1) holds for the pair (G,N) if and only if it holds for (G',N).

Proof
If there exists a sequence of normal subgroups $G \rhd G' \rhd N$ and $G/G' \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, then we can construct G/N-Hilb$(N$-Hilb$(\mathbb{C}^3))$ in three steps:

$$G/G'\text{-Hilb}\big(G'/N\text{-Hilb}(N\text{-Hilb}(\mathbb{C}^3))\big),$$

so by Lemma 7.8 we can take G' instead of G. $\square$

Therefore from now on we assume that no such surjection exists, which means that G/N is cyclic.

LEMMA 7.10

If $G/N\text{-Hilb}(N\text{-Hilb}(\mathbb{C}^3)) \cong G\text{-Hilb}(\mathbb{C}^3)$*, then there is no regular triangle of side* ≥ 2 *in* $N\text{-Hilb}(\mathbb{C}^3)$.

Proof

Let T be a regular triangle in Δ of side ≥ 2, where Δ is triangulated for N-Hilb$(\mathbb{C}^3)$. Then T is triangulated by the regular tesselation, and there always exists a triangle $\Delta_i \subset \Delta$ which does not contain any of the vertices of T. Now consider the action of G/N on N-Hilb$(\mathbb{C}^3)$ and the corresponding triangulation on Δ_i. Since any side of T extends to some vertex e_i, by Proposition 7.6(i) any line of the triangulation for G-Hilb$(\mathbb{C}^3)$ inside Δ_i must be parallel to some side of Δ_i. This implies that the action of G/N into Δ_i has to be of the form $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, so that Δ_i is triangulated again with the regular tesselation. But then $G/N \cong \mathbb{Z}/m \times \mathbb{Z}/m\mathbb{Z}$, which contradicts our assumption. $\square$

LEMMA 7.11

The triangulation of Δ *corresponding to* $N\text{-Hilb}(\mathbb{C}^3)$ *contains only regular triangles of side* 1 *if and only if* $N \cong \frac{1}{r}(1,1,r-2)$, $\frac{1}{r}(1,r-1,0)$, *or* $\frac{1}{7}(1,2,4)$.

Proof

Notice that there are only regular triangles of side 1 if and only if there exists a unique crepant resolution of $\mathbb{C}^3/N$, namely, N-Hilb$(\mathbb{C}^3)$. Indeed, if every regular triangle in the triangulation of Δ is of side 1, then Proposition 7.6(i) implies that every line goes to one of the e_i's, and there is no parallelogram in this triangulation. Therefore, there is no flop from N-Hilb$(\mathbb{C}^3)$. Conversely, if there exists a unique crepant resolution there is no parallelogram in the triangulation of Δ; in particular, there is no regular triangle of side bigger than 1.

Finally, notice that the Abelian groups for which there exist a unique crepant resolution are $\frac{1}{r}(1,1,r-2)$, $\frac{1}{r}(1,r-1,0)$, or $\frac{1}{7}(1,2,4)$. This follows from the fact that either all points are contained in a line or $N \cong \frac{1}{7}(1,2,4)$; otherwise we would have 4 points in Δ with not 3 of them aligned, hence a flop. $\square$

From the three possibilities for N in Lemma 7.11 we can exclude the case $N \cong \frac{1}{7}(1,2,4)$. Indeed, let $\Delta = \bigcup_{i=1}^{7} \Delta_i$ be the triangulation corresponding to N-Hilb$(\mathbb{C}^3)$. Then, on the regular triangle with vertices $\frac{1}{7}(1,2,4)$, $\frac{1}{7}(2,4,1)$, and $\frac{1}{7}(4,1,2)$ in the middle of Δ, the next triangulation created by the action of G/N has to be a regular triangle again (otherwise it would contradict Proposition 7.6(i)), so we are again in the case of Lemma 7.8.

Now consider the case $N \cong \frac{1}{r}(1,1,r-2)$. Then every lattice point $P_j \in \Delta$ distinct from the vertices e_i for $i = 1,2,3$, are on a line L passing through e_3. If we consider the triangulation induced by G/N, by Proposition 7.6(i) there are

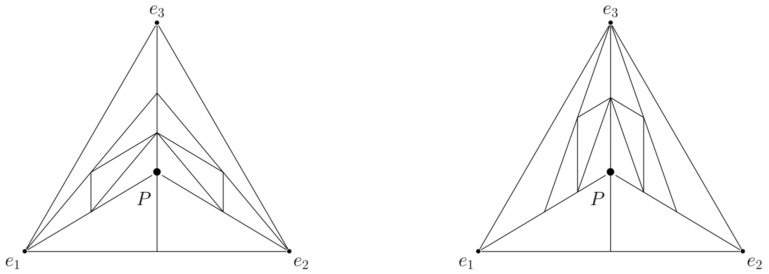


Figure 22

no new lines going out any of the points P_j unless $r = 2$ or 3. Therefore, either $G/N \cong \frac{1}{s}(1,1,s-2)$, so that every new point is again on the line L and G is of type $\frac{1}{n}(1,1,n-2)$, or $N \cong \frac{1}{3}(1,1,1)$, or $N \cong \frac{1}{2}(1,1,0)$, or we are in the case of Lemma 7.9.

Similarly, if $N \cong \frac{1}{r}(1,r-1,0)$, then either $G/N \cong \frac{1}{s}(1,s-1,0)$ for some $s|r$ or $N \cong \frac{1}{2}(1,1,0)$. In any case, we are either in case (2) of Theorem 7.7 or N has order 2 or 3.

Let $N \cong \frac{1}{3}(1,1,1)$, and let P be the point in the center of the triangulation $\Delta = \bigcup_{i=1}^{3} \Delta_i$ of N-Hilb$(\mathbb{C}^3)$ where the 3 lines L_i from the vertices e_i meet ($i = 1,2,3$). Now consider the second triangulation produced by G/N. Because of the "meeting of champions" only one of the lines L_i can be extended, so that the final valency of P is at most 4.

If the valency is 3, then every Δ_i has a basic triangle around P. Since the areas of the basic triangles are the same, it follows that the three vectors at P have the same length, and since two of them form a basis of the two-dimensional lattice $\mathbb{Z}^2_\Delta$, we can conclude that $G/N \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ as in Lemma 7.8. If the valency is 4, then at least 2 of the Δ_i's must have a regular triangle around P, which must be of the same side since they share a generator of the lattice (see Figure 22).

But then there exists a subgroup $G' \cong \frac{1}{3r}(1,1,3r-2)$ such that $G/G' \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ with $m > 1$. Indeed, let M be the middle point of e_1 and e_2. Then every regular triangle inside the triangle e_1PM has e_1 as a vertex and the other two vertices lie on the segment PM. Moreover, these regular triangles have the same area. Then we can see that there is a subgroup $G' \cong \frac{1}{3r}(1,1,3r-2)$ such that these regular triangles for G are basic triangles for G'. If these regular triangles are divided into m^2 basic triangles, then $G/G' \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

In the case $N \cong \frac{1}{2}(1,1,0)$ we have the following lemma, which gives case (3) in Theorem 7.7 and finishes the proof.

LEMMA 7.12
Let $N \cong \frac{1}{2}(1,1,0)$ *and* $Y_2 := G/N\text{-Hilb}(N\text{-Hilb}(\mathbb{C}^3))$. *Then*

$$Y_2 \cong G\text{-Hilb}(\mathbb{C}^3) \Longleftrightarrow G \cong \frac{1}{2r}(1,a,-a-1) \quad \textit{with } a^2 \equiv 1 \pmod{4r}.$$

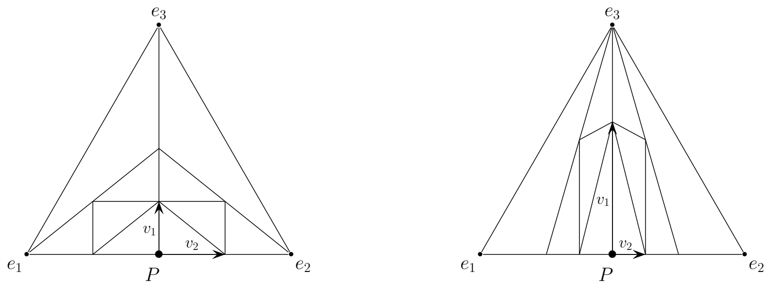


Figure 23. The two possible regular tesselations around $P = \frac{1}{2}(1,1,0)$ and the generators v_1, v_2 of the lattice $\mathbb{Z}^2_\Delta$.

Proof
Let $\Delta = \Delta_1 \cup \Delta_2$ be the triangulation for N-Hilb($\mathbb{C}^3$), and let $P := \frac{1}{2}(1,1,0) \in \Delta$. Suppose that we have the isomorphism as varieties. After the action of G/N on Δ, by Proposition 7.6(i) the triangles Δ_i must contain regular triangles around P. The sides of the regular triangles must pass through some vertex e_i, so there are two possible configurations (see Figure 23).

Notice that any line passing through the point P must go to one of the vertices e_i for $i = 1,2,3$ (otherwise, the sides of two regular triangles would intersect), and in particular there exists the diagonal line $L := e_3P$. Then the vectors v_1, v_2 are sides of a basic triangle; therefore they form a basis of the 2-dimensional lattice $\mathbb{Z}^2_\Delta$.

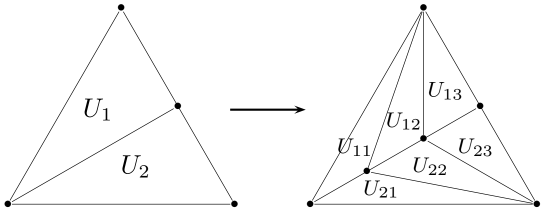
This implies that the lattice is symmetric with respect to the diagonal line L, and in particular the regular triangles around P have the same side. Then it follows that the continued fraction $\frac{2r}{a}$ at the vertex e_3 has to be symmetric with respect to the middle entry, and the boundary of the Newton polygon must contain a lattice point in L. This implies that the expansion of $\frac{2r}{2r-a}$ is also symmetric. Thus if $x^iy^j \in (\mathbb{Z}^2_\Delta)^\vee$, then $x^jy^i \in (\mathbb{Z}^2_\Delta)^\vee$ for any i and j. In other words, we have the condition $a^2 \equiv 1 \pmod{2r}$ in order to have such a symmetric Newton polygon. In addition, the vectors v_1 and v_2 being a basis of $\mathbb{Z}^2_\Delta$ imply that $a^2 \equiv 1 \pmod{4r}$, so the groups G we are looking for are precisely $G \cong \frac{1}{2r}(1,a,-a-1)$ with $a^2 \equiv 1 \pmod{2n}$.

Conversely, if the group is of the form $G \cong \frac{1}{2r}(1,a,-a-1)$ with $a^2 \equiv 1$ (mod $4r$), then $\{v_1, v_2\}$ and $\{v_1, -v_2\}$ form a basis of the lattice of Δ. It follows that the line L has to be part of the triangulation, and again the distribution of points along Δ is symmetric with respect to L, with regular triangles in Δ_1 and Δ_2 around P. The continued fractions at the vertices e_1 and e_2 are the same, and the one for the vertex e_3 is symmetric with respect to the middle term, so by the Craw–Reid method the triangulation of G-Hilb($\mathbb{C}^3$) and G/N-Hilb(N-Hilb($\mathbb{C}^3$)) are the same, and the result follows. □

EXAMPLE 7.13
In this example we show a group G with two subgroups; with one of them there

$$\theta := \begin{array}{|c|c|}\hline \theta_0 & \theta_1 \\ \hline \theta_2 & \theta_3 \\ \hline \theta_4 & \theta_5 \\ \hline \end{array} = \begin{array}{|c|c|}\hline -a-\varepsilon(b_1+b_2) & a \\ \hline -a+\varepsilon b_1 & a \\ \hline -a+\varepsilon b_2 & a \\ \hline \end{array}$$

Figure 24. Stability condition for $\mathbb{Z}/3$-Hilb($\mathbb{Z}/2$-Hilb($\mathbb{C}^3$)).



U_{11}	$\{1,y,y^2,y^3,y^4,y^5\}$
U_{12}	$\{1,y,y^2,y^3,z,yz\}$
U_{13}	$\{1,y,z,yz,z^2,y^2\}$
U_{21}	$\{1,x,x^2,x^3,x^4,x^5\}$
U_{22}	$\{1,x,x^2,x^3,z,xz\}$
U_{23}	$\{1,y,z,xz,z^2,xz^2\}$

Figure 25. Fan for $\mathbb{Z}/3$-Hilb($\mathbb{Z}/2$-Hilb($\mathbb{C}^3$)) and the corresponding G-constellations.

is an isomorphism as moduli spaces (hence as varieties) between G-Hilb($\mathbb{C}^3$) and G/N-Hilb(Y_1) and with the other an isomorphism as varieties but in a different chamber. The stability condition is shown in Figure 24.

Let $G = \frac{1}{6}(1,1,4)$. By taking $N = \frac{1}{2}(1,1,0)$, let $Y_1 := N$-Hilb($\mathbb{C}^3$). Then we have an isomorphism G-Hilb($\mathbb{C}^3$) $\cong G/N$-Hilb(Y_1) as moduli spaces of representations of the McKay quiver.

The crepant resolution Y_1 is covered by $U_1 \cong \mathbb{C}^3_{y^2,z,x/y}$ and $U_2 \cong \mathbb{C}^3_{x^2,z,y/x}$, and $\tau_1^{-1}(0)$ consists of the single $\mathbb{P}^1$ joining the two toric fixed points with coordinates $(x:y)$. In every open set the action of G/N is isomorphic to $\frac{1}{3}(1,2,0)$, and we have that $\tau_1^{-1}(0) = Y_1^{G/N}$. The corresponding G-constellations are shown in Figure 25. The chamber C for G-Hilb($\mathbb{C}^3$) is given by the inequalities

$$\theta_3 > 0, \qquad \theta_5 > 0, \qquad \theta_2+\theta_3 > 0, \qquad \theta_4+\theta_5 > 0,$$
$$\theta_1+\theta_3+\theta_5 > 0, \qquad \theta_0 < 0, \qquad \theta_0+\theta_4 < 0.$$

For G/N-Hilb(N-Hilb($\mathbb{C}^3$)), according to Definition 2.4 the stability is given in Figure 24 where $a, b \in \mathbb{Q}$ and $0 < \varepsilon \ll 1$, which is contained in C.

Now take $N = \frac{1}{3}(1,1,1)$. Then we have that G-Hilb($\mathbb{C}^3$) $\cong G/N$-Hilb(N-Hilb($\mathbb{C}^3$)) are isomorphic as varieties but in different chambers. In this case N-Hilb($\mathbb{C}^3$) has 3 open sets

$$V_1 \cong \mathbb{C}^3_{y^3,\frac{z}{y},\frac{x}{y}}, \qquad V_2 \cong \mathbb{C}^3_{x^3,\frac{z}{x},\frac{y}{x}}, \qquad V_3 \cong \mathbb{C}^3_{\frac{y}{z},\frac{x}{z},z^3},$$

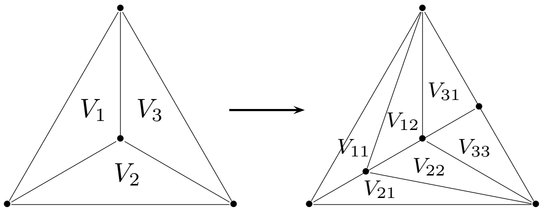
and $G/N \cong \frac{1}{2}(1,1,0)$ in every open set. Then we obtain the distinguished G-constellations shown below. In this case the stability condition is given in Figure 26, where in particular we have $\theta_3 < 0$, so the chamber is different than C.

7.2.3. Some non-Abelian subgroups of SL$(3,\mathbb{C})$

In this section we present some non-Abelian subgroups $G \subset \mathrm{SL}(3,\mathbb{C})$ for which G/N-Hilb(N-Hilb($\mathbb{C}^3$)) and G-Hilb($\mathbb{C}^3$) are nonisomorphic, proving Theorem 1.7(iii).

$$\theta := \begin{array}{|c|c|c|}\hline \theta_0 & \theta_1 & \theta_2 \\ \hline \theta_3 & \theta_4 & \theta_5 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline -a_1 - a_2 - \varepsilon b & a_1 & a_2 \\ \hline -a_1 - a_2 + \varepsilon b & a_1 & a_2 \\ \hline \end{array}$$

Figure 26. Stability condition for $\mathbb{Z}/2$-Hilb($\mathbb{Z}/3$-Hilb($\mathbb{C}^3$)).



V_{11}	$\{1, y, y^2, y^3, y^4, y^5\}$
V_{12}	$\{1, y, y^2, \frac{z}{y}, z, yz\}$
V_{21}	$\{1, x, x^2, x^3, x^4, x^5\}$
V_{22}	$\{1, x, x^2, \frac{z}{x}, z, xz\}$
V_{31}	$\{1, z, z^2, \frac{x}{z}, x, xz\}$
V_{32}	$\{1, z, z^2, \frac{y}{z}, y, yz\}$

Figure 27. Fan for $\mathbb{Z}/2$-Hilb($\mathbb{Z}/3$-Hilb($\mathbb{C}^3$)) and the corresponding G-constellations.

By the calculations presented in Section 6 we know that if $G \subset \mathrm{SO}(3)$ is of type D_{2n} or G_{12} and N is the maximal Abelian subgroup of G, then the fibre over $0 \in \mathbb{C}^3/G$ is different in both spaces. Therefore we conclude that G/N-Hilb(N-Hilb($\mathbb{C}^3$)) is not isomorphic to G-Hilb($\mathbb{C}^3$).

Now consider $G \subset \mathrm{GL}(2,\mathbb{C})$ to be a small non-Abelian subgroup, and let $N = G \cap \mathrm{SL}(2,\mathbb{C})$. Then we can embed G into $\mathrm{SL}(3,\mathbb{C})$ to form a non-Abelian intransitive subgroup (cf. type B in [YY]), and we obtain the following result as a consequence of Proposition 7.5.

COROLLARY 7.14

If $G \subset \mathrm{SL}(3,\mathbb{C})$ *is a non-Abelian small intransitive subgroup and* $N = G \cap \mathrm{SL}(2,\mathbb{C})$*, then* G/N-Hilb(N-Hilb($\mathbb{C}^3$)) *is not isomorphic to* G-Hilb($\mathbb{C}^3$)*.*

Proof

Both sides are crepant resolutions of $\mathbb{C}^3/G$, and the proper transforms of $\mathbb{C}^2/G \subset \mathbb{C}^3/G$ to them are G/N-Hilb(N-Hilb($\mathbb{C}^2$)) and G-Hilb($\mathbb{C}^2$), respectively. The former is not a minimal resolution by Proposition 7.5, but the latter is minimal. This implies that the two crepant resolutions are different. $\square$

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