Moduli of oriented orthogonal sheaves on a nodal curve

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Abstract We study the moduli stack of oriented orthogonal sheaves on a nodal curve and prove a factorization theorem.

1. Introduction

For a complex smooth projective curve C, let $U_C(n,d)$ be the moduli space of semistable vector bundles of rank n and degree d on C. The vector space of global sections of a line bundle on $U_C(n,d)$ is called a space of generalized theta functions. Suppose the curve C degenerates to an irreducible nodal curve C_0, and consider the moduli space $U_{C_0}(n,d)$ of semistable torsion-free sheaves of rank n and degree d on C_0. The factorization theorem says that the space of generalized theta functions on $U_{C_0}(n,d)$ decomposes as a direct sum of spaces of generalized theta functions on the moduli spaces of parabolic vector bundles on the normalization $\widetilde{C_0}$ (see [NR], [Sun], [K2] for precise statements).

In this paper we prove a factorization theorem for the space of generalized theta functions on the moduli stack of oriented orthogonal sheaves. An orthogonal sheaf on C_0 is a torsion-free sheaf E (of rank n) on C_0 with a nondegenerate symmetric bilinear form, and its orientation is a morphism $\wedge^n E \to \mathcal{O}_{C_0}$ satisfying a certain compatibility condition. (Giving an oriented orthogonal locally free sheaf is equivalent to giving a principal SO-bundle.) The main theorem (Theorem 6.1) in this paper describes how the space of global sections of a power of the determinant line bundle on the moduli stack of oriented orthogonal sheaves on C_0 decomposes as a direct sum of spaces of global sections of a line bundle on the moduli stack of parabolic oriented orthogonal bundles on $\widetilde{C_0}$. (Precisely speaking, we restrict our attention to a certain open substack of the moduli stack of oriented orthogonal sheaves C_0.)

The reason why we are interested in the moduli of oriented orthogonal bundles comes from the so-called strange duality phenomena. The most typical strange duality, proven by Belkale [Bel] and Marian and Oprea [MO], is a duality between the space of level r generalized theta functions on the moduli space of

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SL_n-bundles and that of level n generalized theta functions on the moduli space of GL_r-bundles. The author proved the strange duality for symplectic bundles in [A2] and [A3]. In its proof, the factorization theorem for symplectic bundles played an important role. In [Beu], Beauville proved a strange duality between the space of generalized theta functions on the moduli of SO_n-bundles and that on the moduli of $\mathrm{SO}_2(=\mathbb{G}_\mathrm{m})$-bundles. It is tempting to search for a strange duality for SO_n-bundles and SO_m-bundles. We do not know yet even how to formulate such a $(\mathrm{SO}_n, \mathrm{SO}_m)$-strange duality but hope that the factorization theorem for SO-bundles in this paper gives a first step towards it.

The proof of the factorization theorem for SO-bundles in this paper is similar to that for symplectic bundles in [A1], but we comment on a difference. The moduli stack of sheaves on a nodal curve has singularity at points corresponding to nonlocally free sheaves. Whereas the singular locus has, generically, normal singularity in the symplectic bundle case, the singular locus has, generically, normal-crossing singularity in the SO-bundle case. So we not only describe moduli-theoretically the normalization of the moduli stack of SO-sheaves on a nodal curve, but also consider how the normalization of the moduli stack of SO-sheaves is glued to form the normal-crossing singularity of the moduli stack of SO-sheaves. (In the symplectic case, we did not need the gluing argument.) The argument of describing the gluing data is similar to that given in [K2], but not the same. In the SO-case, the notion of ι-transform, introduced in Section 2.2, plays an important role.

Related questions. In order to compute the dimension of the space of generalized theta functions on the moduli of SO-bundles by induction on the genus using a degeneration argument, it is necessary to establish that the dimension of the space of generalized theta functions does not jump as a smooth curve degenerates to a singular one. We do not address this problem in this paper.

Let G be a simple algebraic group over $\mathbb{C}$, and let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated to $\mathfrak{g} := \mathrm{Lie}(G)$. Tsuchiya, Ueno, and Yamada [TUY] established the factorization for the conformal block of $\widehat{\mathfrak{g}}$. When G is simply connected, the conformal blocks are isomorphic to spaces of generalized theta functions on the moduli of parabolic G-bundles (cf. [LS]). When G is not simply connected, like SO, the author does not know what representation-theoretic spaces associated to $\widehat{\mathfrak{g}}$ are isomorphic to spaces of generalized theta functions.

Organization of the paper. In Section 2 we gather what we use in the following sections. In Section 2.1 we define an orthogonal sheaf and its orientation. In Section 2.2 we introduce the ι-transformation, which associates to an orthogonal bundle E with a 1-dimensional isotropic subspace of the fiber $E|_P$ at a point P an orthogonal bundle E^ι together with a 1-dimensional isotropic subspace of the fiber $E^\iota|_P$. In Section 2.3 we gather basic facts about orthogonal Grassmannians. In Section 3 we study the deformation theory of oriented orthogonal sheaves on a formal neighborhood of a node. In Section 4 we study the structure of the moduli stack of oriented orthogonal sheaves on C_0. Using the results in Section 3 we can see immediately its local structure. We find that the singularity of the moduli stack is generically normal crossing. We restrict our attention to a certain normal-crossing open substack and describe its normalization and gluing data in terms of the moduli stack of oriented orthogonal bundles on $\widetilde{C_0}$. In Section 5 we define a certain compactification of an orthogonal group and study the space of global sections of line bundles on it. In Section 6, combining the results in Sections 4 and 5, we prove the factorization theorem for SO-bundles.

The results that lead up to the factorization theorem Theorem 6.1 are Propositions 4.5 and 4.10 and Lemma 2.1(2), Proposition 5.10, and Theorem 5.14.

NOTATION AND CONVENTION

- In this paper, 2 is invertible; that is, every scheme is over $\operatorname{Spec}\mathbb{Z}[1/2]$ and the characteristic of a field is not 2. When we write $\sqrt{-1}$, it is a fixed solution of $x^2 = -1$ in an algebraically closed field.
- For integers $a < b$, we denote by $[a,b]$ the set $\{a, a+1, \dots, b\}$. When it is clear from the context, the ordered set $(a, a+1, \dots, b)$ is also denoted by $[a,b]$. For ordered sets $I = (i_1, \dots, i_a)$ and $J = (j_1, \dots, j_b)$, the ordered set $(i_1, \dots, i_a, j_1, \dots, j_b)$ is denoted by $I \cup J$.
- Let S be a scheme, and let $*$ be an object (such as a sheaf, a scheme, a morphism, etc.) over S. For an S-scheme T, we denote by $(*)_T$ or $*_T$ the base change of $*$ by $T \to S$.
- If $\mathcal{X}$ is a stack over an algebraically closed field k and A is an object of $\mathcal{X}(\operatorname{Spec} k)$, then we write $[A] \in \mathcal{X}$ or $A \in \mathcal{X}$. If $\rho : \mathcal{X} \to \mathcal{Y}$ is a morphism of stacks and A is an object of $\mathcal{X}(S)$ for a k-scheme S, then we denote by $\rho(A)$ the image of A in $\mathcal{Y}(S)$.

2. Preliminaries

2.1. Oriented orthogonal sheaves

An orthogonal sheaf of rank n on a scheme X is a pair (E, γ), where E is a coherent $\mathcal{O}_X$-module such that it is generically locally free of rank n on each component of X, and γ is a nondegenerate symmetric bilinear form $E \otimes E \to \mathcal{O}_X$, where *nondegenerate* means that the induced morphism $E \to \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ is an isomorphism.

An orientation of an orthogonal sheaf (E, γ) of rank n on X is a morphism $\delta : \wedge^n E \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules such that the diagram

$$\begin{array}{ccc} \wedge^n E \otimes \wedge^n E & \xrightarrow{\delta \otimes \delta} & \mathcal{O}_X \otimes \mathcal{O}_X \\ {\scriptstyle \overline{\wedge}^n \gamma}\downarrow & & \downarrow{\scriptstyle \simeq} \\ \mathcal{O}_X & = & \mathcal{O}_X \end{array} \tag{2.1}$$

commutes, where $\overline{\wedge}^n \gamma$ is defined by

$$e_1 \wedge \cdots \wedge e_n \otimes f_1 \wedge \cdots \wedge f_n \mapsto \det\bigl(\gamma(e_i, f_j)\bigr). \tag{2.2}$$

The triple (E, γ, δ) is called an oriented orthogonal sheaf. When E is locally free, we say (oriented) orthogonal bundle instead of (oriented) orthogonal sheaf. An isomorphism between oriented orthogonal sheaves $(E_1, \gamma_1, \delta_1)$ and $(E_2, \gamma_2, \delta_2)$ is an isomorphism $E_1 \simeq E_2$ compatible with the bilinear forms and orientations.

Given two oriented orthogonal sheaves $(E_i, \gamma_i, \delta_i)$ $(i = 1, 2)$ of rank r_i, define the bilinear form γ of $E_1 \oplus E_2$ by

$$\gamma\big((e_1, e_2), (e'_1, e'_2)\big) := \gamma_1(e_1, e'_1) + \gamma_2(e_2, e'_2)$$

and the orientation δ by

$$\bigwedge^{r_1+r_2} (E_1 \oplus E_2) \to \bigwedge^{r_1} E_1 \otimes \bigwedge^{r_2} E_2 \xrightarrow{\delta_1 \otimes \delta_2} \mathcal{O} \otimes \mathcal{O} \simeq \mathcal{O},$$

where the first arrow is the projection. The oriented orthogonal sheaf $(E_1 \oplus E_2, \gamma, \delta)$ is called the direct sum of $(E_1, \gamma_1, \delta_1)$ and $(E_2, \gamma_2, \delta_2)$.

Assume that $X \to B$ is a flat quasi-compact morphism of schemes such that every geometric fiber is reduced and equidimensional. (The situation we have in mind is the family of nodal curves.) An orthogonal sheaf (resp., oriented orthogonal sheaf) of rank n on X/B is an orthogonal sheaf (E, γ) (resp., oriented orthogonal sheaf (E, γ, δ)) on X such that E is flat over B, and for every geometric point b of B, the restriction $E|_{X_b}$ is torsion-free and $(E, \gamma)|_{X_b}$ (resp., $(E, \gamma, \delta)|_{X_b}$) is an orthogonal sheaf (resp., oriented orthogonal sheaf) of rank n on X_b. When $B = \operatorname{Spec} k$ with k a field and there is no confusion, we say simply (oriented) orthogonal sheaf "on X" instead of "on $X/\operatorname{Spec} k$."

2.2. ι-transform

Let X be a smooth projective curve over an algebraically closed field k, and let S be a k-scheme. Fix a finite set $\overrightarrow{P} = \{P_1, \ldots, P_m\}$ of points of X. Let $\mathcal{B}$ be the groupoid whose objects are tuples

$$\big(\mathcal{F}, \gamma, \delta; \mathcal{L}_i \subset \mathcal{F}|_{\{P_i\} \times S} \ (1 \leq i \leq m)\big), \tag{2.3}$$

where $(\mathcal{F}, \gamma, \delta)$ is an oriented orthogonal bundle on $X \times S$ and $\mathcal{L}_i$ is an isotropic line subbundle of $\mathcal{F}|_{\{P_i\} \times S}$. An isomorphism between two objects $(\mathcal{F}, \gamma_{\mathcal{F}}, \delta_{\mathcal{F}}; \mathcal{L}_i \subset \mathcal{F}|_{\{P_i\} \times S} \ (1 \leq i \leq m))$ and $(\mathcal{G}, \gamma_{\mathcal{G}}, \delta_{\mathcal{G}}; \mathcal{M}_i \subset \mathcal{G}|_{\{P_i\} \times S} \ (1 \leq i \leq m))$ is an isomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ of oriented orthogonal bundles such that $(\varphi|_{\{P_i\} \times S})(\mathcal{L}_i) = \mathcal{M}_i$.

Suppose that we are given an object (2.3) of $\mathcal{B}$. Put

$$\mathcal{F}^\flat := \operatorname{Ker}\Big(\mathcal{F} \to \bigoplus_{i=1}^m \frac{\mathcal{F}|_{\{P_i\} \times S}}{\mathcal{L}_i^\perp}\Big) \qquad \text{and} \qquad \mathcal{F}^\sharp := (\mathcal{F}^\flat)^\vee.$$

We have inclusions $\mathcal{F}^\flat \subset \mathcal{F} \simeq \mathcal{F}^\vee \hookrightarrow \mathcal{F}^\sharp$. The bilinear form γ induces a $\operatorname{pr}_X^* \mathcal{O}\big(\sum_{i=1}^m P_i\big)$-valued bilinear form $\gamma^\sharp : \mathcal{F}^\sharp \otimes \mathcal{F}^\sharp \to \operatorname{pr}_X^* \mathcal{O}\big(\sum_{i=1}^m P_i\big)$. Since $\gamma^\sharp(\mathcal{F}^\flat \otimes \mathcal{F}^\sharp) \subset \mathcal{O}_{X \times S}$, the bilinear form $\gamma^\sharp$ induces a bilinear form

$$\bar{\gamma}^\sharp : \frac{\mathcal{F}^\sharp}{\mathcal{F}^\flat} \otimes \frac{\mathcal{F}^\sharp}{\mathcal{F}^\flat} \to \operatorname{pr}^*\Big(\frac{\mathcal{O}\big(\sum_{i=1}^m P_i\big)}{\mathcal{O}_X}\Big).$$

On each section $\{P_i\} \times S$, $\mathcal{F}^\sharp/\mathcal{F}^\flat$ is an orthogonal bundle of rank 2 having $\mathcal{F}/\mathcal{F}^\flat$ as an isotropic line subbundle. Let $\mathcal{T}_i \subset \mathcal{F}^\sharp/\mathcal{F}^\flat$ be the other isotropic line subbundle. We can find a vector bundle $\mathcal{F}^\iota$ such that $\mathcal{F}^\flat \subset \mathcal{F}^\iota \subset \mathcal{F}^\sharp$ and $\mathcal{F}^\iota/\mathcal{F}^\flat = \mathcal{T}_i$ on each $\{P_i\} \times S$. Let γ^ι be the restriction of $\gamma^\sharp$ to $\mathcal{F}^\iota$. Then γ^ι is $\mathcal{O}_{X\times S}$-valued and $(\mathcal{F}^\iota, \gamma^\iota)$ is an orthogonal bundle. We define its orientation δ^ι to be the composite $\det \mathcal{F}^\iota \simeq \det \mathcal{F}^\flat\big(\sum_{i=1}^m \{P_i\} \times S\big) \simeq \det \mathcal{F} \xrightarrow{\delta} \mathcal{O}_{X\times S}$. Put $\mathcal{L}_i^\iota := \mathrm{Ker}(\mathcal{F}^\iota|_{\{P_i\}\times S} \to \mathcal{F}^\sharp|_{\{P_i\}\times S})$. Then $\mathcal{L}_i^\iota$ is an isotropic line subbundle of $\mathcal{F}^\iota|_{\{P_i\}\times S}$. Thus the tuple

$$\big(\mathcal{F}^\iota, \gamma^\iota, \delta^\iota; \mathcal{L}_i^\iota \subset \mathcal{F}^\iota|_{\{P_i\}\times S} (1 \le i \le m)\big) \tag{2.4}$$

is an object of $\mathcal{B}$. We call this tuple the ι-transform over $\overrightarrow{P}$ of the tuple (2.3). The construction of the ι-transform shows that the double ι-transform of an object of $\mathcal{B}$ is naturally isomorphic to the object itself, that is, the ι-transformation gives rise to an involution on $\mathcal{B}$.

There is an isomorphism

$$\mathcal{F}^\flat|_{\{P_i\}\times S}/(\mathcal{F}^\flat|_{\{P_i\}\times S})^\perp \simeq \mathcal{L}_i^\perp/\mathcal{L}_i. \tag{2.5}$$

Since $(\mathcal{F}^\iota)^\flat = \mathcal{F}^\flat$, we have an isomorphism

$$\mathcal{L}_i^{\iota\perp}/\mathcal{L}_i^\iota \simeq \mathcal{L}_i^\perp/\mathcal{L}_i \tag{2.6}$$

of orthogonal bundles.

The ι-transform is locally described as follows. Consider the case $S = \mathrm{Spec}\, k$ and $\overrightarrow{P} = \{P\}$, and let t be a local coordinate at P. For a tuple (2.3), let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a local frame of $\mathcal{F}$ around P such that $(\mathbf{e}_i, \mathbf{e}_{n+1-i}) = 1$ and the 1-dimensional subspace is spanned by $\mathbf{e}_1$. Then $\mathcal{F}^\iota$ is generated by $t^{-1}\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n-1}, t\mathbf{e}_n$.

2.3. Orthogonal Grassmannian

Let k be an algebraically closed field, and let V be an n-dimensional k-vector space with a nondegenerate bilinear form γ. For $m \le [n/2]$, we put

$$\mathrm{OG}_m(V) := \{U \subset V \mid \dim U = m \text{ and } \gamma|_{U\times U} \equiv 0\}.$$

Assume that n is even. Then $\mathrm{OG}_{n/2}(V)$ has two connected components. Let $\delta : \wedge^n V \to k$ be an orientation of (V, γ) (considered as an orthogonal bundle on $\mathrm{Spec}\, k$). We name the two connected components of $\mathrm{OG}_{n/2}(V)$ as $\mathrm{OG}_{n/2}(V)^{(+)}$ and $\mathrm{OG}_{n/2}(V)^{(-)}$ as follows. For $[U \subset V] \in \mathrm{OG}_{n/2}(V)$, define $\epsilon = +$ or $-$ as the diagram

$$\begin{array}{ccc}
\wedge^{n/2} U \otimes \wedge^{n/2} V/U & \xrightarrow{\text{cano.}} & \wedge^n V \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \delta} \\
k & \xrightarrow{\cdot\epsilon(\sqrt{-1})^{n/2}} & k
\end{array}$$

commutes, where υ is defined by

$$e_1 \wedge \cdots \wedge e_{n/2} \otimes \overline{f_1} \wedge \cdots \wedge \overline{f_{n/2}} \mapsto \det\bigl(\gamma(e_i, f_j)\bigr).$$

If $n/2$ is even, we put $\mathrm{OG}_{n/2}(V)_{(\epsilon)} := \mathrm{OG}_{n/2}(V)^{(\epsilon)}$ for $\epsilon = \pm$, and if $n/2$ is odd, then we put $\mathrm{OG}_{n/2}(V)_{(+)} := \mathrm{OG}_{n/2}(V)^{(-)}$ and $\mathrm{OG}_{n/2}(V)_{(-)} := \mathrm{OG}_{n/2}(V)^{(+)}$.

If $\mathcal{V}$ is a rank n vector bundle with a nondegenerate bilinear form on a scheme X, then we can consider the family of orthogonal Grassmannians $\mathrm{OG}_m(\mathcal{V}) \to X$, and, moreover, if $\mathcal{V}$ has an orientation, then the notation $\mathrm{OG}_m(\mathcal{V})^{(\pm)}$ makes sense.

Let E_i be an n-dimensional vector space with a nondegenerate bilinear form γ_i and an orientation δ_i $(i = 1, 2)$. We endow the vector space $E_1 \oplus E_2$ with the bilinear form γ defined by

$$\gamma\bigl((e_1, e_2), (e'_1, e'_2)\bigr) = \gamma_1(e_1, e'_1) - \gamma_2(e_2, e'_2)$$

and the orientation defined by

$$\bigwedge^{2n}(E_1 \oplus E_2) \simeq \bigwedge^{n} E_1 \otimes \bigwedge^{n} E_2 \xrightarrow{\delta_1 \otimes \delta_2} k \otimes k \simeq k \xrightarrow{\cdot(\sqrt{-1})^n} k.$$

If $f : E_1 \to E_2$ is a k-linear isomorphism compatible with bilinear forms and orientations, then the graph $\Gamma_f \subset E_1 \oplus E_2$ lies in the component $\mathrm{OG}_n(E_1 \oplus E_2)_{(+)}$. The following lemma is easy.

LEMMA 2.1

Let $[U \subset E_1 \oplus E_2] \in \mathrm{OG}_n(E_1 \oplus E_2)$.

(1) *We have* $\dim U \cap (E_1 \times \{0\}) = \dim U \cap (\{0\} \times E_2)$.

(2) *We put*

$$\mathrm{OG}_n^{=a}(E_1 \oplus E_2) := \bigl\{U \subset E_1 \oplus E_2 \bigm| \dim U \cap (E_1 \times \{0\}) = a\bigr\}.$$

Let $f : \mathrm{OG}_n^{=a}(E_1 \oplus E_2) \to \mathrm{OG}_a(E_1) \times \mathrm{OG}_a(E_2)$ *be the map that sends* U *to* $(U \cap (E_1 \times \{0\}), U \cap (\{0\} \times E_2))$. *For* $(B_1, B_2) \in \mathrm{OG}_a(E_1) \times \mathrm{OG}_a(E_2)$, *there is an isomorphism*

$$f^{-1}\bigl((B_1, B_2)\bigr) \simeq \mathrm{OG}_{n-2a}^{=0}(B_1^{\perp}/B_1 \oplus B_2^{\perp}/B_2).$$

Moreover, if we define the orientations δ'_1 *and* δ'_2 *of* $B_1^{\perp}/B_1$ *and* $B_2^{\perp}/B_2$ *so that the diagrams*

$$\begin{array}{ccccc}
\wedge^a B_1 \otimes \wedge^a \frac{E_1}{B_1^{\perp}} \otimes \wedge^{n-2a} \frac{B_1^{\perp}}{B_1} & \xrightarrow{\sim} & \wedge^n E_1 & \xrightarrow{\delta_1} & k \\
\Big\downarrow{\scriptstyle \bar{\wedge}^a \gamma_1 \otimes \mathrm{id}} & & & & \Big\downarrow{\scriptstyle \cdot(\sqrt{-1})^a} \\
\wedge^{n-2a} \frac{B_1^{\perp}}{B_1} & & \xrightarrow{\qquad\quad \delta'_1 \qquad\quad} & & k
\end{array}$$

$$\begin{array}{ccccc}
\wedge^a \frac{E_2}{B_2^\perp} \otimes \wedge^a B_2 \otimes \wedge^{n-2a} \frac{B_2^\perp}{B_2} & \xrightarrow{\sim} & \wedge^n E_2 & \xrightarrow{\delta_2} & k \\
\Big\downarrow {\scriptstyle \bar{\wedge}^a \gamma_2 \otimes \mathrm{id}} & & & & \Big\downarrow {\scriptstyle \cdot(\sqrt{-1})^a} \\
\wedge^{n-2a} \frac{B_2^\perp}{B_2} & & \xrightarrow{\qquad \delta_2' \qquad} & & k
\end{array}$$

commute, then

$$f^{-1}\big((B_1, B_2)\big) \cap \mathrm{OG}_n^{=a}(E_1 \oplus E_2)_{(+)} \simeq \mathrm{OG}_{n-2a}^{=0}(B_1^\perp/B_1 \oplus B_2^\perp/B_2)_{(+)}.$$

(3) *Put* $\mathrm{OG}_n^{\leq a}(E_1 \oplus E_2) := \bigcup_{i=0}^{a} \mathrm{OG}_n^{=i}(E_1 \oplus E_2)$. *Then* $\mathrm{OG}_n^{\leq a}(E_1 \oplus E_2)$ *is an open subscheme of* $\mathrm{OG}_n(E_1 \oplus E_2)$*, and the codimension of the complement of* $\mathrm{OG}_n^{\leq a}(E_1 \oplus E_2)$ *in* $\mathrm{OG}_n(E_1 \oplus E_2)$ *is greater than or equal to* $(a+1)^2$*.*

3. Oriented orthogonal sheaves in a neighborhood of a node

Let $(R, \mathfrak{m})$ be a complete Noetherian local ring. Assume that $k := R/\mathfrak{m}$ is algebraically closed. Put $A := R[[x, y]]/(xy - \pi)$, where $\pi \in \mathfrak{m}$.

3.1. Construction of orthogonal sheaves

We recall Faltings's construction of orthogonal sheaves on $\operatorname{Spec} A/\operatorname{Spec} R$ (see [F]).

Given matrices $P = (p_{ij}), Q = (q_{ij}) \in \mathrm{Mat}_{n\times n}(\mathfrak{m})$ with $PQ = QP = \pi \cdot I_n$, we define $(2n \times 2n)$-matrices α, β with entries in A by

$$\alpha = \begin{pmatrix} x \cdot I_n & P \\ Q & y \cdot I_n \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} y \cdot I_n & -P \\ -Q & x \cdot I_n \end{pmatrix}.$$

Then $\alpha\beta = \beta\alpha = 0$, and the complex

$$\cdots \to A^{2n} \xrightarrow{\beta} A^{2n} \xrightarrow{\alpha} A^{2n} \xrightarrow{\beta} A^{2n} \xrightarrow{\alpha} A^{2n} \to \cdots$$

is exact. Put $E(P, Q) := \operatorname{Im}\alpha = \operatorname{Ker}\beta$. Then $E(P, Q)$ is R-flat and $\widetilde{E(P,Q)/\mathfrak{m}E(P,Q)}$ does not have free summands, that is, $E(P,Q)/\mathfrak{m}E(P,Q) \simeq (\widetilde{A/\mathfrak{m}A})^n$, where $\widetilde{A/\mathfrak{m}A} = k[[x]] \oplus k[[y]]$.

We denote by b the standard bilinear form given by I_n on R^n or A^n. Taking the dual of the sequence $A^{2n} \overset{\alpha}{\twoheadrightarrow} E(P,Q) \subset A^{2n}$, we get $(A^{2n})^\vee \hookleftarrow E(P,Q)^\vee \twoheadleftarrow (A^{2n})^\vee$. Identifying $(A^{2n})^\vee$ with A^{2n} by the standard bilinear form and noting ${}^t\alpha = \left(\begin{smallmatrix} x\cdot I_n & {}^tQ \\ {}^tP & y\cdot I_n \end{smallmatrix}\right)$, we obtain a canonical isomorphism $E({}^tQ, {}^tP) \simeq E(P,Q)^\vee$.

Now assume that $P = {}^tQ$; then the composite

$$E(P,Q) \xrightarrow{\mathrm{id}} E({}^tQ, {}^tP) \simeq E(P,Q)^\vee$$

gives rise to a nondegenerate symmetric bilinear form on $E(P,Q)$. We denote this bilinear form by $\gamma_{(P,Q)}$. Explicitly, we have

$$\gamma_{(P,Q)}\left(\alpha\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}, \begin{pmatrix}\mathbf{u}'\\ \mathbf{v}'\end{pmatrix}\right) = b(\mathbf{u}, \mathbf{u}') + b(\mathbf{v}, \mathbf{v}')$$

for $\mathbf{u}, \mathbf{u}', \mathbf{v}, \mathbf{v}' \in A^n$. The pair $(E(P,Q), \gamma_{(P,Q)})$ is an orthogonal sheaf of rank n on $\operatorname{Spec} A/\operatorname{Spec} R$.

3.2. Construction of oriented orthogonal sheaves

Now we consider an orientation of the orthogonal sheaf $(E(P,Q), \gamma_{(P,Q)})$.

Let $\Delta : \wedge^n(A^{2n}) \to A$ be an A-homomorphism. Let $\mathbf{e}_1, \dots, \mathbf{e_n}, \mathbf{f}_1, \dots, \mathbf{f}_n$ be the standard basis of A^{2n}. For $I \in [1,n]^a$ and $J \in [1,n]^b$ with $a+b=n$, we put $C_{I;J} := \Delta(\mathbf{e}_I \wedge \mathbf{f}_J)$, where $\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_a}$ if $I = (i_1, \dots, i_a)$, and similarly for $\mathbf{f}_J$. The homomorphism Δ factors as $\wedge^n(A^{2n}) \xrightarrow{\wedge^n \alpha} \wedge^n E(P,Q) \to A$ if and only if $\Delta(\wedge^{n-1}(A^{2n}) \otimes \operatorname{Im} \beta) = 0$; that is, the images of

$$\Big(y\mathbf{e}_s - \sum_{l=1}^{n} q_{ls}\mathbf{f}_l\Big) \wedge \mathbf{e}_I \wedge \mathbf{f}_J \qquad \text{and} \qquad \Big(x\mathbf{f}_t - \sum_{m=1}^{n} p_{mt}\mathbf{e}_m\Big) \wedge \mathbf{e}_I \wedge \mathbf{f}_J$$

by Δ in A is zero for all $I \in [1,n]^a$ and $J \in [1,n]^b$ with $a+b=n-1$. These are equivalent to

$$yC_{I\cup\{s\};J} = \sum_{l=1}^{n} q_{ls} C_{I;\{l\}\cup J}, \tag{3.1}$$

$$xC_{I;\{t\}\cup J} = \sum_{m=1}^{n} p_{mt} C_{I\cup\{m\};J}. \tag{3.2}$$

When equations (3.1) and (3.2) hold, we define $\delta : \wedge^n E(P,Q) \to A$ by $\Delta = \delta \circ \wedge^n \alpha$. The homomorphism δ becomes an orientation of $(E(P,Q), \gamma_{(P,Q)})$ if and only if

$$\bar{\wedge}^n \gamma_{(P,Q)}\big((\wedge^n\alpha)(\mathbf{e}_{[1,n]}), (\wedge^n\alpha)(\mathbf{e}_{[1,n]})\big) = \delta\big((\wedge^n\alpha)(\mathbf{e}_{[1,n]})\big)^2,$$
$$\bar{\wedge}^n \gamma_{(P,Q)}\big((\wedge^n\alpha)(\mathbf{f}_{[1,n]}), (\wedge^n\alpha)(\mathbf{f}_{[1,n]})\big) = \delta\big((\wedge^n\alpha)(\mathbf{f}_{[1,n]})\big)^2$$

because the sections $(\wedge^n\alpha)(\mathbf{e}_{[1,n]})$ and $(\wedge^n\alpha)(\mathbf{f}_{[1,n]})$ generate $\wedge^n E(P,Q)$ over $\operatorname{Spec} A \setminus \{\text{the closed point}\}$. The above conditions are equivalent to

$$C^2_{[1,n];\emptyset} = x^n \qquad \text{and} \qquad C^2_{\emptyset;[1,n]} = y^n. \tag{3.3}$$

LEMMA 3.1

If (E, γ, δ) is an oriented orthogonal sheaf of rank n on $\operatorname{Spec} A/\operatorname{Spec} R$ such that $E/\mathfrak{m}E$ has no free summand, then n is even.

Proof

We may assume that $R = k$. There is an isomorphism $(E,\gamma) \simeq (E(O,O), \gamma_{(O,O)})$ of orthogonal sheaves (see [F, Theorem 3.7]). By this isomorphism, we regard δ as the orientation of $(E(O,O), \gamma_{(O,O)})$. Then by equations (3.3), $C_{[1,n];\emptyset} \in k[[x,y]]/(xy)$ satisfies $C^2_{[1,n];\emptyset} = x^n$. This implies that n is even. □

In the rest of this subsection we assume that n is even. Assume also that for any $I \in [1,n]^{n/2}$ and $J \in [1,n]^{n/2}$, the equality

$$\operatorname{sgn}\begin{pmatrix} J^c \cup J \\ [1,n] \end{pmatrix} \det Q_{J^c \times I} = \operatorname{sgn}\begin{pmatrix} I \cup I^c \\ [1,n] \end{pmatrix} \det P_{I^c \times J} \tag{3.4}$$

holds, where $Q_{J^c\times I}$ (resp., $P_{I^c\times J}$) is the $((n/2)\times(n/2))$-matrix with entries q_{uv} (resp., p_{uv}) with $u \in J^c$ and $v \in I$ (resp., $u \in I^c$ and $v \in J$). We define $\Delta : \wedge^n(A^{2n}) \to A$ as follows. For $I \in [1,n]^a$ and $J \in [1,n]^b$ with $a+b=n$, if $a \le n/2$, then

$$C_{I;J} = \operatorname{sgn}\begin{pmatrix} J^c \cup J \\ [1,n] \end{pmatrix} y^{(n/2)-a} \det Q_{J^c \times I}, \tag{3.5}$$

and if $b \le n/2$, then

$$C_{I;J} = \operatorname{sgn}\begin{pmatrix} I \cup I^c \\ [1,n] \end{pmatrix} x^{(n/2)-b} \det P_{I^c \times J}, \tag{3.6}$$

where we understand that $\det P_{\emptyset;\emptyset} = \det Q_{\emptyset;\emptyset} = 1$ by convention. Then these $C_{I;J}$'s satisfy (3.1), (3.2), and (3.3), so it induces an orientation of the orthogonal sheaf $(E(P,Q), \gamma_{(P,Q)})$. We denote this orientation by $\delta_{(P,Q)}$.

LEMMA 3.2

Let $K \subset R$ be a nilpotent ideal. Assume that $P = (p_{ij}), Q = (q_{ij}) \in \operatorname{Mat}_{n\times n}(\mathfrak{m})$ satisfy $PQ = QP = \pi \cdot I_n$ and $Q = {}^tP$. Let $(E(P,Q), \gamma_{(P,Q)}, \delta)$ be an oriented orthogonal sheaf on $\operatorname{Spec} A/\operatorname{Spec} R$. Denote by $\bar{P} = (\bar{p}_{ij})$ and $\bar{Q} = (\bar{q}_{ij})$ the images of P and Q in $\operatorname{Mat}_{n\times n}(\mathfrak{m}/K)$. Assume that for any $I \in [1,n]^{n/2}$ and $J \in [1,n]^{n/2}$, the equality

$$\operatorname{sgn}\begin{pmatrix} J^c \cup J \\ [1,n] \end{pmatrix} \det \bar{Q}_{J^c \times I} = \operatorname{sgn}\begin{pmatrix} I \cup I^c \\ [1,n] \end{pmatrix} \det \bar{P}_{I^c \times J}$$

holds, and $\delta|_{\operatorname{Spec} A/KA} = \delta_{(\bar{P},\bar{Q})}$. Then for any $I \in [1,n]^{n/2}$ and $J \in [1,n]^{n/2}$, the equality (3.4) holds, and $\delta = \delta_{(P,Q)}$.

Proof

We may assume that $\mathfrak{m}K = 0$. As before, put $\Delta = \delta \circ \wedge^n \alpha$ and $C_{I;J} = \Delta(\mathbf{e}_I \wedge \mathbf{f}_J)$ for $I = (i_1,\dots,i_a) \in [1,n]^a$ and $J = (j_1,\dots,j_b) \in [1,n]^b$ with $a+b=n$.

If $b \le n/2$, then we have, by assumption,

$$C_{I;J} = \operatorname{sgn}\begin{pmatrix} I \cup I^c \\ [1,n] \end{pmatrix} x^{(n/2)-b} \det P_{I^c\times J} + s^{I;J} + \sum_{i>0} t_i^{I;J} x^i + \sum_{i>0} u_i^{I;J} y^i$$

with $s^{I;J}, t_i^{I;J}, u_i^{I;J} \in K$. If $b < n/2$, then by (3.1) we have

$$\pi \operatorname{sgn}\begin{pmatrix} I \cup I^c \\ [1,n] \end{pmatrix} x^{(n/2)-b-1} \det P_{I^c\times J} + s^{I;J} y + \sum_{i>0} u_i^{I;J} y^{i+1}$$

$$= \sum_{l=1}^{n} q_{l i_a} \operatorname{sgn}\begin{pmatrix} I \cup I^c \\ [1,n] \end{pmatrix} x^{(n/2)-b-1} \det P_{(\{i_a\}\cup I)\times(\{l\}\cup J)}.$$

The first term of the left-hand side is equal to the right-hand side. So

$$s^{I;J} = u_i^{I;J} = 0. \tag{3.7}$$

If $0 < b$, by using (3.2), we obtain

$$s^{I;J} = t_i^{I;J} = 0. \tag{3.8}$$

If $a \le n/2$, then we have, by assumption,

$$C_{I;J} = \operatorname{sgn}\binom{J^c \cup J}{[1,n]} y^{(n/2)-a} \det Q_{J^c \times I} + s^{I;J} + \sum_{i>0} t_i^{I;J} x^i + \sum_{i>0} u_i^{I;J} y^i$$

with $s^{I;J}, t_i^{I;J}, u_i^{I;J} \in K$. Arguing as above, we have

$$s^{I;J} = t_i^{I;J} = 0 \tag{3.9}$$

if $a < n/2$ and

$$s^{I;J} = u_i^{I;J} = 0 \tag{3.10}$$

if $0 < a$. From (3.7), (3.8), (3.9), and (3.10), it follows that P and Q satisfy (3.4), and $C_{I;J}$ is given by (3.5) or (3.6) if $0 < a \le n/2$ or $0 < b \le n/2$. It remains to show that $C_{[1,n];\emptyset} = x^{n/2}$ and $C_{\emptyset;[1,n]} = y^{n/2}$. By (3.7), we have

$$C_{[1,n];\emptyset} = x^{n/2} + \sum_{i>0} t_i x^i$$

with $t_i \in K$. Since $C^2_{[1,n];\emptyset} = x^n$ by (3.3), we have $t_i = 0$. Hence $C_{[1,n];\emptyset} = x^{n/2}$. Likewise we have $C_{\emptyset;[1,n]} = y^{n/2}$. □

3.3. Deformation of an oriented orthogonal sheaf

Let **Art** be the category of Artinian local R-algebras with residue field k. Fix an oriented orthogonal sheaf $\mathbb{E} := (E, \gamma, \delta)$ on $\operatorname{Spec} A/\mathfrak{m}A$. A deformation of $\mathbb{E}$ over $S \in \mathbf{Art}$ is an oriented orthogonal sheaf $\mathbb{E}_1 = (E_1, \gamma_1, \delta_1)$ on $\operatorname{Spec} A \otimes_R S / \operatorname{Spec} S$ together with an isomorphism $\varphi_1 : \mathbb{E}_1|_{\operatorname{Spec} A/\mathfrak{m}A} \simeq \mathbb{E}$. Two deformations $(\mathbb{E}_1; \varphi_1)$ and $(\mathbb{E}_2; \varphi_2)$ of $\mathbb{E}$ over S are said to be equivalent if there is an isomorphism $\theta : \mathbb{E}_1 \simeq \mathbb{E}_2$ such that $\varphi_2 \circ (\theta|_{\operatorname{Spec} A/\mathfrak{m}A}) = \varphi_1$.

Let $\mathcal{D}_{\mathbb{E}} : \mathbf{Art} \to \mathbf{Sets}$ be the functor that associates to S the set of equivalence classes of deformations of $\mathbb{E}$ over S.

We have an isomorphism $E \simeq (A/\mathfrak{m}A)^a \oplus (\widetilde{A/\mathfrak{m}A})^{n-a}$ as $A/\mathfrak{m}A$-modules. Let $E' \subset E$ be the free summand, and let E'' be its orthogonal complement. The composite

$$E' \hookrightarrow E \to E^\vee \to (E')^\vee$$

is an isomorphism. From this, it follows that E is a direct sum of E' and E'' and $\gamma' := \gamma|_{E' \otimes E'}$ is a nondegenerate symmetric bilinear form. The orientation δ factors as

$$\bigwedge^n E \xrightarrow{\mathrm{pr}} \bigwedge^a E' \otimes \bigwedge^{n-a} E'' \to A,$$

where pr is the projection. Choose an isomorphism $\delta' : \wedge^a E' \to A$ so that the diagram

$$
\begin{array}{ccc}
\wedge^a E' \otimes \wedge^a E' & \xrightarrow{\delta'\otimes\delta'} & A \otimes_A A \\
{\scriptstyle \bar{\wedge}^a \gamma'} \downarrow & & \downarrow {\scriptstyle \simeq} \\
A & = & A
\end{array}
\tag{3.11}
$$

commutes. Choose an A-homomorphism $\delta'' : \wedge^{n-a} E'' \to A$ so that

$$\delta = (\delta' \otimes \delta'') \circ \mathrm{pr}. \tag{3.12}$$

Then the diagram

$$
\begin{array}{ccc}
\wedge^{n-a} E'' \otimes \wedge^{n-a} E'' & \xrightarrow{\delta''\otimes\delta''} & A \otimes_A A \\
{\scriptstyle \bar{\wedge}^{n-a} \gamma''} \downarrow & & \downarrow {\scriptstyle \simeq} \\
A & = & A
\end{array}
\tag{3.13}
$$

commutes. It follows from (3.11) and (3.13) that $\mathbb{E}' := (E', \gamma', \delta')$ and $\mathbb{E}'' := (E'', \gamma'', \delta'')$ are oriented orthogonal sheaves, and from (3.12) that $\mathbb{E}$ is a direct sum of $\mathbb{E}'$ and $\mathbb{E}''$. By associating to a deformation of $\mathbb{E}''$ over S the direct sum of the deformation of $\mathbb{E}''$ and the trivial deformation of $\mathbb{E}'$ over S, we obtain a natural transformation $\Phi : \mathcal{D}_{\mathbb{E}''} \to \mathcal{D}_{\mathbb{E}}$. One can check that Φ is smooth and

$$\Phi(k[\epsilon]) : \mathcal{D}_{\mathbb{E}''}(k[\epsilon]) \to \mathcal{D}_{\mathbb{E}}(k[\epsilon])$$

is bijective. Thus the hull of $\mathcal{D}_{\mathbb{E}''}$ and that of $\mathcal{D}_{\mathbb{E}}$ are isomorphic.

In the rest of this subsection we assume that E has no direct summand.

LEMMA 3.3
$\mathbb{E}$ is isomorphic to $(E(O,O), \gamma_{(O,O)}, \delta_{(O,O)})$.

Proof
We may assume that $E = E(O,O)$ and $\gamma = \gamma_{(O,O)}$. Using the equations (3.1), (3.2), and (3.3), we know that $\Delta(:= \delta \circ \wedge^n \alpha)$ is given by $C_{[1,n];\emptyset} = \epsilon_1 x^{n/2}$, $C_{\emptyset;[1,n]} = \epsilon_2 y^{n/2}$ with $\epsilon_i = \pm 1$ and $C_{I;J} = 0$ for other I, J. Let D_i be the $(n \times n)$-matrix $\mathrm{diag}(\epsilon_i, 1, \dots, 1)$. Then the $(2n \times 2n)$-matrix $\left(\begin{smallmatrix} D_1 & O \\ O & D_2 \end{smallmatrix}\right)$ gives an isomorphism $(E(O,O), \gamma_{(O,O)}, \delta) \simeq (E(O,O), \gamma_{(O,O)}, \delta_{(O,O)})$. $\square$

Let us construct the hull of the deformation functor $\mathcal{D}_{\mathbb{E}}$. Consider $(n \times n)$-matrices $\mathbf{P} = (\mathbf{p}_{ij})$ and $\mathbf{Q} := {}^t\mathbf{P}$, where $\mathbf{p}_{ij}$'s are indeterminates. Let $\widetilde{U}$ be the residue ring of $R[[\mathbf{p}_{ij} \mid 1 \le i, j \le n]]$ by the ideal generated by the relation

$$\mathbf{PQ} = \mathbf{QP} = \pi \cdot I_n.$$

Then on $\mathrm{Spec}\, \widetilde{U}[[x,y]]/(xy - \pi)$, we have the orthogonal sheaf $(E(\mathbf{P}, \mathbf{Q}), \gamma_{(\mathbf{P},\mathbf{Q})})$, which is the versal deformation of the orthogonal sheaf $(E(O,O), \gamma_{(O,O)})$ (cf. [F, Theorem 3.7, Remark 3.8, Theorem 3.9]); that is, the natural transformation $h_{\widetilde{U}} \to \mathcal{F}$ is a hull (see [Sch, Definition 2.7]), where $\mathcal{F}$ is the deformation functor of $(E(O,O), \gamma_{(O,O)})$.

Let U be the residue ring of $\widetilde{U}$ by the ideal generated by the relations

$$\operatorname{sgn}\begin{pmatrix} J^c \cup J \\ [1,n] \end{pmatrix} \det \mathbf{Q}_{J^c \times I} = \operatorname{sgn}\begin{pmatrix} I \cup I^c \\ [1,n] \end{pmatrix} \det \mathbf{P}_{I^c \times J}$$

for $I \in [1,n]^{n/2}$ and $J \in [1,n]^{n/2}$. We have the oriented orthogonal sheaf $(E(\mathbf{P},\mathbf{Q}),$ $\gamma_{(\mathbf{P},\mathbf{Q})}, \delta_{(\mathbf{P},\mathbf{Q})})$ on $\operatorname{Spec} U[[x,y]]/(xy-\pi)$. We have a natural transformation $\Psi : h_U \to \mathcal{D}_{\mathbb{E}}$.

PROPOSITION 3.4

We have that Ψ *is a hull of* $\mathcal{D}_{\mathbb{E}}$.

Proof

For any $S \in \mathbf{Art}$, the map $h_{\widetilde{U}}(S) \to \mathcal{F}(S)$ is surjective (see [F, Theorem 3.7]). By this and Lemma 3.2, we know that the map $\mathcal{D}_{\mathbb{E}}(S) \to \mathcal{F}(S)$ forgetting the orientation is injective for all $S \in \mathbf{Art}$, and the diagram

$$\begin{array}{ccc} h_U(S) & \longrightarrow & \mathcal{D}_{\mathbb{E}}(S) \\ \big\downarrow & & \big\downarrow \\ h_{\widetilde{U}}(S) & \longrightarrow & \mathcal{F}(S) \end{array}$$

is Cartesian. From this, using the fact that $h_{\widetilde{U}} \to \mathcal{F}$ is a hull (see [F]), it follows that $\Psi : h_U \to \mathcal{D}_{\mathbb{E}}$ is a hull. □

4. Moduli stack of oriented orthogonal sheaves

Let $(R,\mathfrak{m})$ be a complete Noetherian local ring with residue field $k =: R/\mathfrak{m}$ algebraically closed. Let $C \to B := \operatorname{Spec} R$ be a flat projective morphism whose geometric fibers are connected nodal curves of arithmetic genus g. Put $B_0 :=$ $\operatorname{Spec} R/\mathfrak{m}$. For simplicity, we assume that the closed fiber C_0 is irreducible and has only one node Q. Fix an isomorphism $\widehat{\mathcal{O}_{C,Q}} \simeq R[[x,y]]/(xy-\pi) =: A$ of R-algebras, where $\pi \in \mathfrak{m}$.

DEFINITION 4.1

The moduli stack $\bar{M}_n(C)$ of oriented orthogonal sheaves is the stack such that for an affine B-scheme T, objects of the groupoid $\bar{M}_n(C)(T)$ are oriented orthogonal sheaves of rank n on C_T/T.

REMARK 4.2

The moduli stack $\bar{M}_n(C)$ is not connected.

If $(\mathcal{E},\gamma,\delta)$ is an oriented orthogonal sheaf of rank n on C_0, then

$$\mathcal{E} \otimes \widehat{\mathcal{O}_{C_0,Q}} \simeq \big(k[[x,y]]/(xy)\big)^{n-2a} \oplus (k[[x]] \oplus k[[y]])^{2a}$$

for some $a \geq 0$. Such an oriented orthogonal sheaf is said to be of type a. We write $\bar{M}_n(C_0)$ for $\bar{M}_n(C) \times_B B_0$. We denote by $\bar{M}_n^{\leq a}(C_0)$ the open substack of $\bar{M}_n(C_0)$ parameterizing oriented orthogonal sheaves of type $\leq a$.

To see that the stack $\bar{M}_n(C)$ is an algebraic stack, we construct an atlas of $\bar{M}_n(C)$ using the following well-known result (cf. [AK, (1.1)]).

LEMMA 4.3

Let $f : X \to Y$ be a projective morphism of schemes. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that $\mathcal{G}$ is flat over Y. Then the functor that associates to a Y-scheme S the set $\operatorname{Hom}_{X_S}(\mathcal{F}_S, \mathcal{G}_S)$ is representable by a scheme which is affine and of finite type over Y.

We denote by $\mathbb{H}(\mathcal{F}, \mathcal{G})$ the Y-scheme representing the functor in the above lemma. Fix a B-very ample line bundle $\mathcal{O}_C(1)$ on C with degree d on fibers. For $N > 0$, let $\mathfrak{Q}_N$ be Grothendieck's quote scheme parameterizing quotients of the sheaf $\mathcal{O}_C(-N)^{\oplus n(Nd+1-g)}$ of rank n and degree 0. Let $\mathrm{pr}_C^* \mathcal{O}_C(-N)^{\oplus n(Nd+1-g)} \to \mathcal{E}$ be the universal quotient over $C \times_B \mathfrak{Q}_N$. Let $\mathfrak{Q}_N^o \subset \mathfrak{Q}_N$ be the open subscheme consisting of points t such that $\mathrm{H}^1(C_t, \mathcal{E}_t(N)) = 0$, the natural map $\mathrm{H}^0(C_t, \mathcal{O})^{\oplus n(Nd+1-g)} \to \mathrm{H}^0(C_t, \mathcal{E}_t(N))$ is an isomorphism, and $\mathcal{E}_t$ is torsion-free. Put $\mathcal{E}^o := \mathcal{E}|_{C \times \mathfrak{Q}_N^o}$, and put $\mathbb{H} := \mathbb{H}(\mathcal{E}^o \otimes \mathcal{E}^o, \mathcal{O}) \times_{\mathfrak{Q}_N^o} \mathbb{H}(\wedge^n \mathcal{E}^o, \mathcal{O})$. Denote by $\tilde{\mathcal{E}}$ the pullback of $\mathcal{E}^o$ to $C \times_B \mathbb{H}$. Over $C \times_B \mathbb{H}$, we have the universal bilinear form $\gamma : \tilde{\mathcal{E}} \otimes \tilde{E} \to \mathcal{O}_{C \times_B \mathbb{H}}$ and the universal morphism $\delta : \wedge^n \tilde{\mathcal{E}} \to \mathcal{O}_{C \times_B \mathbb{H}}$. Let $A_N \subset \mathbb{H}$ be the maximum closed subscheme such that the restriction of the triple $(\tilde{\mathcal{E}}, \gamma, \delta)$ to $C \times_B A_N$ is an oriented orthogonal sheaf on $C \times_B A_N / A_N$. Then $\bigsqcup_{N>0} A_N \to \bar{M}_n(C)$ is an atlas of $\bar{M}_n(C)$.

4.1. Local structure

As in Section 3.3, we let **Art** be the category of Artinian local R-algebras with residue field k. Fix an oriented orthogonal sheaf $\overline{\mathcal{E}} = (\mathcal{E}, \gamma, \delta)$ on C_0.

For $S \in \mathbf{Art}$, an object $\overline{\mathcal{E}_1}$ of $\bar{M}_n(C)(\operatorname{Spec} S)$ together with an isomorphism $\overline{\mathcal{E}_1}|_{C_0} \simeq \overline{\mathcal{E}}$ is called a deformation of $\overline{\mathcal{E}}$ over S. Let $\mathcal{D}_{\overline{\mathcal{E}}}$ be the functor that associates to $S \in \mathbf{Art}$ the set of equivalence classes of deformations of $\overline{\mathcal{E}}$ over S. By taking the completion of $\overline{\mathcal{E}}$ at Q, we obtain an oriented orthogonal sheaf $\mathbb{E}$ on $\operatorname{Spec} A/\mathfrak{m}A$. By associating to a deformation of $\overline{\mathcal{E}}$ over S its completion at Q, we have a natural transformation $f : \mathcal{D}_{\overline{\mathcal{E}}} \to \mathcal{D}_{\mathbb{E}}$. By [BL], f is smooth. The hull of $\mathcal{D}_{\mathbb{E}}$ has already been determined in Section 3.3. For us, the following result in the case of type 1 is important.

PROPOSITION 4.4

Assume that the above $\overline{\mathcal{E}}$ is of type 1. For an atlas $V \to \bar{M}_n(C)$, let $p \in V$ be a point over $[\overline{\mathcal{E}}] \in \bar{M}_n(C)$. Then there is an isomorphism

$$\widehat{\mathcal{O}_{V,p}} \simeq R[[x, y, z_1, z_2, \ldots]]/(xy - \pi).$$

Proof
Since $\overline{\mathcal{E}}$ is of type 1, by Proposition 3.4, the hull of the deformation functor of $\mathcal{D}_{\mathbb{E}}$ is $\operatorname{Spec} R[[\mathbf{p}_{ij} \mid 1 \le i, j \le 2]]/J$, where the ideal J is generated by the relations

$$ {}^t\mathbf{P}\mathbf{P} = \mathbf{P}^t\mathbf{P} = \pi \cdot I_2, \qquad \mathbf{p}_{11} = \mathbf{p}_{22}, \qquad \text{and} \qquad \mathbf{p}_{12} = -\mathbf{p}_{21}. $$

These relations are equivalent to

$$ \mathbf{p}_{11}^2 + \mathbf{p}_{12}^2 = \pi, \qquad \mathbf{p}_{11} = \mathbf{p}_{22}, \qquad \text{and} \qquad \mathbf{p}_{12} = -\mathbf{p}_{21}. $$

Hence $R[[\mathbf{p}_{ij} \mid 1 \le i, j \le 2]]/J \simeq R[[x,y]]/(xy - \pi)$. □

4.2. Desingularization of $\bar{M}_n(C_0)$

Let $\mathfrak{n} : \widetilde{C_0} \to C_0$ be the normalization of C_0. Put $\tilde{g} := g - 1$, the genus of $\widetilde{C_0}$. Put $\{P_1, P_2\} := \mathfrak{n}^{-1}(Q)$. Let $M_n(\widetilde{C_0})$ be the moduli stack of oriented orthogonal bundles of rank n on $\widetilde{C_0}$. Let $(\mathcal{F}^u, \gamma^u, \delta^u)$ be the universal family of oriented orthogonal bundles over $\widetilde{C_0} \times M_n(\widetilde{C_0})$. For $i = 1, 2$, put

$$ \mathcal{F}_i^u := \mathcal{F}^u|_{\{P_i\} \times M_n(\widetilde{C_0})}, \qquad \gamma_i^u := \gamma^u|_{\mathcal{F}_i^u \otimes \mathcal{F}_i^u}, \qquad \text{and} \qquad \delta_i^u := \delta^u|_{\wedge^n \mathcal{F}_i^u}. $$

Then $(\mathcal{F}_i^u, \gamma_i^u, \delta_i^u)$ is an oriented orthogonal bundle of rank n on $M_n(\widetilde{C_0})$. Consider the orthogonal Grassmannian bundle

$$ \tau : \mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u) \to M_n(\widetilde{C_0}). $$

The stack $\mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)$ parameterizes rank n oriented orthogonal bundles (F, γ_F, δ_F) on $\widetilde{C_0}$ plus n-dimensional isotropic subspaces $U \subset F|_{P_1} \oplus F|_{P_2}$. It is a disjoint union of $\mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)}$ and $\mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(-)}$. Given $(F, \gamma_F, \delta_F; U \subset F|_{P_1} \oplus F|_{P_2}) \in \mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)}$, if we put

$$ E := \operatorname{Ker}\bigl(\mathfrak{n}_*(F) \to \mathfrak{n}_*(F)|_Q = F|_{P_1} \oplus F|_{P_2} \to (F|_{P_1} \oplus F|_{P_2})/U\bigr), $$

then there are a symmetric bilinear form $\gamma_E : E \otimes E \to \mathcal{O}_{C_0}$ and an $\mathcal{O}_{C_0}$-linear map $\delta_E : \wedge^n E \to \mathcal{O}_{C_0}$ such that the following diagrams commute:

$$ \begin{array}{ccc} \mathfrak{n}_*(F) \otimes \mathfrak{n}_*(F) & \longrightarrow & \mathfrak{n}_*(\mathcal{O}_{\widetilde{C_0}}) \\ \uparrow & & \uparrow \\ E \otimes E & \xrightarrow{\gamma_E} & \mathcal{O}_{C_0} \end{array} \qquad \begin{array}{ccc} \mathfrak{n}_*(\wedge^n F) & \longrightarrow & \mathfrak{n}_*(\mathcal{O}_{\widetilde{C_0}}) \\ \uparrow & & \uparrow \\ \wedge^n E & \xrightarrow{\delta_E} & \mathcal{O}_{C_0} \end{array} $$

You can check easily that (E, γ_E, δ_E) is an oriented orthogonal sheaf of rank n on C_0. By associating to $(F, \gamma_F, \delta_F; U \subset F|_{P_1} \oplus F|_{P_2})$ the oriented orthogonal sheaf (E, γ_E, δ_E) on C_0, we have a morphism

$$ \rho : \mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)} \to \bar{M}_n(C_0) $$

of stacks.

PROPOSITION 4.5
For any smooth morphism $\alpha : V \to \bar{M}_n(C_0)$ with V a scheme, the product $\mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)} \times_{\bar{M}_n(C_0)} V$ is a smooth variety and the projection

$$\mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)} \times_{\bar{M}_n(C_0)} V \to V$$

is a proper birational morphism which is an isomorphism over $V^0 := \alpha^{-1}(\bar{M}_n^{\leq 0}(C_0))$.

For the proof of the proposition, we prepare lemmas.

LEMMA 4.6

Fix integers $m \geq 0$ *and* d. *Let* V *be a* B_0*-scheme, and let* $\mathcal{H}$ *be a coherent sheaf on* $\widetilde{C_0} \times V$. *Then the functor* $\mathcal{R}_{\mathcal{H}}$ *that associates to a* V*-scheme* S *the set*

$$\left\{ (\mathcal{H}_S \xrightarrow{f} \mathcal{F}) \;\middle|\; \begin{array}{c} \mathcal{F}\text{: rank } m \text{ vector bundle on } \widetilde{C_0} \times S, \\ f|_{\widetilde{C_0}\times s} \text{ is generically surjective,} \\ \text{and } \deg(\mathcal{F}|_{\widetilde{C_0}\times s}) = d \text{ for any } s \in S \end{array} \right\} \Big/ \sim \tag{4.1}$$

is representable by a proper scheme $R_{\mathcal{H}}$ *over* V. *Here* $(\mathcal{H}_S \xrightarrow{f} \mathcal{F})$ *and* $(\mathcal{H}_S \xrightarrow{f'} \mathcal{F}')$ *are defined to be equivalent* $\sim$ *if there is an isomorphism* $h : \mathcal{F} \to \mathcal{F}'$ *such that* $h \circ f = f'$.

Proof

We may assume that V is affine.

Case (1): $\mathcal{H}$ *is a vector bundle.*

Put $e := \operatorname{rank} \mathcal{H}$ and $b := \chi(\mathcal{H}^\vee|_{\widetilde{C_0}\times s})$. Let $(\mathcal{H}_S \xrightarrow{f} \mathcal{F})$ be as in (4.1). If we let $\mathcal{Q} := \operatorname{Coker}(f^\vee : \mathcal{F}^\vee \to \mathcal{H}_S^\vee)$, then $\mathcal{Q}$ is flat over S with numerical invariants

$$\chi(\mathcal{Q}|_{\widetilde{C_0}\times s}) = b + d - m(1-\tilde{g}) \qquad \text{and} \qquad \operatorname{rank} \mathcal{Q} = e - m. \tag{4.2}$$

Conversely, given a surjection $\mathcal{H}_S^\vee \xrightarrow{h} \mathcal{Q}$ with numerical invariants (4.2), if we let $\mathcal{F} := (\operatorname{Ker} h)^\vee$, then $(\mathcal{H}_S \xrightarrow{h^\vee} \mathcal{F})$ satisfies the condition in (4.1). So the functor $\mathcal{R}_{\mathcal{H}}$ is isomorphic to Grothendieck's quot functor of quotients of $\mathcal{H}^\vee$, which is representable by a projective V-scheme.

Case (2): *General case.*

Take a resolution $\mathcal{L}_1 \xrightarrow{h} \mathcal{L}_0 \to \mathcal{H} \to 0$ by locally free sheaves. Let $\mathcal{L}_0 \otimes \mathcal{O}_{R_{\mathcal{L}_0}} \xrightarrow{f} \mathcal{F}$ be the universal family parameterized by $R_{\mathcal{L}_0}$. The composite $\mathcal{L}_1 \otimes \mathcal{O}_{R_{\mathcal{L}_0}} \xrightarrow{f\circ(h\otimes \mathrm{id})} \mathcal{F}$ gives a morphism $\sigma : R_{\mathcal{L}_0} \to \mathbb{H}(\mathcal{L}_1 \otimes \mathcal{O}_{R_{\mathcal{L}_0}}, \mathcal{F})$. Let $Z \subset \mathbb{H}(\mathcal{L}_1 \otimes \mathcal{O}_{R_{\mathcal{L}_0}}, \mathcal{F})$ be the closed subscheme parameterizing zero morphisms. Then the functor $\mathcal{R}_{\mathcal{H}}$ is representable by $\sigma^{-1}(Z)$. □

LEMMA 4.7

Let V *be a* B_0*-scheme, let* $\mathcal{H}$ *be a coherent sheaf on* $\widetilde{C_0} \times V$, *and let* $\mathcal{F}$, $\mathcal{L}$ *be vector bundles on* $\widetilde{C_0} \times V$. *Let* $f : \mathcal{H} \to \mathcal{F}$ *be a morphism such that there is a closed subset* $Z \subset \widetilde{C_0} \times V$ *not containing a fiber of* pr_V *with the condition that* f *is an isomorphism over* $\widetilde{C_0} \times V \setminus Z$. *Let* $\varphi : \mathbb{H}(\mathcal{F}, \mathcal{L}) \to \mathbb{H}(\mathcal{H}, \mathcal{L})$ *be the morphism induced by the composition with* f. *Then* φ *is a closed immersion.*

Proof
Since f is an isomorphism over $\widetilde{C_0} \times V \setminus Z$, for any V-scheme S, the map of S-valued points

$$\varphi(S) : \mathbb{H}(\mathcal{F}, \mathcal{L})(S) \to \mathbb{H}(\mathcal{H}, \mathcal{L})(S)$$

is injective. It remains to show that φ is proper. Let $S = \operatorname{Spec} R$ with R a discrete valuation ring over V, and let $\iota : \eta := \operatorname{Spec} K \to S$ be the open immersion, where K is the fractional field of R. Suppose that we are given morphisms $g : \mathcal{H}_S \to \mathcal{L}_S$ on $\widetilde{C_0} \times S$ and $h : \mathcal{F}_\eta \to \mathcal{L}_\eta$ on $\widetilde{C_0} \times \eta$ such that $h \circ f_\eta = g_\eta$. Let l be the composite of morphisms

$$\mathcal{F}_S \to (\mathrm{id}_{\widetilde{C_0}} \times \iota)_*(\mathcal{F}_\eta) \xrightarrow{(\mathrm{id}_{\widetilde{C_0}} \times \iota)_* h} (\mathrm{id}_{\widetilde{C_0}} \times \iota)_*(\mathcal{L}_\eta).$$

Since $\mathcal{F}_S$ and $\mathcal{E}_S$ are isomorphic over $\widetilde{C_0} \times S \setminus (Z \times_V S)$, for any section $\sigma \in \mathcal{F}_S$, the section $l(\sigma)$, regarded as a rational section of $\mathcal{L}_S$, does not have poles along the closed fiber. So $\operatorname{Im} l \subset \mathcal{L}_S$. This means that the valuative criterion of properness holds for φ. $\square$

Proof of Proposition 4.5
Let $\overline{\mathcal{E}} = (\mathcal{E}, \gamma_{\mathcal{E}}, \delta_{\mathcal{E}})$ be the object of the groupoid $\bar{M}_n(C_0)(V)$ that determines the morphism $\alpha : V \to \bar{M}_n(C_0)$. To prove that $\mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)} \times_{\bar{M}_n(C_0)} V$ is a scheme, we need to show that the functor $\mathcal{A}$ that associates to a V-scheme S the set of tuples

$$\left(\overline{\mathcal{F}} = (\mathcal{F}, \gamma_{\mathcal{F}}, \delta_{\mathcal{F}}, \mathcal{U} \subset \mathcal{F}|_{\{P_1\} \times S} \oplus \mathcal{F}|_{\{P_2\} \times S}); \xi\right) \tag{4.3}$$

is representable by a scheme, where $\overline{\mathcal{F}} \in \mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)}(S)$ and ξ is an isomorphism in $\bar{M}_n(C_0)(V)$ between the image of $\overline{\mathcal{F}}$ by ρ and $\overline{\mathcal{E}}$. Put $\mathcal{H} := (\mathfrak{n} \times \mathrm{id}_V)^* \mathcal{E}$. Giving the tuple (4.3) is equivalent to giving a tuple $(\overline{\mathcal{F}}, \theta)$, where $\overline{\mathcal{F}} \in \mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)}(S)$ and θ is a morphism $\mathcal{H}_S \to \mathcal{F}$ such that the composite of morphisms

$$\mathcal{E}_S \xrightarrow{\theta^{\mathrm{ad}}} (\mathfrak{n} \times \mathrm{id}_S)_* \mathcal{F} \to (Q, \mathrm{id}_S)_* \left(\frac{\mathcal{F}|_{\{P_1\} \times S} \oplus \mathcal{F}|_{\{P_2\} \times S}}{\mathcal{U}} \right) \tag{4.4}$$

is zero, and the following diagrams commute:

$$\begin{array}{ccccccc} \mathcal{F} \otimes \mathcal{F} & \xrightarrow{\gamma_{\mathcal{F}}} & \mathcal{O}_{\widetilde{C_0} \times S} & \quad & \wedge^n \mathcal{F} & \xrightarrow{\delta_{\mathcal{F}}} & \mathcal{O}_{\widetilde{C_0} \times S} \\ \uparrow & & \| & & \uparrow & & \| \\ \mathcal{H}_S \otimes \mathcal{H}_S & \longrightarrow & \mathcal{O}_{\widetilde{C_0} \times S}, & & \wedge^n \mathcal{H}_S & \longrightarrow & \mathcal{O}_{\widetilde{C_0} \times S} \end{array} \tag{4.5}$$

where the bottom arrows in the above diagrams are induced by $\gamma_{\mathcal{E}}$ and $\delta_{\mathcal{E}}$, respectively.

Let R be the V-scheme $R_{\mathcal{H}}$ in Lemma 4.6 for $m = n$ and $d = 0$. Let $\mathcal{H}_R \to \mathcal{G}$ be the universal morphism on $\widetilde{C_0} \times R$. The bilinear form $\mathcal{H}_R \otimes \mathcal{H}_R \to \mathcal{O}_{\widetilde{C_0} \times R}$ induced by $\gamma_{\mathcal{E}}$ gives a section $R \xrightarrow{\alpha} \mathbb{H}(\mathcal{H}_R \otimes \mathcal{H}_R, \mathcal{O}_{\widetilde{C_0} \times R})$ of the R-scheme $\mathbb{H}(\mathcal{H}_R \otimes$

$\mathcal{H}_R, \mathcal{O}_{\widetilde{C_0}\times R})$. By Lemma 4.7, we have a closed subscheme $\mathbb{H}(\mathcal{G}\otimes\mathcal{G}, \mathcal{O}_{\widetilde{C_0}\times R}) \subset \mathbb{H}(\mathcal{H}_R\otimes\mathcal{H}_R, \mathcal{O}_{\widetilde{C_0}\times R})$. Put $R_1 := \alpha^{-1}(\mathbb{H}(\mathcal{G}\otimes\mathcal{G}, \mathcal{O}_{\widetilde{C_0}\times R}))$. Similarly the morphism $\wedge^n\mathcal{H}_R \to \mathcal{O}_{\widetilde{C_0}\times R}$ induced by $\delta_{\mathcal{E}}$ gives a section $R \xrightarrow{\beta} \mathbb{H}(\wedge^n\mathcal{H}_R, \mathcal{O}_{\widetilde{C_0}\times R})$ of the R-scheme $\mathbb{H}(\wedge^n\mathcal{H}_R, \mathcal{O}_{\widetilde{C_0}\times R})$. Put $R_2 := \beta^{-1}(\mathbb{H}(\wedge^n\mathcal{F}, \mathcal{O}_{\widetilde{C_0}\times R}))$, where $\mathbb{H}(\wedge^n\mathcal{F}, \mathcal{O}_{\widetilde{C_0}\times R})$ is a closed subscheme of $\mathbb{H}(\wedge^n\mathcal{H}_R, \mathcal{O}_{\widetilde{C_0}\times R})$, again by Lemma 4.7. Put $R' := R_1\cap R_2$. Then, by the definition of R', $\mathcal{G}_{R'}$ has a bilinear form $\gamma_{\mathcal{G}_{R'}}$ and a morphism $\delta_{\mathcal{G}_{R'}} : \wedge^n\mathcal{G}_{R'} \to \mathcal{O}_{\widetilde{C_0}\times R'}$. Since the diagrams (4.5), in which $\mathcal{F}$ and $\mathcal{H}_S$ are replaced by $\mathcal{G}_{R'}$ and $\mathcal{H}_{R'}$, respectively, are commutative, the triple $(\mathcal{G}_{R'}, \gamma_{\mathcal{G}_{R'}}, \delta_{\mathcal{G}_{R'}})$ is an oriented orthogonal bundle on $\widetilde{C_0}\times R'/R'$. Put $\mathcal{G}_i := \mathcal{G}_{R'}|_{\{P_i\}\times R'}$ $(i=1,2)$. Consider the orthogonal Grassmannnian bundle

$$G := \mathrm{OG}_n(\mathcal{G}_1\oplus\mathcal{G}_2)_{(+)} \to R'.$$

Let $\mathcal{U}\subset(\mathcal{G}_1\oplus\mathcal{G}_2)_G$ be the universal isotropic subbundle. The composite of morphisms of $\mathcal{O}_{C_0\times G}$-modules

$$\mathcal{E}_G \to (\mathfrak{n}\times\mathrm{id}_G)_*\mathcal{G}_G \to (Q,\mathrm{id}_G)_*\big((\mathcal{G}_1\oplus\mathcal{G}_2)_G/\mathcal{U}\big) := \mathcal{K}$$

gives a section $G \xrightarrow{\lambda} \mathbb{H}(\mathcal{E}_G, \mathcal{K})$. Then the inverse image by λ of the closed subscheme of $\mathbb{H}(\mathcal{E}_G, \mathcal{K})$ parameterizing zero morphisms represents the functor $\mathcal{A}$. By construction, it is proper over V. It is smooth because it is smooth over the smooth stack $\mathrm{OG}_n(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}$. Since giving an oriented orthogonal bundle on C_0 is equivalent to giving an oriented orthogonal bundle on $\widetilde{C_0}$ plus the gluing data between the fibers over P_1 and P_2 compatible with the bilinear form and orientation, the projection to V is an isomorphism over V^0. □

4.3. The involution ι

Consider the open substack $\mathrm{OG}_n^{\le 1}(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}$ of $\mathrm{OG}_n(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}$ (see Lemma 2.1 for the notation $\mathrm{OG}_n^{\le a}(-)$). We define the closed substack $\mathrm{OG}_n^{=1}(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}$ of $\mathrm{OG}_n^{\le 1}(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}$ as follows. For a B_0-scheme S, an object

$$(\mathcal{F}, \gamma_F, \delta_F; \mathcal{U}\subset\mathcal{F}|_{\{P_1\}\times S}\oplus\mathcal{F}|_{\{P_2\}\times S}) \tag{4.6}$$

of $\mathrm{OG}_n^{\le 1}(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}(S)$ is in $\mathrm{OG}_n^{=1}(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}(S)$ if and only if the morphism $\wedge^n\mathcal{U}\to\wedge^n\mathcal{F}|_{\{P_1\}\times S}$ is zero. (Here note that $\wedge^n\mathcal{U}\to\wedge^n\mathcal{F}|_{\{P_1\}\times S}$ is zero if and only if $\wedge^n\mathcal{U}\to\wedge^n\mathcal{F}|_{\{P_2\}\times S}$ is as well.)

For short, we write $\mathrm{OG}_{(+)}$, $\mathrm{OG}_{(+)}^{\le 1}$, and $\mathrm{OG}_{(+)}^{=1}$ for $\mathrm{OG}_n(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}$, $\mathrm{OG}_n^{\le 1}(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}$, and $\mathrm{OG}_n^{=1}(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}$, respectively.

Suppose we are given an object (4.6) in $\mathrm{OG}_{(+)}^{=1}(S)$. Then $\mathcal{B}_1 := \mathcal{U}\cap(\mathcal{F}|_{\{P_1\}\times S}\oplus 0)$ and $\mathcal{B}_2 := \mathcal{U}\cap(0\oplus\mathcal{F}|_{\{P_2\}\times S})$ are isotropic line subbundles of $\mathcal{F}|_{\{P_1\}\times S}$ and $\mathcal{F}|_{\{P_2\}\times S}$, respectively. We have $\mathcal{U}\subset\mathcal{B}_1^\perp\oplus\mathcal{B}_2^\perp$ and

$$\mathcal{V} := \frac{\mathcal{U}}{\mathcal{B}_1\oplus\mathcal{B}_2} \subset \frac{B_1^\perp}{B_1}\oplus\frac{B_2^\perp}{B_2}$$

is in $\mathrm{OG}_{n-2}\big(\frac{\mathcal{B}_1^\perp}{B_1}\oplus\frac{\mathcal{B}_2^\perp}{B_2}\big)_{(+)}(S)$. Here the orientations of $\frac{\mathcal{B}_1^\perp}{B_1}$ and $\frac{\mathcal{B}_2^\perp}{B_2}$ are given as in Lemma 2.1(2). So we obtain a tuple

$$(4.7) \qquad \overline{\mathcal{F}} := \Big(\mathcal{F}, \gamma_{\mathcal{F}}, \delta_{\mathcal{F}}, \mathcal{B}_i \subset \mathcal{F}|_{\{P_i\}\times S}\ (i=1,2), \mathcal{V} \subset \frac{\mathcal{B}_1^{\perp}}{\mathcal{B}_1} \oplus \frac{\mathcal{B}_2^{\perp}}{\mathcal{B}_2}\Big),$$

where $\mathcal{B}_i$ is an isotropic line subbundle and $\big[\mathcal{V} \subset \frac{\mathcal{B}_1^{\perp}}{\mathcal{B}_1} \oplus \frac{\mathcal{B}_2^{\perp}}{\mathcal{B}_2}\big] \in \mathrm{OG}_{n-2}\big(\frac{\mathcal{B}_1^{\perp}}{\mathcal{B}_1} \oplus \frac{\mathcal{B}_2^{\perp}}{\mathcal{B}_2}\big)_{(+)}(S)$. Conversely, if we are given a tuple (4.7), then by reversing the above procedure, we obtain a tuple (4.6). So giving an object of $\mathrm{OG}^{=1}_{(+)}(S)$ is equivalent to giving a tuple (4.7).

Now let $\overline{\mathcal{F}}$ be the tuple (4.7). Let $(\mathcal{F}^{\iota}, \gamma_{\mathcal{F}^{\iota}}, \delta_{\mathcal{F}^{\iota}}; \mathcal{B}_i^{\iota} \subset \mathcal{F}^{\iota}|_{\{P_i\}\times S}\ (i=1,2))$ be the ι-transform of $(\mathcal{F}, \gamma_{\mathcal{F}}, \delta_{\mathcal{F}}; \mathcal{B}_i \subset \mathcal{F}|_{\{P_i\}\times S}\ (i=1,2))$ over $\{P_1, P_2\}$. By (2.6), we have an isomorphism $\frac{\mathcal{B}_1^{\perp}}{\mathcal{B}_1} \oplus \frac{\mathcal{B}_2^{\perp}}{\mathcal{B}_2} \simeq \frac{\mathcal{B}_1^{\iota\perp}}{\mathcal{B}_1^{\iota}} \oplus \frac{\mathcal{B}_2^{\iota\perp}}{\mathcal{B}_2^{\iota}}$. Let $\mathcal{V}^{\iota}$ be the image of the isotropic subbundle $\mathcal{V}$ of $\frac{\mathcal{B}_1^{\perp}}{\mathcal{B}_1} \oplus \frac{\mathcal{B}_2^{\perp}}{\mathcal{B}_2}$ in $\frac{\mathcal{B}_1^{\iota\perp}}{\mathcal{B}_1^{\iota}} \oplus \frac{\mathcal{B}_2^{\iota\perp}}{\mathcal{B}_2^{\iota}}$ by this isomorphism. After all, from $\overline{\mathcal{F}}$, we obtained a tuple

$$(4.8) \qquad \overline{\mathcal{F}}^{\iota} := \Big(\mathcal{F}^{\iota}, \gamma_{\mathcal{F}^{\iota}}, \delta_{\mathcal{F}^{\iota}}, \mathcal{B}_i^{\iota} \subset \mathcal{F}^{\iota}|_{\{P_i\}\times S}\ (i=1,2), \mathcal{V}^{\iota} \subset \frac{\mathcal{B}_1^{\iota\perp}}{\mathcal{B}_1^{\iota}} \oplus \frac{\mathcal{B}_2^{\iota\perp}}{\mathcal{B}_2^{\iota}}\Big).$$

The assignment $\overline{\mathcal{F}} \mapsto \overline{\mathcal{F}}^{\iota}$ defines a morphism of stacks $\mathrm{OG}^{=1}_{(+)} \to \mathrm{OG}^{=1}_{(+)}$, which we denote also by ι. By construction, there is a natural isomorphism $(\overline{\mathcal{F}}^{\iota})^{\iota} \simeq \overline{\mathcal{F}}$. So ι is an involution on $\mathrm{OG}^{=1}_{(+)}$.

4.4. Description of $\bar{M}_n^{\leq 1}(C_0)$ by gluing

If $\overline{F} = (F, \gamma_F, \delta_F; U \subset F|_{P_1} \oplus F|_{P_2}) \in \mathrm{OG}_n^{=a}(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)}$, then $\rho(\overline{F}) \in \bar{M}_n(C_0)$ is of type a. Hence $\rho^{-1}(\bar{M}^{\leq a}(C_0)) = \mathrm{OG}_n^{\leq a}(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(+)}$. Let $\rho_{\leq 1} : \mathrm{OG}^{\leq 1}_{(+)} \to \bar{M}_n^{\leq 1}(C_0)$ and $\rho_1 : \mathrm{OG}^{=1}_{(+)} \to \bar{M}_n^{\leq 1}(C_0)$ be restrictions of ρ. The group $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{OG}^{=1}_{(+)}$ by the involution ι, and on $\bar{M}_n^{\leq 1}(C_0)$ trivially. The morphism ρ_1 is equivariant with respect to this action.

LEMMA 4.8

There is a natural isomorphism between the morphisms $\rho_1 \circ \iota$ and ρ_1 from $\mathrm{OG}^{\leq 1}_{(+)}$ to $\bar{M}_n^{\leq 1}(C_0)$.

Proof

Let $\overline{\mathcal{F}}$ be an object $\mathrm{OG}^{\leq 1}_{(+)}(S)$. We shall show that the two objects $\rho(\overline{\mathcal{F}})$ and $\rho(\overline{\mathcal{F}}^{\iota})$ in $\bar{M}_n^{\leq 1}(C_0)(S)$ are naturally isomorphic. If $\overline{\mathcal{F}}$ is expressed in the form (4.6), then by the definition of ρ,

$$\rho(\overline{\mathcal{F}}) = \bigg(\mathcal{E} := \mathrm{Ker}\Big((\mathfrak{n} \times \mathrm{id}_S)_* \mathcal{F} \to (Q, \mathrm{id}_S)_* \frac{\oplus_{i=1}^2 \mathcal{F}|_{\{P_i\}\times S}}{\mathcal{U}}\Big), \gamma_{\mathcal{E}}, \delta_{\mathcal{E}}\bigg),$$

where $\gamma_{\mathcal{E}}$ and $\delta_{\mathcal{E}}$ are induced from $\gamma_{\mathcal{F}}$ and $\delta_{\mathcal{F}}$. If we express $\overline{\mathcal{F}}$ in the equivalent form (4.7), then

$$\mathcal{E} \simeq \mathrm{Ker}\Big((\mathfrak{n} \times \mathrm{id}_S)_* \mathcal{F}^{\flat} \to (Q, \mathrm{id}_S)_* \frac{\bigoplus_{i=1}^2 \mathcal{F}^{\flat}|_{\{P_i\}\times S}/(\mathcal{F}^{\flat}|_{\{P_i\}\times S})^{\perp}}{\mathcal{V}}\Big),$$

where $\mathcal{V}$ is considered as a subbundle of $\bigoplus_{i=1}^{2} \mathcal{F}^\flat|_{\{P_i\}\times S}/(\mathcal{F}^\flat|_{\{P_i\}\times S})^\perp$ through the natural isomorphism (cf. (2.5)). By the definition of the involution ι, $\mathcal{F}^\flat = (\mathcal{F}^\iota)^\flat$, and under the natural isomorphisms

$$\frac{\mathcal{B}_1^\perp}{\mathcal{B}_1} \oplus \frac{\mathcal{B}_2^\perp}{\mathcal{B}_2} \simeq \frac{\mathcal{F}^\flat|_{\{P_1\}\times S}}{(\mathcal{F}^\flat|_{\{P_1\}\times S})^\perp} \oplus \frac{\mathcal{F}^\flat|_{\{P_2\}\times S}}{(\mathcal{F}^\flat|_{\{P_2\}\times S})^\perp} \simeq \frac{\mathcal{B}_1^{\iota\perp}}{\mathcal{B}_1^\iota} \oplus \frac{\mathcal{B}_2^{\iota\perp}}{\mathcal{B}_2^\iota},$$

$\mathcal{V}$ and $\mathcal{V}^\iota$ correspond. This show that there is a natural isomorphism of $\mathcal{O}_{C_0\times S}$-modules between $\rho(\overline{\mathcal{F}})$ and $\rho(\overline{\mathcal{F}}^\iota)$. This isomorphism is compatible with bilinear forms and orientations because so it is over $C_0 \times S \setminus \{Q\} \times S$. □

LEMMA 4.9

For $\overline{E} = (E, \gamma_E, \delta_E) \in \bar{M}_n^{\leq 1}(C_0)$ *of type* 1*, if we let* $\operatorname{Spec} k \to \bar{M}_n^{\leq 1}(C_0)$ *be the morphism determined by* $\overline{E}$*, then* $\mathrm{OG}_{(+)}^{=1} \times_{\bar{M}_n^{\leq 1}(C_0)} \operatorname{Spec} k$ *is a scheme consisting of two points. Moreover, these points are interchanged by the involution* ι*.*

Proof

Take $\overline{F} = (F, \gamma_F, \delta_F, \ldots) \in \mathrm{OG}_{(+)}^{=1}$ such that $\rho_1(\overline{F}) = \overline{E}$. Then we have a morphism $\mathfrak{n}^*E \to F^\flat \subset F$. Put $H := \mathfrak{n}^*E/(\text{torsion})$. Since $\chi(H) = \chi(F^\flat)$, we have an isomorphism $H \simeq F^\flat$ and identify them. For each $i = 1, 2$, the quotient $(H^\vee)_{P_i}/H_{P_i}$ of stalks is a 2-dimensional $k(P_i)$-vector space with a nondegenerate symmetric form with values in $\mathcal{O}(P_i) \otimes k(P_i)$; $F_{P_i}/H_{P_i} \subset (H^\vee)_{P_i}/H_{P_i}$ is a 1-dimensional isotropic subspace. Let L_i be the other 1-dimensional isotropic subspace of $(H^\vee)_{P_i}/H_{P_i}$. If $\overline{F^\dagger} = (F^\dagger, \gamma_{F^\dagger}, \delta_{F^\dagger}, \ldots) \in \mathrm{OG}_{(+)}^{=1}$ is another object such that $\rho_1(\overline{F^\dagger}) = \overline{E}$, then $F^\dagger_{P_i}/H_{P_i} = F_{P_i}/H_{P_i}$ or $F^\dagger_{P_i}/H_{P_i} = L_i$. The condition that $\overline{F^\dagger}$ lies in the component $\mathrm{OG}_{(+)}^{=1}$, not in $\mathrm{OG}_n(\mathcal{F}_1^u \oplus \mathcal{F}_2^u)_{(-)}$, implies that $F^\dagger_{P_i}/H_{P_i} = F_{P_i}/H_{P_i}$ for both $i = 1, 2$, or $F^\dagger_{P_i}/H_{P_i} = L_i$ for both $i = 1, 2$. This shows that $\overline{F^\dagger} = \overline{F}$ or $\overline{F^\dagger} = \overline{F}^\iota$. (Note that the isomorphism between $F^\dagger$ and F (or F^ι) is compatible with bilinear forms and orientations because it is so over $\widetilde{C_0} \setminus \{P_1, P_2\}$.) □

For any smooth morphism $\alpha : V \to \bar{M}_n^{\leq 1}(C_0)$ with V a scheme, the product $\mathrm{OG}_{(+)}^{\leq 1} \times_{\bar{M}_n^{\leq 1}(C_0)} V$ is a smooth variety, and $\mathrm{pr} : \mathrm{OG}_{(+)}^{\leq 1} \times_{\bar{M}_n^{\leq 1}(C_0)} V \to V$ is proper by Proposition 4.5. By Lemma 4.9, pr is a finite morphism, so $\mathrm{OG}_{(+)}^{\leq 1} \times_{\bar{M}_n^{\leq 1}(C_0)} V$ is a normalization of V. Moreover, $\mathrm{pr}^{-1}(\mathrm{Sing}(V)) = \mathrm{OG}_{(+)}^{=1} \times_{\bar{M}_n^{\leq 1}(C_0)} V$ since they are both reduced and equal set-theoretically. By Lemma 4.9, we can say that $\bar{M}_n(C_0)$ is constructed from $\mathrm{OG}_{(+)}^{\leq 1}$ by gluing the closed substack $\mathrm{OG}_{(+)}^{=1}$ by the involution ι. If a function on $\mathrm{OG}_{(+)}^{\leq 1} \times_{\bar{M}_n^{\leq 1}(C_0)} V$ takes the same value at $[\overline{F}]$ and $[\overline{F}^\iota]$ for any $[\overline{F}] \in \mathrm{OG}_{(+)}^{=1} \times_{\bar{M}_n^{\leq 1}(C_0)} V$, then it is the pullback of a function on V. Thus we have the following.

PROPOSITION 4.10

Let $\mathcal{L}$ *be a line bundle on* $\bar{M}_n^{\leq 1}(C_0)$*. By Lemma 4.8,* $\rho_1^*\mathcal{L}$ *is an* ι*-equivariant line* *bundle on* $\mathrm{OG}^{=1}_{(+)}$. *Let* $\mathfrak{r}:\mathrm{H}^0(\mathrm{OG}^{\leq 1}_{(+)},\rho^*_{\leq 1}\mathcal{L})\to \mathrm{H}^0(\mathrm{OG}^{=1}_{(+)},\rho^*_1\mathcal{L})$ *be the restriction morphism. Then* $\mathrm{H}^0(\bar{M}^{\leq 1}_n(C_0),\mathcal{L})\simeq \mathfrak{r}^{-1}(\mathrm{H}^0(\mathrm{OG}^{=1}_{(+)},\rho^*_1\mathcal{L})^{\iota-\mathrm{inv}})$.

REMARK 4.11

The restriction map $\mathfrak{r}$ is not injective. A section in $\mathrm{Ker}\,\mathfrak{r}$ vanishes on $\mathrm{OG}^{=1}_{(+)}$, so it is a pullback of a section of $\mathcal{L}$ on $\bar{M}^{\leq 1}_n(C_0)$.

5. Compactification of the orthogonal group via generalized orthogonal isomorphisms

In [K1], Kausz constructed a compactification of the general linear group as a moduli space of generalized isomorphisms. In [A1], the author constructed a compactification of the symplectic group as a moduli space of generalized symplectic isomorphisms. By an almost straightforward modification of the argument in [A1], we can construct a compactification of the orthogonal group as a moduli space of generalized orthogonal isomorphisms. In this section, we state definitions and propositions modified for the orthogonal case without proof.

5.1. bf-morphisms

We first recall the definition of bf-morphisms.

DEFINITION 5.1

Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves on a scheme S. A *bf-morphism from* $\mathcal{E}$ *to* $\mathcal{F}$ is a tuple

$$g=(\mathcal{M},\mu,\mathcal{E}\xrightarrow{g^\sharp}\mathcal{F},\mathcal{M}\otimes\mathcal{E}\xleftarrow{g^\flat}\mathcal{F},r),$$

where $\mathcal{M}$ is a line bundle on S, and μ is a global section of $\mathcal{M}$ such that the following hold.

1. The composed morphism $g^\sharp\circ g^\flat$ and $g^\flat\circ g^\sharp$ are both induced by the morphism $\mu:\mathcal{O}_S\to\mathcal{M}$.
2. For every point $x\in S$ with $\mu(x)=0$, the complex

$$\mathcal{E}|_x\to\mathcal{F}|_x\to(\mathcal{M}\otimes\mathcal{E})|_x\to(\mathcal{M}\otimes\mathcal{F})|_x$$

is exact and the rank of the morphism $\mathcal{E}|_x\to\mathcal{F}|_x$ is r.

The following lemma (cf. [A1, Lemma 2.8], [K1, Lemma 6.1, Proposition 6.2]) is used later to define a generalized orthogonal$_{(\pm)}$ isomorphism.

LEMMA 5.2

Let $\mathcal{A}$, $\mathcal{B}$ *be vector bundles of rank* m, *and let*

$$(\mathcal{L},\lambda,\mathcal{A}\xrightarrow{g^\sharp}\mathcal{B},\mathcal{L}\otimes\mathcal{A}\xleftarrow{g^\flat}\mathcal{B},i)$$

be a bf-morphism of rank i.

(1) *There is a natural isomorphism*

$$\mathcal{L}^{\otimes(m-i)} \otimes \det \mathcal{A} \simeq \det \mathcal{B}.$$

(2) *If* $\lambda = 0$, *then* $\mathrm{Im}(\mathcal{A} \to \mathcal{B}) = \mathrm{Ker}(\mathcal{B} \to \mathcal{L} \otimes \mathcal{A})$ *and* $\mathrm{Ker}(\mathcal{A} \to \mathcal{B}) = \mathrm{Im}(\mathcal{L}^\vee \otimes \mathcal{B} \to \mathcal{A})$, *and they are subbundles of rank* i *and of rank* $m-i$ *of* $\mathcal{B}$ *and* $\mathcal{A}$, *respectively.*

5.2. Definition of generalized orthogonal isomorphisms

Let $(\mathcal{E}, \gamma_\mathcal{E})$ and $(\mathcal{F}, \gamma_\mathcal{F})$ be orthogonal bundles of rank $n = 2r$ or $2r+1$ on a scheme S.

DEFINITION 5.3

A *generalized orthogonal isomorphism from* $\mathcal{E}$ *to* $\mathcal{F}$ is a tuple

$$\begin{aligned} \Phi = \big(&\mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ &\mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \ (0 \le i \le r-1), h : \mathcal{E}_r \xrightarrow{\sim} \mathcal{F}_r\big), \end{aligned} \tag{5.1}$$

where $\mathcal{E} = \mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_r, \mathcal{F}_r, \ldots, \mathcal{F}_1, \mathcal{F}_0 = \mathcal{F}$ are locally free $\mathcal{O}_S$-modules of rank n and the tuples

$$(\mathcal{M}_i, \mu_i, \mathcal{E}_{i+1} \xrightarrow{e_i^\sharp} \mathcal{E}_i, \mathcal{M}_i \otimes \mathcal{E}_{i+1} \xleftarrow{e_i^\flat} \mathcal{E}_i, n-r+i)$$

and

$$(\mathcal{M}_i, \mu_i, \mathcal{F}_{i+1} \xrightarrow{f_i^\sharp} \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \xleftarrow{f_i^\flat} \mathcal{F}_i, n-r+i)$$

are bf-morphisms of rank $n-r+i$ for $0 \le i \le r-1$ such that for each $x \in S$ the following hold.

1. If $\mu_i(x) = 0$ and (f, g) is one of the following pairs of morphisms

$$\mathcal{E}_r|_x \xrightarrow{f} \mathcal{E}_{i+1}|_x \xrightarrow{g} \mathcal{E}_i|_x,$$

$$\mathcal{E}|_x \xrightarrow{f} \left(\left(\bigotimes_{j=0}^{i-1} \mathcal{M}_j\right) \otimes \mathcal{E}_i\right)\Big|_x \xrightarrow{g} \left(\left(\bigotimes_{j=0}^{i} \mathcal{M}_j\right) \otimes \mathcal{E}_{i+1}\right)\Big|_x,$$

$$\mathcal{F}_r|_x \xrightarrow{f} \mathcal{F}_{i+1}|_x \xrightarrow{g} \mathcal{F}_i|_x,$$

$$\mathcal{F}|_x \xrightarrow{f} \left(\left(\bigotimes_{j=0}^{i-1} \mathcal{M}_j\right) \otimes \mathcal{F}_i\right)\Big|_x \xrightarrow{g} \left(\left(\bigotimes_{j=0}^{i} \mathcal{M}_j\right) \otimes \mathcal{F}_{i+1}\right)\Big|_x,$$

then $\mathrm{Im}(g \circ f) = \mathrm{Im}(g)$.

2. We have $(h|_x)(\mathrm{Ker}(\mathcal{E}_r|_x \to \mathcal{E}_0|_x)) \cap \mathrm{Ker}(\mathcal{F}_r|_x \to \mathcal{F}_0|_x) = \{0\}$.
3. The following diagram is commutative:

$$
\begin{array}{ccc}
& \Big\{\Big(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{E}_0\Big)\times_{\mathcal{E}_k}\mathcal{E}_r\Big\}\otimes\Big\{\Big(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{F}_0\Big)\times_{\mathcal{F}_k}\mathcal{F}_r\Big\} & \\
& \alpha\swarrow \qquad\qquad \searrow\beta & \\
\Big(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{E}_0\Big)\otimes\mathcal{E}_0 & & \mathcal{F}_0\otimes\Big(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{F}_0\Big) \\
& \gamma'_{\mathcal{E}}\searrow \qquad\qquad \swarrow\gamma'_{\mathcal{F}} & \\
& \bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee &
\end{array}
\tag{5.2}
$$

where $\gamma'_{\mathcal{E}}$ and $\gamma'_{\mathcal{F}}$ are induced by $\gamma_{\mathcal{E}}$ and $\gamma_{\mathcal{F}}$, respectively, and

$$\alpha = q_k^{\mathcal{E}}\otimes(e_0^\sharp\circ\cdots\circ e_{r-1}^\sharp\circ h^{-1}\circ p_k^{\mathcal{F}}),$$

$$\beta = (f_0^\sharp\circ\cdots\circ f_{r-1}^\sharp\circ h\circ p_k^{\mathcal{E}})\otimes q_k^{\mathcal{F}},$$

where $p_k^{\mathcal{E}}$, $q_k^{\mathcal{E}}$, $p_k^{\mathcal{F}}$, and $q_k^{\mathcal{F}}$ are defined by

$$
\begin{array}{ccc}
\Big(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{E}_0\Big)\times_{\mathcal{E}_k}\mathcal{E}_r & \xrightarrow{p_k^{\mathcal{E}}} & \mathcal{E}_r \\
q_k^{\mathcal{E}}\downarrow & \square & \downarrow e_k^\sharp\circ\cdots\circ e_{r-1}^\sharp \\
\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{E}_0 & \xrightarrow[e_{k-1}^\flat\circ\cdots\circ e_0^\flat]{} & \mathcal{E}_k
\end{array}
\tag{5.3}
$$

and

$$
\begin{array}{ccc}
\Big(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{F}_0\Big)\times_{\mathcal{F}_k}\mathcal{F}_r & \xrightarrow{p_k^{\mathcal{F}}} & \mathcal{F}_r \\
q_k^{\mathcal{F}}\downarrow & \square & \downarrow f_k^\sharp\circ\cdots\circ f_{r-1}^\sharp \\
\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{F}_0 & \xrightarrow[f_{k-1}^\flat\circ\cdots\circ f_0^\flat]{} & \mathcal{F}_k.
\end{array}
\tag{5.4}
$$

DEFINITION 5.4

Two generalized orthogonal isomorphisms

$$\Phi = \big(\mathcal{M}_i,\mu_i,\mathcal{E}_i\to\mathcal{M}_i\otimes\mathcal{E}_{i+1},\mathcal{E}_i\leftarrow\mathcal{E}_{i+1},$$
$$\mathcal{F}_{i+1}\to\mathcal{F}_i,\mathcal{M}_i\otimes\mathcal{F}_{i+1}\leftarrow\mathcal{F}_i(0\le i\le r-1),h:\mathcal{E}_r\to\mathcal{F}_r\big),$$
$$\Phi' = \big(\mathcal{M}'_i,\mu'_i,\mathcal{E}'_i\to\mathcal{M}'_i\otimes\mathcal{E}'_{i+1},\mathcal{E}'_i\leftarrow\mathcal{E}'_{i+1},$$
$$\mathcal{F}'_{i+1}\to\mathcal{F}'_i,\mathcal{M}'_i\otimes\mathcal{F}'_{i+1}\leftarrow\mathcal{F}'_i\ (0\le i\le r-1),h':\mathcal{E}'_r\to\mathcal{F}'_r\big)$$

from $\mathcal{E}$ to $\mathcal{F}$ are defined to be equivalent if there are isomorphisms $\mathcal{M}_i\simeq\mathcal{M}'_i$ $(0\le i\le r-1)$ by which μ_i maps to μ'_i, and isomorphisms $\mathcal{E}_i\simeq\mathcal{E}'_i$ and $\mathcal{F}_i\simeq\mathcal{F}'_i$ $(0\le i\le r)$ such that $\mathcal{E}_0\simeq\mathcal{E}'_0$ and $\mathcal{F}_0\simeq\mathcal{F}'_0$ are the identity and the obvious diagrams are commutative.

DEFINITION 5.5

The functor $\mathcal{KO}(\mathcal{E},\mathcal{F})$ from the category of S-schemes to the category of sets is defined to associate to an S-scheme T the set of equivalence classes of generalized orthogonal isomorphisms from $\mathcal{E}_T$ to $\mathcal{F}_T$.

As an orthogonal analogue of [A1, Proposition 3.13, Corollary 3.16], we have the following.

PROPOSITION 5.6

The functor $\mathcal{KO}(\mathcal{E},\mathcal{F})$ is represented by a scheme $\mathrm{KO}(\mathcal{E},\mathcal{F})$ which is smooth and projective over S.

The difference from the symplectic case is that $\mathrm{KO}(\mathcal{E},\mathcal{F})$ is not connected because the orthogonal group is not connected.

Suppose that we are given orientations δ_E and δ_F of $(\mathcal{E},\gamma_{\mathcal{E}})$ and $(\mathcal{F},\gamma_{\mathcal{F}})$, respectively. Given a generalized orthogonal isomorphism Φ (5.1) from $\mathcal{E}$ and $\mathcal{F}$, put $\mathcal{E}^{(i)} := \bigwedge^n \mathcal{E}_i \otimes \bigotimes_{j=0}^{i-1} \mathcal{M}_j^{\otimes r-j}$ and $\mathcal{F}^{(i)} := \bigwedge^n \mathcal{F}_i \otimes \bigotimes_{j=0}^{i-1} \mathcal{M}_j^{\otimes r-j}$. Then by Lemma 5.2, we have isomorphisms $d_i^{\mathcal{E}} : \mathcal{E}^{(i)} \to \mathcal{E}^{(i+1)}$ and $d_i^{\mathcal{F}} : \mathcal{F}^{(i)} \to \mathcal{F}^{(i+1)}$. Then the composite of morphisms

$$\mathcal{O} \xrightarrow{\delta_{\mathcal{E}}^{-1}} \mathcal{E}^{(0)} \xrightarrow{d_{r-1}^{\mathcal{E}}\circ\cdots\circ d_0^{\mathcal{E}}} \mathcal{E}^{(r)} \xrightarrow{\wedge^n h\otimes \mathrm{id}} \mathcal{F}^{(r)} \xrightarrow{(d_{r-1}^{\mathcal{F}}\circ\cdots\circ d_0^{\mathcal{F}})^{-1}} \mathcal{F}^{(0)} \xrightarrow{\delta_{\mathcal{F}}} \mathcal{O}$$

is $\pm\,\mathrm{id}$. If it is id, then Φ is called a generalized orthogonal$_{(+)}$ isomorphism, and if it is $-\,\mathrm{id}$, then Φ is called a generalized orthogonal$_{(-)}$ isomorphism. $\mathrm{KO}(\mathcal{E},\mathcal{F})$ is a disjoint union of $\mathrm{KO}(\mathcal{E},\mathcal{F})_{(+)}$ and $\mathrm{KO}(\mathcal{E},\mathcal{F})_{(-)}$ parameterizing generalized orthogonal$_{(+)}$ isomorphisms and generalized orthogonal$_{(-)}$ isomorphisms, respectively.

5.3. Relation with the orthogonal Grassmannian

Let $(\mathcal{E},\gamma_{\mathcal{E}},\delta_{\mathcal{E}})$ and $(\mathcal{F},\gamma_{\mathcal{F}},\delta_{\mathcal{F}})$ be oriented orthogonal bundles of rank $n=2r$ or $2r+1$ on a scheme S.

Let

$$\Phi = \big(\mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1},$$
$$\mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \ (0 \le i \le r-1), h : \mathcal{E}_r \xrightarrow{\sim} \mathcal{F}_r\big)$$

be the universal generalized orthogonal isomorphism from $\mathcal{E}_0 = \mathcal{E}_{\mathrm{KO}}$ to $\mathcal{F}_0 = \mathcal{F}_{\mathrm{KO}}$ on $\mathrm{KO}(\mathcal{E},\mathcal{F})$.

Then by Definition 5.3(2), the morphism

$$\beta := (e_0^\sharp \circ \cdots \circ e_{r-1}^\sharp, f_0^\sharp \circ \cdots \circ f_{r-1}^\sharp \circ h) : \mathcal{E}_r \to \mathcal{E}_{\mathrm{KO}} \oplus \mathcal{F}_{\mathrm{KO}}$$

is injective, and its image is a subbundle of $\mathcal{E}_{\mathrm{KO}} \oplus \mathcal{F}_{\mathrm{KO}}$. By Definition 5.3(3), this subbundle is isotropic. Hence $\beta(\mathcal{E}_r) \subset \mathcal{E}_{\mathrm{KO}} \oplus \mathcal{F}_{\mathrm{KO}}$ gives rise to a morphism $g : \mathrm{KO}(\mathcal{E},\mathcal{F}) \to \mathrm{OG}_n(\mathcal{E}\oplus\mathcal{F})$. For $\epsilon = +$or $-$, the component $\mathrm{KO}(\mathcal{E},\mathcal{F})_{(\epsilon)}$ maps to $\mathrm{OG}_n(\mathcal{E}\oplus\mathcal{F})_{(\epsilon)}$ (see Section 2.3 for which component of $\mathrm{OG}_n(\mathcal{E}\oplus\mathcal{F})$ is called $\mathrm{OG}_n(\mathcal{E}\oplus\mathcal{F})_{(+)}$ or $\mathrm{OG}_n(\mathcal{E}\oplus\mathcal{F})_{(-)}$).

By the same proof as in [A1, Lemma 4.2], we have the following.

LEMMA 5.7
Let $0 \to \mathcal{U} \to \mathrm{pr}_S^*(\mathcal{E} \oplus \mathcal{F}) \to \mathcal{Q} \to 0$ *be the universal sequence on* $\mathrm{OG}_n(\mathcal{E} \oplus \mathcal{F})$*. Then there is a natural isomorphism*

$$g^* \det \mathcal{Q} \simeq \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes(r-i)}. \tag{5.5}$$

5.4. Geometry of strata

Let $(\mathcal{E}, \gamma_\mathcal{E})$ and $(\mathcal{F}, \gamma_\mathcal{F})$ be orthogonal bundles of rank $n = 2r$ or $2r+1$ on a scheme S over an algebraically closed field. Let

$$\begin{aligned}\Phi = \big(&\mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1},\\ &\mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \ (0 \le i \le r-1), h : \mathcal{E}_r \xrightarrow{\sim} \mathcal{F}_r\big)\end{aligned} \tag{5.6}$$

be the universal generalized orthogonal isomorphism from $\mathcal{E}_0 = (\mathcal{E})_{\mathrm{KO}(\mathcal{E},\mathcal{F})}$ to $\mathcal{F}_0 = (\mathcal{F})_{\mathrm{KO}(\mathcal{E},\mathcal{F})}$.

DEFINITION 5.8
For a subset $I \subset [0, r-1]$, we denote by $\mathrm{KO}(\mathcal{E},\mathcal{F})_I$ the subscheme $\bigcap_{i\in I}\{\mu_i = 0\} \subset \mathrm{KO}(\mathcal{E},\mathcal{F})$.

DEFINITION 5.9
For a subset $I = \{i_1 < \cdots < i_l\} \subset \{0, \ldots, r-1\}$, let $\mathcal{F}\mathrm{l}_I(\mathcal{E})$ be the functor from the category of S-schemes to the category of sets that associates to an S-scheme T the set of filtrations

$$0 \subset \mathbb{F}_{i_l}(\mathcal{E}_T) \subset \mathbb{F}_{i_{l-1}}(\mathcal{E}_T) \subset \cdots \subset \mathbb{F}_{i_1}(\mathcal{E}_T) \subset \mathcal{E}_T$$

of isotropic subbundles indexed by I with rank $\mathbb{F}_{i_j}(\mathcal{E}_T) = r - i_j$. We understand that $\mathbb{F}_{i_{l+1}}(\mathcal{E}_T) = 0$.

We denote by $\mathrm{Fl}_I(\mathcal{E})$ the S-scheme that represents $\mathcal{F}\mathrm{l}_I(\mathcal{E})$.

Put $\mathbf{Fl}_I := \mathrm{Fl}_I(\mathcal{E}) \times_S \mathrm{Fl}_I(\mathcal{F})$, put $\widetilde{\mathcal{E}} := (\mathcal{E})_{\mathbf{Fl}_I}$, and put $\widetilde{\mathcal{F}} := (\mathcal{F})_{\mathbf{Fl}_I}$.

Let

$$0 \subset \mathbb{F}_{i_l}(\widetilde{\mathcal{E}}) \subset \cdots \subset \mathbb{F}_{i_1}(\widetilde{\mathcal{E}}) \subset \widetilde{\mathcal{E}} \qquad \text{and} \qquad 0 \subset \mathbb{F}_{i_l}(\widetilde{\mathcal{F}}) \subset \cdots \subset \mathbb{F}_{i_1}(\widetilde{\mathcal{F}}) \subset \widetilde{\mathcal{F}} \tag{5.7}$$

be the pullbacks to $\mathbf{Fl}_I$ of the universal filtrations of $\mathcal{E}$ and $\mathcal{F}$ on $\mathrm{Fl}_I(\mathcal{E})$ and $\mathrm{Fl}_I(\mathcal{F})$, respectively. The nondegenerate symmetric bilinear forms $\gamma_\mathcal{E}$ and $\gamma_\mathcal{F}$ induce nondegenerate symmmetric bilinear forms

$$\widetilde{\gamma}_\mathcal{E} : \mathbb{F}_{i_1}(\widetilde{\mathcal{E}})^\perp/\mathbb{F}_{i_1}(\widetilde{\mathcal{E}}) \otimes \mathbb{F}_{i_1}(\widetilde{\mathcal{E}})^\perp/\mathbb{F}_{i_1}(\widetilde{\mathcal{E}}) \to \mathcal{O}_{\mathbf{Fl}_I},$$

$$\widetilde{\gamma}_\mathcal{F} : \mathbb{F}_{i_1}(\widetilde{\mathcal{F}})^\perp/\mathbb{F}_{i_1}(\widetilde{\mathcal{F}}) \otimes \mathbb{F}_{i_1}(\widetilde{\mathcal{F}})^\perp/\mathbb{F}_{i_1}(\widetilde{\mathcal{F}}) \to \mathcal{O}_{\mathbf{Fl}_I}$$

and nondegenerate bilinear forms

$$\widetilde{\gamma}_{\mathcal{E},i_j} : \mathbb{F}_{i_{j+1}}(\widetilde{\mathcal{E}})^{\perp}/\mathbb{F}_{i_j}(\widetilde{\mathcal{E}})^{\perp} \otimes \mathbb{F}_{i_j}(\widetilde{\mathcal{E}})/\mathbb{F}_{i_{j+1}}(\widetilde{\mathcal{E}}) \to \mathcal{O}_{\mathbf{Fl}_I},$$

$$\widetilde{\gamma}_{\mathcal{F},i_j} : \mathbb{F}_{i_{j+1}}(\widetilde{\mathcal{F}})^{\perp}/\mathbb{F}_{i_j}(\widetilde{\mathcal{F}})^{\perp} \otimes \mathbb{F}_{i_j}(\widetilde{\mathcal{F}})/\mathbb{F}_{i_{j+1}}(\widetilde{\mathcal{F}}) \to \mathcal{O}_{\mathbf{Fl}_I} \quad (1 \le j \le l).$$

Then we have a scheme $Q(\widetilde{\gamma}_{\mathcal{E},i_j}, \widetilde{\gamma}_{\mathcal{F},i_j})$ that is smooth projective over S, which is a compactification of $\mathrm{PGL}(\mathbb{F}_{i_j}(\widetilde{\mathcal{F}})/\mathbb{F}_{i_{j+1}}(\widetilde{\mathcal{F}}), \mathbb{F}_{i_{j+1}}(\widetilde{\mathcal{E}})^{\perp}/\mathbb{F}_{i_j}(\widetilde{\mathcal{E}})^{\perp})$ (see [A1, p. 22] for the definition of $Q(\widetilde{\gamma}_{\mathcal{E},i_j}, \widetilde{\gamma}_{\mathcal{F},i_j})$).

Then the stratum $\mathrm{KO}(\mathcal{E},\mathcal{F})_I$ is described as follows.

PROPOSITION 5.10

There is an isomorphism

$$\mathrm{KO}(\mathcal{E},\mathcal{F})_I \to \mathrm{KO}(\mathbb{F}_{i_1}(\widetilde{\mathcal{E}})^{\perp}/\mathbb{F}_{i_1}(\widetilde{\mathcal{E}}), \mathbb{F}_{i_1}(\widetilde{\mathcal{F}})^{\perp}/\mathbb{F}_{i_1}(\widetilde{\mathcal{F}})) \times_{\mathbf{Fl}_I} Q \tag{5.8}$$

of S-schemes, where $Q = Q(\widetilde{\pi}_{\mathcal{E},i_1}, \widetilde{\pi}_{\mathcal{F},i_1}) \times_{\mathbf{Fl}_I} \cdots \times_{\mathbf{Fl}_I} Q(\widetilde{\pi}_{\mathcal{E},i_l}, \widetilde{\pi}_{\mathcal{F},i_l})$. In particular, we have an isomorphism

$$\mathrm{KO}(\mathcal{E},\mathcal{F})_{[0,r-1]} \simeq \mathbf{Fl}_{[0,r-1]}. \tag{5.9}$$

NOTATION 5.11

For tuples $(a_1,\dots,a_r)$ and $(b_1,\dots,b_r)$ of integers, we denote by $\mathcal{O}(a_1,\dots,a_r; b_1,\dots,b_r)$ the line bundle

$$\bigotimes_{j=1}^{r} \left(\mathbb{F}_{r+1-j}(\widetilde{\mathcal{E}})^{\perp}/\mathbb{F}_{r-j}(\widetilde{\mathcal{E}})^{\perp}\right)^{\otimes a_j} \otimes \bigotimes_{j=1}^{r} \left(\mathbb{F}_{r+1-j}(\widetilde{\mathcal{F}})^{\perp}/\mathbb{F}_{r-j}(\widetilde{\mathcal{F}})^{\perp}\right)^{\otimes b_j}$$

on $\mathbf{Fl}_{[0,r-1]} (= \mathrm{Fl}_{[0,r-1]}(\mathcal{E}) \times_S \mathrm{Fl}_{[0,r-1]}(\mathcal{F}))$.

We often identify $\mathrm{KO}(\mathcal{E},\mathcal{F})_{[0,r-1]}$ with $\mathbf{Fl}_{[0,r-1]}$ by the isomorphism (5.9).

LEMMA 5.12

There are natural isomorphisms

$$\mathcal{M}_0|_{\mathrm{KO}(\mathcal{E},\mathcal{F})_{[0,r-1]}} \simeq \mathcal{O}(\mathbf{e}_r;\mathbf{e}_r),$$

and for $1 \le j \le r-1$,

$$\mathcal{M}_j|_{\mathrm{KO}(\mathcal{E},\mathcal{F})_{[0,r-1]}} \simeq \mathcal{O}(\mathbf{e}_{r-j} - \mathbf{e}_{r-j+1}; \mathbf{e}_{r-j} - \mathbf{e}_{r-j+1})$$

of line bundles on $\mathrm{KO}(\mathcal{E},\mathcal{F})_{[0,r-1]} \simeq \mathbf{Fl}_{[0,r-1]}$, where

$$\mathbf{e}_i := (0,\dots,0,\overset{ith}{1},0,\dots,0).$$

Proof

This is the same as the proof of [A1, Lemma 5.6]. □

Now suppose that the orthogonal bundles $(\mathcal{E},\gamma_{\mathcal{E}})$ and $(\mathcal{F},\gamma_{\mathcal{F}})$ are given orientations $\delta_{\mathcal{E}}$ and $\delta_{\mathcal{F}}$, respectively. Define orientations $\widetilde{\delta}_{\mathcal{E}}$ and $\widetilde{\delta}_{\mathcal{F}}$ of $(\mathbb{F}_{i_1}(\widetilde{\mathcal{E}})^{\perp}/\mathbb{F}_{i_1}(\widetilde{\mathcal{E}}), \widetilde{\gamma}_{\mathcal{E}})$ and $(\mathbb{F}_{i_1}(\widetilde{\mathcal{F}})^{\perp}/\mathbb{F}_{i_1}(\widetilde{\mathcal{F}}), \widetilde{\gamma}_{\mathcal{F}})$ so that the diagrams

$$\begin{array}{ccccc}
\wedge^a \mathbb{F}_{i_1}(\widetilde{\mathcal{E}}) \otimes \wedge^a \frac{\widetilde{\mathcal{E}}}{\mathbb{F}_{i_1}(\widetilde{\mathcal{E}})^\perp} \otimes \wedge^{n-2a} \frac{\mathbb{F}_{i_1}(\widetilde{\mathcal{E}})^\perp}{\mathbb{F}_{i_1}(\widetilde{\mathcal{E}})} & \xrightarrow{\sim} & \wedge^n \widetilde{\mathcal{E}} & \xrightarrow{\delta_{\mathcal{E}}} & \mathcal{O}_{\mathbf{Fl}_I} \\
\downarrow{\scriptstyle \bar{\wedge}^a \gamma_{\mathcal{E}} \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \cdot(\sqrt{-1})^a} \\
\wedge^{n-2a} \frac{\mathbb{F}_{i_1}(\widetilde{\mathcal{E}})^\perp}{\mathbb{F}_{i_1}(\widetilde{\mathcal{E}})} & & \xrightarrow{\quad \widetilde{\delta}_{\mathcal{E}} \quad} & & \mathcal{O}_{\mathbf{Fl}_I}
\end{array}$$

$$\begin{array}{ccccc}
\wedge^a \frac{\widetilde{\mathcal{F}}}{\mathbb{F}_{i_1}(\widetilde{\mathcal{F}})^\perp} \otimes \wedge^a \mathbb{F}_{i_1}(\widetilde{\mathcal{F}}) \otimes \wedge^{n-2a} \frac{\mathbb{F}_{i_1}(\widetilde{\mathcal{F}})^\perp}{\mathbb{F}_{i_1}(\widetilde{\mathcal{F}})} & \xrightarrow{\sim} & \wedge^n \widetilde{\mathcal{F}} & \xrightarrow{\delta_{\mathcal{F}}} & \mathcal{O}_{\mathbf{Fl}_I} \\
\downarrow{\scriptstyle \bar{\wedge}^a \gamma_{\mathcal{F}} \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \cdot(\sqrt{-1})^a} \\
\wedge^{n-2a} \frac{\mathbb{F}_{i_1}(\widetilde{\mathcal{F}})^\perp}{\mathbb{F}_{i_1}(\widetilde{\mathcal{F}})} & & \xrightarrow{\quad \widetilde{\delta}_{\mathcal{F}} \quad} & & \mathcal{O}_{\mathbf{Fl}_I}
\end{array}$$

commute, where $a = r - i_1$. For $\epsilon = +$ or $-$, put $\mathrm{KO}(\mathcal{E},\mathcal{F})_{I(\epsilon)} := \mathrm{KO}(\mathcal{E},\mathcal{F})_I \cap \mathrm{KO}(\mathcal{E},\mathcal{F})_{(\epsilon)}$.

In the isomorphism (5.8), we have

$$\mathrm{KO}(\mathcal{E},\mathcal{F})_{I(\epsilon)} \to \mathrm{KO}\big(\mathbb{F}_{i_1}(\widetilde{\mathcal{E}})^\perp/\mathbb{F}_{i_1}(\widetilde{\mathcal{E}}), \mathbb{F}_{i_1}(\widetilde{\mathcal{F}})^\perp/\mathbb{F}_{i_1}(\widetilde{\mathcal{F}})\big)_{(\epsilon)} \times_{\mathbf{Fl}_I} Q. \tag{5.10}$$

When $n = 2r$ and $0 \in I$, $\mathrm{Fl}_I(\mathcal{E})$ (resp., $\mathrm{Fl}_I(\mathcal{F})$) is a disjoint union $\mathrm{Fl}_I(\mathcal{E})^{(+)} \sqcup \mathrm{Fl}_I(\mathcal{E})^{(-)}$ (resp., $\mathrm{Fl}_I(\mathcal{F})_{(+)} \sqcup \mathrm{Fl}_I(\mathcal{F})_{(-)}$), where $\mathrm{Fl}_I(\mathcal{E})^{(\epsilon)}$ (resp., $\mathrm{Fl}_I(\mathcal{F})_{(\epsilon)}$) maps to the component $\mathrm{OG}_r(\mathcal{E})^{(\epsilon)}$ (resp., $\mathrm{OG}_r(\mathcal{F})_{(\epsilon)}$) by the natural map. If we put

$$Q^{(\epsilon_1)}_{(\epsilon_2)} = \big(\mathrm{Fl}_I(\mathcal{E})^{(\epsilon_1)} \times_S \mathrm{Fl}_I(\mathcal{F})_{(\epsilon_2)}\big) \times_{\mathbf{Fl}_I} Q$$

for $\epsilon_i = \pm$, then in the isomorphism (5.10), we have

$$\mathrm{KO}(\mathcal{E},\mathcal{F})_{I(+)} \simeq Q^{(+)}_{(+)} \sqcup Q^{(-)}_{(-)} \quad \text{and} \quad \mathrm{KO}(\mathcal{E},\mathcal{F})_{I(-)} \simeq Q^{(+)}_{(-)} \sqcup Q^{(-)}_{(+)}. \tag{5.11}$$

5.5. Global sections

We retain the notation of the preceding subsection, but we assume that $S = \operatorname{Spec} k$ with k an algebraically closed field of characteristic zero. We write E and F instead of $\mathcal{E}$ and $\mathcal{F}$.

In this section we describe vector spaces of global sections of a line bundle on $\mathrm{KO}(E,F)_{I(+)}$ in terms of vector spaces of global sections of line bundles on flag varieties. To simplify notation, we write X_I for $\mathrm{KO}(E,F)_{I(+)}$. When $n = 2r$ and $0 \in I$, we have a decomposition $X_I = X_{I^{(+)}} \sqcup X_{I^{(-)}}$, where $X_{I^{(\epsilon)}}$ corresponds to $Q^{(\epsilon)}_{(\epsilon)}$ in the isomorphism (5.11). The inclusion $X_I \to \mathrm{KO}(E,F)_{(+)}$ and $X_{I^{(\epsilon)}} \to \mathrm{KO}(E,F)_{(+)}$ are denoted by κ_I and $\kappa_{I^{(\epsilon)}}$, respectively.

The algebraic group $\mathrm{SO}(E) \times \mathrm{SO}(F)$ acts on $\mathrm{KO}(E,F)_{(+)}$ from the left. The closed subschemes X_I (and $X_{I^{(\epsilon)}}$ in case $n = 2r$ and $0 \in I$) are stable with respect to the action. The line bundles $\mathcal{M}_i$ $(0 \le i \le r-1)$ on $\mathrm{KO}(E,F)_{(+)}$ have $(\mathrm{SO}(E) \times \mathrm{SO}(F))$-linearization. Therefore the vector spaces $\mathrm{H}^0(X_I, \kappa_I^* \otimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i})$ are $(\mathrm{SO}(E) \times \mathrm{SO}(F))$-modules.

DEFINITION 5.13

For a tuple $\vec{c} = (c_0, \ldots, c_{r-1}) \in \mathbb{Z}^r$, the set $A^{\mathrm{even}}(\vec{c})_I$ is defined to consist of tuples $(q_1, \ldots, q_r) \in \mathbb{Z}^r$ such that

$$q_1 \geq \cdots \geq q_{r-1} \geq |q_r|, \tag{5.12}$$

$$c_j \geq \sum_{k=1}^{r-j} |q_k| \quad \text{for } j \in [0, r-1] \setminus I, \tag{5.13}$$

$$c_j = \sum_{k=1}^{r-j} q_k \quad \text{for } j \in I. \tag{5.14}$$

The set $A^{\mathrm{odd}}(\vec{c})_I$ is defined to consist of tuples $(q_1, \ldots, q_r) \in \mathbb{Z}^r$ such that $q_1 \geq \cdots \geq q_{r-1} \geq q_r \geq 0$ and the conditions (5.13) and (5.14) hold.

Now we state the irreducible decomposition of $\mathrm{H}^0(X_I, \kappa_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i})$.

THEOREM 5.14

Assume that $n = 2r$, $0 \notin I$ *and* $\vec{c} = (c_0, \ldots, c_{r-1}) \in \mathbb{Z}^r$. *We have a decomposition into distinct irreducible* $(\mathrm{SO}(E) \times \mathrm{SO}(F))$*-modules:*

$$\mathrm{H}^0\Big(X_I, \kappa_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i}\Big) = \bigoplus_{\vec{q} \in A^{\mathrm{even}}(\vec{c})_I} V_{\vec{q}}.$$

The irreducible $(\mathrm{SO}(E) \times \mathrm{SO}(F))$*-submodule* $V_{\vec{q}}$ *is contained in the subspace* $\mathrm{H}^0\big(X_I, \kappa_I^*\big(\bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} |q_j|}\big)\big)$, *and the composite of morphisms*

$$V_{\vec{q}} \hookrightarrow \mathrm{H}^0\Bigg(X_I, \kappa_I^*\Big(\bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} |q_j|}\Big)\Bigg) \xrightarrow{\text{restr.}} \mathrm{H}^0\big(X_{[0,r-1]^{(+)}}, \mathcal{O}(\vec{q}; \vec{q})\big)$$

is an isomorphism if $q_r \geq 0$, *and the composite of morphisms*

$$V_{\vec{q}} \hookrightarrow \mathrm{H}^0\Bigg(X_I, \kappa_I^*\Big(\bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} |q_j|}\Big)\Bigg) \xrightarrow{\text{restr.}} \mathrm{H}^0\big(X_{[0,r-1]^{(-)}}, \mathcal{O}(\vec{q}^{\natural}; \vec{q}^{\natural})\big)$$

is an isomorphism if $q_r \leq 0$, *where* $\vec{q}^{\natural} = (q_1, \ldots, q_{r-1}, -q_r)$ *(see Notation 5.11 for* $\mathcal{O}(\vec{a}, \vec{b})$*).*

REMARK 5.15

Under the assumption of Theorem 5.14, we have a diagram

$$\begin{array}{ccccc} X_{[0,r-1]^{(+)}} & \hookrightarrow & X_{[1,r-1]} & \hookrightarrow & X_I \\ & \searrow^{\beta} & \downarrow{\scriptstyle\alpha} & & \\ & & \mathbf{Fl}_{[1,r-1]} & & \end{array}$$

If $q_r = 0$, then the line bundle $\kappa^*_{[1,r-1]}\big(\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes\sum_{j=1}^{r-i}|q_j|}\big)$ on $X_{[1,r-1]}$ is a pull-back of a line bundle, say, $\mathcal{L}_{\vec{q}}$, on $\mathbf{Fl}_{[1,r-1]}$. The morphisms

$$\mathrm{H}^0(\mathbf{Fl}_{[1,r-1]},\mathcal{L}_{\vec{q}}) \xrightarrow{\alpha^*} \mathrm{H}^0\left(X_{[1,r-1]},\kappa^*_{[1,r-1]}\Big(\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes\sum_{j=1}^{r-i}|q_j|}\Big)\right)$$

and

$$\mathrm{H}^0(\mathbf{Fl}_{[1,r-1]},\mathcal{L}_{\vec{q}}) \xrightarrow{\beta^*} \mathrm{H}^0\big(X_{[0,r-1]^{(+)}},\mathcal{O}(\vec{q};\vec{q})\big)$$

are isomorphisms since α is a $\mathbb{P}^1$-bundle and β is an isomorphism. So the restriction map $\mathrm{H}^0\big(X_{[1,r-1]},\kappa^*_{[1,r-1]}\big(\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes\sum_{j=1}^{r-i}|q_j|}\big)\big) \to \mathrm{H}^0(X_{[0,r-1]^{(+)}},\mathcal{O}(\vec{q};\vec{q}))$ is an isomorphism. Therefore in Theorem 5.14, when $q_r = 0$, the composite of morphisms

$$\begin{aligned} V_{\vec{q}} \quad &\hookrightarrow \quad \mathrm{H}^0\left(X_I,\kappa_I^*\Big(\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes\sum_{j=1}^{r-i}|q_j|}\Big)\right) \\ &\xrightarrow{\text{restr.}} \mathrm{H}^0\left(X_{[1,r-1]},\kappa^*_{[1,r-1]}\Big(\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes\sum_{j=1}^{r-i}|q_j|}\Big)\right) \xrightarrow{(\alpha^*)^{-1}} \mathrm{H}^0(\mathbf{Fl}_{[1,r-1]},\mathcal{L}_{\vec{q}}) \end{aligned}$$

is an isomorphism.

THEOREM 5.16

Assume that $n = 2r$, assume that $0 \in I$, and assume that $\vec{c} = (c_0,\dots,c_{r-1}) \in \mathbb{Z}^r$. Let $\epsilon = +$ or $-$. We have a decomposition into distinct irreducible $(\mathrm{SO}(E)\times\mathrm{SO}(F))$*-modules:*

$$\mathrm{H}^0\Big(X_{I^{(\epsilon)}},\kappa^*_{I^{(\epsilon)}}\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes c_i}\Big) = \bigoplus_{\vec{q}\in A^{\mathrm{even}}(\vec{c})_I} V_{\vec{q}}.$$

The irreducible $(\mathrm{SO}(E)\times\mathrm{SO}(F))$*-submodule $V_{\vec{q}}$ is contained in the subspace $\mathrm{H}^0\big(X_{I^{(\epsilon)}},\kappa^*_{I^{(\epsilon)}}\big(\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes\sum_{j=1}^{r-i}q_j}\big)\big)$, and the composite of morphisms*

$$V_{\vec{q}} \hookrightarrow \mathrm{H}^0\left(X_{I^{(\epsilon)}},\kappa^*_{I^{(\epsilon)}}\Big(\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes\sum_{j=1}^{r-i}q_j}\Big)\right) \xrightarrow{\text{restr.}} \mathrm{H}^0\big(X_{[0,r-1]^{(\epsilon)}},\mathcal{O}(\vec{q};\vec{q})\big)$$

is an isomorphism.

THEOREM 5.17

Assume that $n = 2r+1$, and assume that $\vec{c} = (c_0,\dots,c_{r-1}) \in \mathbb{Z}^r$. We have a decomposition into distinct irreducible $(\mathrm{SO}(E)\times\mathrm{SO}(F))$*-modules:*

$$\mathrm{H}^0\Big(X_I,\kappa_I^*\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes c_i}\Big) = \bigoplus_{\vec{q}\in A^{\mathrm{odd}}(\vec{c})_I} V_{\vec{q}}.$$

The irreducible ($\mathrm{SO}(E)\times\mathrm{SO}(F)$*)-submodule $V_{\vec{q}}$ is contained in the subspace $\mathrm{H}^0\big(X_I,\kappa_I^*\big(\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes\sum_{j=1}^{r-i}q_j}\big)\big)$, and the composites of morphisms*

$$V_{\vec{q}} \hookrightarrow \mathrm{H}^0\Big(X_I, \kappa_I^*\Big(\bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} q_j}\Big)\Big) \xrightarrow{\text{restr.}} \mathrm{H}^0\big(X_{[0,r-1]}, \mathcal{O}(\vec{q};\vec{q})\big)$$

is an isomorphism.

6. Factorization theorem

Assume that $n=2r$ or $2r+1\geq 3$, and assume that $\mathrm{char} k = 0$. Fix a positive integer l called a level. In this section we prove a factorization theorem, which describes the vector space $\mathrm{H}^0(\bar{M}_n^{\leq 1}(C_0), \mathcal{D}^{\otimes l})$, where $\mathcal{D}$ is the determinant line bundle, in terms of spaces of global sections of line bundles on moduli of parabolic oriented orthogonal bundles on $\widetilde{C_0}$.

6.1. The stack $\mathbf{Fl}_{\vec{q}}$

For a sequence $\vec{q}=(l\geq q_1\geq\cdots\geq q_r\geq 0)$ of integers, we define the stack $\mathbf{Fl}_{\vec{q}}$ over $M_n(\widetilde{C_0})$ as follows. If $n=2r$ and $q_r>0$, then $\mathbf{Fl}_{\vec{q}} := \bigsqcup_{\epsilon=+,-} \mathrm{Fl}_{[0,r-1]}(\mathcal{F}_1^u)^{(\epsilon)} \times \mathrm{Fl}_{[0,r-1]}(\mathcal{F}_2^u)_{(\epsilon)}$. If $n=2r$ and $q_r=0$, then $\mathbf{Fl}_{\vec{q}} := \mathrm{Fl}_{[1,r-1]}(\mathcal{F}_1^u)\times \mathrm{Fl}_{[1,r-1]}(\mathcal{F}_2^u)$. If $n=2r+1$, then $\mathbf{Fl}_{\vec{q}} := \mathrm{Fl}_{[0,r-1]}(\mathcal{F}_1^u)\times \mathrm{Fl}_{[0,r-1]}(\mathcal{F}_2^u)$.

We define the involution $\iota : \mathrm{Fl}_{\vec{q}} \to \mathrm{Fl}_{\vec{q}}$ as follows. In the case where $n=2r$ and $q_r>0$, the stack $\mathrm{Fl}_{\vec{q}}$ parameterizes tuples

$$\big(F,\gamma,\delta;\mathbb{F}_\bullet(F|_{P_i})\ (i=1,2)\big), \tag{6.1}$$

where (F,γ,δ) is an oriented orthogonal bundle on $\widetilde{C_0}$ and $\mathbb{F}_\bullet(F|_{P_i})$ is a filtration

$$0\subset \mathbb{F}_{r-1}(F|_{P_i})\subset\cdots\subset\mathbb{F}_0(F|_{P_i})\subset F|_{P_i}$$

by isotropic subspaces with $\dim \mathbb{F}_a(F|_{P_i}) = r-a$ such that $(\mathbb{F}_\bullet(F|_{P_1}),\mathbb{F}_\bullet(F|_{P_2}))\in \mathrm{Fl}_{[0,r-1]}(F|_{P_1})^{(\epsilon)}\times \mathrm{Fl}_{[0,r-1]}(F|_{P_2})_{(\epsilon)}$ for $\epsilon=\pm$. For a tuple (6.1), let $(F^\iota,\gamma^\iota,\delta^\iota, \mathbb{F}_{r-1}(F^\iota|_{P_i})\ (i=1,2))$ be the ι-transform of $(F,\gamma,\delta,\mathbb{F}_{r-1}(F|_{P_i})\ (i=1,2))$ over $\{P_1,P_2\}$. We define the filtration

$$\big(\mathbb{F}_{r-1}(F^\iota|_{P_i})\subset\big)\mathbb{F}_{r-2}(F^\iota|_{P_i})\subset\cdots\subset\mathbb{F}_0(F^\iota|_{P_i})\subset F^\iota|_{P_i}$$

so that $\mathbb{F}_j(F^\iota|_{P_i})/\mathbb{F}_{r-1}(F^\iota|_{P_i})$ and $\mathbb{F}_j(F|_{P_i})/\mathbb{F}_{r-1}(F|_{P_i})$ correspond through the natural isomorphism $\mathbb{F}_{r-1}(F^\iota|_{P_i})^\perp/\mathbb{F}_{r-1}(F^\iota|_{P_i})\simeq \mathbb{F}_{r-1}(F|_{P_i})^\perp/\mathbb{F}_{r-1}(F|_{P_i})$ (cf. (2.6)). Then $(\mathbb{F}_\bullet(F^\iota|_{P_1}),\mathbb{F}_\bullet(F^\iota|_{P_2}))\in Fl_{[0,r-1]}(F|_{P_1})^{(\epsilon')}\times Fl_{[0,r-1]}(F|_{P_2})_{(\epsilon')}$, where $\{\epsilon,\epsilon'\}=\{+,-\}$. By associating the tuple $(F^\iota,\gamma^\iota,\delta^\iota;\mathbb{F}_\bullet(F^\iota|_{P_i})\ (i=1,2))$ to the tuple (6.1), we can define a morphism $\iota:\mathrm{Fl}_{\vec{q}}\to\mathrm{Fl}_{\vec{q}}$. In the case where $n=2r$ and $q_r=0$, or where $n=2r+1$, we define $\iota:\mathrm{Fl}_{\vec{q}}\to\mathrm{Fl}_{\vec{q}}$ similarly, that is, ignore in the above procedure the zeroth filter in the case $n=2r$ and $q_r=0$, and $(\epsilon),(\epsilon')$ in the case $n=2r+1$.

6.2. The line bundle $\mathcal{L}_{\vec{q}}$

We denote by π the projection $\mathrm{Fl}_{\vec{q}}\to M_n(\widetilde{C_0})$. We put $\tilde{\mathcal{F}}^u := (\mathrm{id}_{\widetilde{C_0}}\times\pi)^*\mathcal{F}^u$ and $\tilde{\mathcal{F}}_i^u := \pi^*\mathcal{F}_i^u$.

We define the line bundle $\mathcal{L}_{\vec{q}}$ on $Fl_{\vec{q}}$ as follows.

In the case where $n=2r$ and $q_r>0$, or where $n=2r+1$, if $0\subset\mathbb{F}_{r-1}(\tilde{\mathcal{F}}_i^u)\subset\cdots\subset\mathbb{F}_0(\tilde{\mathcal{F}}_i^u)\subset\tilde{\mathcal{F}}_i^u$ is the universal filtration, then

$$\mathcal{L}_{\vec{q}}:=\bigotimes_{i=1,2}\bigotimes_{j=1}^{r}\big(\mathbb{F}_{r-j+1}(\tilde{\mathcal{F}}_i^u)^{\perp}/\mathbb{F}_{r-j}(\tilde{\mathcal{F}}_i^u)^{\perp}\big)^{q_j}. \tag{6.2}$$

In the case where $n=2r$ and $q_r=0$, if $0\subset\mathbb{F}_{r-1}(\tilde{\mathcal{F}}_i^u)\subset\cdots\subset\mathbb{F}_1(\tilde{\mathcal{F}}_i^u)\subset\tilde{\mathcal{F}}_i^u$ is the universal filtration, then

$$\mathcal{L}_{\vec{q}}:=\bigotimes_{i=1,2}\bigotimes_{j=1}^{r-1}\big(\mathbb{F}_{r-j+1}(\tilde{\mathcal{F}}_i^u)^{\perp}/\mathbb{F}_{r-j}(\tilde{\mathcal{F}}_i^u)^{\perp}\big)^{q_j}.$$

Consider the line bundle $\pi^*(\det\mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)}\otimes\mathcal{L}_{\vec{q}}$ on $\mathrm{Fl}_{\vec{q}}$, where $\tilde{p}:\widetilde{C_0}\times M_n(\widetilde{C_0})\to M_n(\widetilde{C_0})$ is the projection.

Put $(\tilde{\mathcal{F}}^u)^{\flat}:=\mathrm{Ker}\big(\tilde{\mathcal{F}}^u\to\bigoplus_{i=1,2}\tilde{\mathcal{F}}_i^u/\mathbb{F}_{r-1}(\tilde{\mathcal{F}}_i^u)^{\perp}\big)$, where $\tilde{\mathcal{F}}_i^u$ is considered as a sheaf on $\{P_i\}\times Fl_{\vec{q}}$. If $q_1=l$, then

$$\begin{aligned}\pi^*(\det\mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)}\otimes\mathcal{L}_{\vec{q}}\simeq{}&\big(\det\mathbb{R}\tilde{p}'_*(\tilde{\mathcal{F}}^u)^{\flat}\big)^{\otimes(-l)}\\&\otimes\bigotimes_{i=1,2}\bigotimes_{j=2}^{r}\big(\mathbb{F}_{r-j+1}(\tilde{\mathcal{F}}_i^u)^{\perp}/\mathbb{F}_{r-j}(\tilde{\mathcal{F}}_i^u)^{\perp}\big)^{q_j},\end{aligned} \tag{6.3}$$

where $\tilde{p}':\widetilde{C_0}\times\mathrm{Fl}_{\vec{q}}\to\mathrm{Fl}_{\vec{q}}$ is the projection. (Ignore the term for $j=r$ in the case where $n=2r$ and $q_r=0$.) The right-hand side of (6.3) has a natural ι-linearization. Therefore the line bundle $\pi^*(\det\mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)}\otimes\mathcal{L}_{\vec{q}}$ is naturally an ι-equivariant line bundle on $\mathrm{Fl}_{\vec{q}}$ if $q_1=l$. So ι acts on the vector space $\mathrm{H}^0(\mathrm{Fl}_{\vec{q}},\pi^*(\det\mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)}\otimes\mathcal{L}_{\vec{q}})$ if $q_1=l$.

6.3. Statement of factorization theorem

Let $(\mathcal{E}^u,\gamma_{\mathcal{E}^u},\delta_{\mathcal{E}^u})$ be the universal oriented orthogonal sheaf over $C_0\times\bar{M}_n(C_0)$. We denote by p the projection $C_0\times\bar{M}_n(C_0)\to\bar{M}_n(C_0)$. Put $\mathcal{D}:=(\det\mathbb{R}p_*\mathcal{E}^u)^{\vee}$, the determinant line bundle.

To state the factorization theorem in a concise form, we understand that ι acts on the vector space $\mathrm{H}^0(\mathrm{Fl}_{\vec{q}},\pi^*(\det\mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)}\otimes\mathcal{L}_{\vec{q}})$ trivially if $q_1<l$.

THEOREM 6.1

There is a natural isomorphism

$$\mathrm{H}^0\big(\bar{M}_n^{\leq 1}(C_0),\mathcal{D}^{\otimes l}\big)\simeq\bigoplus_{\vec{q}}\mathrm{H}^0\big(\mathrm{Fl}_{\vec{q}},\pi^*(\det\mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)}\otimes\mathcal{L}_{\vec{q}}\big)^{\iota-\mathrm{inv}},$$

where $\vec{q}=(q_1,\dots,q_r)$ runs through all sequences of integers such that $l\geq q_1\geq\cdots\geq q_r\geq 0$ (see the paragraph after Definition 4.1 for the notation $\bar{M}_n^{\leq 1}(C_0)$).

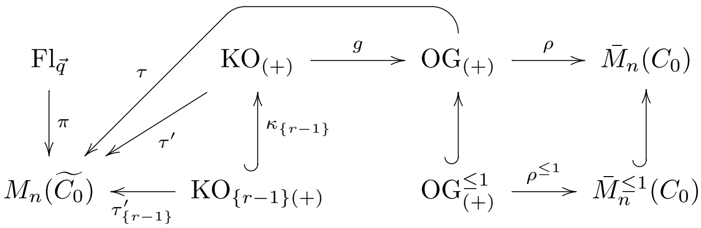
6.4. Proof of factorization theorem

We give a proof for $n=2r$. The case $n=2r+1$ is similar. Recall that τ denotes the projection $\mathrm{OG}_n(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}\to M_n(\widetilde{C_0})$. Put $\mathcal{F}_i^{u\dagger}=\tau^*\mathcal{F}_i^u$. Let $\mathcal{F}_1^{u\dagger}\oplus\mathcal{F}_2^{u\dagger}\to\mathcal{Q}$ be the universal quotient bundle over $\mathrm{OG}_n(\mathcal{F}_1^u\oplus\mathcal{F}_2^u)_{(+)}$. As in Section 4.3, we write $\mathrm{OG}_{(+)}$, $\mathrm{OG}^{\leq 1}_{(+)}$, and $\mathrm{OG}^{=1}_{(+)}$ for $\mathrm{OG}_n(\mathcal{F}^u_1 \oplus \mathcal{F}^u_2)_{(+)}$, $\mathrm{OG}^{\leq 1}_n(\mathcal{F}^u_1 \oplus \mathcal{F}^u_2)_{(+)}$, and $\mathrm{OG}^{=1}_n(\mathcal{F}^u_1 \oplus \mathcal{F}^u_2)_{(+)}$, respectively.

Consider the $\mathrm{KO}_{(+)}$-bundle $\tau' : \mathrm{KO}(\mathcal{F}^u_1, \mathcal{F}^u_2)_{(+)} \to M_n(\widetilde{C_0})$. Put $\mathcal{F}^{u\ddagger}_i = \tau'^* \mathcal{F}^u_i$. Let

$$\big(\mathcal{M}_i, \mu_i, \mathcal{F}^{u\ddagger}_{1,i} \to \mathcal{M}_i \otimes \mathcal{F}^{u\ddagger}_{1,i+1}, \mathcal{F}^{u\ddagger}_{1,i} \leftarrow \mathcal{F}^{u\ddagger}_{1,i+1},$$
$$\mathcal{F}^{u\ddagger}_{2,i+1} \to \mathcal{F}^{u\ddagger}_{2,i}, \mathcal{M}_i \otimes \mathcal{F}^{u\ddagger}_{2,i+1} \leftarrow \mathcal{F}^{u\ddagger}_{2,i+1} \ (0 \leq i \leq r-1), h : \mathcal{F}^{u\ddagger}_{1,r} \xrightarrow{\sim} \mathcal{F}^{u\ddagger}_{2,r}\big),$$

be the universal generalized orthogonal morphism over $\mathrm{KO}(\mathcal{F}^u_1, \mathcal{F}^u_2)_{(+)}$. Recall that for a subset $I \subset [0, r-1]$, $\mathrm{KO}(\mathcal{F}^u_1, \mathcal{F}^u_2)_{I(+)}$ denotes the locus $\bigcap_{i \in I} \{\mu_i = 0\}$. For short, we write $\mathrm{KO}_{(+)}$ and $\mathrm{KO}_{I(+)}$ for $\mathrm{KO}(\mathcal{F}^u_1, \mathcal{F}^u_2)_{(+)}$ and $\mathrm{KO}(\mathcal{F}^u_1, \mathcal{F}^u_2)_{I(+)}$, respectively. The restricted morphism $\tau'|_{\mathrm{KO}_{I(+)}}$ is denoted by τ'_I. Our situation is summarized in the following diagram:



To prove the factorization theorem, we first amplify Proposition 4.10. It follows from Lemma 2.1(2) and Proposition 5.10 that $\mathrm{OG}^{=1}_{(+)}$ and $\mathrm{KO}_{\{r-1\}(+)} \setminus \bigcup_{0 \leq i < r-1} \mathrm{KO}_{\{i\}(+)}$ parameterize the same objects. Thus restriction of g gives an isomorphism $\mathrm{KO}_{\{r-1\}(+)} \setminus \bigcup_{0 \leq i < r-1} \mathrm{KO}_{\{i\}(+)} \to \mathrm{OG}^{=1}_{(+)}$. Through this isomorphism, the involution on $\mathrm{OG}^{=1}_{(+)}$ gives rise to an involution on $\mathrm{KO}_{\{r-1\}(+)} \setminus \bigcup_{0 \leq i < r-1} \mathrm{KO}_{\{i\}(+)}$, which we denote also by ι.

LEMMA 6.2

The involution ι on $\mathrm{KO}_{\{r-1\}(+)} \setminus \bigcup_{0 \leq i < r-1} \mathrm{KO}_{\{i\}(+)}$ extends to $\mathrm{KO}_{\{r-1\}(+)}$.

Proof

We construct an involution on $\mathrm{KO}_{\{r-1\}(+)}$, which is an extension of ι. By Proposition 5.10, $\mathrm{KO}_{\{r-1\}(+)}$ parameterizes tuples

$$\big(F, \gamma, \delta, L_i \subset F|_{P_i} \ (i = 1, 2), \Phi\big), \tag{6.4}$$

where (F, γ, δ) is an oriented orthogonal bundle on $\widetilde{C_0}$, L_i is an isotropic line of $F|_{P_i}$, and $\Phi \in \mathrm{KO}(L_1^\perp / L_1, L_2^\perp / L_2)_{(+)}$. Given a tuple (6.4), let $(F^\iota, \gamma^\iota, \delta^\iota, L^\iota_i \subset F^\iota|_{P_i} \ (i = 1, 2))$ be the ι-transform of $(F, \gamma, \delta, L_i \subset F|_{P_i} \ (i = 1, 2))$ over $\{P_1, P_2\}$. Since $L_i^\perp / L_i \simeq L_i^{\iota\perp} / L^\iota_i$ by (2.6), Φ determines a generalized orthogonal morphism $\Phi^\iota \in \mathrm{KO}(L_1^{\iota\perp}/L^\iota_1, L_2^{\iota\perp}/L^\iota_2)_{(+)}$. By associating to a tuple (6.4) the tuple $(F^\iota, \gamma^\iota, \delta^\iota, L^\iota_i \subset F^\iota|_{P_i} \ (i = 1, 2), \Phi^\iota)$, we obtain an involution on $\mathrm{KO}_{\{r-1\}(+)}$, which is clearly an extension of ι. □

By abuse of notation, the extension to $\mathrm{KO}_{\{r-1\}(+)}$ of the involution ι is also denoted by ι. For a line bundle $\mathcal{L}$ on $\bar{M}_n(C_0)$, we have isomorphisms

$$\mathrm{H}^0(\mathrm{OG}^{\leq 1}_{(+)}, \rho^*\mathcal{L}) \simeq \mathrm{H}^0(\mathrm{OG}_{(+)}, \rho^*\mathcal{L}) \simeq \mathrm{H}^0\big(\mathrm{KO}_{(+)}, (\rho \circ g)^*\mathcal{L}\big) \tag{6.5}$$

because the codimension of the complement of $\mathrm{OG}^{\leq 1}_{(+)}$ in $\mathrm{OG}_{(+)}$ is equal to or greater than 2 by Lemma 2.1(3), and g is proper birational. Note also that for a line bundle $\mathcal{N}$ on $\mathrm{KO}_{\{r-1\}(+)}$, the map

$$\mathrm{H}^0(\mathrm{KO}_{\{r-1\}(+)}, \mathcal{N}) \to \mathrm{H}^0\Big(\mathrm{KO}_{\{r-1\}(+)} \setminus \bigcup_{0 \leq i < r-1} \mathrm{KO}_{\{i\}(+)}, \mathcal{N}\Big) \tag{6.6}$$

is injective. Let $\tilde{\mathfrak{r}}$ be the restriction map

$$\mathrm{H}^0\big(\mathrm{KO}_{(+)}, (\rho \circ g)^*\mathcal{L}\big) \to \mathrm{H}^0\big(\mathrm{KO}_{\{r-1\}(+)}, (\rho \circ g \circ \kappa_{\{r-1\}})^*\mathcal{L}\big).$$

Note that the target of this map has an ι-action. By (6.5) and the injectivity of (6.6), we can amplify Proposition 4.10 as follows:

$$\mathrm{H}^0\big(\bar{M}^{\leq 1}_n(C_0), \mathcal{L}\big) \simeq \tilde{\mathfrak{r}}^{-1}\big(\mathrm{H}^0(\mathrm{KO}_{\{r-1\}(+)}, (\rho \circ g \circ \kappa_{\{r-1\}})^*\mathcal{L})^{\iota-\mathrm{inv}}\big). \tag{6.7}$$

LEMMA 6.3

There is an isomorphism

$$(\rho \circ g)^*\mathcal{D}^{\otimes l} \simeq \tau'^*(\det \mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)} \otimes \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes l(r-i)}.$$

Proof

As in [A1, Lemma 7.5], we can prove that

$$\rho^*\mathcal{D}^{\otimes l} \simeq \tau^*(\det \mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)} \otimes (\det \mathcal{Q})^{\otimes l}.$$

Composing this isomorphism with (5.5), we get the result. □

By Theorem 5.14, we have a decomposition

$$(\det \mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)} \otimes \tau'_*\Big(\bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes l(r-i)}\Big) = \bigoplus_{\vec{q} \in \mathbb{B}} \mathcal{V}_{\vec{q}},$$

where $\mathbb{B}$ consists of all sequences $\vec{q} = (q_1, \dots, q_r)$ of integers such that $l \geq q_1 \geq \cdots \geq q_{r-1} \geq |q_r| \geq 0$. Similarly we have

$$(\det \mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)} \otimes \tau'_{\{r-1\}*}\Big(\bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes l(r-i)}\Big|_{\mathrm{KO}_{\{r-1\}(+)}}\Big) = \bigoplus_{\vec{q} \in \mathbb{B}'} \mathcal{V}_{\vec{q}},$$

where $\mathbb{B}'$ consists of all sequences $\vec{q} = (q_1, \dots, q_r)$ of integers such that $l = q_1 \geq \cdots \geq q_{r-1} \geq |q_r| \geq 0$. Using the projection formula, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0\big(\mathrm{KO}_{(+)}, (\rho\circ g)^*\mathcal{D}^{\otimes l}\big) & \xrightarrow{\quad\simeq\quad} & \bigoplus_{\vec{q}\in\mathbb{B}} \mathrm{H}^0\big(M_n(\widetilde{C_0}), \mathcal{V}_{\vec{q}}\big) \\ \Big\downarrow \mathfrak{r} & & \Big\downarrow \\ \mathrm{H}^0\big(\mathrm{KO}_{\{r-1\}(+)}, (\rho\circ g\circ\kappa_{\{r-1\}})^*\mathcal{D}^{\otimes l}\big) & \xrightarrow[(\spadesuit)]{\simeq} & \bigoplus_{\vec{q}\in\mathbb{B}'} \mathrm{H}^0\big(M_n(\widetilde{C_0}), \mathcal{V}_{\vec{q}}\big), \end{array}$$

where the right vertical arrow is the projection. The vector space $\mathrm{H}^0(\mathrm{KO}_{\{r-1\}(+)}, (\rho\circ g\circ\kappa_{\{r-1\}})^*\mathcal{D}^{\otimes l})$ has an ι-action, so we get an ι-action on $\bigoplus_{\vec{q}\in\mathbb{B}'} \mathrm{H}^0(M_n(\widetilde{C_0}), \mathcal{V}_{\vec{q}})$. By (6.7), we have an isomorphism

$$\begin{aligned} &\mathrm{H}^0\big(\bar{M}_n^{\le 1}(C_0), \mathcal{D}^{\otimes l}\big) \\ &\simeq \bigoplus_{\vec{q}\in\mathbb{B}\setminus\mathbb{B}'} \mathrm{H}^0\big(M_n(\widetilde{C_0}), \mathcal{V}_{\vec{q}}\big) \oplus \Big(\bigoplus_{\vec{q}\in\mathbb{B}'} \mathrm{H}^0\big(M_n(\widetilde{C_0}), \mathcal{V}_{\vec{q}}\big)\Big)^{\iota-\mathrm{inv}}. \end{aligned} \tag{6.8}$$

Now we analyze the ι-action on $\bigoplus_{\vec{q}\in\mathbb{B}'} \mathrm{H}^0(M_n(\widetilde{C_0}), \mathcal{V}_{\vec{q}})$. There is no reason that the action preserves the direct summands. Let $\mathbb{B}_{\ge 0}$ be the subset of $\mathbb{B}$ consisting of $\vec{q} = (q_1, \dots, q_r)$ with $q_r \ge 0$. Put $\mathbb{B}'_{\ge 0} = \mathbb{B}' \cap \mathbb{B}_{\ge 0}$. For $\vec{q} \in \mathbb{B}_{\ge 0}$, we put $H_{\vec{q}} := \mathrm{H}^0(M_n(\widetilde{C_0}), \mathcal{V}_{\vec{q}})$ if $q_r = 0$, and $H_{\vec{q}} := \mathrm{H}^0(M_n(\widetilde{C_0}), \mathcal{V}_{\vec{q}}) \oplus \mathrm{H}^0(M_n(\widetilde{C_0}), \mathcal{V}_{\vec{q}^\natural})$ if $q_r > 0$, where $\vec{q}^\natural = (q_1, \dots, q_{r-1}, -q_r)$. We define a partial order $\preceq$ on $\mathbb{B}_{\ge 0}$ so that $\vec{q'} \preceq \vec{q}$ if and only if $\sum_{j=1}^{r-i} q'_j \le \sum_{j=1}^{r-i} q_j$ for $0 \le i \le r-1$.

LEMMA 6.4

For $\vec{q} \in \mathbb{B}'_{\ge 0}$, the subspace $\bigoplus_{\mathbb{B}'_{\ge 0}\ni\vec{q'}\preceq\vec{q}} H_{\vec{q'}}$ of $\bigoplus_{\vec{q'}\in\mathbb{B}'_{\ge 0}} H_{\vec{q'}}$ is stable with respect to the ι-action.

Proof

By Theorem 5.14, $\bigoplus_{\mathbb{B}'_{\ge 0}\ni\vec{q'}\preceq\vec{q}} H_{\vec{q'}}$ is isomorphic to the subspace

$$\mathrm{H}^0\Big(\mathrm{KO}_{\{r-1\}(+)}, \tau'^*(\det \mathbb{R}\tilde{p}_*\mathcal{F}^u)^{\otimes(-l)} \otimes \bigotimes \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} q_j}\Big)$$

through the isomorphism $(\spadesuit)$. This subspace is clearly stable with respect to the ι-action. $\square$

We use the following result of linear algebra whose proof is left to the reader.

LEMMA 6.5

Let S be a finite set with a partial order $\preceq$. Assume that we are given an involution ι on a vector space $\bigoplus_{s\in S} V_s$ such that for any $t \in S$, $\iota\big(\bigoplus_{s'\preceq t} V_{s'}\big) = \bigoplus_{s'\preceq t} V_{s'}$. We denote by $\bar{\iota}$ the induced involution on the graded part $V_t \simeq \bigoplus_{s'\preceq t} V_{s'} / \bigoplus_{s'\prec t} V_{s'}$. Then the composite of morphisms $\big(\bigoplus_{s\in S} V_s\big)^{\iota-\mathrm{inv}} \hookrightarrow \bigoplus_{s\in S} V_s \xrightarrow{\text{projection}} \bigoplus_{s\in S} (V_s)^{\bar{\iota}-\mathrm{inv}}$ is an isomorphism.

By Lemma 6.4, we have an induced involution $\bar{\iota}$ on the graded part $H_{\vec{q}} \simeq \bigoplus_{\vec{q}' \preceq \vec{q}} H_{\vec{q}'} / \bigoplus_{\vec{q}' \prec \vec{q}} H_{\vec{q}'}$. By Lemma 6.5, we obtain, from (6.8), the isomorphism

$$\mathrm{H}^0\big(\bar{M}_n^{\leq 1}(C_0), \mathcal{D}^{\otimes l}\big) \simeq \bigoplus_{\vec{q} \in \mathbb{B}_{\geq 0} \setminus \mathbb{B}'_{\geq 0}} H_{\vec{q}} \oplus \bigoplus_{\vec{q} \in \mathbb{B}'_{\geq 0}} (H_{\vec{q}})^{\bar{\iota}-\mathrm{inv}}. \tag{6.9}$$

By Theorem 5.14 and Remark 5.15, we have an isomorphism

$$H_{\vec{q}} \simeq \mathrm{H}^0\big(\mathrm{Fl}_{\vec{q}}, \pi^*(\det \mathbb{R}\tilde{p}_* \mathcal{F}^u)^{\otimes(-l)} \otimes \mathcal{L}_{\vec{q}}\big). \tag{6.10}$$

Moreover, by the definition of the involutions, the $\bar{\iota}$-action on the left-hand side of (6.10) is nothing but the ι-action on the right-hand side of (6.10) defined in Section 6.2 if $\vec{q} \in \mathbb{B}'_{\geq 0}$. This completes the proof of Theorem 6.1. □

Complements

The factorization formula in Theorem 6.1 involves ι-inv. We can formulate the factorization theorem in such a way that ι-inv does not appear by considering moduli of vector bundles with a degenerate symmetric bilinear form. Let $M'_n(\widetilde{C_0})$ be the moduli stack parameterizing triples

$$\big(G, \gamma : G \otimes G \to \mathcal{O}_{\widetilde{C_0}}, \delta : \wedge^n G \to \mathcal{O}_{\widetilde{C_0}}(-P_1 - P_2)\big),$$

where G is a vector bundle of rank n on $\widetilde{C_0}$, γ is a symmetric bilinear form with $(G^\vee/G)_{P_i} \simeq k_{P_i}^{\oplus 2}$ $(i = 1, 2)$, and δ is an isomorphism such that the diagram

$$\begin{array}{ccc} \wedge^n G \otimes \wedge^n G & \xrightarrow{\delta \otimes \delta} & \mathcal{O}_{\widetilde{C_0}}(-P_1 - P_2) \otimes \mathcal{O}_{\widetilde{C_0}}(-P_1 - P_2) \\ {\scriptstyle \bar{\wedge}^n \gamma} \downarrow & & \downarrow \\ \mathcal{O}_{\widetilde{C_0}} & \xrightarrow{\simeq} & \mathcal{O}_{\widetilde{C_0}} \otimes \mathcal{O}_{\widetilde{C_0}} \end{array} \tag{6.11}$$

commutes. Note that for $[(G, \gamma, \delta)] \in M'_n(\widetilde{C_0})$, we have $\dim(G|_{P_i})^\perp = 2$. For a sequence $\vec{t} = (l \geq t_2 \geq \cdots \geq t_r \geq 0)$, we define the moduli stack $\mathbf{Fl}'_{\vec{t}}$ and the line bundle $\mathcal{A}_{\vec{t}}$ on it as follows. If $n = 2r$ and $t_r > 0$, or $n = 2r + 1$ (resp., if $n = 2r$ and $t_r = 0$), then $\mathbf{Fl}'_{\vec{t}}$ parameterizes $(G, \gamma, \delta) \in M'_n(\widetilde{C_0})$ together with filtrations, for $i = 1, 2$,

$$G|_{P_i} \supset G_r^{(i)} \supset \cdots \supset G_2^{(i)} \supset G_1^{(i)} = (G|_{P_i})^\perp \supset 0$$

$$\big(\text{resp., } G|_{P_i} \supset G_{r-1}^{(i)} \supset \cdots \supset G_2^{(i)} \supset G_1^{(i)} = (G|_{P_i})^\perp \supset 0\big),$$

where $G_j^{(i)}$ is an isotropic $(j+1)$-dimensional subspace. Let $\mathcal{G}$ be the universal bundle on $\widetilde{C_0} \times \mathbf{Fl}'_{\vec{t}}$, and let

$$\mathcal{G}|_{\{P_i\} \times \mathbf{Fl}'_{\vec{t}}} \supset \mathcal{G}_r^{(i)} \supset \cdots \supset \mathcal{G}_2^{(i)} \supset \mathcal{G}_1^{(i)} = (\mathcal{G}|_{\{P_i\} \times \mathbf{Fl}'_{\vec{t}}})^\perp \supset 0$$

$$\big(\text{resp., } \mathcal{G}|_{\{P_i\} \times \mathbf{Fl}'_{\vec{t}}} \supset \mathcal{G}_{r-1}^{(i)} \supset \cdots \supset \mathcal{G}_2^{(i)} \supset \mathcal{G}_1^{(i)} = (\mathcal{G}|_{\{P_i\} \times \mathbf{Fl}'_{\vec{t}}})^\perp \supset 0\big)$$

be the universal filtration. The line bundle $\mathcal{A}_{\vec{t}}$ on $\mathbf{Fl}'_{\vec{t}}$ is defined to be

$$(\det \mathbb{R}\,\mathrm{pr}_* \mathcal{G})^{\otimes(-l)} \otimes \bigotimes_{i=1,2} \bigotimes_{j=2}^{r} (\mathcal{G}_{j-1}^{(i)\perp}/\mathcal{G}_{j}^{(i)\perp})^{\otimes t_j}$$

$$\Bigl(\text{resp., } (\det \mathbb{R}\,\mathrm{pr}_* \mathcal{G})^{\otimes(-l)} \otimes \bigotimes_{i=1,2} \bigotimes_{j=2}^{r-1} (\mathcal{G}_{j-1}^{(i)\perp}/\mathcal{G}_{j}^{(i)\perp})^{\otimes t_j}\Bigr),$$

where pr is the projection $\widetilde{C_0} \times \mathbf{Fl}'_{\vec{t}} \to \mathbf{Fl}'_{\vec{t}}$.

For a sequence $\vec{q} = (l = q_1 \geq \cdots \geq q_r \geq 0)$, we have an isomorphism

$$\mathrm{H}^0\bigl(\mathrm{Fl}_{\vec{q}}, \pi^*(\det \mathbb{R}\tilde{p}_* \mathcal{F}^u)^{\otimes(-l)} \otimes \mathcal{L}_{\vec{q}}\bigr)^{\iota-\mathrm{inv}} \simeq \mathrm{H}^0(\mathbf{Fl}', \mathcal{A}_{\vec{t}}),$$

where $\vec{t} = (l \geq q_2 \geq \cdots \geq q_r \geq 0)$. So we can rewrite Theorem 6.1 as follows.

THEOREM 6.6

There is a natural isomorphism

$$\mathrm{H}^0\bigl(\bar{M}_n^{\leq 1}(C_0), \mathcal{D}^{\otimes l}\bigr) \simeq \bigoplus_{\vec{q}} \mathrm{H}^0\bigl(\mathrm{Fl}_{\vec{q}}, \pi^*(\det \mathbb{R}\tilde{p}_* \mathcal{F}^u)^{\otimes(-l)} \otimes \mathcal{L}_{\vec{q}}\bigr)$$
$$\oplus \bigoplus_{\vec{t}} \mathrm{H}^0(\mathbf{Fl}'_{\vec{t}}, \mathcal{A}_{\vec{t}}),$$

where $\vec{q} = (q_1, \ldots, q_r)$ runs through all sequences of integers such that $l > q_1 \geq \cdots \geq q_r \geq 0$, and $\vec{t} = (t_2, \ldots, t_r)$ such that $l \geq t_2 \geq \cdots \geq t_r \geq 0$.

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