Strongly symmetric smooth toric varieties

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Abstract We investigate toric varieties defined by arrangements of hyperplanes and call them strongly symmetric. The smoothness of such a toric variety translates to the fact that the arrangement is crystallographic. As a result, we obtain a complete classification of this class of toric varieties. Further, we show that these varieties are projective and describe associated toric arrangements in these varieties.

1. Introduction

In this paper we investigate toric varieties which are defined by fans of arrangements of hyperplanes, thereby generalizing the definition and construction of toric varieties which are associated to classical root systems. Toric varieties arising from root systems had previously been considered, investigated, and used by De Concini and Procesi [DCP1], Voskresenskij and Klyachko [VK], Procesi [Pro], Dolgachev and Lunts [DL], Stembridge [Ste], Klyachko [Kly], and Brion and Joshua [BJ]. Recently Batyrev and Blume [BB2], [BB1] found generalizations of the Losev–Manin moduli spaces by investigating the functor of toric varieties associated with Weyl chambers. The so-called crystallographic arrangements are generalizations of the classical root systems and their Weyl chamber structure. In this paper we establish a one-to-one correspondence between crystallographic arrangements and toric varieties which are smooth and projective, and which have the property of being strongly symmetric (see Definition 3.1), a property which has not been used in the previous papers mentioned above.

Crystallographic arrangements where originally used in the theory of pointed Hopf algebras: classical Lie theory leads to the notion of Weyl groups which are special reflection groups characterized by a certain integrality and which are therefore also called *crystallographic* reflection groups. A certain generalization of the universal enveloping algebras of Lie algebras yields Hopf algebras to which one can associate *root systems* and *Weyl groupoids* (see [Hec], [HS], [AHS]). The case of *finite Weyl groupoids* has recently been treated including a complete classification in a series of papers (see [CH2], [CH1], [CH3], [CH4], [CH5]).

The theorems needed for the classification reveal an astonishing connection. It turns out that finite Weyl groupoids correspond to certain simplicial arrangements called *crystallographic* (see [Cu]). Let $\mathcal{A}$ be a simplicial arrangement of

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finitely many real hyperplanes in a Euclidean space V, and let R be a set of nonzero covectors such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$. Assume that $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$. The pair $(\mathcal{A}, R)$ is called crystallographic (see [Cu, Definition 2.3] or Definition 2.1) if for any chamber K the elements of R are integer linear combinations of the covectors defining the walls of K. For example, crystallographic Coxeter groups give rise to crystallographic arrangements in this sense, but there are many others.

Thus the main feature of crystallographic arrangements is the integrality. But integrality is also the fundamental property of a fan in toric geometry. Indeed, the set of closed chambers of a rational simplicial arrangement is a fan which is strongly symmetric. A closer look reveals that the property *crystallographic* corresponds to the smoothness of the variety. We obtain the following (see Theorem 4.3).

THEOREM 1.1

There is a one-to-one correspondence between crystallographic arrangements and strongly symmetric smooth fans.

Thus the classification of finite Weyl groupoids (see [CH5]) gives the following.

COROLLARY 1.2

Any strongly symmetric smooth complete toric variety is isomorphic to a product of

(1) *varieties of dimension two corresponding to triangulations of a convex n-gon by nonintersecting diagonals (see Section 6),*

(2) *varieties of dimension $r > 2$ corresponding to the reflection arrangements of type A_r, B_r, C_r, and D_r, or out of a series of $r - 1$ further varieties,*

(3) 74 *further "sporadic" varieties.*

To each crystallographic arrangement $\mathcal{A}$, we construct a polytope P such that the toric variety of P is isomorphic to the toric variety corresponding to $\mathcal{A}$. Thus we obtain that the variety is projective (see Section 5). Further, the strong symmetry of the fan Σ associated to $\mathcal{A}$ gives rise to a system $\{Y^E\}_{E \in L(\mathcal{A})}$ of smooth strongly symmetric toric varieties $Y^E \subseteq X_\Sigma$. (Here $L(\mathcal{A})$ is the poset of intersections of hyperplanes of $\mathcal{A}$.) This system mirrors the arrangement $\mathcal{A}$ in X_Σ in a remarkable way (see Section 7.2) and will be called the associated toric arrangement. The intersections $Y^H \cap T$ with the torus T of X_Σ for $H \in \mathcal{A}$ are subtori of T and form a toric arrangement.

This note is organized as follows. After recalling the notions of fans and arrangements of hyperplanes in Section 2, we collect some results on strongly symmetric fans in Section 3. We then prove the main theorem (the correspondence) in Section 4. In Section 5 we construct a polytope for each crystallographic arrangement. In Section 6 we compare the well-known classifications of smooth complete surfaces (specified for the centrally symmetric case) and the corresponding arrangements of rank two. In the following section we discuss the toric arrangements associated to the crystallographic arrangements. The last section consists of further remarks on irreducibility, blowups, and automorphisms.

2. Preliminaries

Let us first recall the notions of hyperplane arrangements and of fans for normal toric varieties.

For subsets A in a real vector space V of dimension r and a subset B of its dual V^* we set

$$A^\perp = \{b \in V^* \mid b(a) = 0 \ \forall a \in A\},$$
$$B^\vee = \{a \in V \mid b(a) \geq 0 \ \forall b \in B\},$$
$$B^\perp = \{a \in V \mid b(a) = 0 \ \forall b \in B\}.$$

An *open* or *closed simplicial cone* σ is a subset $\sigma \subseteq V$ such that there exist linearly independent $n_1, \dots, n_d$, $d \in \mathbb{N}$, with

$$\sigma = \langle n_1, \dots, n_d \rangle_{\mathbb{R}_{>0}} := \mathbb{R}_{>0} n_1 + \cdots + \mathbb{R}_{>0} n_d$$

or

$$\sigma = \langle n_1, \dots, n_d \rangle_{\mathbb{R}_{\geq 0}} := \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_d,$$

respectively.

2.1. Fans and toric varieties

Given a lattice N in V of rank r, its dual lattice $M = \operatorname{Hom}(N, \mathbb{Z})$ is viewed as a lattice in V^*. A subset $\sigma \subseteq V$ is called a (closed) *strongly convex rational polyhedral cone* if there exist $n_1, \dots, n_d \in N$ such that

$$\sigma = \langle n_1, \dots, n_d \rangle_{\mathbb{R}_{\geq 0}} \qquad \text{and} \qquad \sigma \cap -\sigma = \{0\}.$$

We say that $n_1, \dots, n_d$ are *generators* of σ. By abuse of notation we will call such a cone simply an "N-cone."

We call σ *simplicial* if σ is a closed simplicial cone. If σ is generated by a subset of a $\mathbb{Z}$-basis of N, then we say that σ is *smooth*. Let σ be an N-cone. We write $\langle \sigma \rangle_{\mathbb{R}} := \sigma + (-\sigma)$ for the subspace spanned by σ. The *dimension* $\dim(\sigma)$ of σ is the dimension of $\langle \sigma \rangle_{\mathbb{R}}$.

Identifying $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ with V, we consider fans Σ in $N_{\mathbb{R}}$ of strongly convex rational polyhedral cones as defined in the standard theory of toric varieties (see [Oda], [CLS]).

A *face* of σ is the intersection of σ with a supporting hyperplane, $\sigma \cap m^\perp$, $m \in V^*$, $m(a) \geq 0$ for all $a \in \sigma$. Faces of codimension 1 are called *facets*.

A *fan* in N is a nonempty collection of N-cones Σ such that

(1) any face τ of a cone $\sigma \in \Sigma$ is contained in Σ;
(2) any intersection $\sigma_1 \cap \sigma_2$ of two cones $\sigma_1, \sigma_2 \in \Sigma$ is a face of σ_1 and σ_2.

For $k \in \mathbb{N}$ we write $\Sigma(k) = \{\sigma \in \Sigma \mid \dim(\sigma) = k\}$. For $S \subseteq \Sigma$ we write $\operatorname{Supp} S = \bigcup_{\sigma \in S} \sigma$ for the *support* of S.

The fan Σ and its associated toric variety X_Σ (over the ground field $\mathbb{C}$) are called *simplicial* if any cone of Σ is simplicial. It is well known that X_Σ for finite Σ is nonsingular (smooth) if and only if each cone σ of Σ is smooth. Moreover, X_Σ is complete (compact) if and only if Σ is finite and $\operatorname{Supp} \Sigma = N_{\mathbb{R}}$.

2.2. Crystallographic arrangements

Let $\mathcal{A}$ be a simplicial arrangement in $V = \mathbb{R}^r$; that is, $\mathcal{A} = \{H_1, \dots, H_n\}$ where $H_1, \dots, H_n$ are distinct linear hyperplanes in V and every component of $V \setminus \bigcup_{H \in \mathcal{A}} H$ is an open simplicial cone. Let $\mathcal{K}(\mathcal{A})$ be the set of connected components of $V \setminus \bigcup_{H \in \mathcal{A}} H$; they are called the *chambers* of $\mathcal{A}$.

For each H_i, $i = 1, \dots, n$, we choose an element $x_i \in V^*$ such that $H_i = x_i^\perp$. Let

$$R = \{\pm x_1, \dots, \pm x_n\} \subseteq V^*.$$

For each chamber $K \in \mathcal{K}(\mathcal{A})$ set

$$W^K = \bigl\{H \in \mathcal{A} \bigm| \dim(H \cap \overline{K}) = r - 1\bigr\},$$
$$B^K = \bigl\{\alpha \in R \bigm| \alpha^\perp \in W^K, \{\alpha\}^\vee \cap K = K\bigr\} \subseteq R.$$

Here, $\overline{K}$ denotes the closure of K. The elements of W^K are the *walls* of K and B^K "is" the set of normal vectors of the walls of K pointing to the inside. Note that

$$\overline{K} = \bigcap_{\alpha \in B^K} \{\alpha\}^\vee$$

and that B^K is a basis of V^* because $\mathcal{A}$ is simplicial. Moreover, if $\alpha_1^\vee, \dots, \alpha_r^\vee$ is the dual basis to $B^K = \{\alpha_1, \dots, \alpha_r\}$, then

$$K = \Bigl\{\sum_{i=1}^r a_i \alpha_i^\vee \Bigm| a_i > 0 \text{ for all } i = 1, \dots, r\Bigr\}. \tag{2.1}$$

DEFINITION 2.1

Let $\mathcal{A}$ be a simplicial arrangement, and let $R \subseteq V^*$ be a finite set such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$. For $K \in \mathcal{K}(\mathcal{A})$ set

$$R_+^K = R \cap \sum_{\alpha \in B^K} \mathbb{R}_{\geq 0} \alpha.$$

We call $(\mathcal{A}, R)$ a *crystallographic arrangement* if for all $K \in \mathcal{K}(\mathcal{A})$,

(I) $R \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha$.

REMARK 2.2

Notice that one can in fact prove that if $(\mathcal{A}, R)$ is crystallographic, then $R \subseteq \pm \sum_{\alpha \in B^K} \mathbb{N}_0 \alpha$ (see [Cu]).

3. Strong symmetry of fans

DEFINITION 3.1

We call a fan Σ in V *strongly symmetric* if it is complete and if there exist hyperplanes $H_1, \dots, H_n$ in V such that

$$\operatorname{Supp}\Sigma(r-1) = H_1 \cup \cdots \cup H_n.$$

We write $\mathcal{A}(\Sigma) := \{H_1, \dots, H_n\}$. We call a toric variety X_Σ *strongly symmetric* if Σ is strongly symmetric.

We call a fan Σ *centrally symmetric* if $\Sigma = -\Sigma$. We call a toric variety X_Σ *centrally symmetric* if Σ is centrally symmetric.

REMARK 3.2

One could also call a strongly symmetric fan *strongly complete* because for any $\tau \in \Sigma$ the collection of $\sigma \cap \langle\tau\rangle_\mathbb{R}$, $\sigma \in \Sigma$, is a complete fan in $\langle\tau\rangle_\mathbb{R}$ as a subfan of Σ.

LEMMA 3.3

Let τ be an $(r-1)$-dimensional cone in $\mathbb{R}^r$, and let $H_1, \dots, H_n$ be hyperplanes in $\mathbb{R}^r$. If $\tau \subseteq H_1 \cup \cdots \cup H_n$, then $\tau \subseteq H_i$ for some $1 \le i \le n$.

Proof

We construct inductively sets $T_i \subseteq \tau$ with $i + r - 1$ elements such that each subset B, $|B| = r - 1$, is linearly independent: let $T_0 := \{n_1, \dots, n_{r-1}\}$, where $n_1, \dots, n_{r-1} \in \tau$ are linearly independent and span $\langle\tau\rangle_\mathbb{R}$. Given T_i, let

$$\Xi_i := \bigl\{\langle v_1, \dots, v_{r-2}\rangle \bigm| v_1, \dots, v_{r-2} \in T_i\bigr\}$$

be the set of subspaces generated by $r-2$ elements of T_i. Since τ has dimension $r-1$, $\bigcup_{U \in \Xi_i} U \neq \langle\tau\rangle_\mathbb{R}$. For any $w \in \tau \smallsetminus \bigcup_{U \in \Xi_i} U$, $T_{i+1} := T_i \cup \{w\}$ has the required property.

Now consider the $(r-1)n$ elements of $T_{(r-1)(n-1)}$. Let ℓ be the maximal number of elements in any H_i. Then $\ell \ge r-1$. Then there is a $1 \le i \le n$ such that $r-1$ of these elements lie in H_i. These are linearly independent and belong to τ, so $\tau \subseteq \langle\tau\rangle_\mathbb{R} \subseteq H_i$. □

LEMMA 3.4

Let Σ be an r-dimensional fan. Then the following are equivalent:

(1) *Σ is complete, and for all $\tau \in \Sigma(r-1)$, $\sigma \in \Sigma$,*

$$\sigma \cap \langle\tau\rangle_\mathbb{R} \in \Sigma;$$

(2) *the fan Σ is strongly symmetric.*

Proof

Assume (1). Let $\tau \in \Sigma(r-1)$. Since Σ is complete, $\langle\tau\rangle_\mathbb{R} \subseteq \operatorname{Supp}\Sigma$. Thus $\langle\tau\rangle_\mathbb{R} = \bigcup_{\sigma\in\Sigma} \langle\tau\rangle_\mathbb{R} \cap \sigma$. By (1), $\Sigma' := \{\langle\tau\rangle_\mathbb{R} \cap \sigma \mid \sigma \in \Sigma\}$ is a subfan of Σ. Further, $\operatorname{Supp}\Sigma'(r-1) = \operatorname{Supp}\Sigma' = \langle\tau\rangle_\mathbb{R}$ because Σ' is complete in $N \cap \langle\tau\rangle_\mathbb{R}$ and the maximal cones in Σ' have dimension $r-1$. Hence for each τ of codimension 1, $\langle\tau\rangle_{\mathbb{R}}$ is a union of elements of $\Sigma(r-1)$. This implies $\operatorname{Supp}\Sigma(r-1)=\bigcup_{\tau\in\Sigma(r-1)}\langle\tau\rangle_{\mathbb{R}}$ (finite union by definition of complete).

Now assume $\operatorname{Supp}\Sigma(r-1)=H_1\cup\cdots\cup H_n$ for some hyperplanes $H_1,\ldots,H_n$. Let $\tau\in\Sigma(r-1)$ and $\sigma\in\Sigma$. Then by Lemma 3.3, $\langle\tau\rangle_{\mathbb{R}}=H_i$ for some $1\le i\le n$, and there exist $\eta_1,\ldots,\eta_k\in\Sigma(r-1)$ with $H_i=\eta_1\cup\cdots\cup\eta_k$. But $\sigma\cap H_i=\bigcup_{j=1}^k\sigma\cap\eta_j$, so $\overset{\circ}{\sigma}\cap H_i=\emptyset$; that is, H_i is a supporting hyperplane and $\sigma\cap H_i$ is a face of σ and thus an element of Σ. □

LEMMA 3.5

Let Σ be an r-dimensional strongly symmetric fan. Then the set of all intersections of closed chambers of $\mathcal{A}(\Sigma)$ is Σ. In particular, Σ is centrally symmetric.

Proof

Let $\sigma\in\Sigma(r)$. Then the facets of σ are in $\operatorname{Supp}\Sigma(r-1)=H_1\cup\cdots\cup H_n$ and $\overset{\circ}{\sigma}\subseteq\mathbb{R}^r\smallsetminus\operatorname{Supp}\Sigma(r-1)$. Since Σ is complete, $\Sigma(r)$ is the set of closed chambers of $\mathcal{A}$. □

DEFINITION 3.6

Let Σ be a fan in N, $\delta\in\Sigma$, and write $\kappa:V\to V/\langle\delta\rangle_{\mathbb{R}}$ for the canonical projection. Then

$$\operatorname{Star}(\delta)=\left\{\overline{\sigma}=\kappa(\sigma)\subseteq V/\langle\delta\rangle_{\mathbb{R}}\;\middle|\;\delta\subseteq\sigma\in\Sigma\right\}$$

is a fan in $N(\delta):=\kappa(N)$ (cf. [CLS, Example 3.2.7]). Its toric variety is isomorphic to the orbit closure $V(\delta)$ in X_Σ.

LEMMA 3.7

Let Σ be an r-dimensional fan. Then the following are equivalent:

(1) *the fan Σ is strongly symmetric;*
(2) *the fan* $\operatorname{Star}(\sigma)$ *is strongly symmetric for all $\sigma\in\Sigma$.*

Proof

We use Lemma 3.4. Assume (1). Let $\sigma\in\Sigma$, and consider a cone $\overline{\tau}\in\operatorname{Star}(\sigma)$ of codimension one. Then $\langle\overline{\tau}\rangle_{\mathbb{R}}=\overline{\langle\tau\rangle_{\mathbb{R}}}\subseteq V/\langle\sigma\rangle_{\mathbb{R}}$, and hence for any cone $\overline{\pi}\in\operatorname{Star}(\sigma)$ we have $\overline{\pi}\cap\langle\overline{\tau}\rangle_{\mathbb{R}}=\overline{\pi\cap\langle\tau\rangle_{\mathbb{R}}}\in\operatorname{Star}(\sigma)$, because $\pi\cap\langle\tau\rangle_{\mathbb{R}}$ is a cone in Σ containing σ; thus $\operatorname{Star}(\sigma)$ is strongly symmetric.

Since $\Sigma=\operatorname{Star}(\{0\})$, (1) follows from (2). □

PROPOSITION 3.8

Let Σ be an r-dimensional complete fan. Then the following are equivalent:

(1) *the fan Σ is strongly symmetric;*
(2) *the fan* $\operatorname{Star}(\sigma)$ *is centrally symmetric for all $\sigma\in\Sigma$;*
(3) *the fan* $\operatorname{Star}(\delta)$ *is centrally symmetric for all $\delta\in\Sigma(r-2)$.*

Proof
The implication (1) $\Rightarrow$ (2) follows from Lemmas 3.5 and 3.7; (2) $\Rightarrow$ (3) is obvious.

Suppose that $\mathrm{Star}(\delta)$ is centrally symmetric for any $\delta \in \Sigma(r-2)$. We have to show that for any $\tau_0 \in \Sigma(r-1)$, $H := \langle \tau_0 \rangle_{\mathbb{R}} \subseteq S := \operatorname{Supp} \Sigma(r-1)$. Suppose $H \nsubseteq S$. Let $\{\tau_0, \ldots, \tau_k\} = \{\tau \in \Sigma(r-1) \mid \tau \subseteq H\}$. Then

$$\tau_0 \cup \cdots \cup \tau_k \subsetneqq H.$$

Let p be a point of the relative border $\partial(\tau_0 \cup \cdots \cup \tau_k)$ in H. Then there is an i with $p \in \partial\tau_i$ and a $\delta \in \Sigma(r-2)$, $\delta \subseteq \tau_i$, such that $p \in \delta \subseteq \tau_i \subseteq \langle \tau_i \rangle_{\mathbb{R}} = H$. We have $\overline{\tau_i} \in \mathrm{Star}(\delta)$, $\overline{\tau_i} \subseteq \overline{H}$, and $\dim \overline{H} = 1$. Because $\mathrm{Star}(\delta)$ is centrally symmetric, $-\overline{\tau_i} \in \mathrm{Star}(\delta)$. Then $-\overline{\tau_i} = \overline{\tau'}$ for some $\delta \subseteq \tau' \in \Sigma(r-1)$ with $\overline{\tau'} \subseteq \overline{H}$. Then $\tau' \subseteq H$, $\delta \subseteq \tau_i \cap \tau'$, and $\tau' \neq \tau_i$. Hence $\delta = \tau_i \cap \tau'$ because $\dim(\delta) = r-2$. But then $p \notin \partial(\tau_0 \cup \cdots \cup \tau_k)$, contradicting the assumption. $\square$

EXAMPLE 3.9
There are of course fans which are centrally symmetric but not strongly symmetric. Here is such an example which is smooth: let R be the standard basis of $\mathbb{R}^3$, and let Σ_R be the fan as defined in Lemma 4.1. Blowing up along two opposite cones $\sigma, -\sigma \in \Sigma_R$ preserves the central symmetry, but the resulting fan is not strongly symmetric.

In the case of smooth strongly symmetric fans, we obtain the following.

LEMMA 3.10
Let Σ be a smooth strongly symmetric fan in N, let $\sigma \in \Sigma$, and let $E := \langle \sigma \rangle_{\mathbb{R}}$. Then $N \cap E$ is a lattice of rank $\dim(\sigma)$ *and $\Sigma^E := \{\eta \cap E \mid \eta \in \Sigma\} \subseteq \Sigma$ is a smooth strongly symmetric fan in $N \cap E$.*

Proof
Using a $\mathbb{Z}$-basis of σ one finds that $N \cap E$ is a sublattice of N of rank $\dim(\sigma)$ and that the inclusion $N \cap E \hookrightarrow N$ is split. Consider first a $\sigma \in \Sigma(r-1)$, and let $E := \langle \sigma \rangle_{\mathbb{R}}$. By Lemma 3.4, $\eta \cap E \in \Sigma$ for all $\eta \in \Sigma$. Thus Σ^E is a subfan of Σ, and it is complete since $\operatorname{Supp} \Sigma = V$. Write $\operatorname{Supp} \Sigma(r-1) = E \cup H_2 \cup \cdots \cup H_n$ for hyperplanes $H_2, \ldots, H_n$. Then

$$\operatorname{Supp} \Sigma^E(r-2) = (H_2 \cup \cdots \cup H_n) \cap E = (H_2 \cap E) \cup \cdots \cup (H_n \cap E);$$

that is, Σ^E is strongly symmetric. The claim is true for arbitrary $\sigma \in \Sigma$ by induction on $\dim(\sigma)$. $\square$

4. The correspondence

LEMMA 4.1
Let $(\mathcal{A}, R)$ be a crystallographic arrangement in V. Set

$$M_R := \sum_{\alpha \in R} \mathbb{Z}\alpha \cong \mathbb{Z}^r,$$

and let N_R be the dual lattice to M_R. Then the set Σ_R of all intersections of closed chambers of $\mathcal{A}$ is a strongly symmetric smooth fan in N_R.

Proof

It is clear that Σ_R is a strongly symmetric fan. Let $\sigma \in \Sigma_R$ be of maximal dimension; that is, $\sigma = \overline{K}$ for a chamber $K \in \mathcal{K}(\mathcal{A})$. By equation (2.1), σ is generated by the basis of N_R dual to B^K; hence σ is smooth. $\square$

LEMMA 4.2

Let Σ be a strongly symmetric smooth fan in $N \subseteq V = \mathbb{R}^r$. Then there exists a set $R \subseteq V^*$ such that $(\mathcal{A}, R)$ is a crystallographic arrangement, where

$$\mathcal{A} = \mathcal{A}(\Sigma) = \big\{ \langle \tau \rangle_{\mathbb{R}} \;\big|\; \tau \in \Sigma(r-1) \big\}.$$

Proof

Since Σ is strongly symmetric, $\mathcal{A}$ is a finite set of hyperplanes, and by Lemma 3.5, the set of all intersections of closed chambers of $\mathcal{A}$ is Σ. Further,

$$\bigcup_{\sigma \in \Sigma(r)} \mathring{\sigma} = V \setminus \bigcup_{H \in \mathcal{A}} H$$

since each facet of a $\sigma \in \Sigma(r)$ is contained in a hyperplane of $\mathcal{A}$ and since Σ is complete. The cones $\mathring{\sigma}$ in the above union are open simplicial cones, because σ is smooth; hence $\mathcal{A}$ is a simplicial arrangement.

Let $\sigma \in \Sigma$ be a cone of maximal dimension. Since σ is smooth, there exists a unique $\mathbb{Z}$-basis of N generating σ. We will prematurely denote by B^{K_σ} its dual basis, where K_σ is the chamber with $\overline{K_\sigma} = \sigma$.

Now set R to be the union of all the B^{K_σ} for $\sigma \in \Sigma(r)$. Clearly,

$$R \subseteq \sum_{\alpha \in B^{K_\sigma}} \mathbb{Z}\alpha,$$

since each B^{K_σ} is a $\mathbb{Z}$-basis of $M = \mathrm{Hom}(N, \mathbb{Z})$ and $R \subseteq M$.

It remains to show that for each hyperplane $H = \langle \tau \rangle_{\mathbb{R}} \in \mathcal{A}$, $\tau \in \Sigma(r-1)$, there is a vector $x \in R$ such that $R \cap H^\perp = \{\pm x\}$.

Let $\sigma \in \Sigma(r)$ containing τ, and let x be the element with $\{x\} = B^{K_\sigma} \cap H^\perp$. In particular, x is primitive. Assume $\lambda x \in R$ for a $\lambda \in \mathbb{Z}$. Then there exists a $\sigma' \in \Sigma$ with $\lambda x \in B^{K_{\sigma'}}$. Thus $\lambda = \pm 1$ since $B^{K_{\sigma'}}$ is a $\mathbb{Z}$-basis of M. $\square$

THEOREM 4.3

The map $(\mathcal{A}, R) \mapsto \Sigma_R$ from the set of crystallographic arrangements to the set of strongly symmetric smooth fans is a bijection.

Proof

This is Lemmas 4.1 and 4.2. $\square$

COROLLARY 4.4

A complete classification of strongly symmetric smooth toric varieties is now known.

Proof

This is [CH5, Theorem 1.1]. □

DEFINITION 4.5

We denote the toric variety of the fan Σ_R by $X(\mathcal{A}, R)$ or $X(\mathcal{A})$ and call it the toric variety of the arrangement $(\mathcal{A}, R)$.

REMARK 4.6

For a fixed crystallographic arrangement $(\mathcal{A}, R)$, choosing another lattice than M_R may result in a strongly symmetric fan which is not smooth. Further, the correspondence $(\mathcal{A}, R) \mapsto \Sigma_R$ extends by its definition to a correspondence between rational simplicial arrangements and simplicial strongly symmetric fans. However, there exist rational simplicial noncrystallographic arrangements; that is, there is a basis with respect to which all covectors of the hyperplanes have rational coordinates, although there is no lattice M for which the corresponding fan is smooth. The smallest example in dimension three has 12 hyperplanes and is denoted $\mathcal{A}(12, 1)$ in [Grü] (cf. the catalogue in [Grü] with the list in [CH4]).

REMARK 4.7

Any smooth complete fan in N can be visualized by a triangulation of the sphere $S = V \smallsetminus \{0\}/\mathbb{R}_{>0}$ (see [Oda, Section 1.7]). Such a fan is centrally symmetric if and only if its triangulation is invariant under the reflection $p \leftrightarrow -p$ of S, and the strong symmetry of the fan Σ_R of a crystallographic arrangement $(\mathcal{A}, R)$ means that its triangulation is induced by the hyperplane sections $H \cap S$, $H \in \mathcal{A}$.

In particular, in dimension 3 Tsuchihashi's characterization by admissible N-weights (see [Oda, Corollary 1.32]) for strongly symmetric fans agrees with the classification in [CH4]. For higher dimensions the correspondence to Weyl groupoids produces similar conditions if one considers certain products of reflections.

For a geometric interpretation of the strong symmetry of $X(\mathcal{A})$, see Remark 7.9.

EXAMPLE 4.8

The crystallographic arrangement with the largest number of hyperplanes in dimension three has 37 hyperplanes. Figure 1 is a projective image of this *sporadic* arrangement: The triangles correspond to the maximal cones; one hyperplane is the line at infinity.

We further obtain a new proof of [BC, Proposition 5.3].

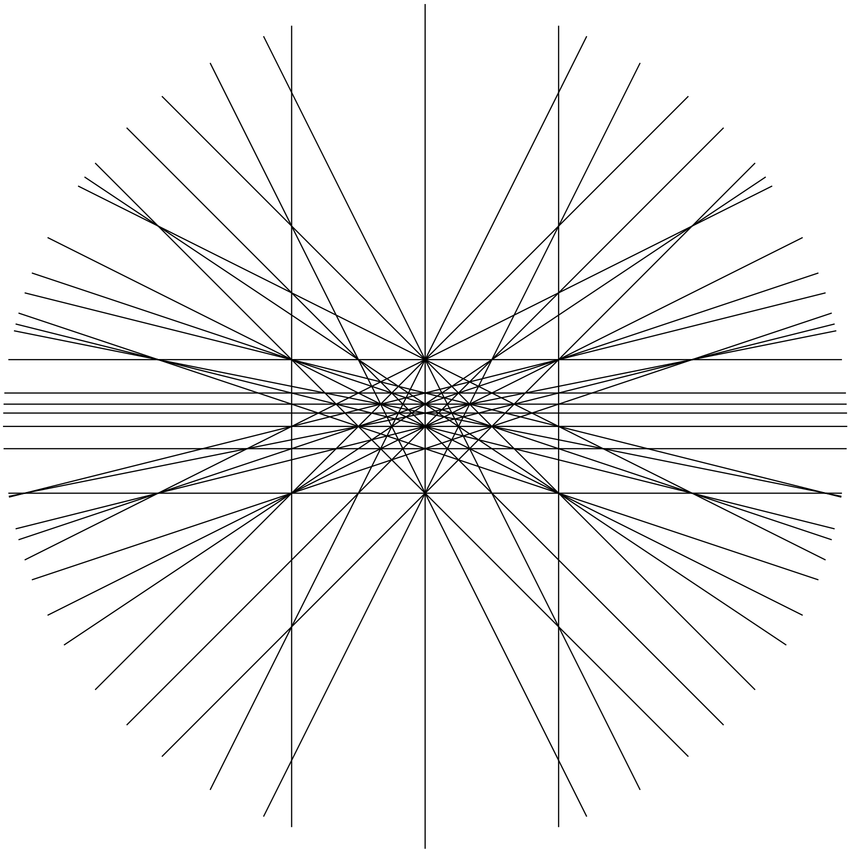

Figure 1. The largest crystallographic arrangement in dimension three (see Example 4.8)

COROLLARY 4.9

Let $\mathcal{A}$ be a crystallographic arrangement, and let E be an intersection of hyperplanes of $\mathcal{A}$. Then the restriction $\mathcal{A}^E$ of $\mathcal{A}$ to E,

$$\mathcal{A}^E := \{E \cap H \mid H \in \mathcal{A}, E \not\subseteq H\},$$

is a crystallographic arrangement.

Proof

This follows from Theorem 4.3, the fact that subfans of smooth fans are smooth, and Lemma 3.10. □

5. Projectivity

Let $(\mathcal{A}, R)$ be a crystallographic arrangement, and let N, M, V, V^* be as in Section 4, $\Sigma := \Sigma_R$. We first prove that $X(\mathcal{A}) = X_\Sigma$ is projective by constructing a polytope P such that $X_P \cong X_\Sigma$.

PROPOSITION 5.1

Let $\mathcal{A}$ be a crystallographic arrangement. For a chamber K let

$$\rho^K := \frac{1}{2} \sum_{\alpha \in R_+^K} \alpha.$$

Then the set $\{\rho^K \mid K \in \mathcal{K}(\mathcal{A})\}$ is the set of vertices of an integral convex polytope P in $(1/2)M$.

Proof

For each chamber K define a simplicial cone by

$$S^K := \rho^K - \langle \alpha \mid \alpha \in B^K \rangle_{\mathbb{R}_{\geq 0}}.$$

Let P be the polytope

$$P := \bigcap_{K \in \mathcal{K}(\mathcal{A})} S^K.$$

Let K be a chamber. We prove that ρ^K is a vertex of P by showing $\rho^K \in P$. Let K' be a chamber. Notice first that for $\alpha \in R$ we have

$$\alpha \in R_+^K \Longleftrightarrow -\alpha \in R \setminus R_+^K,$$

which implies $R_+^{K'} \setminus R_+^K = -R_+^K \setminus R_+^{K'}$. Thus

$$\begin{aligned} \rho^K &= \rho^{K'} - \frac{1}{2} \sum_{\alpha \in R_+^{K'} \setminus R_+^K} \alpha + \frac{1}{2} \sum_{\alpha \in R_+^K \setminus R_+^{K'}} \alpha \\ &= \rho^{K'} - \sum_{\alpha \in R_+^{K'} \setminus R_+^K} \alpha \in S^{K'}. \end{aligned}$$

□

REMARK 5.2

The set $\{\rho^K \mid K \in \mathcal{K}(\mathcal{A})\}$ of Proposition 5.1 is the orbit of one fixed ρ^K under the action of the Weyl groupoid $\mathcal{W}(\mathcal{A})$ since for a simple root $\alpha \in B^K$ we have $\sigma_\alpha(\rho^K) = \rho^K - \alpha$ (see [CH2]).

COROLLARY 5.3

Let $\mathcal{A}$ be a crystallographic arrangement. Then X_Σ is a projective variety isomorphic to X_P, where P is the polytope of Proposition 5.1.

Proof

This is Proposition 5.1 and [Oda, Theorem 2.22]. □

We now describe an explicit immersion of X_Σ into $\mathbb{P}_1^R \cong \mathbb{P}_1^{2n}$.

DEFINITION 5.4

For any $\sigma \in \Sigma$, $\alpha \in R$ let

$$s_\alpha(\sigma) = \begin{cases} +1 & \text{if } \alpha(\sigma) = \mathbb{R}_{\geq 0}, \\ 0 & \text{if } \alpha(\sigma) = \{0\}, \\ -1 & \text{if } \alpha(\sigma) = \mathbb{R}_{\leq 0}, \end{cases}$$

and let $s(\sigma) = (s_\alpha(\sigma))_{\alpha \in R}$.

DEFINITION 5.5

Let $2n = |R|$, let V' be a $2n$-dimensional vector space over $\mathbb{R}$, and let $(e_\alpha)_{\alpha \in R}$ be a basis of V'^*. Further, let $M' := \mathbb{Z}\{e_\alpha \mid \alpha \in R\} \subseteq V'^*$ be the lattice generated by this basis, and let N' be the dual lattice. Then $\mathcal{A}' := \{e_\alpha^\perp \mid \alpha \in R\}$ is a Boolean arrangement, and we call the corresponding fan $\Sigma' := \Sigma(\mathcal{A}')$ a *Boolean fan*. Notice that

$$X_{\Sigma'} \cong \mathbb{P}_1^{2n}.$$

Consider the homomorphism $M' \to M$, $e_\alpha \mapsto \alpha$ for $\alpha \in R$, and consider its dual

$$\varphi : N \to N', \qquad n \mapsto (\alpha(n))_{\alpha \in R}.$$

LEMMA 5.6

Choose a chamber K. Then with respect to the basis B^{K^*} of N the map φ is represented by a matrix of the form

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ * & \cdots & * \\ \vdots & & \vdots \end{pmatrix}.$$

It follows that φ is a split monomorphism and in particular $N'/\varphi(N)$ is torsion free.

LEMMA 5.7

(1) *The map φ is a map of fans $(N, \Sigma) \to (N', \Sigma')$.*
(2) *For any $\sigma' \in \Sigma'$, $\varphi(V) \cap \sigma' \in \Sigma$.*

Proof

(1) Let $\sigma \in \Sigma$, and let $\sigma' \in \Sigma'$ be the cone with $s(\sigma') = s(\sigma)$. Then $\varphi(\sigma) \subseteq \sigma'$.
(2) If $\sigma' \in \Sigma'$ is maximal, let $s(\sigma') = (\varepsilon_1, \dots, \varepsilon_{2n})$ with $\varepsilon_\nu \in \{\pm 1\}$, and let

$$\tau = \bigcap_\nu \{x \in V \mid \varepsilon_\nu \alpha_\nu(x) \geq 0\}.$$

Then $\tau \in \Sigma$ and $\tau = \varphi^{-1}(\sigma')$. If σ' is arbitrary, then $\sigma' = \sigma'_1 \cap \dots \cap \sigma'_k$ for maximal σ'_i and then $\varphi^{-1}(\sigma') = \bigcap \varphi^{-1}(\sigma'_i) \in \Sigma$. □

COROLLARY 5.8

The induced toric morphism $f = \varphi_* : X_\Sigma \to X_{\Sigma'}$ is proper, and $X_\Sigma \twoheadrightarrow f(X_\Sigma)$ is the normalization of the closed (reduced) image.

Proof

See [Oda, Proposition 1.14]. $\square$

PROPOSITION 5.9

The map $X_\Sigma \to X_{\Sigma'}$ is a closed embedding of nonsingular toric varieties.

Proof

Let σ be a maximal cone, let K be the corresponding chamber, and let $B^K \subseteq R$ be the basis of M. If $\sigma' \in \Sigma'$ is the cone with $s(\sigma) = s(\sigma')$ ($\sigma = \varphi(V) \cap \sigma'$), then the dual cone to σ' is

$$\sigma'^\vee = \langle e_\alpha \in R \mid s_\alpha(\sigma') = 1 \rangle_{\mathbb{R}_{\geq 0}}.$$

The map $\sigma'^\vee \cap M' \to \langle B^K \rangle_{\mathbb{Z}_{\geq 0}}$ is surjective, so $\mathbb{C}[\sigma'^\vee \cap M'] \to \mathbb{C}[\langle B^K \rangle_{\mathbb{Z}_{\geq 0}}]$ is a surjective homomorphism of $\mathbb{C}$-algebras giving rise to the closed embedding

$$f|_{U_\sigma} : U_\sigma \to U'_{\sigma'},$$

where $f = \varphi_*$ as in Corollary 5.8. Because U_σ is dense in X_Σ, the closure of $f(U_\sigma)$ equals $f(X_\Sigma)$; hence $f(U_\sigma) = f(X_\Sigma) \cap U'_{\sigma'}$. It follows that $f(X_\Sigma)$ is smooth and that $X_\Sigma \to f(X_\Sigma)$ is an isomorphism. The injectivity of f follows from that of $f|_{U_\sigma}$ because then $f|_{\mathrm{orb}(\sigma)}$ is an injective map $\mathrm{orb}(\sigma) \to \mathrm{orb}(\sigma')$ for each cone σ of the orbit decomposition of X_Σ. $\square$

6. Remarks on surfaces

For 2-dimensional fans of complete toric surfaces, obviously strongly symmetric is the same as centrally symmetric. The classification of smooth complete toric surfaces (see [Oda, Corollary 1.29]) can be specialized as follows. It turns out that this classification coincides with the classification of crystallographic arrangements of rank two (see [CH1], [CH3]).

Let Σ be the fan of a smooth complete toric surface with rays $\rho_1, \dots, \rho_s$ ordered counterclockwise with primitive generators $n_1, \dots, n_s$. There are integers $a_1, \dots, a_s$ such that

$$n_{j-1} + n_{j+1} + a_j n_j = 0$$

for $1 \leq j \leq s$ where $n_{s+1} := n_1$, $n_0 := n_s$. The integers a_j are the self-intersection numbers of the divisors D_j associated to the rays ρ_j. The circular weighted graph $\Gamma(\Sigma)$ has as its vertices on S^1 the rays ρ_j with weights a_j. These weights satisfy the identity

$$\begin{pmatrix} 0 & -1 \\ 1 & -a_s \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & -a_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

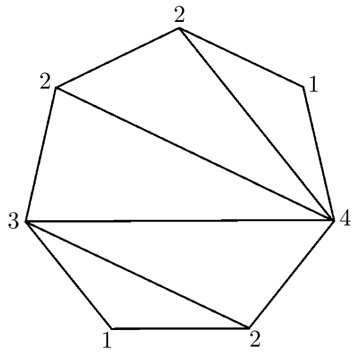

Figure 2. Triangulation of a t-gon

Conversely, to any circular weighted graph with this identity there is a smooth complete toric surface with this graph, unique up to toric isomorphisms.

All these surfaces are obtained from the basic surfaces $\mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1$, and the Hirzebruch surfaces $\mathbb{F}_a$, $a \geq 2$, by a finite succession of blowups. If the surface X_Σ is centrally symmetric, then the number s of rays is even, $s = 2t$, and $a_{t+j} = a_j$ for $1 \leq j \leq t$. In this case

$$\begin{pmatrix} 0 & -1 \\ 1 & -a_t \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & -a_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is “dual” to the formula of the classification of crystallographic arrangements of rank two (see [CH1]).

Note further that sequences $a_1, \dots, a_t$ satisfying this formula are in bijection with triangulations of a convex t-gon by nonintersecting diagonals. The numbers in Figure 2 are $-a_1, \dots, -a_t$; these are certain entries of the Cartan matrices of the corresponding Weyl groupoid (see [CH3] for more details). Attaching a triangle to the t-gon corresponds to a double blowup on the variety.

One can subdivide a smooth complete 2-dimensional fan Σ by filling in the opposite $-\rho$ of each ray ρ in order to get a complete centrally symmetric fan Σ_C. However, Σ_C need not be smooth as in Example 6.1. But by inserting further pairs $\rho, -\rho$ of rays one can desingularize the surface X_{Σ_C} in an even succession of blowups to obtain a smooth complete centrally symmetric surface $X_{\tilde{\Sigma}}$ with a surjective toric morphism $X_{\tilde{\Sigma}} \to X_\Sigma$.

EXAMPLE 6.1

Let Σ be the fan of the Hirzebruch surface $\mathbb{F}_a$, $a \geq 2$, with the primitive generators

$$n_1 = (1,0), \qquad n_2 = (0,1), \qquad n_3 = (-1,a), \qquad n_4 = (0,-1).$$

The fan Σ_C is then obtained by adding the rays spanned by $(-1,0)$ and $(1,-a)$. This fan is no longer smooth. After filling in the rays spanned by $(1,-\nu)$ for $1 \leq \nu < a$, we obtain a smooth complete centrally symmetric fan $\tilde{\Sigma}$ with $2a$ rays. In the case $a = 2$ its circular graph has the weights $(-1,-2,-1,-2;-1,-2,-1,-2)$. (This corresponds to the reflection arrangement of type B and C.)

EXAMPLE 6.2

In good cases the centrally symmetric fan Σ_C may already be smooth. As an example let Σ be the fan of $\mathbb{P}_2$ spanned by $(1,0)$, $(0,1)$, and $(-1,-1)$. Then the fan Σ_C is spanned, in counterclockwise order, by

$$(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1).$$

This is the fan of the blowup $\tilde{\mathbb{P}}_2$ of $\mathbb{P}_2$ at the three fixed points of the torus action. The corresponding arrangement is the reflection arrangement of type A_2. Its circular graph has the weights

$$(-1,-1,-1;-1,-1,-1).$$

The same surface can be obtained by blowing up $\mathbb{P}_1 \times \mathbb{P}_1$ in two points corresponding to the enlargement of the weighted graph $(0,0,0,0)$ by inserting -1 after the first and third places (see [Oda, Corollary 1.29]).

Notice that $\mathbb{P}_1 \times \mathbb{P}_1$ corresponds to the reducible reflection arrangement of type $A_1 \times A_1$. One should also note here that $\tilde{\mathbb{P}}_2$ and $\mathbb{P}_1 \times \mathbb{P}_1$ are the only toric del Pezzo surfaces which are centrally symmetric.

7. Parabolic subgroupoids and toric arrangements

If $(\mathcal{A},R)$ is a crystallographic arrangement in V and E is an intersection of hyperplanes of $\mathcal{A}$, then by Corollary 4.9 the restriction $\mathcal{A}^E$ is again crystallographic. The dual statement is that $\mathrm{Star}(\delta)$ for $\delta \in \Sigma_R$ is the fan of a crystallographic arrangement which corresponds to a parabolic subgroupoid (see below). Both constructions may be translated to the corresponding toric varieties in a compatible way. This gives rise to posets of toric varieties which we call *toric arrangements* (see Section 7.2).

7.1. Star fans and parabolic subgroupoids

Let $(\mathcal{A},R)$ be a crystallographic arrangement, let Σ_R be the corresponding smooth strongly symmetric fan in $\mathbb{R}^r$, and let $\delta \in \Sigma$, $E := \langle\delta\rangle_{\mathbb{R}}$, and $d := \dim(E)$. Let $R_E := R \cap E^\perp$, let

$$\mathcal{A}_E := \{\overline{\alpha^\perp} \subseteq V/E \mid \alpha \in R_E\},$$

and notice that $\overline{\alpha^\perp}$ are hyperplanes in V/E because $\alpha \in E^\perp$. Remark also that $\mathcal{A}_E$ depends only on E. By [CH4, Corollary 2.5], R_E is a set of real roots of a parabolic subgroupoid of $\mathcal{W}(\mathcal{A}(\Sigma))$ (see [HW, Definition 2.3] for the precise definition of a parabolic subgroupoid). Here, $\mathcal{W}(\mathcal{A}(\Sigma))$ is the Weyl groupoid of the Cartan scheme given by the crystallographic arrangement $\mathcal{A}(\Sigma)$ as described in [Cu, Proposition 4.5]. Thus $(\mathcal{A}_E,R_E)$ is a crystallographic arrangement. It corresponds to the fan $\mathrm{Star}(\delta)$:

PROPOSITION 7.1

Let $(\mathcal{A},R)$ be a crystallographic arrangement, and let δ be a d-dimensional cone of the fan Σ_R. Then the orbit closure $V(\delta) \subseteq X(\mathcal{A})$ of $\mathrm{orb}(\delta)$ corresponds to the *crystallographic arrangement*

$$\mathcal{A}_E = \{\overline{H} \subseteq V/E \mid H \in \mathcal{A}\} = \big\{\langle\overline{\tau}\rangle_{\mathbb{R}} \,\big|\, \overline{\tau} \in \operatorname{Star}(\delta)(r-d-1)\big\},$$

where $E = \langle\delta\rangle_{\mathbb{R}}$ *as above.*

Proof

Let $\overline{H}$ be in the left set. Then $\delta \subseteq E \subseteq H$; thus there exists a $\tau \in \Sigma(r-1)$ with $\delta \subseteq \tau \subseteq H$. Hence $\langle\overline{\tau}\rangle_{\mathbb{R}}$ is in the right-hand set.

Now let $\langle\overline{\tau}\rangle_{\mathbb{R}}$ be in the right-hand set. Then $E \subseteq \langle\tau\rangle_{\mathbb{R}} \subseteq H$ for an $H \in \mathcal{A}$, and so $\langle\overline{\tau}\rangle_{\mathbb{R}} \subseteq \overline{H}$. But since these have the same dimension, they are equal. □

COROLLARY 7.2

Let Σ *be a strongly symmetric fan in* $\mathbb{R}^r$*, and let* $\delta, \delta' \in \Sigma$ *with* $\langle\delta\rangle_{\mathbb{R}} = \langle\delta'\rangle_{\mathbb{R}}$*. Then* $\operatorname{Star}(\delta) = \operatorname{Star}(\delta')$ *and* $V(\delta) \cong V(\delta')$*; even so,* $V(\delta) \neq V(\delta')$*.*

Proof

As in Proposition 7.1, $\operatorname{Star}(\delta)$ depends only on $\langle\delta\rangle_{\mathbb{R}}$ because $\operatorname{Star}(\delta)$ is strongly symmetric. Note that here smoothness is not used. □

COROLLARY 7.3

Let Σ *be a smooth strongly symmetric fan in* $\mathbb{R}^r$*, let* $\mathcal{W}(\mathcal{A}(\Sigma))$ *be the corresponding Weyl groupoid, and let* $\delta \in \Sigma$*. Then the Weyl groupoid* $\mathcal{W}(\mathcal{A}(\operatorname{Star}(\delta)))$ *is equivalent to a connected component of a parabolic subgroupoid of* $\mathcal{W}(\mathcal{A}(\Sigma))$*.*

7.2. Associated toric arrangements

As before let Σ be the fan of a crystallographic arrangement $(\mathcal{A}, R)$, and as in [OT, Definition 2.1] let $L(\mathcal{A})$ be the poset of nonempty intersections of elements of $\mathcal{A}$. By Lemma 3.10, for any $E \in L(\mathcal{A})$ we are given the strongly symmetric smooth subfan

$$\Sigma^E = \{\sigma \cap E \mid \sigma \in \Sigma\} = \{\sigma \in \Sigma \mid \sigma \subseteq E\}$$

of Σ. Let X^E denote its toric variety. The inclusion $\iota : N^E = N \cap E \hookrightarrow N$ is then a sublattice and compatible with the fans Σ^E and Σ and induces a toric morphism

$$f^E : X^E \to X(\mathcal{A}) = X_\Sigma.$$

LEMMA 7.4

The map f^E *is a closed immersion with image* $Y^E \subseteq X(\mathcal{A})$ *of dimension* $\dim E$*.*

Proof

The subspace E is spanned by any cone $\tau \in \Sigma^E$ of maximal dimension $s := \dim E$. Using a $\mathbb{Z}$-basis of τ as in the proof of Lemma 3.10 one finds that $N^E = N \cap E$ is a sublattice of N of rank s and that the inclusion $\iota : N^E \hookrightarrow N$ is split. The induced map $\iota_{\mathbb{R}}$ sends a cone σ to itself and thus gives rise to a proper toric morphism f^E. Let M^E be the dual lattice of N^E and $\sigma \in \Sigma^E$. Using the duals of bases of N^E and N, one finds that the induced dual map $\iota^* : M \cap \sigma^\vee \to M^E \cap \sigma^\vee$ is surjective. Then

$$f^E|_{U_\sigma^E} : U_\sigma^E \to U_\sigma$$

is a closed immersion, where $U_\sigma^E \subseteq X^E$ and $U_\sigma \subseteq X_\Sigma$ denote the open affine spectra defined by $M^E \cap \sigma^\vee$ and $M \cap \sigma^\vee$, respectively. As in the proof of Proposition 5.9 we conclude that f^E is globally a closed immersion. □

REMARK 7.5

Note that Y^E is not invariant under the torus action on X_Σ but is a strongly symmetric smooth toric variety on its own with torus $T^E = N^E \otimes \mathbb{C}^* \subseteq T$.

PROPOSITION 7.6

With the above notation the subvarieties $Y^E \subseteq X_\Sigma$ have the following properties.

(i) *Each Y^E, $E \in L(\mathcal{A})$, is invariant under the involution of X_Σ defined by the central symmetry of Σ.*

(ii) *For each cone $\sigma \in \Sigma$,*

$$Y^E \cap \operatorname{orb}(\sigma) = \begin{cases} \operatorname{orb}^E(\sigma) & \text{if } \sigma \subseteq E, \\ \emptyset & \text{if } \sigma \not\subseteq E, \end{cases}$$

and

$$Y^E \cap V(\sigma) = \begin{cases} V^E(\sigma) & \text{if } \sigma \subseteq E, \\ \emptyset & \text{if } \sigma \not\subseteq E, \end{cases}$$

where $\operatorname{orb}^E(\sigma)$ (resp., $V^E(\sigma)$) denote the images of the orbit of σ (resp., its closure in X^E).

(iii) *When $F, E \in L(\mathcal{A})$ with $F \subseteq E$, then the composition $X^F \hookrightarrow X^E \hookrightarrow X_\Sigma$ is the inclusion $X^F \hookrightarrow X_\Sigma$.*

(iv) *For any $E, F \in L(\mathcal{A})$, $Y^{E\cap F} = Y^E \cap Y^F$.*

(v) *The intersections $Y^E \cap T$ of Y^E with the torus T of X_Σ are the subtori $T^E = N^E \otimes \mathbb{C}^*$ of T of dimension $\dim(E)$ and constitute a toric arrangement.*

DEFINITION 7.7

We call the system $\{Y^E\}_{E\in L(\mathcal{A})}$ the associated *toric arrangement* of the strongly symmetric smooth toric variety $X(\mathcal{A})$.

REMARK 7.8

Proposition 7.6(iv) shows that the assignment $E \mapsto Y^E$ is an isomorphism of posets.

REMARK 7.9

Proposition 7.6 yields a geometric interpretation of the strong symmetry of $X(\mathcal{A})$ by its toric arrangement. For any hyperplane $H \in \mathcal{A}$ the union of the curves $V(\tau)$,

$\tau \subseteq H$, $\dim(\tau) = r-1$, is the set of fixed points of $X(\mathcal{A})$ under the action of the subtorus $T^H = N^H \otimes \mathbb{C}^* = Y^H \cap T$ of T. This union meets the hypersurface Y^H exactly in the set of its fixed points under the action of its torus T^H and does not meet any other $Y^{H'}$.

The same holds for any $E \in L(\mathcal{A})$ for Y^E and the varieties $V(\tau)$, $\tau \subseteq E$, $\dim(\tau) = \dim E$, inside any other Y^F, $E \subseteq F \in L(\mathcal{A})$.

Proof of Proposition 7.6
(i) Follows from the fact that f^E is induced by the map ι between strongly symmetric fans.

Assertion (ii) follows from the orbit decompositions of Y^E and $V(\sigma)$ and the fact that f^E maps $\mathrm{orb}^E(\sigma)$ into $\mathrm{orb}(\sigma)$, because $\iota_{\mathbb{R}}(\sigma) = \sigma$ for $\sigma \in \Sigma^E$. If $\sigma \not\subseteq E$, no $\mathrm{orb}^E(\tau)$, $\tau \subseteq E$, can meet $\mathrm{orb}(\sigma)$. If $\sigma \subseteq E$, $\mathrm{orb}^E(\sigma) = \mathrm{orb}(\sigma) \cap Y^E$.

Assertion (iii) follows directly from the definition of the morphisms f^E.

Assertion (iv) It is sufficient to assume that F is a hyperplane $H \in \mathcal{A}$ with $E \not\subseteq H$. Let $s = \dim E$. Then $\dim Y^{E\cap H} = s-1$ and $Y^{E\cap H} \subseteq Y^E \cap Y^H$. Suppose that there is a point $x \in Y^E \cap Y^H$ and $x \notin Y^{E\cap H}$. Then let $\sigma \in \Sigma$ be a maximal cone with $x \in \mathrm{orb}(\sigma)$. Then $Y^{E\cap H} \cap \mathrm{orb}(\sigma) \subsetneq Y^E \cap Y^H \cap \mathrm{orb}(\sigma)$. By property (ii), $\sigma \subseteq E \cap H$ and

$$\begin{aligned} Y^{E\cap H} \cap \mathrm{orb}(\sigma) &= \mathrm{orb}^{E\cap H}(\sigma), \\ Y^E \cap \mathrm{orb}(\sigma) &= \mathrm{orb}^E(\sigma), \\ Y^H \cap \mathrm{orb}(\sigma) &= \mathrm{orb}^H(\sigma) \end{aligned}$$

are subtori of $\mathrm{orb}(\sigma)$ of dimensions $s-1-\dim(\sigma)$, $s-\dim(\sigma)$, $r-1-\dim(\sigma)$, and $\dim(\mathrm{orb}(\sigma)) = r - \dim(\sigma)$. It follows that $Y^E \cap Y^H \cap \mathrm{orb}(\sigma)$ is a subtorus of dimension $s-1-\dim(\sigma)$, too. Hence $Y^{E\cap H} \cap \mathrm{orb}(\sigma) = Y^E \cap Y^H \cap \mathrm{orb}(\sigma)$, a contradiction.

(v) follows from (ii) for the special case $T = \mathrm{orb}(\{0\})$. Then the definition of a toric arrangement as in [DCP2] is satisfied. □

Property (ii) of Proposition 7.6 also includes that the intersections $Y^E \cap V(\sigma)$ are smooth, irreducible, and proper of dimension $\dim E - \dim \sigma$. Moreover, we have the following.

PROPOSITION 7.10
With the above notation:

(1) *For any fixed orbit closure* $V(\tau) \subseteq X(\mathcal{A})$ *the intersections* $Y^E \cap V(\tau)$, $\tau \subseteq E$ *constitute the toric arrangement* $\{Y^{E/\langle\tau\rangle_{\mathbb{R}}}\}$ *of the variety* $V(\tau)$ *corresponding to the crystallographic arrangement* $\mathcal{A}_D$, $D = \langle\tau\rangle_{\mathbb{R}}$ *with fan* $\mathrm{Star}(\tau)$ *as in Proposition 7.1.*

(2) *The intersections* $Y^E \cap \mathrm{orb}(\tau)$, $\tau \subseteq E$, *form a toric arrangement of subtori in each orbit* $\mathrm{orb}(\tau)$ *of* $X(\mathcal{A})$.

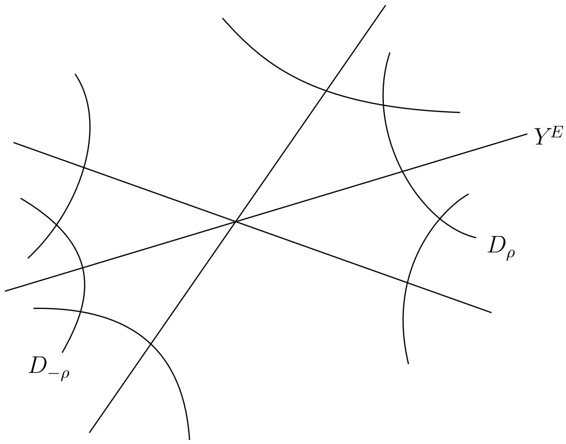

Figure 3. Example 7.11

Proof
Let $D = \langle\tau\rangle_{\mathbb{R}} \subseteq E$, and let $\overline{E} = E/D$. Under the isomorphism $X_{\mathrm{Star}(\tau)} \cong V(\tau)$ an orbit $\mathrm{orb}(\sigma) \subseteq V(\tau)$, $\tau \subseteq \sigma$, is identified with the orbit $\mathrm{orb}(\overline{\sigma})$ with $\overline{\sigma} \subseteq V/D$ the image of σ. Likewise, an orbit $\mathrm{orb}^E(\sigma)$ in X^E with $\tau \subseteq \sigma \subseteq E$ can be identified with the orbit $\mathrm{orb}^{\overline{E}}(\overline{\sigma})$ in the variety $X_{\mathrm{Star}(\tau)^{\overline{E}}} \cong V^E(\tau)$ in X^E. It follows that the embeddings $X^E \hookrightarrow X(\mathcal{A})$ and $X_{\mathrm{Star}(\tau)^{\overline{E}}} \hookrightarrow X_{\mathrm{Star}(\tau)} = V(\tau)$ are compatible and thus that $Y^E \cap V(\tau)$ is the image of the latter.

Assertion (2) follows from Proposition 7.6(v) since $\mathrm{orb}(\tau)$ is the torus of $V(\tau)$. $\square$

EXAMPLE 7.11
The system $\{Y^E\}_{E\in L(\mathcal{A})}$ for strongly symmetric toric surfaces has the following special features (see Figure 3). Here each E is a line of $\mathcal{A}$:

(1) for $\rho \subseteq E$, $Y^E \cap D_\rho = \mathrm{orb}^E(\rho)$ is a point $p_\rho \in \mathrm{orb}(\rho)$;
(2) $Y^E \smallsetminus (D_\rho \cup D_{-\rho})$ is the torus $T^E \cong k^*$ of Y^E;
(3) $Y^E \cap D_{\rho'} = \emptyset$ for $\rho' \not\subseteq E$;
(4) $Y^E \cap Y^F = \{1\} \subseteq T$ for any $E, F \in L(\mathcal{A})$.

Notice here that all the divisors D_ρ and Y^E are isomorphic to $\mathbb{P}_1$ and that the intersections are transversal.

There is an interesting formula for the divisor classes of the curves Y^E in terms of the toric divisors D_ρ as follows. Keeping the notation of Section 6, let $a_1, \ldots, a_{2t}$ be a chosen order of the weights of the circular graph of the surface $X(\mathcal{A})$ with corresponding divisors $D_1, \ldots, D_{2t}$, and let $Y_1 = Y^E$ in the case $E := \langle n_1\rangle_{\mathbb{R}}$.

Then the standard sequence $0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \operatorname{Pic} X(\mathcal{A}) \to 0$ can be represented by the exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{(Q,-Q)} \mathbb{Z}^t \oplus \mathbb{Z}^t \xrightarrow{\left(\begin{smallmatrix} A & I \\ 0 & I \end{smallmatrix}\right)} \mathbb{Z}^{t-2} \oplus \mathbb{Z}^t \longrightarrow 0,$$

where $Q^\vee = (n_1, \ldots, n_t)$ is the matrix of the first t primitive elements and $A^\vee$ is the matrix

$$A^\vee = \begin{pmatrix} a_1 & 1 & & & -1 \\ 1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ -1 & & & 1 & a_t \end{pmatrix}$$

of rank $t-2$ expressing the relations $n_{j-1} + a_j n_j + n_{j+1} = 0$. To deduce the formula for Y_1 we choose n_1, n_t as the basis of the lattice N. Then

$$Q^\vee = \begin{pmatrix} 1 & x_2 & \cdots & x_{t-1} & 0 \\ 0 & y_2 & \cdots & y_{t-1} & 1 \end{pmatrix},$$

and $y_2 = 1$ since $A \cdot Q = 0$.

PROPOSITION 7.12

With the above notation,

$$Y_1 \sim D_2 + \sum_{\nu=3}^{t-1} y_\nu D_\nu + D_t \sim D_{t+2} + \sum_{\nu=3}^{t-1} y_\nu D_{\nu+2} + D_{2t} \tag{7.1}$$

up to rational equivalence, and Y_1 has self-intersection $Y_1^2 = 0$.

REMARK 7.13

Choosing n_1, n_t as a basis, the columns of $Q^\vee$ become the positive roots of the associated Weyl groupoid at the object corresponding to Y_1.

The formula for the other $Y_\nu = Y^E$, $n_\nu \in E$, follows by cyclic permutation of the indices. Note that the classes of $D_2, \ldots, D_t$ are part of a basis of $\operatorname{Pic} X(\mathcal{A})$. The formula can be derived as follows. If Y_1 is equivalent to $\sum c_\nu D_\nu$, the intersection numbers $D_\nu^2 = a_\nu$, $D_\mu D_\nu \in \{0,1\}$ for $\mu \neq \nu$ and

$$Y_1 D_\nu = \begin{cases} 1, & \nu \in \{1, t+1\}, \\ 0 & \text{else}, \end{cases}$$

yield a system of equations for the coefficients $c_2, \ldots, c_{2t}$. This system has a unique solution modulo $(Q, -Q)$ such that $c_1 = 0$, $c_2 = 1$, which is

$$(c_2, \ldots, c_{2t}) = (y_2, \ldots, y_{t-1}, 1, 0, \ldots, 0) \mod (Q, -Q).$$

For that one has to use the relations between the weights $a_1, \ldots, a_t$ (see Section 6). The proof for $Y_1^2 = 0$ follows from the second equivalence of equation (7.1).

REMARK 7.14

The relations between the weights $a_1, \dots, a_{2t}$ naturally lead to the Grassmannian and to cluster algebras of type A (see [CH3] for more details).

8. Further remarks

8.1. Reducibility

An arrangement $(\mathcal{A}, V)$ is called *reducible* if there exist arrangements $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ such that $V = V_1 \oplus V_2$ and

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 := \{H \oplus V_2 \mid H \in \mathcal{A}_1\} \cup \{V_1 \oplus H \mid H \in \mathcal{A}_2\}$$

(cf. [OT, Definition 2.15]). It is easy to see that a crystallographic arrangement $(\mathcal{A}, V)$ is reducible if and only if the corresponding Cartan scheme is reducible in the sense of [CH2, Definition 4.3], that is, the generalized Cartan matrices are decomposable. For the fan Σ corresponding to $\mathcal{A}$, reducibility translates to the fact that there are fans Σ_1 and Σ_2 such that

$$\Sigma = \Sigma_1 \times \Sigma_2 = \{\sigma \times \tau \mid \sigma \in \Sigma_1, \tau \in \Sigma_2\}.$$

Notice that by Lemma 3.10 the fans Σ_1 and Σ_2 are strongly symmetric and smooth as well.

8.2. Inserting one hyperplane and blowups

In higher dimension, the situation is much more complicated. There are only finitely many crystallographic arrangements for each rank $r > 2$. Whether the insertion of new hyperplanes corresponds to a series of blowups is unclear. The case of a single new hyperplane may be explained in the following way.

PROPOSITION 8.1

Let $(\mathcal{A}, R)$ and $(\mathcal{A}', R')$ be crystallographic arrangements of rank r with $\mathcal{A}' = \mathcal{A} \dot\cup \{H\}$. Then the toric morphism $X_{\Sigma'} \to X_\Sigma$ induced by the subdivision is a blowup along 2-dimensional torus invariant subvarieties of X_Σ.

Proof

Let $\sigma \in \Sigma := \Sigma_R$ be a maximal cone with $H \cap \overset{\circ}{\sigma} \neq \emptyset$. We prove that H star subdivides σ. The hyperplane H divides σ into two parts σ_1' and σ_2' which intersect in a codimension one cone τ'. Note that $|\sigma(1)| = r$, $|\sigma_1'(1) \cup \sigma_2'(1)| = r + 1$; thus there is exactly one ray ρ' involved which is not in Σ. Let $\rho_1 \subseteq \sigma_1', \rho_2 \subseteq \sigma_2'$ be the rays which are not subsets of τ', and let $\tau \subseteq \sigma$ be the cone generated by ρ_1, ρ_2. Then $H \cap \tau = \rho'$. But by Corollary 4.9, $\mathcal{A}'^{\langle\tau\rangle_\mathbb{R}}$ is a crystallographic arrangement in which $\langle\rho_1\rangle_\mathbb{R}$, $\langle\rho'\rangle_\mathbb{R}$, $\langle\rho_2\rangle_\mathbb{R}$ are subsequent hyperplanes. By Section 6 we obtain that ρ is generated by the sum of the generators of ρ_1', ρ_2'. $\square$

8.3. Automorphisms

Let Σ be a strongly symmetric smooth fan, and let $(\mathcal{A}, R)$ be the corresponding crystallographic arrangement.

DEFINITION 8.2
If $\mathcal{A}$ comes from the connected simply connected Cartan scheme $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$, and $a \in A$, then we call

$$\mathrm{Aut}(\mathcal{C}, a) := \{w \in \mathrm{Hom}(a, b) \mid b \in A, R^a = R^b\}$$

the *automorphism group of* $\mathcal{C}$ *at* a. This is a finite subgroup of $\mathrm{Aut}(\mathbb{Z}^r) \cong \mathrm{Aut}(M)$ because the number of all morphisms is finite.

Since $\mathcal{C}$ is connected, $\mathrm{Aut}(\mathcal{C}, a) \cong \mathrm{Aut}(\mathcal{C}, b)$ for all $a, b \in A$. The choice of $a \in A$ corresponds to the choice of a chamber and thus of an isomorphism $\mathbb{Z}^r \cong M$. Every element of $\mathrm{Aut}(\mathcal{C}, a)$ clearly induces a toric automorphism of Σ. The groups $\mathrm{Aut}(\mathcal{C}, a)$ have been determined in [CH5] (see [CH5, Theorem 3.18 and Appendix A.3]). However, sometimes there are elements of $\mathrm{Aut}(\Sigma)$ which are not induced by an element of $\mathrm{Aut}(\mathcal{C}, a)$. For example, we always have the toric automorphism

$$N \to N, \qquad v \mapsto -v,$$

but there is a sporadic Cartan scheme of rank three with trivial automorphism group.

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