Missing terms in the weighted Hardy–Sobolev inequalities and its application

Hiroshi Ando and Toshio Horiuchi

Abstract Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ containing the origin. In the present paper we establish the weighted Hardy–Sobolev inequalities with sharp remainders. For example, when $\alpha = 1 - n/p$ and $1 < p < +\infty$ hold, we establish the following inequality.

There exist positive numbers $\Lambda_{n,p,\alpha}, C$, and R such that we have

$$\int_\Omega |\nabla u|^p |x|^{\alpha p}\,dx \geq \Lambda_{n,p,\alpha} \int_\Omega \frac{|u(x)|^p}{|x|^n} A_1(|x|)^{-p}\,dx + C \int_\Omega \frac{|u(x)|^p}{|x|^n} A_1(|x|)^{-p} A_2(|x|)^{-2}\,dx \tag{0.1}$$

for any $u \in W^{1,p}_{\alpha,0}(\Omega)$. Here $A_1(|x|) = \log \frac{R}{|x|}$, and $A_2(|x|) = \log A_1(|x|)$. This is called the critical Hardy–Sobolev inequality with a sharp remainder involving a singular weight $A_1(|x|)^{-p} A_2(|x|)^{-2}$, in the sense that the improved inequality holds for this weight but fails for any weight more singular than this one. Here $\Lambda_{n,p,\alpha}$ is a sharp constant independent of each function u. Further we establish the Hardy–Sobolev inequalities in the subcritical case $(\alpha > 1 - n/p)$ and the supercritical case $(\alpha < 1 - n/p)$.

As an application, we use our improved inequality to determine exactly when the first eigenvalues of the weighted eigenvalue problems for the operators represented by $-\operatorname{div}(|x|^{\alpha p}|\nabla u|^{p-2}\nabla u) - \mu/|x|^n A_1(|x|)^{-p}|u|^{p-2}u$ (the critical case) will tend to zero as μ increases to $\Lambda_{n,p,\alpha}$. This also gives us sufficient conditions for the operators to have the positive first eigenvalue in a certain nontrivial functional framework, and we study the eigenvalue problem in the borderline case.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with $0 \in \Omega$ and $n \geq 1$. In this paper, we shall establish the weighted Hardy–Sobolev inequalities with sharp remainders. Assume temporally that $1 < p < \infty$ and $\alpha > 1 - n/p$, which is basic and called the *subcritical case*. Then we first consider a variant of the Hardy–Sobolev inequalities given by

$$\int_\Omega |\nabla u|^p |x|^{\alpha p}\,dx \geq \Lambda_{n,p,\alpha} \int_\Omega \frac{|u(x)|^p}{|x|^p} |x|^{\alpha p}\,dx \tag{1.1}$$

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for any $u \in W^{1,p}_{\alpha,0}(\Omega)$, where $\Lambda_{n,p,\alpha} = |\frac{n-p+p\alpha}{p}|^p$ is the best constant independent of each u. Here the best constant $\Lambda_{n,p,\alpha}$ in the subcritical case is given by the infimum of

$$\frac{\int_\Omega |\nabla u|^p |x|^{\alpha p}\,dx}{\int_\Omega \frac{|u(x)|^p}{|x|^p}|x|^{\alpha p}\,dx} \quad \text{for } u \in W^{1,p}_{\alpha,0}(\Omega) \setminus \{0\}.$$

Moreover there exists no extremal function in $W^{1,p}_{\alpha,0}(\Omega)$ which attains the infimum of this quantity. Roughly speaking, the candidates of the extremals are too singular at the origin to belong to the admissible class $W^{1,p}_{\alpha,0}(\Omega)$. Hence, it is natural to consider that there exist "missing terms" in the right-hand side of (1.1). In view of this, we shall investigate the Hardy–Sobolev inequalities represented by (1.1) and improve them by finding out missing terms. More precisely, in the subcritical case ($\alpha > 1 - n/p$ and $1 < p < +\infty$), we shall achieve an optimal improvement of the inequality (1.1) by adding a second term involving the singular weight $(\log \frac{1}{|x|})^{-2}$, in the sense that the improved inequality holds for this weight but fails for any weight more singular than this one. To make clear the sharpness of the inequalities, we shall introduce a family of test functions involving logarithmic functions. We also treat the cases $\alpha = 1 - n/p$ and $\alpha < 1 - n/p$, which are called the critical case and the supercritical case, respectively.

To make clear the relations of these new inequalities to the known ones, let us first recall briefly the following weighted Sobolev inequalities.

There is a positive number $S(p,q,\alpha,\beta,n)$ depending only on p, q, α, β, and n such that for any $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\,dx \geq S(p,q,\alpha,\beta,n)\Big(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\,dx\Big)^{p/q}, \tag{1.2}$$

where $n \geq 1$ and p, q, α, β are real parameters satisfying

$$0 \leq \frac{1}{p} - \frac{1}{q} = \frac{1-\alpha+\beta}{n}, \quad -\frac{n}{q} < \beta \leq \alpha, 1 \leq p < +\infty.$$

Here $S(p,q,\alpha,\beta,n)$ is the best constant given by the infimum of

$$\frac{\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\,dx}{\big(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\,dx\big)^{p/q}} \quad \text{for } u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \setminus \{0\}. \tag{1.3}$$

These inequalities are often called the *Caffarelli–Kohn–Nirenberg-type*. Actually in [4] they proved more general multiplicative inequalities. In [13] we also studied the inequalities (1.2) and obtained some results on the best constant, the existence and nonexistence of extremal functions, and their qualitative properties. Here we remark that if we consider the variational problem (1.3) in the radial function space instead of $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$, then the extremal functions are explicitly obtained by solving the corresponding Euler–Lagrange equations. Therefore the best constants, denoted by $S_R(p,q,\alpha,\beta,n)$, are known for $\alpha > 1 - n/p$ and $\alpha - 1 \leq \beta \leq \alpha$, and they readily satisfy $\lim_{\beta \to \alpha - 1 + 0} S_R(p,q,\alpha,\beta,n) = \Lambda_{n,p,\alpha}$. In an upcoming paper [17] we will establish the Caffarelli–Kohn–Nirenberg-type inequalities in both of the critical ($\alpha = 1 - n/p$) and supercritical ($\alpha < 1 - n/p$)

cases, and we will see that $\lim_{\beta\to\alpha-1+0} S(p,q,\alpha,\beta,n) = \Lambda_{n,p,\alpha}$ holds for any $\alpha \in \mathbb{R}$ as well. Hence the weighted Hardy inequalities are naturally regarded as the limit of the Caffarelli–Kohn–Nirenberg-type inequalities as $\beta \to \alpha - 1 + 0$. When $p = 2$ and $\alpha > 1 - n/2$, this was shown by Catrina and Wang [5]. They studied the inequality (1.2) with $p = 2$ and $\alpha > 1 - n/2$ intensively and obtained interesting results (see also [6]).

We also believe that the improved Hardy inequalities are not only of interest by themselves but are also applicable to various problems. To illustrate this, we will exploit our results to study the eigenvalue problem for the operators represented by

$$(1.4) \qquad -\operatorname{div}(|x|^{\alpha p}|\nabla u|^{p-2}\nabla u) - \frac{\mu}{|x|^{p-p\alpha}}|u|^{p-2}u \quad \text{(the subcritical case).}$$

First we use our improved inequality to determine exactly when the first eigenvalues of the weighted eigenvalue problems for the operators (1.4) will tend to 0 as μ increases to the best constant. This also gives us sufficient conditions for the operators considered to have the positive first eigenvalue in a certain nontrivial functional framework, and it seems to be interesting to study more precisely the borderline case. In this regard, we will give some basic but nontrivial results, assuming that $p = 2$, $\alpha = 1 - n/2$ (the critical case), and $\Omega = B_1 = \{x : |x| < 1\}$. Let us consider the operators on $C_0^\infty(B_1)$ given by

$$(1.5) \qquad P_\mu u = -\operatorname{div}(|x|^{2-n}\nabla u) - \mu \frac{u}{|x|^n\left(\log \frac{R}{|x|}\right)^2} \quad \text{(the critical case),}$$

where $R > 1$ and $\mu \le 1/4$. Here note that the best constant $\Lambda_{n,2,2-n/2} = 1/4$ in this case. Then we will study the following eigenvalue problem with a parameter $\lambda \ge 0$:

$$(1.6) \qquad \begin{cases} P_\mu u = \lambda |x|^a u & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $a > -n$.

When $\mu < 1/4$, this problem can be dealt with in a usual functional framework, namely, in a Hilbert space $W^{1,2}_{2-n/2,0}(B_1)$ defined in Section 2. But if $\mu = 1/4$, then, apart from the previous case, the presence of the potential term with the best constant in the weighted Hardy–Sobolev inequalities prevents us from treating this problem in $W^{1,2}_{2-n/2,0}(B_1)$. With the the aid of the improved Hardy inequalities, this fact leads us to the use of a Hilbert space $V = V^{1,2}_{2-n/2,0}(B_1)$ (see Definition 2.7) closely related to the operator $P_{1/4}$. Then we will see that $P_{1/4} : V \to V'$ (the dual) actually becomes continuous, linear, and self-adjoint. Further we will see the unboundedness of the first eigenfunction u_1 in addition to some (classical) results on the eigenvalue problem such as the existence of a sequence of eigenvalues and the positivity of the first eigenvalue as Theorem 2.3 in Section 2.

In the subcritical case ($\alpha > 1 - n/2$) for $p = 2$, this topic was actually treated by Garcia-Azorero, Peral, and Primo [10]. They studied semilinear elliptic problems in the borderline case involving weights of the same type. We do not go further into this topic in the present paper. It seems interesting for us to study various nonlinear elliptic problems in the critical and the supercritical cases.

In the rest of this section let us give a brief historical remark. We use the following notation from now on.

DEFINITION 1.1

For a given positive number R, we set for $t > 0$ and $k = 2, 3, \dots$

$$(1.7) \qquad A_1(t) := \log \frac{R}{t}, \qquad A_k(t) := \log A_{k-1}(t), \qquad e_1 := e, \qquad e_k := e^{e_{k-1}}.$$

When no weight function is involved, such improved Hardy inequalities are already studied by many authors. Adimurthi, Nirmalendu, Chaudhuri, and Ramaswamy [1] have proved the following.

PROPOSITION 1.1

There exists a constant $C > 0$, depending on $n \geq 2, 1 < p < n$ and $R > \sup_\Omega(|x|e^{2/p})$ such that for $u \in W_0^{1,p}(\Omega)$

$$(1.8) \qquad \int_\Omega |\nabla u|^p \, dx \geq \Big(\frac{n-p}{p}\Big)^p \int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx + C \int_\Omega \frac{|u(x)|^p}{|x|^p} A_1(|x|)^{-2} \, dx.$$

When $p = 2$, we have proved in [9] the existence of *finitely many sharp missing terms* of the Hardy–Sobolev inequality (1.1). In an upcoming paper [3] we will establish a similar result in the weighted case. For the case of the Laplacian (Rellich-type inequality) we have proved in [7] and [8] the following.

PROPOSITION 1.2

Let $n \geq 3$, $0 \in \Omega$ and Ω is a bounded domain in $\mathbb{R}^n$.

(1) Noncritical case ($1 < p < n/2$)

Then there exists $K = K(n) > 0$ and $C = C(n) > 0$ such that if $R > K \sup_\Omega |x|$, then

$$(1.9) \qquad \int_\Omega |\Delta u|^p \, dx \geq \lambda_{n,p} \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} \, dx + C \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} A_1(|x|)^{-2} \, dx$$

for any $u \in W_0^{2,p}(\Omega)$ and $\lambda_{n,p} = (n - 2p/p)^p (np - n/p)^p$.

(2) Critical case ($p = n/2$)

Then there exists $K^* = K^*(n) > 0$ and $C^* = C^*(n) > 0$ such that if $R > K^* \sup_\Omega |x|$, then

$$(1.10) \qquad \int_\Omega |\Delta u|^{n/2} \, dx \geq \lambda_{n/2} \int_\Omega \frac{|u(x)|^{n/2}}{|x|^n A_1(|x|)^{n/2}} \, dx + C^* \int_\Omega \frac{|u(x)|^{n/2}}{|x|^n A_1(|x|)^{n/2+1}} \, dx$$

for any $u \in W_0^{2,n/2}(\Omega)$ and $\lambda_{n/2} = (n - 2/\sqrt{n})^n$.

As for the weighted case, a similar improved inequality was introduced in [15, Theorem 11.1] to study a blow-up profile of a minimal solution for a certain class of quasilinear elliptic equations. See also [14] for the higher-order case. The present work is strongly motivated by our recent research in this field.

This paper is organized in the following way. In Section 2 we describe our main results on Hardy–Sobolev inequalities including an application to a nonlinear eigenvalue problem. In Section 3 we prepare lemmas that are needed in the proof of the main results stated in Section 2. In Section 4 we prove Theorem 2.1 and Corollary 2.1, which are stated in Section 2. In Sections 5 and 6 we give proofs to Theorem 2.2 and Theorem 2.3, respectively, and in Section 7 we describe some related results in the linear case without a proof.

2. Main result and its application

2.1. Main result

First we modify the classical Sobolev spaces so that we can treat the weighted Hardy inequalities and variational problems in the subsequent sections. Let us begin with defining weighted Sobolev spaces.

Let $1 \le p < +\infty$. First we assume that α is a real number satisfying $\alpha > -n/p$. Let $L^p(\Omega, |x|^{p\alpha})$ denote the space of Lebesgue measurable functions, defined on a bounded domain $\Omega \subset \mathbb{R}^n$, for which

$$\|u\|_{L^p(\Omega,|x|^{p\alpha})} = \Big(\int_\Omega |u|^p |x|^{\alpha p}\, dx \Big)^{1/p} < +\infty. \tag{2.1}$$

$W^{1,p}_{\alpha,0}(\Omega)$ is given by the completion of $C_0^\infty(\Omega)$ with respect to the norm defined by

$$\|u\|_{W^{1,p}_{\alpha,0}(\Omega)} = \| |\nabla u| \|_{L^p(\Omega,|x|^{p\alpha})}. \tag{2.2}$$

Then $W^{1,p}_{\alpha,0}(\Omega)$ becomes a Banach space with the norm $\| \cdot \|_{W^{1,p}_{\alpha,0}(\Omega)}$.

To study the Hardy–Sobolev inequality in the supercritical case, we prepare the following. For any $\alpha \in \mathbb{R}$, by $\dot{W}^{1,p}_{\alpha,0}(\Omega)$ we denote the completion of $C_0^\infty(\Omega \setminus \{0\})$ with respect to the norm defined by

$$\|u\|_{\dot{W}^{1,p}_{\alpha,0}(\Omega)} = \| |\nabla u| \|_{L^p(\Omega,|x|^{p\alpha})} + \|u\|_{L^p(\Omega,|x|^{p(\alpha-1)})}. \tag{2.3}$$

Then $\dot{W}^{1,p}_{\alpha,0}(\Omega)$ is also a Banach space with the norm $\| \cdot \|_{\dot{W}^{1,p}_{\alpha,0}(\Omega)}$. Here we note that if $1 - n/p < \alpha$, then $\dot{W}^{1,p}_{\alpha,0}(\Omega)$ coincides with $W^{1,p}_{\alpha,0}(\Omega)$ by the Hardy inequality (2.6). But if $\alpha \le 1 - n/p$, then $\dot{W}^{1,p}_{\alpha,0}(\Omega)$, roughly speaking, consists of functions vanishing at the origin, because $|x|^{p(\alpha-1)} \notin L^1_{\mathrm{loc}}(\Omega)$.

We also define a Banach space of radial functions. For a ball $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ we set

$$\begin{cases} R^{1,p}_{\alpha,0}(B_r) = \{u \in W^{1,p}_{\alpha,0}(B_r) : u \text{ is a radial function}\}, \\ \|u\|_{R^{1,p}_{\alpha,0}(B_r)} = \|u\|_{W^{1,p}_{\alpha,0}(B_r)}. \end{cases} \tag{2.4}$$

Now we give the definition of the Hardy–Sobolev best constant.

DEFINITION 2.1

For $1<p<+\infty$ we set

$$\Lambda_{n,p,\alpha}=\begin{cases}\left|\frac{n-p+\alpha p}{p}\right|^p, & \text{if } \alpha\neq 1-\frac{n}{p},\\ \left(\frac{p-1}{p}\right)^p, & \text{if } \alpha=1-\frac{n}{p}.\end{cases}\tag{2.5}$$

We recall the notation given by Definition 1.1.

For a given positive number R, we set for $t>0$ and $k=2,3,\dots$

$$A_1(t):=\log\frac{R}{t},\qquad A_k(t):=\log A_{k-1}(t),\qquad e_1:=e,\qquad e_k:=e^{e_{k-1}}.$$

Using this notation, we are in a position to state our main result.

THEOREM 2.1

Let $n\geq 1$, $0\in\Omega$, and Ω is a bounded domain in $\mathbb{R}^n$.

(1) Subcritical case ($\alpha>1-n/p$, $1<p<+\infty$)

There exist $K=K(n)>1$ and $C=C(n)>0$ such that if $R>K\sup_\Omega|x|$, then

$$\int_\Omega|\nabla u|^p|x|^{\alpha p}\,dx\geq\Lambda_{n,p,\alpha}\int_\Omega\frac{|u(x)|^p}{|x|^p}|x|^{\alpha p}\,dx+C\int_\Omega\frac{|u(x)|^p}{|x|^p}A_1(|x|)^{-2}|x|^{\alpha p}\,dx\tag{2.6}$$

for any $u\in W^{1,p}_{\alpha,0}(\Omega)$.

(2) Critical case ($\alpha=1-n/p$, $1<p<+\infty$)

Then there exist $K=K(n)>1$ and $C=C(n)>0$ such that if $R>K\sup_\Omega|x|$, then

$$\int_\Omega|\nabla u|^p|x|^{\alpha p}\,dx\geq\Lambda_{n,p,\alpha}\int_\Omega\frac{|u(x)|^p}{|x|^n}A_1(|x|)^{-p}\,dx+C\int_\Omega\frac{|u(x)|^p}{|x|^n}A_1(|x|)^{-p}A_2(|x|)^{-2}\,dx\tag{2.7}$$

for any $u\in W^{1,p}_{\alpha,0}(\Omega)$.

(3) Supercritical case ($\alpha<1-n/p$, $1<p<+\infty$)

Then there exist $K=K(n)>0$ and $C=C(n)>0$ such that if $R>K\sup_\Omega|x|$, then

$$\int_\Omega|\nabla u|^p|x|^{\alpha p}\,dx\geq\Lambda_{n,p,\alpha}\int_\Omega\frac{|u(x)|^p}{|x|^p}|x|^{\alpha p}\,dx+C\int_\Omega\frac{|u(x)|^p}{|x|^p}A_1(|x|)^{-2}|x|^{\alpha p}\,dx\tag{2.8}$$

for any $u\in\dot W^{1,p}_{\alpha,0}(\Omega)$.

REMARK 2.1

The constant $\Lambda_{n,p,\alpha}$ is the best possible. Further weight functions of the terms in the right-hand side are sharp, which will be shown in Section 4 (cf. [2, Theorem 3.4]).

REMARK 2.2

If we restrict ourselves to the case that $\alpha = 0$, then the so-called rearrangement argument works effectively. Therefore in this case we may assume that Ω is a ball and $u \in W^{1,p}_{\alpha,0}(\Omega)$ is radial. But in the other cases, the rearrangement is not valid in general. Hence we employ a polar coordinate system instead, and we establish direct estimates from below for the left-hand side; then it results in the radial case.

REMARK 2.3

When $p \geq 2$, the proof seems to be straightforward, even if it needs many steps. On the other hand, when $1 < p < 2$, the proof requires additional technical lemmas, which are inspired by [1].

REMARK 2.4

If Ω is an unbounded domain of $\mathbb{R}^n$, then the weight functions $A_k(t)\,(k = 1, 2, \ldots)$ are not defined, and it seems that there exists no missing term. In fact, it will be shown in the upcoming paper [17] that there exists no missing term provided that $\Omega = \mathbb{R}^n$ and $p = 2$. When $\Omega = \mathbb{R}^n$ and $p \neq 2$, somewhat weaker nonexistence results will be given there.

2.2. Applications

In this section we state two applications of our improved inequality. One is an immediate consequence of Theorem 2.1, and the other is concerned with a nonlinear eigenvalue problem. Let us prepare more notation.

DEFINITION 2.2

If $\alpha \neq 1 - n/p$, then we define

$$F_{p,\alpha} = \Big\{ f : \Omega \to \mathbb{R}^+ \;\Big|\; f \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\}), \limsup_{|x| \to 0} \frac{|x|^{p(1-\alpha)} f(x)}{\big(\log \frac{1}{|x|}\big)^{-2}} < \infty \Big\}.$$

If $\alpha = 1 - n/p$, then we define

$$F_{p,\alpha} = \Big\{ f : \Omega \to \mathbb{R}^+ \;\Big|\; f \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \setminus \{0\}), \limsup_{|x| \to 0} \frac{|x|^n \big(\log \frac{1}{|x|}\big)^p f(x)}{\big(\log\big(\log \frac{1}{|x|}\big)\big)^{-2}} < \infty \Big\}.$$

Then as a corollary to Theorem 2.1, we immediately have the following. The proof will be given in Section 4.4.

COROLLARY 2.1

Assume that $1 < p < +\infty$.

(1) *Assume that* $f \in F_{p,\alpha}$, *then there exists* $\lambda(f) > 0$ *such that for a sufficiently large* $R > 0$ *we have*

$$\int_\Omega |\nabla u|^p |x|^{\alpha p}\,dx$$
$$\geq \Lambda_{n,p,\alpha} \times \begin{cases} \int_\Omega \frac{|u(x)|^p}{|x|^p} |x|^{\alpha p}\,dx + \lambda(f)\int_\Omega |u(x)|^p f(x)\,dx, & \text{if } \alpha \neq 1-\frac{n}{p}, \\ \int_\Omega \frac{|u(x)|^p}{|x|^n} A_1(|x|)^{-p}\,dx + \lambda(f)\int_\Omega |u(x)|^p f(x)\,dx, & \text{if } \alpha = 1-\frac{n}{p}, \end{cases}$$

where $u \in W^{1,p}_{\alpha,0}(\Omega)$ *if* $1-n/p \leq \alpha$, *and* $u \in \dot{W}^{1,p}_{\alpha,0}(\Omega)$ *if* $\alpha < 1-n/p$. *Here* $\lambda(f)$ *and* R *are independent of each* u.

(2) *Assume that* $f \notin F_{p,\alpha}$. *Further, assume that*

$$\begin{cases} \lim_{|x|\to 0} |x|^{p(1-\alpha)} f(x)\big(\log\frac{1}{|x|}\big)^2 = \infty, & \text{if } \alpha \neq 1-\frac{n}{p} \\ \lim_{|x|\to 0} |x|^n f(x)\big(\log\frac{1}{|x|}\big)^p \big(\log\big(\log\frac{1}{|x|}\big)\big)^2 = \infty, & \text{if } \alpha = 1-\frac{n}{p}. \end{cases}$$

Then no inequality of type in the assertion 1 can hold.

Next we define quasilinear degenerate elliptic operators and a class of singular weight functions.

DEFINITION 2.3

For $1 < p < +\infty$ and we set

$$L_{p,\alpha} u = -\operatorname{div}(|x|^{p\alpha} |\nabla u|^{p-2} \nabla u). \tag{2.9}$$

DEFINITION 2.4

If $\alpha \neq 1-n/p$, then we define

$$\mathcal{F}_{p,\alpha} = \Big\{ f : \Omega \to \mathbb{R}^+ \;\Big|\; f \in L^\infty_{\rm loc}(\bar{\Omega}\setminus\{0\}),\ \lim_{|x|\to 0} |x|^{p(1-\alpha)} f(x) = 0 \Big\}.$$

If $\alpha = 1-n/p$, then we define

$$\mathcal{F}_{p,\alpha} = \Big\{ f : \Omega \to \mathbb{R}^+ \;\Big|\; f \in L^\infty_{\rm loc}(\bar{\Omega}\setminus\{0\}),\ \lim_{|x|\to 0} |x|^n \Big(\log\frac{1}{|x|}\Big)^p f(x) = 0 \Big\}.$$

Then in the subcritical and the critical cases, we study the nonlinear eigenvalue problems given by

$$\begin{cases} L_{p,\alpha} u - \frac{\mu}{|x|^{p-p\alpha}} |u|^{p-2} u = \lambda |u|^{p-2} u f \ \text{in } \Omega, & \text{if } \alpha > 1-\frac{n}{p}, \\ L_{p,\alpha} u - \frac{\mu}{|x|^n} A_1(|x|)^{-p} |u|^{p-2} u = \lambda |u|^{p-2} u f \ \text{in } \Omega, & \text{if } \alpha = 1-\frac{n}{p}, \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.10}$$

where $f \in \mathcal{F}_{p,\alpha}$. For $0 \leq \mu < \Lambda_{n,p,\alpha}$ and $\lambda \in \mathbb{R}$, we look for weak solutions $u \in W^{1,p}_{\alpha,0}(\Omega)$ of these problems, and we study the asymptotic behavior of the first eigenvalues for different singular weights as μ increases to $\Lambda_{n,p,\alpha}$, after which the operator $L_{p,\alpha}$ is no more bounded from below. Here we define a weak solution in the following way.

DEFINITION 2.5

$u \in W^{1,p}_{\alpha,0}(\Omega)$ is said to be a weak solution of (2.10) if and only if for any $\phi \in C^1(\bar{\Omega})$

with $\phi = 0$ on $\partial\Omega$

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,|x|^{p\alpha}\,dx\\ &\quad=\begin{cases}\int_\Omega\big(\frac{\mu}{|x|^p}|u|^{p-2}u\phi|x|^{p\alpha}-\lambda|u|^{p-2}u\phi f\big)\,dx, & \text{if } \alpha>1-\frac{n}{p}\\ \int_\Omega\big(\frac{\mu}{|x|^n}A_1(|x|)^{-p}|u|^{p-2}u\phi|x|^{p\alpha}-\lambda|u|^{p-2}u\phi f\big)\,dx, & \text{if } \alpha=1-\frac{n}{p}.\end{cases}\end{aligned}$$

Then we have

THEOREM 2.2

Assume that $\alpha \geq 1-\frac{n}{p}$, $1<p<+\infty$ and $f\in\mathcal{F}_{p,\alpha}$. Then we have the following.

(1) *The above problem admits a positive weak solution $u\in W^{1,p}_{\alpha,0}(\Omega)$ for all $0\leq\mu<\Lambda_{n,p,\alpha}$, corresponding to the first eigenvalue $\lambda=\lambda^1_\mu(f)>0$.*

(2) *As μ increases to $\Lambda_{n,p,\alpha}$, $\lambda^1_\mu(f)\to\lambda^1(f)\geq 0$ for all $f\in\mathcal{F}_{p,\alpha}$.*

(3) *The limit $\lambda^1(f)>0$ if $f\in F_{p,\alpha}$.*

(4) *If $f\notin F_{p,\alpha}$ and if*

$$\begin{cases}\lim_{|x|\to 0}|x|^{p-\alpha p}f(x)\big(\log\frac{1}{|x|}\big)^2=+\infty, & \text{if } \alpha>1-\frac{n}{p},\\ \lim_{|x|\to 0}|x|^n f(x)\big(\log\frac{1}{|x|}\big)^p\big(\log\big(\log\frac{1}{|x|}\big)\big)^2=+\infty, & \text{if } \alpha=1-\frac{n}{p},\end{cases}$$

then the limit $\lambda^1(f)=0$.

This is proved in Section 5.

Next we further study the borderline case ($\mu=\Lambda_{n,p,\alpha}$), assuming that $\Omega=B_1=\{x\in\mathbb{R}^n;|x|<1\}$, $p=2$, and $\alpha=1-n/2$ (the critical case). Let us recall the notation in Section 1.

DEFINITION 2.6

For $R>1$ and $\mu\leq 1/4$, we define for any $u\in C_0^\infty(B_1)$

$$P_\mu u=-\operatorname{div}(|x|^{2-n}\nabla u)-\mu\frac{u}{|x|^n A_1(|x|)^2}\quad\text{(the critical case).}\tag{2.11}$$

Here note that the best constant satisfies $\Lambda_{n,2,2-n/2}=1/4$ in this case. Now we define a norm which is suitable for the operator $P_{1/4}$ as follows.

DEFINITION 2.7

For any $u\in C_0^\infty(B_1)$

$$\|u\|^2_{V^{1,2}_{2-n/2,0}(B_1)}=\int_{B_1}|\nabla u|^2|x|^{2-n}\,dx-\frac{1}{4}\int_{B_1}\frac{u^2}{|x|^nA_1(|x|)^2}\,dx.\tag{2.12}$$

By the aid of the improved weighted Hardy inequality (2.8), we see that $\|\cdot\|_{V^{1,2}_{2-n/2,0}(B_1)}$ defines a norm.

DEFINITION 2.8

By $V^{1,2}_{2-n/2,0}(B_1)$ we denote the completion of $C_0^\infty(B_1)$ with respect to this norm.

For the sake of simplicity we set $V = V^{1,2}_{2-n/2,0}(B_1)$. Then V clearly becomes a Hilbert space with inner product

$$(u,v)_V = \int_{B_1} \nabla u \cdot \nabla v \, |x|^{2-n} \, dx - \frac{1}{4} \int_{B_1} \frac{uv}{|x|^n A_1(|x|)^2} \, dx,$$

for any $u, v \in V$.

We remark that V is not imbedded into $W^{1,2}_{2-n/2,0}(B_1)$. In fact we give a function U in Section 6 such that $U \in V$ but $U \notin W^{1,2}_{2-n/2,0}(B_1)$. Now we study the following eigenvalue problem in this framework.

For $\lambda \geq 0$, $\mu \leq 1/4$, $a > -n$ and $u \in V$,

$$\begin{cases} P_\mu u = \lambda |x|^a u & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{2.13}$$

First we remark that the weight function $|x|^a$ belongs to $F_{2,1-n/2}$ provided that $a > -n$. Then it follows from Theorem 2.2 that the problem admits a positive first eigenvalue and corresponding eigenfunctions in $W^{1,2}_{2-n/2,0}(B_1)$ for any $\mu < 1/4$. Now we assume that $\mu = 1/4$. Then we have the following.

THEOREM 2.3

Assume that $\mu = 1/4$ and $a > -n$. Then we have the following.

(1) *There is a sequence of eigenvalues $\{\lambda_k\}$ with $0 < \lambda_k \to +\infty$ as $k \to \infty$. The first eigenvalue λ_1 is simple and the corresponding eigenfunction has a constant sign in B_1.*

(2) *By $u_1 \in V = V^{1,2}_{2-n/2,0}(B_1)$ we denote the first positive eigenfunction. Then $u_1 \notin L^\infty(B_1)$.*

The second assertion will be proved by constructing an unbounded subsolution to the eigenvalue problem.

3. Preliminary lemmas

In this section we prepare lemmas which are needed to prove our main results. First we prepare elementary inequalities.

LEMMA 3.1

For $1 < p \leq 2$ and $M \geq 1$, we have

$$|1+X|^p - 1 - pX \geq c(p) \begin{cases} M^{p-2} X^2, & |X| \leq M, \\ |X|^p, & |X| \geq M. \end{cases} \tag{3.1}$$

Here $c(p)$ is a positive number independent of each X and $M \geq 1$.

Proof

When $X > -1$, this follows from Taylor expansion. If we choose $c(p)$ sufficiently small, then it remains valid for $X \leq -1$. □

The following elementary lemma is useful to show the sharpness of our improved inequalities.

LEMMA 3.2

For $1 < p$ *we have*

$$|1+X|^p - 1 - pX \le C(p)\frac{X^2}{1+X^2}(1+|X|^p). \tag{3.2}$$

Here $C(p)$ *is a positive number independent of each* $X \in \mathbb{R}$.

Proof

Noting that

$$\lim_{X\to 0}\frac{|1+X|^p - 1 - pX}{X^2} = \frac{p(p-1)}{2}, \qquad \lim_{X\to\pm\infty}\frac{|1+X|^p - 1 - pX}{|X|^p} = 1,$$

the assertion is clear. □

Then we give a series of 1-dimensional weighted Hardy's inequalities. Let us recall the notation (1.7); that is, $A_1(r) = \log\frac{R}{r}, A_2(r) = \log A_1(r)$ and so on.

LEMMA 3.3

Assume that $R > 1$, $q < 1$, *and* $\nu = 1 - q/2$. *Then for any* $h \in C^1((0,1])$ *with* $h(1) = 0$, $\int_0^1 |h'(r)|^2 A_1(r)^q r\, dr < +\infty$ *and* $\lim_{r\to 0} h(r)^2 A_1(r)^{q-1} = 0$, *we have*

$$\int_0^1 |h'(r)|^2 A_1(r)^q r\, dr \ge \nu^2 \int_0^1 |h(r)|^2 A_1(r)^{q-2}\frac{dr}{r}. \tag{3.3}$$

More precisely, for any subinterval $[a,b] \subset [0,1]$ *we have*

$$\begin{aligned}\int_a^b |h'(r)|^2 A_1(r)^q r\, dr \ge \nu^2 &\int_a^b |h(r)|^2 A_1(r)^{q-2}\frac{dr}{r}\\ &- \nu\Big(\frac{h(b)^2}{A_1(b)^{1-q}} - \frac{h(a)^2}{A_1(a)^{1-q}}\Big).\end{aligned} \tag{3.4}$$

Proof

Let $h(r) = A_1(r)^\nu w(r)$. Then $w(0) = w(1) = 0$, and

$$\begin{aligned}|h'(r)|^2 &= A_1(r)^{(\nu-1)2}\Big(-\frac{\nu}{r}w(r) + w'(r)A_1(r)\Big)^2\\ &\ge \Big(\frac{\nu}{r}\Big)^2 |h(r)|^2 A_1(r)^{-2} - \frac{\nu}{r}\Big(\frac{d}{dr}w^2(r)\Big)A_1(r)^{2\nu-1}.\end{aligned} \tag{3.5}$$

Hence for $2\nu - 1 + q = 0$, we have

$$\int_0^1 |h'(r)|^2 A_1(r)^q r\, dr \ge \nu^2 \int_0^1 |h(r)|^2 A_1(r)^{q-2}\frac{dr}{r}.$$

The rest of the proof is now clear. □

LEMMA 3.4

Assume that $R > e$. *Then for any* $h \in C((0,1])$ *with* $h(1) = 0$, $\int_0^1 |h'(r)|^2 \times A_1(r) r\, dr < +\infty$ *and* $\lim_{r\to 0} h(r)^2 A_2(r)^{-1} = 0$, *we have*

$$\int_0^1 |h'(r)|^2 r A_1(r)\, dr \geq \frac{1}{4}\int_0^1 \frac{|h(r)|^2}{A_1(r)\cdot A_2(r)^2}\frac{dr}{r}. \tag{3.6}$$

More precisely, for any subinterval $[a,b] \subset [0,1]$ *we have*

$$\begin{aligned}\int_a^b |h'(r)|^2 r A_1(r)\, dr \geq \frac{1}{4}\int_a^b \frac{|h(r)|^2}{A_1(r)\cdot A_2(r)^2}\frac{dr}{r} \\ -\frac{1}{2}\Big(\frac{h(b)^2}{A_2(b)} - \frac{h(a)^2}{A_2(a)}\Big).\end{aligned} \tag{3.7}$$

Proof

Let $h(r) = A_2(r)^\nu w(r)$. Then $w(0) = w(1) = 0$, and

$$\begin{aligned}|h'(r)|^2 &= A_2(r)^{(\nu-1)2}\Big(-\frac{\nu}{r}\frac{w(r)}{A_1(r)} + w'(r)\log\big(A_1(r)\big)\Big)^2 \\ &\geq \Big(\frac{\nu}{r}\Big)^2 \frac{|h(r)|^2}{A_1(r)^2 A_2(r)^2} - \frac{\nu}{r}\Big(\frac{d}{dr}w^2(r)\Big)\frac{A_2(r)^{2\nu-1}}{A_1(r)}.\end{aligned} \tag{3.8}$$

Then we have

$$\begin{aligned}\int_0^1 |h'(r)|^2 r A_1(r)\, dr \geq \nu^2 \int_0^1 \frac{|h(r)|^2}{A_1(r)\cdot A_2(r)^2}\frac{dr}{r} \\ - \nu \int_0^1 \Big(\frac{d}{dr}w^2(r)\Big) A_2(r)^{2\nu-1}\, dr.\end{aligned}$$

Putting $\nu = 1/2$ we have the desired inequality, and the rest of the proof is also clear. □

Here we have the following definition.

DEFINITION 3.1

A function $\varphi \in C^1([0,1])$ is said to belong to $G([0,1])$ if and only if $\varphi(0) = \varphi(1) = 0$ and $\varphi'(0)\cdot\varphi'(1) \neq 0$.

DEFINITION 3.2

For $\varphi \in G([0,1])$ and $M > 1$ we define three subsets of $[0,1]$ as follows:

$$\begin{cases} A(\varphi, M) = \big\{r \in [0,1] \,\big|\, |\varphi'(r)| \leq M\frac{|\varphi(r)|}{r}\big\} \\ B(\varphi, M) = \big\{r \in [0,1] \,\big|\, |\varphi'(r)| > M\frac{|\varphi(r)|}{r}\big\} \\ C(\varphi, M) = \big\{r \in [0,1] \,\big|\, |\varphi'(r)| = M\frac{|\varphi(r)|}{r}\big\}. \end{cases} \tag{3.9}$$

Clearly we see $[0,1] = A(\varphi, M) \cup B(\varphi, M)$. The set $C(\varphi, M)$ coincides with the set of critical points of $\log(|\varphi| r^{\pm M})$. Moreover $1 \in B(\varphi, M)$ and $0 \in A(\varphi, M)$ for any $M > 1$, since $\varphi(1) = 0$ and $\lim_{r\to +0}\varphi(r)/r = \varphi'(0)$.

Then we prepare an approximation lemma.

LEMMA 3.5

Let $M>1$, and let $\varphi \in G([0,1]) \cap C^2([0,1])$. Assume that $\varphi \geq 0$. Then there exists a sequence of functions $\varphi_k \in G([0,1]) \cap C^2([0,1])$ such that $\varphi_k > 0$ in $(0,1)$, $\varphi_k \to \varphi$ in $C^1([0,1])$ as $k \to +\infty$, and $C(\varphi_k, M)$ consists of finite points for any k.

Proof

Take $\varphi \in G([0,1]) \cap C^2([0,1])$. By the definition of $G([0,1])$ we have $\varphi'(0) \cdot \varphi'(1) \neq 0$; hence, there is some positive number $\delta \in (0,1/2)$ such that $\varphi' \neq 0$ on $[0,\delta] \cup [1-\delta,1]$. Choose $\xi \in C_0^\infty(0,1)$ such that $\xi \geq 0$ and $\xi = 1$ on $I_\delta = [\delta, 1-\delta]$. Now we define for a positive ε

$$\varphi^\varepsilon = \varphi + \varepsilon\xi.$$

For a sufficiently small ε we immediately see $\varphi^\varepsilon \in G([0,1]) \cap C^2([0,1])$, $(\varphi^\varepsilon)' \neq 0$ on $[0,\delta] \cup [1-\delta,1]$ and $\varphi^\varepsilon = \varphi + \varepsilon$ on I_δ. Moreover we have for a small $\delta > 0$

$$([0,\delta] \cup [1-\delta,1]) \cap C(\varphi^\varepsilon, M) = \emptyset.$$

In fact, if $x \in C(\varphi^\varepsilon, M)$, then we should have $r\left|\frac{(\varphi^\varepsilon)'}{\varphi^\varepsilon}\right| = M > 1$. But this is impossible for a small $\delta > 0$ because of $\lim_{r\to 0} r\frac{(\varphi^\varepsilon)'}{\varphi^\varepsilon} = 1$ and $\lim_{r\to 1} r\frac{(\varphi^\varepsilon)'}{\varphi^\varepsilon} = +\infty$. For the sake of simplicity, we assume that $C(\varphi, M)$ consists only of $\{r\varphi' - M\varphi = 0\}$.

Put

$$\psi = r\varphi' - M\varphi. \tag{3.10}$$

Then we have $C(\varphi, M) = \psi^{-1}(0)$. Since $\psi \in C^1$, a set of all regular values of ψ is dense in $\mathbb{R}$. Therefore there exists a sequence of regular values $\{\varepsilon_k M\}_{k=1}^\infty$ satisfying $\varepsilon_k \to 0$ as $k \to \infty$. Since we may assume $C(\varphi, M) \subset I_\delta$, we see

$$\psi^{-1}(\varepsilon_k M) = \bigl\{ r \in I_\delta \bigm| r(\varphi + \varepsilon_k)' = M(\varphi + \varepsilon_k)\bigr\}. \tag{3.11}$$

After all we define

$$\varphi_k = \varphi + \varepsilon_k \xi, \quad k = 1, 2, \ldots. \tag{3.12}$$

Then $\{\varphi_k\}_{k=1}^\infty$ has the desired property. □

We also prepare various estimates which will be useful in the proof of Theorem 2.1 in the case $1 < p < 2$.

LEMMA 3.6

Assume that $1 < p < 2$ and $R > 1$. Then for any $\varepsilon > 0$ there is a positive number M such that

$$\int_{B(\varphi,M)} \frac{|\varphi||\varphi'|}{A_1(r)}\, dr \leq \varepsilon \int_{B(\varphi,M)} |\varphi|^{2-p} |\varphi'|^p r^{p-1}\, dr \tag{3.13}$$

holds for any $\varphi \in G([0,1])$.

Proof

We may assume that $\varphi > 0$. Then by the definition we have $r\frac{|\varphi'|}{\varphi} > M$ on $B(\varphi, M)$. Hence we immediately have

$$\varphi^{2-p}|\varphi'|^p r^{p-1} = \varphi|\varphi'| \cdot \Big(r\frac{|\varphi'|}{\varphi}\Big)^{p-1} \geq M^{p-1}\varphi|\varphi'|, \quad \text{on } B(\varphi, M). \tag{3.14}$$

Therefore it suffices to choose M so that

$$M^{1-p}(\log R)^{-1} \leq \varepsilon. \tag{3.15}$$

□

LEMMA 3.7

Assume that $1 < p < 2$ *and* $R > 1$. *Then*

$$\begin{aligned} \int_{A(\varphi,M)} |\varphi'(r)|^2 r\,dr \geq \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{rA_1(r)^2}\,dr \\ + \frac{1}{2}\int_{B(\varphi,M)} \frac{|\varphi|^2}{rA_1(r)^2}\,dr - \int_{B(\varphi,M)} \frac{|\varphi||\varphi'|}{A_1(r)}\,dr \end{aligned} \tag{3.16}$$

holds for any $\varphi \in G([0,1])$.

Proof

By the density argument it suffices to prove this for $\varphi \in C^\infty$. Moreover we may assume that the set $C(\varphi, M)$ consists of finitely many points. If not, from the previous lemma we can approximate φ uniformly in the C^1-topology by a sequence of $\varphi_k \in G([0,1])$ with having this property. Therefore we may assume that the set $A(\varphi, M)$ is a union of finite number of disjoint intervals. Namely,

$$\begin{cases} A(\varphi, M) = \bigcup_{k=0}^m [a_{2k}, a_{2k+1}], \\ B(\varphi, M) = \bigcup_{k=0}^{m-1} (a_{2k+1}, a_{2(k+1)}) \cup (a_{2m+1}, a_{2m+2}] \end{cases} \tag{3.17}$$

with $a_0 = 0$ and $a_{2m+2} = 1$. Then it follows from Lemma 3.3 that

$$\begin{aligned} \int_{A(\varphi,M)} |\varphi'(r)|^2 r\,dr &\geq \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{rA_1(r)^2}\,dr \\ &\quad - \frac{1}{2}\sum_{k=0}^{m}\Big(\frac{\varphi(a_{2k+1})^2}{A_1(a_{2k+1})} - \frac{\varphi(a_{2k})^2}{A_1(a_{2k})}\Big) \\ &= \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{rA_1(r)^2}\,dr + \frac{1}{2}\sum_{k=1}^{m+1}\Big(\frac{\varphi(a_{2k})^2}{A_1(a_{2k})} - \frac{\varphi(a_{2k-1})^2}{A_1(a_{2k-1})}\Big) \\ &= \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{rA_1(r)^2}\,dr + \frac{1}{2}\int_{B(\varphi,M)} \frac{d}{dr}\Big(\frac{\varphi(r)^2}{A_1(r)}\Big)\,dr \\ &\geq \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{rA_1(r)^2}\,dr \\ &\quad + \frac{1}{2}\int_{B(\varphi,M)} \frac{|\varphi|^2}{rA_1(r)^2}\,dr - \int_{B(\varphi,M)} \frac{|\varphi||\varphi'|}{A_1(r)}\,dr. \end{aligned} \tag{3.18}$$

Thus we have the desired estimate. □

In a quite similar way we have the following.

LEMMA 3.8
Assume that $1 < p < 2$ *and* $R > 1$. *Then*

$$\int_{A(\varphi,M)} |\varphi'(r)|^2 r A_1(r)\,dr \geq \frac{1}{4}\int_{A(\varphi,M)} \frac{|\varphi|^2}{rA_1(r)A_2(r)^2}\,dr + \frac{1}{2}\int_{B(\varphi,M)} \frac{|\varphi|^2}{rA_1(r)A_2(r)^2}\,dr - \int_{B(\varphi,M)} \frac{|\varphi||\varphi'|}{A_2(r)}\,dr \tag{3.19}$$

holds for any $\varphi \in G([0,1])$.

4. Proof of Theorem 2.1

We shall give the proof of Theorem 2.1 through subsections and the proof will be finished in Section 4.3. The proof of Corollary 2.1 will be given in Section 4.4. More precisely, in Section 4.1, we shall begin with reducing the assertions in Theorem 2.1 to the corresponding 1-dimensional variational problems consisting of three different types. In Section 4.2 we shall solve these variational problems in each case. Then in Section 4.3, the sharpness of the constant $\Lambda_{n,p,a}$ and the optimality of the logarithmic weight functions will be shown by constructing suitable test functions.

4.1. Step 1 (Reduction to 1-dimensional case)

We start with treating subcritical case ($1 < p < +\infty, \alpha > 1 - n/p$). We study a variational problem defined by

$$J(p,\alpha,n,\Omega) = \inf\Big[\int_\Omega |\nabla u|^p |x|^{\alpha p}\,dx : u \in W^{1,p}_{\alpha,0}(\Omega), \int_\Omega |u|^p |x|^{\alpha p - p}\,dx = 1\Big]. \tag{4.1}$$

Here we can assume that u is nonnegative and smooth as usual. By a polar coordinate system, we rewrite (4.1) to obtain

$$J(p,\alpha,n,\Omega) = \inf\Big[\int_{S^{n-1}} dS_\omega \int_0^\infty |\nabla u|^p r^{\alpha p + n - 1}\,dr : 0 \leq u \in W^{1,p}_{\alpha,0}(\Omega), \int_{S^{n-1}} dS_\omega \int_0^\infty u^p r^{\alpha p - p + n - 1}\,dr = 1\Big]. \tag{4.2}$$

By $(u,v)_{S^{n-1}}$ for $u, v \in L^2(S^{n-1})$ we denote the inner product on $L^2(S^{n-1})$ with the measure dS_ω. By a polar coordinate system, the Laplacian Δ is represented by $r^{1-n}\partial_r(r^{n-1}\partial_r \cdot) + \Delta_{S^{n-1}}/r^2$. Here $\Delta_{S^{n-1}}$ is the Laplace Beltrami operator on the unit sphere. Then we have

$$|\nabla u|^2 = |\partial_r u(r\omega)|^2 + \frac{1}{r^2}|\Lambda u|^2 \geq |\partial_r u(r\omega)|^2. \tag{4.3}$$

Here the spherical gradient operator Λ is defined by

$$(\Delta_{S^{n-1}} u, v)_{S^{n-1}} = (\Lambda u, \Lambda v)_{S^{n-1}} \quad \text{for } u \in C^2(S^{n-1}). \tag{4.4}$$

Now we reduce the variational problem (4.2) to the corresponding 1-dimensional one. To do so we need more notation. Let B^i and B^e be open balls with centers at the origin such as $B^i \subset \Omega \subset B^e$. We introduce a variational problem defined by

$$\begin{aligned} j(p,\alpha,n,d_e) = \inf\Big[\int_0^{d_e} & |\partial_r v(r)|^p r^{\alpha p+n-1}\,dr : \\ & 0 \leq v \in R^{1,p}_{\alpha,0}(B^e), \int_0^{d_e} v(r)^p r^{\alpha p-p+n-1}\,dr = 1\Big], \end{aligned} \tag{4.5}$$

where $d_e = \mathrm{diam}(B^e)/2$. Note that $v \in R^{1,p}_{\alpha,0}(B_{d_e})$ satisfies $v(d_e) = 0$, and this problem is independent of the value d_e. We shall show $j(p,\alpha,n,d) \geq \Lambda_{n,p,\alpha}$ for any $d > 0$ in the second step.

On the other hand, it is an easy task to see that $J(p,\alpha,n,\Omega) \geq j(p,\alpha,n,d_e)$. In fact putting $v_\omega(r) = u(r\omega)$ for $\omega \in S^{n-1}$, we immediately have

$$\begin{aligned} \int_\Omega |\nabla u|^p |x|^{p\alpha}\,dx &\geq \int_{S^{n-1}} dS_\omega \int_0^{d_e} |\partial_r v_\omega(r)|^p\, r^{n-1+p\alpha}\,dr \\ &\geq j(p,\alpha,n,d_e) \int_{S^{n-1}} dS_\omega \int_0^{d_e} v_\omega(r)^p r^{\alpha p-p+n-1}\,dr \\ &= j(p,\alpha,n,d_e) \int_\Omega |u|^p |x|^{\alpha p-p}\,dx. \end{aligned} \tag{4.6}$$

Then from the invariance of $j(p,\alpha,n,d)$ with respect to d and the canonical inclusion $W^{1,p}_{0,\alpha}(B^i) \subset W^{1,p}_{0,\alpha}(\Omega) \subset W^{1,p}_{0,\alpha}(B^e)$ it follows that $J(p,\alpha,n,\Omega) \geq j(p,\alpha,n,d_e)$. Therefore in the proof we can assume that u is radial and Ω is a unit ball.

We proceed to the supercritical case $(1 < p < +\infty, \alpha < 1 - n/p)$. Since $\alpha p - p < -n$, $|x|^{p\alpha-p} \notin L^1_{\mathrm{loc}}(\Omega)$. Therefore we employ $\dot{W}^{1,p}_{\alpha,0}(\Omega)$ instead of $W^{1,p}_{\alpha,0}(\Omega)$. For $0 \leq u \in \dot{W}^{1,p}_{\alpha,0}(\Omega) \cap C_0^\infty(\Omega \setminus \{0\})$ we set

$$u(x) = |x|^l w, \quad w \geq 0, w \in C_0^\infty(\Omega \setminus \{0\}). \tag{4.7}$$

By l we denote the integer part of $2 - \alpha - n/p$; namely, l is the positive integer satisfying $1 - \alpha - n/p < l \leq 2 - \alpha - n/p$. Then there is a positive number C such that we have for any $u \in \dot{W}^{1,p}_{\alpha,0}(\Omega) \cap C_0^\infty(\Omega \setminus \{0\})$

$$|\nabla w|^p |x|^{p(\alpha+l)} \leq C(|\nabla u|^p |x|^{p\alpha} + |u|^p |x|^{p(\alpha-1)}).$$

Hence, we see that $w \in W^{1,p}_{\alpha+l,0}(\Omega) \cap C_0^\infty(\Omega \setminus \{0\})$. By the aid of polar coordinate system again, we have

$$\begin{aligned} \big|\nabla(|x|^l w)\big|^p &= \big||x|^{2l}|\nabla w|^2 + l^2|x|^{2(l-1)}w^2 + 2|x|^{2(l-1)}wx\cdot\nabla w\big|^{p/2} \\ &\geq \big(r^{2l}(\partial_r w)^2 + l^2 r^{2(l-1)}w^2 + 2r^{2l-1}w\partial_r w\big)^{p/2} \\ &= |r^l\partial_r w + lr^{l-1}w|^p = |\partial_r(r^l w)|^p. \end{aligned} \tag{4.8}$$

Hence we see

$$\int_\Omega |\nabla u|^p|x|^{p\alpha}\,dx \geq \int_{S^{n-1}} dS_\omega \int_0^{d_e} \big|\partial_r\big(r^l w(r\omega)\big)\big|^p\, r^{\alpha p+n-1}\,dr. \tag{4.9}$$

Then we introduce a variational problem given by

$$\begin{aligned} k(p,\alpha,n,d_e) = \inf\Big[&\int_0^{d_e} \big|\partial_r\big(r^l v(r)\big)\big|^p r^{\alpha p+n-1}\,dr : \\ &0\leq v\in R^{1,p}_{\alpha+l,0}(B^e),\ \int_0^{d_e} v(r)^p r^{\alpha p+lp-p+n-1}\,dr = 1\Big], \end{aligned} \tag{4.10}$$

where $d_e = \mathrm{diam}(B^e)/2$ and note that $\alpha p + lp - p + n - 1 > -1$. We shall show that $k(p,\alpha,n,d) \geq \Lambda_{n,p,\alpha}$ for any $d>0$ in the second step. As before, it is an easy task to see that $J(p,\alpha,n,\Omega) \geq k(p,\alpha,n,d_e)$ in this case. In fact putting $v_\omega(r) = u(r\omega)$ for $\omega\in S^{n-1}$, we immediately have

$$\int_\Omega |\nabla u|^p|x|^{p\alpha}\,dx \geq k(p,\alpha,n,d_e)\int_\Omega |u|^p|x|^{\alpha p-p}\,dx. \tag{4.11}$$

Again from the invariance of $k(p,\alpha,n,d)$ with respect to d and the inclusion $W^{1,p}_{\alpha,0}(B^i)\subset W^{1,p}_{\alpha,0}(\Omega)\subset W^{1,p}_{\alpha,0}(B^e)$ it follows that $J(p,\alpha,n,\Omega)\geq k(p,\alpha,n,d_e)$. Therefore we can assume that u is radial and Ω is a unit ball.

Lastly we consider the critical case $(1<p<+\infty, \alpha = 1-n/p)$. In this case we introduce the following 1-dimensional problem:

$$\begin{aligned} l(p,\alpha,n,d_e,R) = \inf\Big[&\int_0^{d_e} |\partial_r v(r)|^p r^{p-1}\,dr : \\ &0\leq v\in R^{1,p}_{\alpha,0}(B^e),\ \int_0^{d_e} \frac{|v(r)|^p}{r}A_1(r)^{-p}\,dr = 1\Big], \end{aligned} \tag{4.12}$$

where R is any constant satisfying $R>d_e$. Then we shall show $l(p,\alpha,n,d,R)\geq \Lambda_{n,p,\alpha}$ for any $d>0$ and $R>d$ in the second step. As before, it is an easy task to see that $J(p,\alpha,n,\Omega)\geq l(p,\alpha,n,d_e,R)$ in this case. In fact putting $v_\omega(r) = u(r\omega)$ for $\omega\in S^{n-1}$, we immediately have

$$\int_\Omega |\nabla u|^p|x|^{p-n}\,dx \geq l(p,\alpha,n,d_e,R)\int_\Omega \frac{|u(x)|^p}{|x|^n}A_1(|x|)^{-p}\,dx. \tag{4.13}$$

Here we note that the last integral is a decreasing function of the constant R satisfying the condition $d_e<R$. Then again from the invariance of $l(p,\alpha,n,d,R)$ with respect to d and $R>d$ and from the inclusion $W^{1,p}_{\alpha,0}(B^i)\subset W^{1,p}_{\alpha,0}(\Omega)\subset W^{1,p}_{\alpha,0}(B^e)$ it follows that $J(p,\alpha,n,\Omega)\geq l(p,\alpha,n,d_e,R)$. Therefore in the proof we can assume that u is radial and Ω is a unit ball.

REMARK 4.1

If we assume $\alpha = 0$, then we can employ the symmetric rearrangement of domains and functions. Briefly, we recall the decreasing rearrangement. For a domain Ω we define a ball Ω^* such that $|\Omega^*| = |\Omega|$ with center at the origin. For a measurable function $u : \Omega \to \mathbb{R}$ we denote by $u^* : \Omega^* \to [0, \infty]$ the symmetric decreasing rearrangement of u. It is well known that the symmetric rearrangement does not change the L^p-norm. Then it follows from the Hardy–Littlewood inequality that $\int_\Omega \frac{|u|^p}{|x|^p}\, dx \le \int_{\Omega^*} \frac{u^{*p}}{|x|^p}\, dx$. Moreover it follows from the Pólya–Szegö principle that if u is also weakly differentiable with $|\nabla u| \in L^p(\Omega)$ for some $p \ge 1$, then u^* has the same properties, and $\int_\Omega |\nabla u|^p\, dx \le \int_{\Omega^*} |\nabla u^*|^p\, dx$. Therefore we can assume that u is radial and Ω is a ball in the proof of the main results, if $\alpha = 0$. A similar reduction is also possible provided that $1 - n/p < \alpha \le 0$. For the details, see [13, Lemma 2.3 $(p = q)$]. For the symmetric decreasing rearrangement, see [16]–[18], and [19] (see also [14, Lemmas 3.1–3.3]).

4.2. Step 2 (1-dimensional variational problems)

In this subsection we effectively employ a method of change of unknown functions not only to solve 1-dimensional variational problems but also to show the existence of sharp remainders of the 1-dimensional Hardy–Sobolev inequalities, which are best possible up to constant times. Though we shall see in Section 4.3 the sharpness of $\Lambda_{n,p,\alpha}$ as well as other weight functions, we note that the sharpness of $\Lambda_{n,p,\alpha}$ itself is also seen from the classical Hardy's inequality given in Remark 4.2 in the end of this subsection.

First in the subcritical case $(1 < p < +\infty, \alpha > 1 - n/p)$ we study the problem $j(p, \alpha, n, d_e)$, where $d_e = \mathrm{diam}(B^e)/2$. Since this problem is independent of the value d_e, we assume $d_e = 1$, and we show that $j(p, \alpha, n, 1) \ge \Lambda_{n,p,\alpha}$. Let us recall

$$j(p,\alpha,n,1) = \inf\Big[\int_0^1 |\partial_r v(r)|^p r^{\alpha p+n-1}\, dr : \\ 0 \le v \in R^{1,p}_{\alpha,0}(B_1), \int_0^1 v(r)^p r^{\alpha p - p + n - 1}\, dr = 1\Big], \tag{4.14}$$

where $B_1 = \{x \in \mathbb{R}^n \mid |x| < 1\}$ is a unit ball.

For a radial $v \in C_0^\infty(B_1)$, we define

$$v(r) = h(r) r^{-\delta}, \quad \delta = \alpha - \Big(1 - \frac{n}{p}\Big) > 0, r = |x|. \tag{4.15}$$

Here we may assume that $v > 0$ in B_1 without loss of generality. Then we have

$$|\nabla v(r)|^p = \big|\nabla\big(h(r) r^{-\delta}\big)\big|^p = \delta^p h^p \Big|1 - \frac{r h'}{\delta h}\Big|^p r^{-\alpha p - n}. \tag{4.16}$$

Then, noting $\delta^p = \Lambda_{n,p,\alpha}$, we have

$$\int_{B_1} |\nabla v|^p |x|^{\alpha p}\, dx - \Lambda_{n,p,\alpha} \int_{B_1} \frac{|v(x)|^p}{|x|^p} |x|^{\alpha p}\, dx \\ = \omega_n \Lambda_{n,p,\alpha} \int_0^1 h^p(r) \Big\{\Big|1 - \frac{r h'}{\delta h}\Big|^p - 1\Big\} \frac{dr}{r}. \tag{4.17}$$

For the moment we assume that $p \geq 2$. By a fundamental inequality $|1+t|^p \geq 1 + pt + c(p)|t|^2$ $(t \in \mathbb{R}, p \geq 2, c(p);$ a small positive number), we obtain

$$\begin{aligned}
&\int_{B_1} |\nabla v|^p |x|^{\alpha p}\, dx - \Lambda_{n,p,\alpha} \int_{B_1} \frac{|v(x)|^p}{|x|^p} |x|^{\alpha p}\, dx \\
&\geq -\omega_n \frac{\Lambda_{n,p,\alpha}}{\delta} \int_0^1 p h^{p-1} h'\, dr + c(p)\omega_n \frac{\Lambda_{n,p,\alpha}}{\delta^2} \int_0^1 h^{p-2}(h')^2 r\, dr \\
&= c(p)\omega_n \frac{\Lambda_{n,p,\alpha}}{\delta^2} \int_0^1 h^{p-2}(r)|h'(r)|^2 r\, dr \quad (\text{note that } h(0) = h(1) = 0) \\
&= c(p)\omega_n \frac{4}{p^2} \frac{\Lambda_{n,p,\alpha}}{\delta^2} \int_0^1 \left|\left(h^{\frac{p}{2}}(r)\right)'\right|^2 r\, dr.
\end{aligned} \tag{4.18}$$

Using Lemma 3.3 ($\nu = 1/2, q = 0$), we get

$$\begin{aligned}
\int_0^1 \left|\left(h^{p/2}(r)\right)'\right|^2 r\, dr &\geq \frac{1}{4} \int_0^1 \left(\frac{h^{p/2}(r)}{r A_1(r)}\right)^2 r\, dr \\
&= \frac{1}{4\omega_n} \int_{B_1} \frac{|v(x)|^p}{|x|^p} A_1(r)^{-2} |x|^{\alpha p}\, dx.
\end{aligned} \tag{4.19}$$

Combining this with (4.18) we get the inequality $j(p,\alpha,n,1) \geq \Lambda_{n,p,\alpha}$ and the inequality (2.6) where $C = c(p)\Lambda_{n,p,\alpha}/\delta^2 4/p^2$. In step 3 we shall show the sharpness of these inequalities.

Now we take care of the case that $1 < p < 2$. In this case we need technical lemmas simply because of the lack of uniform estimate of $(1+t)^p$ from below. To overcome this, we borrow a basic idea from [1] and modify it to apply to our case, where the so-called *decreasing rearrangement argument* does not work. Suppose that M is sufficiently large, which will be specified later. For $v \in C_0^\infty(B_1), v > 0$, radial, again we set $v(r) = h(r) r^{-\delta}$, $\delta = \alpha - 1 + n/p > 0$.

By virtue of the definitions of $A(\varphi, M)$, $B(\varphi, M)$, and $C(\varphi, M)$ replacing φ and M by h and δM, respectively, we have $[0,1] = A(h, \delta M) \cup B(h, \delta M)$. By the density argument we can assume that $h \in G([0,1])$; that is, $h'(0) \cdot h'(1) \neq 0$. Moreover from Lemma 3.5 we can assume that $h > 0$ in $(0,1)$ and $C(h, \delta M)$ consists of finite points. For $h \in G([0,1])$ and $M > 1$ we define three subsets of $[0,1]$ as follows:

$$\begin{cases}
A(h,\delta M) = \{r \in [0,1] \mid r|h'(r)| \leq \delta M |h(r)|\} \\
B(h,\delta M) = \{r \in [0,1] \mid r|h'(r)| > \delta M |h(r)|\} \\
C(h,\delta M) = \{r \in [0,1] \mid r|h'(r)| = \delta M |h(r)|\}.
\end{cases} \tag{4.20}$$

Then, Lemma 3.1 implies

$$\begin{aligned}
&\int_{B_1} |\nabla v|^p |x|^{\alpha p}\, dx - \Lambda_{n,p,\alpha} \int_{B_1} \frac{|v(x)|^p}{|x|^p} |x|^{\alpha p}\, dx \\
&\quad = \omega_n \Lambda_{n,p,\alpha} \int_0^1 h^p(r) \left\{ \left|1 - \frac{r h'}{\delta h}\right|^p - 1 \right\} \frac{dr}{r}
\end{aligned}$$

$$\begin{aligned}
&\geq -\omega_n \frac{\Lambda_{n,p,\alpha}}{\delta} \int_0^1 p h^{p-1} h' \, dr \\
&\quad + \omega_n \Lambda_{n,p,\alpha} \int_{A(h,\delta M)} h^p c(p) M^{p-2} \Big(\frac{rh'}{\delta h}\Big)^2 \frac{1}{r} \, dr \qquad (4.21) \\
&\quad + \omega_n \Lambda_{n,p,\alpha} \int_{B(h,\delta M)} h^p c(p) \Big(\frac{rh'}{\delta h}\Big)^p \frac{1}{r} \, dr \\
&= \omega_n \Lambda_{n,p,\alpha} c(p) M^{p-2} \delta^{-2} \int_{A(h,\delta M)} h^{p-2} (h')^2 r \, dr \\
&\quad + \omega_n \Lambda_{n,p,\alpha} c(p) \delta^{-p} \int_{B(h,\delta M)} (h')^p r^{p-1} \, dr.
\end{aligned}$$

Then, noting Lemma 3.7 and $A(h, \delta M) = A(h^{p/2}, p\delta M/2)$, we have

$$\begin{aligned}
\int_{A(h,\delta M)} h^{p-2} (h')^2 r \, dr &= \frac{4}{p^2} \int_{A(h,\delta M)} \big((h^{p/2})'\big)^2 r \, dr \\
&\geq \frac{4}{p^2} \Big(\frac{1}{4} \int_{A(h,\delta M)} \frac{h^p}{r A_1(r)^2} \, dr + \frac{1}{2} \int_{B(h,\delta M)} \frac{h^p}{r A_1(r)^2} \, dr \qquad (4.22) \\
&\quad - \frac{p}{2} \int_{B(h,\delta M)} \frac{h^{p-1} |h'|}{A_1(r)} \, dr \Big).
\end{aligned}$$

Similar to the proof in Lemma 3.6 we can estimate the last term to obtain

$$\frac{p}{2} \int_{B(h,\delta M)} \frac{h^{p-1} |h'|}{A_1(r)} \, dr \leq \frac{p}{2} \frac{1}{(\delta M)^{p-1} \log R} \int_{B(h,\delta M)} |h'|^p r^{p-1} \, dr. \qquad (4.23)$$

Here we simply use the fact that $r|h'| > \delta M h$ holds on the set $B(h, \delta M)$. Combining this with (4.21) and (4.22), choosing M large enough, we have the desired inequality.

Second in the supercritical case $(1 < p < +\infty, \alpha < 1 - n/p)$ we study the problem $k(p, \alpha, n, d_e)$, where $d_e = \mathrm{diam}(B^e)/2$. Since this problem is independent of the value d_e, we assume that $d_e = 1$, and we show that $k(p, \alpha, n, 1) \geq \Lambda_{n,p,\alpha}$. Let us recall

$$\begin{aligned}
k(p, \alpha, n, 1) = \inf \Big[&\int_0^1 \big|\partial_r \big(r^l v(r)\big)\big|^p r^{\alpha p + n - 1} \, dr : \qquad (4.24) \\
&0 \leq v \in R^{1,p}_{\alpha+l,0}(B_1), \int_0^1 v(r)^p r^{\alpha p + lp - p + n - 1} \, dr = 1 \Big],
\end{aligned}$$

where $B_1 = \{x \in \mathbb{R}^n \mid |x| < 1\}$ is a unit ball.

For $v \in C_0^\infty(B_1), v > 0$, radial, we define

$$v(r) = h(r) r^{-l-\delta}, \quad \delta = \alpha - \Big(1 - \frac{n}{p}\Big) < 0. \qquad (4.25)$$

Since $-1 - \delta = -l + (1 - \alpha - n/p) < 0$, we have $h(0) = h(1) = 0$. Then we have

$$\big|\nabla \big(r^l v(r)\big)\big|^p = \big|\nabla \big(h(r) r^{-\delta}\big)\big|^p = (-\delta)^p h^p \Big|1 - \frac{rh'}{\delta h}\Big|^p r^{-\alpha p - n}. \qquad (4.26)$$

Note $(-\delta)^p = \Lambda_{n,p,\alpha}$. Then

$$\begin{aligned}(4.27)\qquad &\int_{B_1} \big|\nabla(|x|^l v)\big|^p |x|^{\alpha p}\,dx - \Lambda_{n,p,\alpha}\int_{B_1} \frac{||x|^l v(x)|^p}{|x|^p}|x|^{\alpha p}\,dx \\ &= \omega_n \Lambda_{n,p,\alpha}\int_0^1 h^p(r)\Big\{\Big|1-\frac{rh'}{\delta h}\Big|^p - 1\Big\}\frac{dr}{r}.\end{aligned}$$

Noting that $-\delta > 0$ in this case, we have the result by a similar argument to that of the subcritical case. Therefore we proceed to the critical case.

Lastly we consider the critical case $(1 < p < +\infty, \alpha = 1 - n/p)$. Recall the 1-dimensional problem:

$$\begin{aligned}(4.28)\qquad l(p,\alpha,n,d_e,R) = \inf\Big[&\int_0^{d_e} |\partial_r v(r)|^p r^{p-1}\,dr : \\ &0 \le v \in R^{1,p}_{\alpha,0}(B^e), \int_0^{d_e} \frac{|v(r)|^p}{r} A_1(r)^{-p}\,dr = 1\Big],\end{aligned}$$

where R is any constant satisfying $R > d_e$. Then we have to show $l(p,\alpha,n,d,R) \ge \Lambda_{n,p,\alpha}$ for any $d > 0$ and $R > d$. As before we assume that $d_e = 1$. Here we define for any nonnegative radial function $v \in C_0^\infty(B_1)$,

$$(4.29)\qquad v(r) = A_1(r)^\mu h(r), \quad \mu = \frac{p-1}{p}.$$

Then we see $h(0) = h(1) = 0$ and

$$(4.30)\qquad |v'(r)|^p = \mu^p A_1(r)^{-1} h(r)^p r^{-p}\Big|1 - \frac{rA_1(r)h'(r)}{\mu h(r)}\Big|^p.$$

For the moment, we assume that $p \ge 2$. By an elementary inequality we have

$$|1+t|^p \ge 1 + pt + c(p)|t|^2 \quad (t \in \mathbb{R}, p \ge 2, c(p);\ \text{a small positive number}).$$

Using this and noting that $h(0) = h(1) = 0$ and $\mu^p = \Lambda_{n,p,\alpha}$, we obtain

$$\begin{aligned}&\int_0^1 |v'|^p r^{p-1}\,dr - \Lambda_{n,p,\alpha}\int_0^1 |v(r)|^p \frac{A_1(r)^{-p}}{r}\,dr \\ &= \Lambda_{n,p,\alpha}\int_0^1 h(r)^p \frac{A_1(r)^{-1}}{r}\Big(\Big|1 - \frac{rA_1(r)h'(r)}{\mu h(r)}\Big|^p - 1\Big)\,dr \\ (4.31)\qquad &\ge \frac{\Lambda_{n,p,\alpha}}{\mu}\int_0^1 ph^{p-1}h'\,dr + c(p)\frac{\Lambda_{n,p,\alpha}}{\mu^2}\int_0^1 h(r)^{p-2}\big(h'(r)\big)^2 rA_1(r)\,dr \\ &= c(p)\frac{\Lambda_{n,p,\alpha}}{\mu^2}\int_0^1 h(r)^{p-2}(r)|h'(r)|^2 rA_1(r)\,dr \\ &= c(p)\frac{4}{p^2}\frac{\Lambda_{n,p,\alpha}}{\mu^2}\int_0^1 \big|\big(h(r)^{p/2}(r)\big)'\big|^2 rA_1(r)\,dr.\end{aligned}$$

Using Lemma 3.4, we get

$$
\begin{aligned}
\int_0^1 \bigl|\bigl(h(r)^{p/2}\bigr)'\bigr|^2 rA_1(r)\,dr &\ge \frac{1}{4}\int_0^1 \frac{h(r)^p}{rA_1(r)A_2(r)^2}\,dr \\
&= \frac{1}{4}\int_0^1 \frac{|v(r)|^p}{rA_1(r)^pA_2(r)^2}\,dr.
\end{aligned} \tag{4.32}
$$

Combining this with (4.31) we get the desired inequality $l(p,\alpha,n,1,R) \ge \Lambda_{n,p,\alpha}$ and the inequality (2.7), where $C = c(p)\Lambda_{n,p,\alpha}/\mu^2 1/p^2$. In step 3 we shall show the sharpness of these inequalities.

Next we take care of the case that $1 < p < 2$. In this case we need technical lemmas again, which are quite similar to the previous case. Therefore we give necessary lemmas only instead of a complete proof. Suppose that M is sufficiently large, which will be specified later. We retain the notation; namely, for $v \in C_0^\infty(B_1), v > 0$, radial, again we set $v(r) = h(r)A_1(r)^\mu$. Then we modify the definition of the sets $A(\varphi,M)$, $B(\varphi,M)$, and $C(\varphi,M)$ as follows.

DEFINITION 4.1

For $\varphi \in G([0,1])$ and $M > 1$ we define three subsets of $[0,1]$ as follows:

$$
\begin{cases}
A(\varphi,M) = \bigl\{r \in [0,1] \bigm| |\varphi'(r)| \le M\frac{|\varphi(r)|}{rA_1(r)}\bigr\} \\
B(\varphi,M) = \bigl\{r \in [0,1] \bigm| |\varphi'(r)| > M\frac{|\varphi(r)|}{rA_1(r)}\bigr\} \\
C(\varphi,M) = \bigl\{r \in [0,1] \bigm| |\varphi'(r)| = M\frac{|\varphi(r)|}{rA_1(r)}\bigr\}.
\end{cases} \tag{4.33}
$$

Then, replacing φ and M by h and μM, respectively, we have $[0,1] = A(h,\mu M) \cup B(h,\mu M)$. By the density argument we can assume $h \in G([0,1])$; that is, $h'(0) \cdot h'(1) \ne 0$. Moreover from Lemma 3.5 we can assume that $h > 0$ in $(0,1)$ and $C(h,\mu M)$ consists of finite points. By Lemma 3.1, we have

$$
\begin{aligned}
&\int_0^1 |v'|^p r^{p-1}\,dr - \Lambda_{n,p,\alpha}\int_0^1 |v(r)|^p \frac{A_1(r)^{-p}}{r}\,dr \\
&= \Lambda_{n,p,\alpha}\int_0^1 h(r)^p \frac{A_1(r)^{-1}}{r}\Bigl(\Bigl|1 - \frac{rA_1(r)h'(r)}{\mu h(r)}\Bigr|^p - 1\Bigr)\,dr \\
&\ge -\frac{\Lambda_{n,p,\alpha}}{\mu}\int_0^1 ph^{p-1}h'\,dr \\
&\qquad + \Lambda_{n,p,\alpha}\int_{A(h,\mu M)} h^p c(p)M^{p-2}\Bigl(\frac{rA_1(r)h'}{\mu h}\Bigr)^2 \frac{A_1(r)^{-1}}{r}\,dr \\
&\qquad + \Lambda_{n,p,\alpha}\int_{B(h,\mu M)} h^p c(p)\Bigl|\frac{rA_1(r)h'}{\mu h}\Bigr|^p \frac{A_1(r)^{-1}}{r}\,dr \\
&= \Lambda_{n,p,\alpha}c(p)M^{p-2}\mu^{-2}\int_{A(h,\mu M)} h^{p-2}|h'|^2 rA_1(r)\,dr \\
&\qquad + \Lambda_{n,p,\alpha}c(p)\mu^{-p}\int_{B(h,\mu M)} |h'|^p r^{p-1}A_1(r)^{p-1}\,dr.
\end{aligned} \tag{4.34}
$$

Note $A(h,\mu M) = A(h^{p/2}, p/2\mu M)$ and $B(h,\mu M) = B(h^{p/2}, p/2\mu M)$. Then applying Lemma 3.8 to $\varphi = h^{p/2}$ and $A(h^{p/2}, p/2\mu M), B(h^{p/2}, p/2\mu M)$ in place of $A(\varphi, M), B(\varphi, M)$, respectively, we have

$$
\begin{aligned}
\int_{A(h,\mu M)} h^{p-2}(h')^2 r A_1(r)\,dr &= \frac{4}{p^2}\int_{A(h,\mu M)} \big((h^{\frac{p}{2}})'\big)^2 r A_1(r)\,dr \\
&\geq \frac{4}{p^2}\Big(\frac{1}{4}\int_{A(h,\mu M)} \frac{h(r)^p}{rA_1(r)A_2(r)^2}\,dr \\
&\quad + \frac{1}{2}\int_{B(h,\mu M)} \frac{h(r)^p}{rA_1(r)A_2(r)^2}\,dr \\
&\quad - \frac{p}{2}\int_{B(h,\mu M)} \frac{h(r)^{p-1}|h'(r)|}{A_2(r)}\,dr\Big).
\end{aligned}
\tag{4.35}
$$

From an easy variant of Lemma 3.6 we can estimate the last term to obtain

$$
\begin{aligned}
&\frac{p}{2}\int_{B(h,\mu M)} \frac{h^{p-1}|h'|}{A_2(r)}\,dr \\
&\quad \leq \frac{p}{2}\frac{1}{(\mu M)^{p-1}\log(\log R)}\int_{B(h,\mu M)} |h'|^p A_1(r)^{p-1} r^{p-1}\,dr.
\end{aligned}
\tag{4.36}
$$

Here we simply use the fact that $rA_1(r)|h'| > \mu M h$ holds on the set $B(h,\mu M)$. Combining this with (4.34) and (4.35), choosing M large enough, we have the desired inequality.

REMARK 4.2

We recall the following classical Hardy inequalities.

PROPOSITION 4.1

Assume that $a \neq 1$ and $1 < p < +\infty$. For any nonnegative function $f \in C_0^\infty([0,\infty))$, it holds that

$$
\int_0^\infty x^{-a}F(x)^p\,dx \leq \Big(\frac{p}{|a-1|}\Big)^p \int_0^\infty x^{-a+p} f(x)^p\,dx,
\tag{4.37}
$$

where

$$
\begin{cases} F(x) = \int_0^x f(t)\,dt & \text{for } a > 1, \\ F(x) = \int_x^\infty f(t)\,dt & \text{for } a < 1. \end{cases}
\tag{4.38}
$$

From this proposition we easily see that in the noncritical case, $\Lambda_{n,p,\alpha}$ becomes the best constant.

4.3. Step 3 (Sharpness)

We construct a family of radial functions in $R^{1,p}_{\alpha,0}(B_1)$ or $\dot{R}^{1,p}_{\alpha,0}(B_1)$ which we will essentially use to show the sharpness of $\Lambda_{n,p,\alpha}$ and the optimality of weight functions $A_1(r)^{-2}$ and $A_2(r)^{-2}$. These test functions contain two positive parameters ε and β, and we assume that $\varepsilon \to 0$ and β is large and specified later.

DEFINITION 4.2 (NONCRITICAL CASE; $\alpha \neq 1 - n/p$)
For $\varepsilon > 0$ and $\beta > \max(0, p-2)$ let us define a radial function $v_{\varepsilon,\beta}(r) = v_{\varepsilon,\beta}(|x|)$ by

$$v_{\varepsilon,\beta}(r) = r^{1-\alpha-\frac{n}{p}} h_{\varepsilon,\beta}(r), \tag{4.39}$$

where

$$h_{\varepsilon,\beta}(r) = \begin{cases} A_1(r)^{1-1/p-\beta/p}, & 0 < r < \varepsilon, \\ A_1(r)^{1-1/p} A_1(\varepsilon)^{-\beta/p} \frac{\log \frac{1}{r}}{\log \frac{1}{\varepsilon}}, & \varepsilon < r < 1. \end{cases} \tag{4.40}$$

Here $A_1(r) = \log R/r$ and $R > 1$.

In the critical case we shall employ the following.

DEFINITION 4.3 (CRITICAL CASE; $\alpha = 1 - n/p$)
For $\varepsilon > 0$ and $\beta > \max(0, p-2)$ let us define a radial function $w_{\varepsilon,\beta}(r) = w_{\varepsilon,\beta}(|x|)$ by

$$w_{\varepsilon,\beta}(r) = A_1(r)^{1-1/p} k_{\varepsilon,\beta}(r), \tag{4.41}$$

where

$$k_{\varepsilon,\beta}(r) = \begin{cases} A_2(r)^{1-1/p-\beta/p}, & 0 < r < \varepsilon, \\ A_2(r)^{1-1/p} A_2(\varepsilon)^{-\beta/p} \frac{\log(\log \frac{e}{r})}{\log(\log \frac{e}{\varepsilon})}, & \varepsilon < r < 1 \end{cases} \tag{4.42}$$

and $A_2(r) = \log A_1(r)$ and $R > e$.

Here we note that $h_{\varepsilon,\beta}(0) = k_{\varepsilon,\beta}(0) = 0$ and $h_{\varepsilon,\beta}(1) = k_{\varepsilon,\beta}(1) = 0$ hold.

First we shall show the sharpness of the constant $\Lambda_{n,p,\alpha}$ and the weight functions $A_1(r)^{-2}$ and $A_2(r)^{-2}$. To this end we set in the noncritical case

$$\begin{cases} I(u) = \int_{B_1} |\nabla u|^p |x|^{\alpha p}\, dx - \Lambda_{n,p,\alpha} \int_{B_1} \frac{|u(x)|^p}{|x|^p} |x|^{\alpha p}\, dx, \\ J(u,\gamma) = \int_{B_1} \frac{|u(x)|^p}{|x|^p} |x|^{\alpha p} A_1(r)^{-\gamma}\, dx, \\ K(u) = \int_{B_1} \frac{|u(x)|^p}{|x|^p} |x|^{\alpha p}\, dx \end{cases} \tag{4.43}$$

and in the critical case

$$\begin{cases} I_c(u) = \int_{B_1} |\nabla u|^p |x|^{p-n}\, dx - \Lambda_{n,p,\alpha} \int_{B_1} \frac{|u(x)|^p}{|x|^n} A_1(r)^{-p}\, dx, \\ J_c(u,\gamma) = \int_{B_1} \frac{|u(x)|^p}{|x|^n} A_1(r)^{-p} A_2(r)^{-\gamma}\, dx, \\ K_c(u) = \int_{B_1} \frac{|u(x)|^p}{|x|^n} A_1(r)^{-p}\, dx. \end{cases} \tag{4.44}$$

Then we shall show that

$$\lim_{\varepsilon \to 0} \frac{I(v_{\varepsilon,\beta})}{J(v_{\varepsilon,\beta},\gamma)} = 0, \quad \text{if } \gamma \in (0,2) \text{ and } \beta > p - \gamma, \tag{4.45}$$

and

$$\lim_{\varepsilon \to 0} \frac{I_c(w_{\varepsilon,\beta})}{J_c(w_{\varepsilon,\beta},\gamma)} = 0, \quad \text{if } \gamma \in (0,2) \text{ and } \beta > p - \gamma. \tag{4.46}$$

Admitting this for the moment, let us show the sharpness of the weight functions and the constant $\Lambda_{n,p,\alpha}$. Since it holds from Hardy's inequality that

$$\frac{I(v_{\varepsilon,\beta}(r))}{J(v_{\varepsilon,\beta}(r),2)}, \frac{I_c(w_{\varepsilon,\beta}(r))}{J_c(w_{\varepsilon,\beta}(r),2)} \geq {}^{\exists}C > 0 \tag{4.47}$$

for any $\varepsilon > 0$, we see that the weight functions $A_1(r)^{-2}$ and $A_2(r)^{-2}$ are sharp. Moreover, since

$$\frac{J(v_{\varepsilon,\beta},\gamma)}{K(v_{\varepsilon,\beta},\gamma)}, \frac{J_c(w_{\varepsilon,\beta},\gamma)}{K_c(w_{\varepsilon,\beta},\gamma)} \leq 1, \tag{4.48}$$

we also see that the constant $\Lambda_{n,p,\alpha}$ is sharp.

Now we show these properties (4.45) and (4.46). For the sake of simplicity we put $v_\varepsilon = v_{\varepsilon,\beta}(r), h_\varepsilon = h_{\varepsilon,\beta}(r)$, $w_\varepsilon = w_{\varepsilon,\beta}(r)$, $k_\varepsilon = k_{\varepsilon,\beta}(r)$, and $\mu = 1-\alpha-n/p$, respectively. Direct calculation gives

$$\begin{cases} v'_\varepsilon = \mu r^{\mu-1} h_\varepsilon \big(1 + \frac{rh'_\varepsilon}{\mu h_\varepsilon}\big), \\ w'_\varepsilon = \frac{p-1}{p} A_1^{-\frac{1}{p}} \frac{k_\varepsilon}{r} \big(-1 + \frac{p}{p-1} \frac{rA_1 k'_\varepsilon}{k_\varepsilon}\big). \end{cases} \tag{4.49}$$

Then we have

$$\begin{cases} I(v_\varepsilon) = \omega_n \Lambda_{n,p,\alpha} \int_0^1 h_\varepsilon^p \big(\big|1 + \frac{rh'_\varepsilon}{\mu h_\varepsilon}\big|^p - 1\big) \frac{dr}{r}, \\ I_c(w_\varepsilon) = \omega_n \Lambda_{n,p,\alpha} \int_0^1 k_\varepsilon^p \big(\big|1 - \frac{p}{p-1} \frac{rA_1 k'_\varepsilon}{k_\varepsilon}\big|^p - 1\big) \frac{dr}{rA_1}. \end{cases} \tag{4.50}$$

Next we shall show (4.45) in the noncritical case. Let us set $X = rh'_\varepsilon/\mu h_\varepsilon$. By the definition we see

$$X = \frac{rh'_\varepsilon}{\mu h_\varepsilon} = \begin{cases} \frac{p-1-\beta}{n+p\alpha-p} \frac{1}{A_1}, & 0 < r < \varepsilon, \\ \frac{p-1}{n+p\alpha-p} \frac{1}{A_1} + \frac{p}{n+p\alpha-p} \frac{1}{\log \frac{1}{r}}, & \varepsilon < r < 1. \end{cases} \tag{4.51}$$

Using Lemma 3.2 and noting $\int_0^1 h_\varepsilon^p X \, dr/r = \frac{1}{\mu} \int_0^1 h_\varepsilon^{p-1} h'_\varepsilon \, dr = 0$, we have

$$I(v_\varepsilon) \leq \omega_n \Lambda_{n,p,\alpha} C(p) \int_0^1 h_\varepsilon^p \frac{X^2}{1+X^2}(1+|X|^p)\frac{dr}{r}. \tag{4.52}$$

By $R(v_\varepsilon)$ we denote the integral in the right-hand side. Then by subdividing the interval $[0,1]$ into $[0,\varepsilon]$ and $[\varepsilon,1]$ for a sufficiently small $\varepsilon > 0$ we immediately get $R(v_\varepsilon) \leq R_1 + R_2$, where

$$R_1 = \int_0^\varepsilon h_\varepsilon^p \frac{X^2}{1+X^2}(1+|X|^p)\frac{dr}{r} \leq C \int_0^\varepsilon \frac{h_\varepsilon^p}{rA_1^2} \, dr = C\frac{A_1(\varepsilon)^{p-2-\beta}}{\beta-(p-2)}. \tag{4.53}$$

Similarly we have

$$R_2 = \int_\varepsilon^1 h_\varepsilon^p \frac{X^2}{1+X^2}(1+|X|^p)\frac{dr}{r} \leq C \int_\varepsilon^1 \frac{h_\varepsilon^p}{rA_1^2}\Big(1 + \frac{1}{\big(\log \frac{1}{r}\big)^p}\Big) \, dr. \tag{4.54}$$

Here

$$\begin{aligned} \int_\varepsilon^1 \frac{h_\varepsilon^p}{rA_1^2} \, dr &= \int_\varepsilon^1 \frac{A_1^{p-3} A_1(\varepsilon)^{-\beta}}{r} \Big(\frac{\log \frac{1}{r}}{\log \frac{1}{\varepsilon}}\Big)^p \, dr \leq \int_\varepsilon^1 \frac{A_1^{2p-3} A_1(\varepsilon)^{-\beta}}{r\big(\log \frac{1}{\varepsilon}\big)^p} \, dr \\ &= \frac{A_1(\varepsilon)^{-\beta}}{2(p-1)\big(\log \frac{1}{\varepsilon}\big)^p}\big(A_1(\varepsilon)^{2p-2} - A_1(1)^{2p-2}\big) \leq C\frac{A_1(\varepsilon)^{p-2-\beta}}{\beta-(p-2)}. \end{aligned} \tag{4.55}$$

Here we use the fact that $\lim_{r\to 0}\frac{\log\frac{1}{r}}{A_1}=1$. If $p\neq 2$, we have in a similar way

$$\begin{aligned}\int_\varepsilon^1 \frac{h_\varepsilon^p}{rA_1^2}\frac{1}{\left(\log\frac{1}{r}\right)^p}\,dr &= \int_\varepsilon^1 \frac{A_1^{p-3}A_1(\varepsilon)^{-\beta}}{r\left(\log\frac{1}{\varepsilon}\right)^p}\,dr\\ &= \frac{1}{2-p}\frac{A_1(\varepsilon)^{-\beta}}{\left(\log\frac{1}{\varepsilon}\right)^p}\left(A_1(1)^{p-2}-A_1(\varepsilon)^{p-2}\right)\\ &\le C\begin{cases}A_1(\varepsilon)^{-2-\beta}, & \text{if } p>2,\\ A_1(\varepsilon)^{-p-\beta}, & \text{if } 1<p<2.\end{cases}\end{aligned}\tag{4.56}$$

Here we note that $-\beta-p<p-2-\beta<0$ holds for any $p>1$ and $\beta>\max(p-2,0)$. Lastly if $p=2$ we have

$$\begin{aligned}\int_\varepsilon^1 \frac{h_\varepsilon^p}{rA_1^2}\frac{1}{\left(\log\frac{1}{r}\right)^p}\,dr &= \int_\varepsilon^1 \frac{A_1^{-1}A_1(\varepsilon)^{-\beta}}{r\left(\log\frac{1}{\varepsilon}\right)^2}\,dr\\ &= \frac{A_1(\varepsilon)^{-\beta}}{\left(\log\frac{1}{\varepsilon}\right)^2}\left(A_2(\varepsilon)-A_2(1)\right)\\ &\le CA_1(\varepsilon)^{-\beta-\beta'}\quad (\text{for any } \beta'\in(0,2)).\end{aligned}\tag{4.57}$$

After all we get

$$I(v_\varepsilon)\le C\frac{A_1(\varepsilon)^{p-2-\beta}}{\beta-(p-2)}+o\left(A_1(\varepsilon)^{p-2-\beta}\right).\tag{4.58}$$

Similarly we have for $\gamma\in(0,2)$ and $\beta>p-\gamma$

$$\begin{aligned}J(v_\varepsilon,\gamma) &= \omega_n\int_0^1 v_\varepsilon^p A_1(r)^{-\gamma}r^{\alpha p-p+n-1}\,dr=\omega_n\int_0^1 h_\varepsilon^p A_1(r)^{-\gamma}\frac{dr}{r}\\ &= \omega_n\int_0^\varepsilon A_1(r)^{p-1-\beta-\gamma}\frac{dr}{r}+\omega_n\int_\varepsilon^1 A_1(r)^{p-1-\gamma}A_1(\varepsilon)^{-\beta}\left(\frac{\log\frac{1}{r}}{\log\frac{1}{\varepsilon}}\right)^p\frac{dr}{r}.\end{aligned}\tag{4.59}$$

The first term in the last line equals $A_1(\varepsilon)^{p-\gamma-\beta}/\beta-(p-\gamma)$. The second is positive; hence, we have

$$J(v_\varepsilon,\gamma)\ge\omega_n\frac{A_1(\varepsilon)^{p-\gamma-\beta}}{\beta-(p-\gamma)}.\tag{4.60}$$

Then we have for $0<\gamma<2,\beta>p-\gamma$

$$\lim_{\varepsilon\to 0}\frac{I(v_{\varepsilon,\beta})}{J(v_{\varepsilon,\beta},\gamma)}\le C\frac{\beta-(p-\gamma)}{\beta-(p-2)}\lim_{\varepsilon\to 0}A_1(\varepsilon)^{\gamma-2}\left(1+o(1)\right)=0.\tag{4.61}$$

REMARK 4.3

The second term of $J(v_\varepsilon,\gamma)$ can be estimated similarly:

$$\begin{aligned}\int_\varepsilon^1 A_1(r)^{p-1-\gamma}A_1(\varepsilon)^{-\beta}\left(\frac{\log\frac{1}{r}}{\log\frac{1}{\varepsilon}}\right)^p\frac{dr}{r} &\le \int_\varepsilon^1 A_1(r)^{2p-1-\gamma}\frac{A_1(\varepsilon)^{-\beta}}{\left(\log\frac{1}{\varepsilon}\right)^p}\frac{dr}{r}\\ &\le \frac{A_1(\varepsilon)^{p-\beta-\gamma}}{2p-\gamma}.\end{aligned}$$

REMARK 4.4
If $p \geq 2$, then we can also put $\beta = p - \gamma > 0$ in the last step.

In quite the same way as we have for $\beta > p$,

$$K(v_\varepsilon) \geq \omega_n \frac{A_1(\varepsilon)^{p-\beta}}{\beta - p}. \tag{4.62}$$

Next we shall show (4.46) in the critical case. Let us set $X = -p/p-1 r A_1(r) k_\varepsilon'/k_\varepsilon$. By the definition we see

$$X = \begin{cases} \frac{p-1-\beta}{p-1} \frac{1}{A_2}, & 0 < r < \varepsilon, \\ \frac{1}{A_2} + \frac{p}{p-1} \frac{A_1}{\log \frac{e}{r} \log(\log \frac{e}{r})}, & \varepsilon < r < 1. \end{cases} \tag{4.63}$$

Using Lemma 3.2 and noting $\int_0^1 k_\varepsilon^p X\, dr/rA_1(r) = -p/p-1 \int_0^1 k_\varepsilon^{p-1} k_\varepsilon'\, dr = 0$, we have

$$I_c(w_\varepsilon) \leq \omega_n \Lambda_{n,p,\alpha} C(p) \int_0^1 k_\varepsilon^p \frac{X^2}{1+X^2} (1 + |X|^p) \frac{dr}{rA_1}. \tag{4.64}$$

By $R(w_\varepsilon)$ we denote the integral in the right-hand side. Then by subdividing the interval $[0,1]$ into $[0,\varepsilon]$ and $[\varepsilon, 1]$ for a sufficiently small $\varepsilon > 0$ we immediately get $R(w_\varepsilon) \leq R_1 + R_2$, where

$$R_1 = \int_0^\varepsilon \frac{k_\varepsilon^p X^2 (1 + |X|^p)}{1+X^2} \frac{dr}{rA_1} \leq C \int_0^\varepsilon \frac{k_\varepsilon^p}{rA_1A_2^2}\, dr = C \frac{A_2(\varepsilon)^{p-2-\beta}}{\beta - (p-2)}. \tag{4.65}$$

Now let us note $\lim_{r\to 0} \frac{\log(\log \frac{e}{r})}{A_2} = 1$. Similarly we have

$$\begin{aligned} R_2 &= \int_\varepsilon^1 \frac{k_\varepsilon^p X^2 (1 + |X|^p)}{1+X^2} \frac{dr}{rA_1} \\ &\leq C \int_\varepsilon^1 \frac{k_\varepsilon^p}{rA_1A_2^2} \Big(1 + \frac{1}{\big(\log\big(\log \frac{e}{r}\big)\big)^p}\Big)\, dr. \end{aligned} \tag{4.66}$$

Then

$$\begin{aligned} \int_\varepsilon^1 \frac{k_\varepsilon^p}{rA_1A_2^2}\, dr &= \int_\varepsilon^1 \frac{A_2^{p-3} A_2(\varepsilon)^{-\beta}}{rA_1} \Big(\frac{\log\big(\log \frac{e}{r}\big)}{\log\big(\log \frac{e}{\varepsilon}\big)}\Big)^p\, dr \\ &\leq \int_\varepsilon^1 \frac{A_2^{2p-3} A_2(\varepsilon)^{-\beta}}{rA_1 \big(\log\big(\log \frac{e}{\varepsilon}\big)\big)^p}\, dr \\ &= \frac{A_2(\varepsilon)^{-\beta} (A_2(\varepsilon)^{2p-2} - A_2(1)^{2p-2})}{2(p-1)\big(\log\big(\log \frac{e}{\varepsilon}\big)\big)^p} \leq C \frac{A_2(\varepsilon)^{p-2-\beta}}{\beta - (p-2)}. \end{aligned} \tag{4.67}$$

If $p \neq 2$, we have in a similar way

$$\begin{aligned} \int_\varepsilon^1 \frac{k_\varepsilon^p}{rA_1A_2^2} \frac{1}{\big(\log\big(\log \frac{e}{\varepsilon}\big)\big)^p}\, dr &= \int_\varepsilon^1 \frac{A_2^{p-3} A_2(\varepsilon)^{-\beta}}{rA_1 \big(\log\big(\log \frac{e}{\varepsilon}\big)\big)^p}\, dr \\ &= \frac{1}{2-p} \frac{A_2(\varepsilon)^{-\beta}}{\big(\log\big(\log \frac{e}{\varepsilon}\big)\big)^p} \big(A_2(1)^{p-2} - A_2(\varepsilon)^{p-2}\big) \end{aligned} \tag{4.68}$$

$$\leq C \begin{cases} A_2(\varepsilon)^{-2-\beta}, & \text{if } p>2, \\ A_2(\varepsilon)^{-p-\beta}, & \text{if } 1<p<2. \end{cases}$$

Here we note that $-\beta - p < p-2-\beta < 0$ holds for any $p > 1$ and $\beta > \max(p-2, 0)$. Lastly if $p = 2$ we have

$$\begin{aligned} \int_\varepsilon^1 \frac{k_\varepsilon^p}{rA_1A_2^2} \frac{1}{\bigl(\log\bigl(\log\frac{e}{r}\bigr)\bigr)^p}\,dr &= \int_\varepsilon^1 \frac{A_2^{-1}A_2(\varepsilon)^{-\beta}}{rA_1\bigl(\log\bigl(\log\frac{e}{r}\bigr)\bigr)^2}\,dr \\ &= \frac{A_1(\varepsilon)^{-\beta}}{\bigl(\log\frac{1}{\varepsilon}\bigr)^2}\bigl(A_2(e) - A_2(1)\bigr) \\ &\leq CA_2(\varepsilon)^{-\beta-\beta'} \quad (\text{for any } \beta' \in (0,2)). \end{aligned} \tag{4.69}$$

After all we get

$$I_c(w_\varepsilon) \leq C\frac{A_2(\varepsilon)^{p-2-\beta}}{\beta-(p-2)} + o\bigl(A_2(\varepsilon)^{p-2-\beta}\bigr). \tag{4.70}$$

Similarly we have for $\gamma \in (0,2)$ and $\beta > p-\gamma$

$$\begin{aligned} J_c(w_\varepsilon,\gamma) &= \omega_n \int_0^1 k_\varepsilon^p A_1(r)^{-1}A_2(r)^{-\gamma}\frac{dr}{r} \\ &= \omega_n\int_0^\varepsilon \frac{A_2(r)^{p-1-\beta-\gamma}}{rA_1(r)}\,dr \\ &\quad + \omega_n\int_\varepsilon^1 A_2(r)^{p-1-\gamma}A_2(\varepsilon)^{-\beta}\Bigl(\frac{\log\bigl(\log\frac{e}{r}\bigr)}{\log\bigl(\log\frac{e}{\varepsilon}\bigr)}\Bigr)^p\frac{dr}{r}. \end{aligned} \tag{4.71}$$

The first term in the last line equals $A_2(\varepsilon)^{p-\gamma-\beta}/\beta-(p-\gamma)$. The second term is positive; hence, we have

$$J_c(v_\varepsilon,\gamma) \geq \omega_n\frac{A_2(\varepsilon)^{p-\gamma-\beta}}{\beta-(p-\gamma)}. \tag{4.72}$$

Then we have for $0<\gamma<2, \beta>p-\gamma$

$$\lim_{\varepsilon\to 0}\frac{I_c(w_{\varepsilon,\beta}(r))}{J_c(w_{\varepsilon,\beta}(r),\gamma)} \leq C\frac{\beta-(p-\gamma)}{\beta-(p-2)}\lim_{\varepsilon\to 0}A_1(\varepsilon)^{\gamma-2}\bigl(1+o(1)\bigr) = 0. \tag{4.73}$$

REMARK 4.5

The second term of $J(v_\varepsilon,\gamma)$ can be estimated as before:

$$\begin{aligned} \int_\varepsilon^1 A_2(r)^{p-1-\gamma}A_2(\varepsilon)^{-\beta}\Bigl(\frac{\log\bigl(\log\frac{e}{r}\bigr)}{\log\bigl(\log\frac{e}{\varepsilon}\bigr)}\Bigr)^p\frac{dr}{r} &\leq \int_\varepsilon^1 A_2(r)^{2p-1-\gamma}\frac{A_2(\varepsilon)^{-\beta}}{\bigl(\log\bigl(\log\frac{e}{\varepsilon}\bigr)\bigr)^p}\frac{dr}{r} \\ &\leq C\frac{A_2(\varepsilon)^{p-\beta-\gamma}}{2p-\gamma}. \end{aligned}$$

REMARK 4.6

If $p \geq 2$, then we can also put $\beta = p-\gamma > 0$ in the last step.

In quite the same way as we have for $\beta > p$,

$$K_c(w_\varepsilon) \geq \omega_n \frac{A_2(\varepsilon)^{p-\beta}}{\beta - p}. \tag{4.74}$$

4.4. Proof of Corollary 2.1

Assume that $\alpha \neq 1 - n/p$. If $f \in F_{p,\alpha}$, then

$$\limsup_{|x|\to 0} f(x)|x|^{p(1-\alpha)}\Big(\log \frac{R}{|x|}\Big)^2 < \infty \tag{4.75}$$

and hence for sufficiently small ϵ, in B_ϵ,

$$f(x) < C|x|^{-p(1-\alpha)}\Big(\log \frac{R}{|x|}\Big)^{-2}. \tag{4.76}$$

Outside B_ϵ, f is a bounded function and hence C can be chosen so that this inequality holds in Ω. Then the assertion will follow from this inequality.

If $f \notin F_{p,\alpha}$, $\alpha \neq 1 - n/p$, and if $|x|^{p(1-\alpha)} f(x)\big(\log \frac{R}{|x|}\big)^2$ tends to ∞ as $|x| \to 0$, then we can write $f(x) = \frac{h(x)}{|x|^{p(1-\alpha)}A_1(|x|)^2}$, where $h(x)$ tends to infinity as x tends to 0. Then from the calculation of Theorem 2.1, for $\epsilon > 0$ sufficiently small we get

$$\begin{aligned}
&\int_{B_1} \frac{|v_{\epsilon,\beta}|^p h(x)}{|x|^{p(1-\alpha)}A_1(|x|)^2}\,dx \\
&\quad\geq \omega_n \int_0^\epsilon |v_{\epsilon,\beta}|^p A_1(r)^{-2} r^{n-1-p(1-\alpha)}\,dr \min_{|x|\leq\varepsilon} h(x) \\
&\quad\geq \int_0^\epsilon A_1(r)^{p-3-\beta} r^{-1}\,dr \min_{|x|\leq\varepsilon} h(x) \\
&\quad= O\Big(\log \frac{R}{\epsilon}\Big)^{p-2-\beta} \min_{|x|\leq\varepsilon} h(x).
\end{aligned} \tag{4.77}$$

Since $\min_{|x|\leq\varepsilon} h(x)$ tends to ∞ as $\epsilon \to 0$, noting (4.59), we conclude that

$$\frac{I(v_{\varepsilon,\beta})}{J(v_{\varepsilon,\beta}, f)} \to 0 \quad \text{for } J(v_{\varepsilon,\beta}, f) = \int_\Omega \big|v_{\varepsilon,\beta}(|x|)\big|^p f(x)\,dx \tag{4.78}$$

as $\epsilon \to 0$ and the inequality in the assertion 1 (noncritical case) cannot hold for such $f \notin F_{p,\alpha}$. Since the argument is quite similar, we shall omit the proof in the critical case. □

5. Proof of Theorem 2.2

Let us recall the theorem.

THEOREM 5.1

Assume that $\alpha \geq 1 - n/p$, $1 < p < +\infty$ and $f \in \mathcal{F}_{p,\alpha}$. Then we have the following.

(1) *The problem (2.11) admits a positive weak solution $u \in W^{1,p}_{\alpha,0}(\Omega)$ for all $0 \leq \mu < \Lambda_{n,p,\alpha}$, corresponding to the first eigenvalue $\lambda = \lambda^1_\mu(f) > 0$.*

(2) *As* μ *increases to* $\Lambda_{n,p,\alpha}, \lambda^1_\mu(f) \to \lambda^1(f) \ge 0$ *for all* $f \in \mathcal{F}_{p,\alpha}$.
(3) *The limit* $\lambda^1(f) > 0$ *if* $f \in F_{p,\alpha}$.
(4) *If* $f \notin F_{p,\alpha}$ *and if*

$$\begin{cases} \lim_{|x|\to 0} |x|^{p-\alpha p} f(x)\big(\log \frac{1}{|x|}\big)^2 = +\infty, & \text{if } \alpha > 1-\frac{n}{p}, \\ \lim_{|x|\to 0} |x|^n f(x)\big(\log \frac{1}{|x|}\big)^p \big(\log\big(\log \frac{1}{|x|}\big)\big)^2 = +\infty, & \text{if } \alpha = 1-\frac{n}{p}, \end{cases}$$

then the limit $\lambda^1(f) = 0$.

To prove the Theorem 5.1, we need the following.

LEMMA 5.1
Let $(\varphi_m)_{m\in\mathbb{N}} \subset L^p(\Omega, |x|^\alpha), \alpha > -n, 1 \le p < \infty$, *be such that, as* $m \to \infty$,

(i) $\varphi_m \rightharpoonup \varphi$ *weakly in* $L^p(\Omega, |x|^\alpha)$,
(ii) $\varphi_m(x) \to \varphi(x)$ *a.e. in* Ω.

Then

$$\lim_{m\to\infty} (\|\varphi_m\|^p_{L^p(\Omega,|x|^\alpha)} - \|\varphi_m - \varphi\|^p_{L^p(\Omega,|x|^\alpha)}) = \|\varphi\|^p_{L^p(\Omega,|x|^\alpha)}. \tag{5.1}$$

For the proof see [18, Theorem 1.9] for example.

LEMMA 5.2
Here

$$\begin{aligned}(|\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u_m - u) \\ \ge C(p) \begin{cases} |\nabla(u_m - u)|^p, & \text{if } p \ge 2, \\ \frac{|\nabla(u_m-u)|^2}{(|\nabla u_m| + |\nabla u|)^{2-p}}, & \text{if } 1 < p \le 2 \end{cases}\end{aligned} \tag{5.2}$$

for some $C(p) > 0$.

For the proof see [15, Lemma 2.3] for example.

REMARK 5.1
The proof of Theorem 5.1 is organized in the following way. First in the proof of assertion 1, u will be characterized as a solution of a minimizing problem for the functional (5.3) subject to the constraint (5.4). Then using Lemmas 5.1 and 5.2, we show that u is not trivial and satisfies the Euler–Lagrange equation in a weak sense. The rest of the assertion follows from Corollary 2.1.

Proof of Theorem 5.1
We treat the subcritical case only, because the argument is quite similar in the critical case. We define the functional

$$E_\mu(u) = \int_\Omega \Big(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p}\Big)|x|^{p\alpha}\, dx. \tag{5.3}$$

By the Hardy inequality, one can check that $E_\mu(\cdot)$ is continuous, Gateaux differentiable, and coercive on $W^{1,p}_{\alpha,0}(\Omega)$. Let us set $W = W^{1,p}_{\alpha,0}(\Omega)$. We minimize this functional over the manifold

$$M_f = \Big\{ u \in W \;\Big|\; \int_\Omega |u(x)|^p f(x)\,dx = 1 \Big\}, \tag{5.4}$$

and let $\lambda^1_\mu > 0$ be the infimum. Then we can choose $(u_m)_{m\in\mathbb{N}} \subset M_f$ a minimizing sequence of E_μ such that $E_\mu(u_m) \to \lambda^1_\mu$ and $E_\mu|'_{M_f}(u_m) \to 0$. The coercivity of E_μ implies that $(u_m)_{m\in\mathbb{N}}$ is a bounded sequence and hence we have for a subsequence, as $k \to \infty$, $u_{m_k} \rightharpoonup u$ weakly in W. Moreover it follows from Definition 2.4 and Theorem 2.1 that the embedding $W \to L^p(\Omega, f)$ is not only continuous but also compact. Here the continuity is clear from the existence of the continuous inclusions $W \subset L^p(\Omega, |x|^{\alpha p-p}) \subset L^p(\Omega, f)$. We also note that by virtue of the assumption on f, the compactness follows from the standard argument; hence, we omit the proof. For the detailed proof, see [12, Theorem 1 and the proof of compactness p. 383] for example (cf. [15, Proposition 4.2], [13, Proposition 1.2]).

Since W is compactly embedded in $L^p(\Omega, f)$, it follows that M_f is weakly closed and hence $u \in M_f$. Therefore we have

$$\begin{cases} \nabla u_{m_k} \rightharpoonup \nabla u & \text{weakly in } (L^p(\Omega, |x|^{p\alpha}))^n, \\ u_{m_k} \rightharpoonup u & \text{weakly in } L^p(\Omega, |x|^{p\alpha-p}), \\ u_{m_k} \to u & \text{strongly in } L^p(\Omega, |x|^{p\alpha}), \\ u_{m_k} \to u & \text{strongly in } L^p(\Omega, f). \end{cases} \tag{5.5}$$

Also u_{m_k} in $\mathcal{D}'(\Omega)$

$$L_{p,\alpha} u_{m_k} = \frac{\mu}{|x|^{p-\alpha p}} |u_{m_k}|^{p-2} u_{m_k} + \lambda_{m_k} |u_{m_k}|^{p-2} u_{m_k} f + f_{m_k}, \tag{5.6}$$

where $f_{m_k} \to 0$ in W' and $\lambda_{m_k} \to {}^\exists\lambda$ as $m_k \to \infty$. By a standard argument we show that

$$L_{p,\alpha} u = \frac{\mu}{|x|^{p-\alpha p}} |u|^{p-2} u + \lambda |u|^{p-2} u f, \quad \text{in } \mathcal{D}'(\Omega). \tag{5.7}$$

For a sufficiently small $\rho > 0$, let $\phi \in C^\infty(\Omega)$ such that for $B_\rho \cap \operatorname{supp}\phi = \emptyset$ and

$$\phi = \begin{cases} 0, & |x| < \rho, \\ 1, & |x| > 2\rho. \end{cases} \tag{5.8}$$

Using the test function $v_{m_k} = \phi(u_{m_k} - u) \in W$, we have

$$\begin{aligned} &\int_\Omega (|\nabla u_{m_k}|^{p-2} \nabla u_{m_k} - |\nabla u|^{p-2} \nabla u) \nabla v_{m_k} |x|^{p\alpha}\,dx \\ &\quad = \lambda_{m_k} \int_\Omega |u_{m_k}|^{p-2} u_{m_k} v_{m_k} f\,dx + \mu \int_\Omega |u_{m_k}|^{p-2} u_{m_k} v_{m_k} f\,dx \\ &\qquad + \langle f_{m_k}, v_{m_k} \rangle - \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v_{m_k} |x|^{p\alpha}\,dx. \end{aligned} \tag{5.9}$$

From (5.5) one deduces that $v_{m_k} \rightharpoonup 0$ weakly in W and $L^p(\Omega, |x|^{\alpha p-p})$ and strongly in $L^p(\Omega, f)$ and $L^p(\Omega)$. Here we also note that v_{m_k} $(k = 1, 2, \ldots)$ vanish uniformly near the origin. Hence for $m_k \to \infty$ and by expanding ∇v_{m_k} we get

$$\lim_{m_k\to\infty} \int_\Omega e_{m_k}\phi = 0, \tag{5.10}$$

where $e_{m_k} = (|\nabla u_{m_k}|^{p-2}\nabla u_{m_k} - |\nabla u|^{p-2}\nabla u)\nabla(u_{m_k} - u)$.

Fix θ with $0<\theta<1$; then

$$\begin{aligned}\int_\Omega e_{m_k}^\theta &= \int_\Omega e_{m_k}^\theta \phi + \int_\Omega e_{m_k}^\theta (1-\phi)\\ &\le \Big(\int_\Omega e_{m_k}\phi\Big)^\theta \Big(\int_\Omega \phi\Big)^{1-\theta} + \Big(\int_\Omega e_{m_k}\Big)^\theta \Big(\int_\Omega (1-\phi)\Big)^{1-\theta}.\end{aligned} \tag{5.11}$$

From (5.10) and (5.11) we get

$$\lim_{m_k\to\infty} \int_\Omega e_{m_k}^\theta \le C_\theta \rho^{n(1-\theta)}.$$

Letting ρ tend to zero implies that for any subsequence $e_{m_k'}$ of e_{m_k}

$$e_{m_k'} \to 0 \quad \text{a.e. in } \Omega. \tag{5.12}$$

We apply Lemma 5.2 to (5.12) and get

$$\nabla u_{m_k'}(x) \to \nabla u(x) \quad \text{a.e. } x\in\Omega. \tag{5.13}$$

For simplicity we retain $\{u_{m_k}\}$ instead of $\{u_{m_k'}\}$. Then we apply Lemma 5.1 to u_{m_k} and also to ∇u_{m_k} to obtain

$$\begin{aligned}\lambda_\mu^1 &= \|\nabla(u_{m_k}-u)\|^p_{L^p(\Omega,|x|^{\alpha p})} - \mu\|u_{m_k}-u\|^p_{L^p(\Omega,|x|^{\alpha p-p})}\\ &\quad + \|\nabla u\|^p_{L^p(\Omega,|x|^{\alpha p})} - \mu\|u\|^p_{L^p(\Omega,|x|^{\alpha p-p})} + o(1)\\ &\ge (\Lambda_{n,p,\alpha}-\mu)\|u_{m_k}-u\|^p_{L^p(\Omega,|x|^{\alpha p-p})} + \lambda_\mu^1 + o(1),\end{aligned} \tag{5.14}$$

where $o(1)\to 0$ as $m_k\to\infty$. As $\mu < \Lambda_{n,p,\alpha}$ we conclude that $\|u_{m_k} - u\|^p_{L^p(\Omega,|x|^{\alpha p-p})} \to 0$ as $m_k\to\infty$ and also $\|\nabla(u_{m_k}-u)\|^p_{L^p(\Omega,|x|^{\alpha p})}\to 0$ as $m_k\to\infty$ and hence we have $E_\mu(u) = \lambda_\mu^1$ and $\lambda=\lambda_\mu^1$. Then by the Euler–Lagrange equation of E_μ, u is a distribution solution of (5.7) and since $u\in W$, it is a weak solution to the eigenvalue problem (2.10), corresponding to $\lambda = \lambda_\mu^1$.

Moreover, if $f\in F_{p,\alpha}$, by Corollary 2.1, we have as $\mu\to\Lambda_{n,p,\alpha}$

$$\lambda_\mu^1(f)\to\lambda^1(f) = \inf_{u\in W(\Omega)\setminus\{0\}} \frac{\int_\Omega \big(|\nabla u|^p - \Lambda_{n,p,\alpha}\frac{|u(x)|^p}{|x|^p}\big)|x|^{\alpha p}\,dx}{\int_\Omega |u(x)|^p f(x)\,dx} > 0. \tag{5.15}$$

If $f\notin F_{p,\alpha}$ and satisfies the condition in (4), then again by Corollary 2.1 we have $\lambda^1(f)=0$. □

6. Proof of Theorem 2.3

Let us recall the eigenvalue problem and the theorem. We consider the problem defined by

$$\begin{cases} P_\mu u = -\operatorname{div}(|x|^{2-n}\nabla u) - \mu\frac{u}{|x|^n A_1(|x|)^2} = \lambda|x|^a u & \text{in } B_1,\\ u=0 \quad \text{on } \partial B_1,\end{cases} \tag{6.1}$$

where $a > -n$, $\lambda \ge 0$ and $\mu \le 1/4$. Then we have the following.

THEOREM 6.1

Assume that $\mu = 1/4$ and $a > -n$. Then we have the following.

(1) *There is a sequence of eigenvalues $\{\lambda_k\}$ with $0 < \lambda_k \to +\infty$ as $k \to \infty$. The first eigenvalue λ_1 is simple and the corresponding eigenfunction has a constant sign in B_1.*

(2) *By $u_1 \in V = V^{1,2}_{2-n/2,0}(B_1)$ we denote the first positive eigenfunction. Then $u_1 \notin L^\infty(B_1)$.*

Before we give a proof of this theorem, we make clear the functional framework. A Hilbert space $V = V^{1,2}_{2-n/2,0}(B_1)$ is given as the completion of $C_0^\infty(B_1)$ with respect to the norm defined by

$$\|u\|^2_{V^{1,2}_{2-n/2,0}(B_1)} = \int_{B_1} |\nabla u|^2 |x|^{2-n}\, dx - \frac{1}{4}\int_{B_1} \frac{u^2}{|x|^n A_1(|x|)^2}\, dx. \tag{6.2}$$

By the aid of the improved weighted Hardy inequality (2.7), we see that $\|\cdot\|_{V^{1,2}_{2-n/2,0}(B_1)}$ defines a norm. Then V clearly becomes a Hilbert space with inner product

$$(u,v)_V = \int_{B_1} \nabla u \cdot \nabla v\, |x|^{2-n}\, dx - \frac{1}{4}\int_{B_1} \frac{uv}{|x|^n A_1(|x|)^2}\, dx,$$

for any $u, v \in V$. Here we note that $L^2(B_1, |x|^a)$ is a Hilbert space as well (see (2.1) for the definition). By V' and $\langle\cdot,\cdot\rangle_{V',V}$ we denote the dual of V and the duality product, respectively. To characterize V we introduce a Hilbert space $\widetilde{V}$ which is isometric to V.

DEFINITION 6.1

By $\widetilde{V}$ we denote the completion of $C_0^\infty(B_1)$ with respect to this norm defined by

$$\|u\|_{\widetilde{V}} = \Big(\int_{B_1} |\nabla u|^2 |x|^{2-n} A_1(|x|)\, dx\Big)^{1/2} \quad \text{for any } u \in C_0^\infty(B_1). \tag{6.3}$$

Then $\widetilde{V}$ clearly becomes a Hilbert space. Now we consider a map T given by

$$T(u) = A_1(|x|)^{-1/2} u \quad \text{for any } u \in C_0^\infty(B_1). \tag{6.4}$$

A direct calculation gives us for any $u \in C_0^\infty(B_1)$ and $v = T(u)$,

$$P_{1/4} u = -A_1(|x|)^{-1/2} \operatorname{div}\big(|x|^{2-n} A_1(|x|) \nabla v\big). \tag{6.5}$$

Then we have following.

LEMMA 6.1

The map $T : V \to \widetilde{V}$ is an isometry. Namely, we have for any $u \in V$ and $v = T(u)$,

$$\|u\|_V = \|v\|_{\widetilde{V}}. \tag{6.6}$$

Proof

First we assume that $u \in C_0^\infty(B_1)$. Then $v = T(u)$ is smooth possibly except for the origin and $v(0) = 0$. Multiplying u to the both sides of (6.5) and using integration by parts, we have

$$\begin{aligned}\|u\|_V^2 &= \langle P_{\frac{1}{4}} u, u\rangle_{V',V}\\ &= \big\langle -A_1(|x|)^{-1/2}\operatorname{div}\big(|x|^{2-n}A_1(|x|)\nabla v\big), A_1(|x|)^{1/2}v\big\rangle_{V',V}\\ &= \int_{B_1} |\nabla v|^2|x|^{2-n}A_1(|x|)\,dx = \|v\|_{\widetilde{V}}^2.\end{aligned}$$

Here we used the fact $v(0) = 0$. Then by the density argument, we see that (6.1) holds for any $u \in V$. □

We remark that V is not embedded into $W^{1,2}_{2-n/2,0}(B_1)$. To see this fact let us set

$$U(x) = A_1(|x|)^{1/2} - A_1(1)^{1/2} \quad \text{for } R > 1.$$

Then we easily see that $U \notin W^{1,2}_{2-n/2,0}(B_1)$. On the other hand, we have $U \in V$. To see this, setting $w = T(U) = 1 - A_1(1)^{1/2}A_1(|x|)^{-1/2}$ we show $w \in \widetilde{V}$. In fact $|\nabla w| \le Cr^{-1}A_1(r)^{-3/2}$ for some positive number C and $r = |x|$, and hence

$$\|w\|_{\widetilde{V}}^2 \le C' \int_0^1 r^{-1}A_1(r)^{-2}\,dr < +\infty,$$

where C' is some positive number. Since w satisfies the boundary condition, $w \in \widetilde{V}$. Hence we can conclude that $U \in V$ from Lemma 6.1.

Proof of Theorem 2.3

If $a > -n$, then it follows from Theorem 2.2 and its proof that V is compactly embedded into $L^2(B_1, |x|^a)$. It is easy to see that $P_{1/4} : V \to V'$ is a continuous, linear, and self-adjoint isomorphism. Moreover $\langle P_{1/4}u, v\rangle_{V',V} = (u, v)_V$. Then by the compactness result mentioned above, the restriction of $P_{1/4}^{-1}$ to $(L^2(B_1, |x|^a))'$ is a compact map on $L^2(B_1, |x|^a)$. Therefore the assertion (1) is now clear from the classical theory of self-adjoint operators in Hilbert spaces. In particular, there exist a positive first eigenvalue λ_1 and corresponding first eigenfunction u_1, which is unique up to a multiplication by constants. We proceed to proof of assertion (2). Without a loss of generality we assume that u_1 is nonnegative. The second assertion will be proved by constructing an unbounded subsolution to the eigenvalue problem. Let us set

$$W(x) = C_1\big(A_1(r)^{1/2} - C_2 r^{n+a}\big) \quad \text{for } r = |x| \text{ and } R > 1. \tag{6.7}$$

Here C_1 and C_2 are positive numbers, which will be specified later. First we note that

$$P_{\frac{1}{4}} A_1(r)^{1/2} = 0.$$

Then we have

$$(6.8)\qquad P_{1/4}W(x)=C_1C_2\Big((n+a)^2+\frac{1}{4}A_1(r)^{-2}\Big)r^a \quad \text{for } r=|x| \text{ and } R>1.$$

For any $\varepsilon>0$, there is some number $C_1>0$ such that we have

$$(6.9)\qquad P_{1/4}W(x)\le \lambda_1 r^a\varepsilon.$$

Here we may put $C_1=\varepsilon\lambda_1 C_2^{-1}((n+a)^2+1/4A_1(1)^{-2})^{-1}$. For any $\rho>0$,

$$W(x)|_{r=\rho}=C_1\big(A_1(\rho)^{1/2}-C_2\rho^{n+a}\big).$$

Let us set $C_2=C_2(\rho)=\rho^{-n-a}A_1(\rho)^{1/2}$. Then we have

$$(6.10)\qquad W=0 \quad \text{on } \partial B_\rho.$$

We need the following.

LEMMA 6.2 (WEAK COMPARISON PRINCIPLE)
Let μ satisfy $\mu\le 1/4$. Let $u,v\in V$ such that

$$\begin{cases} P_\mu u\ge P_\mu v & \textit{in } B_1,\\ u\ge v & \textit{on } \partial B_1.\end{cases}$$

Then $u\ge v$ in B_1.

LEMMA 6.3
Here $u_1>0$ in B_1.

Admitting these for the moment we prove the assertion (2). From this lemma, for some $\varepsilon>$ and some $\rho>0$ we have

$$(6.11)\qquad P_{1/4}u_1=\lambda_1|x|^a u_1\ge \lambda_1|x|^a\varepsilon \quad \text{for } x\in B_\rho.$$

From (6.9) and (6.10) we have

$$(6.12)\qquad \begin{cases} P_{1/4}u_1\ge P_{1/4}W & \text{in } B_\rho,\\ u_1\ge W=0 & \text{on } \partial B_\rho.\end{cases}$$

Then by a weak comparison principle we conclude that

$$(6.13)\qquad u_1\ge W \quad \text{in } B_\rho.$$

Since W is unbounded in B_ρ, the assertion (2) is proved. □

Proof of Lemma 6.2
Let us set $\varphi=u-v$ and $\varphi^-=-\min(u-v,0)$. Noting that $P_\mu\varphi\ge 0$, $0\le\varphi^-\in V$, and $\varphi^-=0$ on ∂B_1, we have

$$0\le\langle P_\mu\varphi,\varphi^-\rangle_{V',V}=-\int_{B_1}|\nabla\varphi^-|^2|x|^{2-n}\,dx+\mu\int_{B_1}\frac{(\varphi^-)^2}{|x|^n A_1(|x|)^2}\,dx.$$

Therefore by the (improved) Hardy inequality we immediately have $\varphi^-\equiv 0$ in B_1. This proves the assertion. □

Proof of Lemma 6.3
We shall employ a weak comparison principle and Harnack's inequality. Let G be a truncation of the potential given by

$$G = \min\bigl(1, 4^{-1}|x|^{-n} A_1(|x|)^{-2}\bigr).$$

Then let $\tilde{u}$ be the solution to the boundary problem:

$$\begin{cases} -\operatorname{div}(|x|^{2-n}\nabla \tilde{u}) = G u_1 \quad \text{in } B_1, \\ \tilde{u} = 0 \quad \text{on } \partial B_1. \end{cases} \tag{6.14}$$

Since G is bounded, we see that $Gu_1 \in (W^{1,2}_{2-n/2,0}(B_1))'$. Hence $\tilde{u} \in W^{1,2}_{2-n/2,0}(B_1)$. Then by Harnack's inequality (see, e.g., [11]) we have $\tilde{u} > 0$ in B_1. On the other hand, we see

$$\begin{cases} -\operatorname{div}(|x|^{2-n}\nabla \tilde{u}) = G u_1 \le -\operatorname{div}(|x|^{2-n}\nabla u_1) \quad \text{in } B_1, \\ \tilde{u} = u_1 = 0 \quad \text{on } \partial B_1. \end{cases} \tag{6.15}$$

Therefore it follows from a weak comparison principle that $u_1 \ge \tilde{u} > 0$. □

7. Further result

When $p = 2$, we have established in [9] the existence of *finitely many sharp missing terms* of the Hardy–Sobolev inequality.

PROPOSITION 7.1
Let $n \ge 2$, $k \ge 1$, and $R \ge e_k \sup_\Omega |x|$. For any $u \in W^{1,2}_{0,0}(\Omega)$, there exist sharp remainder terms such that

$$\begin{aligned} \int_\Omega |\nabla u(x)|^2\, dx \ge \Bigl(\frac{n-2}{2}\Bigr)^2 \int_\Omega \frac{u(x)^2}{|x|^2}\, dx \\ + \frac{1}{4}\int_\Omega \frac{u(x)^2}{|x|^2}\bigl[A_1(|x|)^{-2} + \bigl(A_1(|x|)A_2(|x|)\bigr)^{-2} + \cdots \\ + \bigl(A_1(|x|)A_2(|x|)\ldots A_k(|x|)\bigr)^{-2}\bigr]\, dx, \end{aligned} \tag{7.1}$$

where

$$A_1(t) := \log\frac{R}{t}, \qquad A_k(t) := \log A_{k-1}(t), \qquad e_1 := e,$$
$$e_k := e^{e_{k-1}} \quad (t > 0 \text{ and } k \ge 2).$$

This can be improved in the following way.

PROPOSITION 7.2
Let $n \ge 2$, $k \ge 1$, and $R \ge e_k \sup_\Omega |x|$.

(1) Subcritical case ($\alpha > 1 - n/2$)

For any $u \in W^{1,2}_{\alpha,0}(\Omega)$, *there exist sharp remainder terms such that*

$$\begin{aligned}\int_\Omega |\nabla u(x)|^2 |x|^{2\alpha}\,dx \geq \Lambda_{n,2,\alpha} \int_\Omega \frac{u(x)^2}{|x|^2} |x|^{2\alpha}\,dx \\ + \frac{1}{4}\int_\Omega \frac{u(x)^2}{|x|^2}\big[A_1(|x|)^{-2} + \big(A_1(|x|)A_2(|x|)\big)^{-2} + \cdots \\ + \big(A_1(|x|)A_2(|x|)\dots A_k(|x|)\big)^{-2}\big]|x|^{2\alpha}\,dx.\end{aligned} \tag{7.2}$$

(2) Critical case ($\alpha = 1 - n/2$)
For any $u \in W^{1,2}_{\alpha,0}(\Omega)$, *there exist sharp remainder terms such that*

$$\begin{aligned}\int_\Omega |\nabla u(x)|^2 |x|^{2\alpha}\,dx \geq \frac{1}{4}\int_\Omega \frac{u(x)^2}{|x|^2}\big[A_1(|x|)^{-2} + \big(A_1(|x|)A_2(|x|)\big)^{-2} + \cdots \\ + \big(A_1(|x|)\dots A_k(|x|)\big)^{-2}\big]|x|^{2\alpha}\,dx.\end{aligned} \tag{7.3}$$

(3) Supercritical case ($\alpha < 1 - n/2$)
For any $u \in \dot{W}^{1,2}_{\alpha,0}(\Omega)$, *there exist sharp remainder terms such that*

$$\begin{aligned}\int_\Omega |\nabla u(x)|^2 |x|^{2\alpha}\,dx \geq \Lambda_{n,2,\alpha} \int_\Omega \frac{u(x)^2}{|x|^2} |x|^{2\alpha}\,dx \\ + \frac{1}{4}\int_\Omega \frac{u(x)^2}{|x|^2}\big[A_1(|x|)^{-2} + \big(A_1(|x|)A_2(|x|)\big)^{-2} + \cdots \\ + \big(A_1(|x|)A_2(|x|)\dots A_k(|x|)\big)^{-2}\big]|x|^{2\alpha}\,dx.\end{aligned} \tag{7.4}$$

The proof is done in a straightforward way by using similar lemmas in this paper. This can also be proved as a corollary to the inequality with infinitely many sharp missing terms which will be treated in an upcoming paper [3]. Therefore let us omit the proof here.

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Ando: Department of Mathematics, Faculty of Science, Ibaraki University, Mito, Ibaraki, 310, Japan; hando@mx.ibaraki.ac.jp

Horiuchi: Department of Mathematics, Faculty of Science, Ibaraki University, Mito, Ibaraki, 310, Japan; horiuchi@mx.ibaraki.ac.jp