Aharonov–Bohm effect in resonances of magnetic Schrödinger operators in two dimensions

Hideo Tamura

Abstract We study the Aharonov–Bohm (AB) effect through resonances for magnetic scattering in two dimensions. The scattering system consists of three scatterers, one bounded obstacle, and two scalar potentials with compact supports at large separation, where the obstacle is placed between two supports and the support of the magnetic field is completely shielded by the obstacle. The field does not influence particles from a classical mechanical point of view, but quantum particles are influenced by the corresponding vector potential which does not necessarily vanish outside the obstacle. This quantum phenomenon is called the AB effect. The resonances are shown to be generated near the real axis by the trajectories oscillating between two supports of the scalar potentials as the distances between the three scatterers go to infinity. The location is described in terms of the backward amplitudes for scattering by each of the scalar potentials and by the obstacle, and it depends heavily on the magnetic flux of the field.

1. Introduction

In quantum mechanics, a vector potential is said to have a direct significance to particles moving in a magnetic field. This is called the Aharonov–Bohm (AB) effect and is known as one of the most remarkable quantum phenomena (see [3]). In this work we study the AB quantum effect in resonances of magnetic Schrödinger operators in two dimensions.

We always work in the two-dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$ and write

$$H(A,V) = (-i\nabla - A)^2 + V = \sum_{j=1}^{2}(-i\partial_j - a_j)^2 + V, \quad \partial_j = \partial/\partial x_j,$$

for the Schrödinger operator with the scalar potential $V : \boldsymbol{R}^2 \to \boldsymbol{R}$ and the vector potential $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. The magnetic field $b : \boldsymbol{R}^2 \to \boldsymbol{R}$ associated with A is defined by

$$b(x) = \nabla \times A(x) = \partial_1 a_2 - \partial_2 a_1,$$

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and the quantity defined as the integral $\alpha = (2\pi)^{-1}\int b(x)\,dx$ is called the magnetic flux of b, where the integration with no domain attached is taken over the whole space. We often use this abbreviation throughout the entire discussion. The Hamiltonian $H(A,V)$ describes the energy operator for the quantum system of particles subjected to the electrostatic potential $V(x)$ and to the magnetic field $b(x)$.

We now define the operator. Let $\Omega = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}$ be the exterior domain of a bounded domain $\mathcal{O}$ with the smooth boundary $\partial\mathcal{O}$, $\overline{\mathcal{O}}$ being the closure of $\mathcal{O}$. We assume that $\mathcal{O}$ is simply connected and

$$\mathcal{O} \subset B = \{|x| < 1\} \tag{1.1}$$

with the origin in $\mathcal{O}$. For $d \in \boldsymbol{R}^2$, $|d| \gg 1$, we consider the Hamiltonian

$$H_d = H(A, V_d) = (-i\nabla - A)^2 + V_d$$

on $L^2(\Omega)$, where the potential $V_d(x)$ takes the form

$$V_d(x) = V_{-d}(x) + V_{+d}(x) = V_-(x - d_-) + V_+(x - d_+), \quad d = d_+ - d_-, \tag{1.2}$$

with

$$d_+ = (1-\kappa)d, \qquad d_- = -\kappa d, \quad 0 < \kappa < 1.$$

We further assume that $A(x) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is smooth over $\overline{\Omega}$ ($A \in C^\infty(\overline{\Omega} \to \boldsymbol{R}^2)$) and falls off at infinity. We denote by $b = \nabla \times A$ the magnetic field associated with A and by α the magnetic flux of b. We make the assumption that $V_\pm$ is a smooth function with compact support and that b vanishes on Ω:

$$V_\pm \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}), \qquad b = 0 \quad \text{on } \Omega. \tag{1.3}$$

For brevity, we take A to be the AB potential defined by

$$A(x) = \alpha(-x_2/|x|^2, x_1/|x|^2) = \alpha(-\partial_2 \log|x|, \partial_1 \log|x|). \tag{1.4}$$

Then A generates the solenoidal field

$$b = \nabla \times A = \alpha(\partial_1^2 + \partial_2^2)\log|x| = 2\pi\alpha\delta(x)$$

which has the support only at the origin and α as the magnetic flux. We also assume that $V_\pm$ has support in the unit disk B. The second assumption in (1.3) means that the field b is entirely shielded by the obstacle $\mathcal{O}$, although the corresponding vector potential A does not necessarily vanish over Ω. If $|d| \gg 1$ is large enough, then

$$\operatorname{supp} V_{\pm d} \subset B_{\pm d} = \{|x - d_\pm| < 1\} \subset \Omega \tag{1.5}$$

and the self-adjoint extension in $L^2(\Omega)$ of $H_d = H(A, V_d)$ is realized by imposing the zero boundary conditions. We denote by the same notation H_d this self-adjoint operator with domain $\mathcal{D}(H_d) = H^2(\Omega) \cap H_0^1(\Omega)$, where $H^2(\Omega)$ and $H_0^1(\Omega)$ stand for the usual Sobolev spaces over Ω. If we take another vector potential $\tilde{A} : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ defining the same field $b = \nabla \times \tilde{A}$, then we can show that $\tilde{A}$ takes the form $\tilde{A} = A + \nabla g$ over Ω for some real smooth function $g \in C^\infty(\overline{\Omega})$, and hence

$H(\tilde{A}, V_d)$ turns out to be unitarily equivalent to $H_d = H(A, V_d)$. Here we note that the direction $\hat{d} = d/|d|$ and the ratio κ are fixed with the meaning ascribed above.

We denote by $R(\zeta; T) = (T - \zeta)^{-1}$ the resolvent of a self-adjoint operator T acting on $L^2(\boldsymbol{R}^2)$ or $L^2(\Omega)$. It is known (see [13]) that H_d has no positive eigenvalues and the continuous spectrum occupied by $(0, \infty)$ is absolutely continuous. We further know that the resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2(\Omega) \to L^2(\Omega), \quad \zeta = E + i\eta, E > 0, \eta > 0,$$

is meromorphically continued from the upper half plane of the complex plane to a region (independent of d) in the lower half plane across the positive real axis where the continuous spectrum of H_d is located (see the arguments after the proof of Lemma 3.1). Then $R(\zeta; H_d)$ with $\operatorname{Im} \zeta \le 0$ is well defined as an operator from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$ in the sense that $qR(\zeta; H_d)q : L^2(\Omega) \to L^2(\Omega)$ is bounded for every $q \in C_0^\infty(\overline{\Omega})$, where $L^2_{\mathrm{comp}}(W)$ denotes the space of square integrable functions with compact support in the closure $\overline{W}$ of a region $W \subset \boldsymbol{R}^2$ and $L^2_{\mathrm{loc}}(W)$ denotes the space of locally square integrable functions over $\overline{W}$. This can be shown by an application of the complex scaling method (see [5], [17]) and by the analytic Fredholm theorem (see [15, Theorem VI.14]). We use the same notation $R(\zeta; H_d)$ to denote this meromorphic function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. In fact, we can show that $R(\zeta; H_d)$ admits the meromorphic continuation to the region $\{\zeta \in \boldsymbol{C} : \operatorname{Re} \zeta > 0, \operatorname{Im} \zeta < 0\}$, but the argument here is restricted only to a neighborhood of the positive real axis. The resonances of H_d are defined as the poles of $R(\zeta; H_d)$ in the lower half plane (the unphysical sheet). Our aim is to study how the resonances are generated near the real axis by the trajectories oscillating between $\operatorname{supp} V_{-d}, \operatorname{supp} V_{+d}$, and $\mathcal{O}$ as $|d| \to \infty$. By assumption, the magnetic field b vanishes outside $\mathcal{O}$, which implies that b does not have any influences on classical particles moving in Ω. On the other hand, the vector potential A does not necessarily vanish there, but it has a direct significance to the movement of quantum particles according to the AB effect. A special emphasis is placed on analyzing how the AB effect influences the location of the resonances.

The obtained results are formulated in terms of the backward amplitudes by the potentials $V_\pm$ and by the obstacle $\mathcal{O}$. Let $K_0 = -\Delta$ be the free Hamiltonian, and let $K_\pm$ be the Schrödinger operator defined by

$$K_\pm = K_0 + V_\pm = -\Delta + V_\pm, \quad \mathcal{D}(K_\pm) = H^2(\boldsymbol{R}^2). \tag{1.6}$$

We denote by $f_\pm(\omega \to \theta; E)$ the amplitude for scattering from the incident direction $\omega \in S^1$ to the final one θ at energy $E > 0$ for the pair $(K_0, K_\pm)$. As is stated at the beginning of Section 4, these amplitudes admit the analytic extensions $f_\pm(\omega \to \theta; \zeta)$ in a complex neighborhood of the positive real axis as a function of E. We further denote by $f_0(\omega \to \theta; E)$ the scattering amplitude at energy $E > 0$ for the pair (K_0, H), where H is defined as

$$H = H(A, 0), \quad \mathcal{D}(H) = H^2(\Omega) \cap H^1_0(\Omega). \tag{1.7}$$

The precise representation for $f_0(\omega \to \theta; E)$ is given by Lemma 3.2, and this amplitude is also seen to admit the analytic extension $f_0(\omega \to \theta; \zeta)$ in a complex neighborhood of the positive real axis.

The results heavily depend on the magnetic flux α. We first consider the case where α is not a half integer. We fix $E_0 > 0$ and take $0 < \delta_0 \ll 1$ small enough but independently of d. Then we set

$$D_d = \Big\{\zeta \in \boldsymbol{C} : |\operatorname{Re}\zeta - E_0| < \delta_0, |\operatorname{Im}\zeta| < E_0^{1/2}\Big(1 + \frac{2\delta_0}{E_0}\Big)\Big(\frac{\log|d|}{|d|}\Big)\Big\} \tag{1.8}$$

and define the function $h(\zeta; d)$ by

$$h(\zeta; d) = (e^{2ik|d|}/|d|)\cos^2(\alpha\pi) f_-(-\hat{d} \to \hat{d}; \zeta) f_+(\hat{d} \to -\hat{d}; \zeta) \tag{1.9}$$

over D_d, where $k = \zeta^{1/2}$ is taken in such a way that $\operatorname{Re} k > 0$ for $\operatorname{Re}\zeta > 0$. We always use k with this meaning. Since

$$2\operatorname{Im} k = 2\operatorname{Im}(\operatorname{Re}\zeta + i\operatorname{Im}\zeta)^{1/2} = \operatorname{Im}\zeta/(\operatorname{Re}\zeta)^{1/2} + O(|\operatorname{Im}\zeta|^3) \tag{1.10}$$

for $\zeta \in D_d$ and since

$$(\operatorname{Re}\zeta)^{1/2} = E_0^{1/2}\big(1 + (\operatorname{Re}\zeta - E_0)/(2E_0) + O(\delta_0^2)\big) \tag{1.11}$$

with $|\operatorname{Re}\zeta - E_0| < \delta_0$, it follows that

$$|d|^{1+\delta_0/E_0} < |e^{2ik|d|}| < |d|^{1+3\delta_0/E_0} \tag{1.12}$$

at the bottom $\zeta = \operatorname{Re}\zeta - iE_0^{1/2}(1 + 2\delta_0/E_0)((\log|d|)/|d|)$ of D_d. This implies that the curve defined by $|h(\zeta; d)| = 1$, $|\operatorname{Re}\zeta - E_0| < \delta_0$, is completely contained in D_d. Since $|d_-| = \kappa|d| < |d|$ and $|d_+| = (1-\kappa)|d| < |d|$, we can take $\delta_0 > 0$ so small that

$$\big|e^{2ik|d_+|}/|d_+|\big| = O(|d|^{-\kappa/2}), \qquad \big|e^{2ik|d_-|}/|d_-|\big| = O(|d|^{-(1-\kappa)/2}) \tag{1.13}$$

in D_d. Thus the choice of δ_0 depends on κ as well as on E_0. The bounds above guarantee that there are no resonances generated by the trajectories oscillating between $\mathcal{O}$ and $\operatorname{supp} V_{\pm d}$ in D_d. This will be seen in the course of the proof of the theorems.

We now consider the equation

$$h(\zeta; d) = 1 \tag{1.14}$$

in D_d. We can show (see Lemma 4.6) that it has a finite number of the solutions

$$\big\{\zeta_j(d)\big\}_{1 \le j \le N_d}, \quad \zeta_j(d) \in D_d, \operatorname{Re}\zeta_1(d) < \cdots < \operatorname{Re}\zeta_{N_d}(d),$$

N_d being dependent on d, and that each solution $\zeta_j(d)$ behaves like

$$\operatorname{Im}\zeta_j(d) \sim -E_0^{1/2}(\log|d|)/|d|, \qquad \operatorname{Re}\big(\zeta_{j+1}(d) - \zeta_j(d)\big) \sim 2\pi E_0^{1/2}/|d|,$$

as $|d| \to \infty$. We are in a position to state the first theorem.

THEOREM 1.1

Let the notation be as above. Assume that the magnetic flux α is not a half integer and that the backward amplitudes $f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0)$, $\hat{d} = d/|d|$, at energy E_0 do *not vanish. Then we can take* $\delta_0 > 0$ *so small that the neighborhood* D_d *defined by (1.8) has the following property. For any* $\varepsilon > 0$ *small enough, there exists* $d_\varepsilon \gg 1$ *such that for* $|d| > d_\varepsilon$, H_d *has the resonances* $\{\zeta_{\mathrm{res},j}(d)\}$, $1 \le j \le N_d$, *in* D_d, *which satisfy*

$$|\zeta_{\mathrm{res},j}(d) - \zeta_j(d)| < \varepsilon/|d|, \quad \operatorname{Re}\zeta_{\mathrm{res},1}(d) < \cdots < \operatorname{Re}\zeta_{\mathrm{res},N_d}(d),$$

and the resolvent $R_d(\zeta) = R(\zeta; H_d)$ *is analytic over the domain*

$$D_d \setminus \{\zeta_{\mathrm{res},1}(d), \dots, \zeta_{\mathrm{res},N_d}(d)\}$$

as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ *to* $L^2_{\mathrm{loc}}(\Omega)$.

Here we note that the assumption $f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0) \neq 0$ implies $f_\pm(\pm\hat{d} \to \mp\hat{d}; E) \neq 0$ for E with $|E - E_0| < \delta_0$ by choosing δ_0 even smaller, if necessary.

Next we deal with the case where α is a half integer. Let $d_\pm$ be as in (1.2). We again fix $E_0 > 0$ and take $0 < \delta_0 \ll 1$ small enough. We set

$$D_{\pm d} = \Big\{\zeta : |\operatorname{Re}\zeta - E_0| < \delta_0, |\operatorname{Im}\zeta| < E_0^{1/2}\Big(1 + \frac{2\delta_0}{E_0}\Big)\Big(\frac{\log|d_\pm|}{|d_\pm|}\Big)\Big\} \tag{1.15}$$

and define the function $h_\pm(\zeta; d)$ by

$$h_\pm(\zeta; d) = (e^{2ik|d_\pm|}/|d_\pm|) f_0(\mp\hat{d} \to \pm\hat{d}; \zeta) f_\pm(\pm\hat{d} \to \mp\hat{d}; \zeta)$$

over $D_{\pm d}$. If $0 < \kappa < 1/2$, then $|d_-| < |d_+|$ and we can take $\delta_0 > 0$ so small that the curve defined by $|h_+(\zeta; d)| = 1$, $|\operatorname{Re}\zeta - E_0| < \delta_0$, is completely contained in D_{+d} and that

$$\big|e^{2ik|d_-|}/|d_-|\big| = O(|d|^{-(1/2-\kappa)}) \tag{1.16}$$

at the bottom of D_{+d}. If $1/2 < \kappa < 1$, then $|e^{2ik|d_+|}/|d_+|| = O(|d|^{-(\kappa - 1/2)})$ at the bottom of D_{-d}. The solutions

$$\{\zeta_j^{(\pm)}(d)\}_{1 \le j \le N_{\pm d}}, \quad \zeta_j^{(\pm)}(d) \in D_{\pm d}, \operatorname{Re}\zeta_1^{(\pm)}(d) < \cdots < \operatorname{Re}\zeta_{N_{\pm d}}^{(\pm)}(d)$$

of the equation $h_\pm(\zeta; d) = 1$ can be shown to have properties similar to those of the equation $h(\zeta; d) = 1$ with natural modifications. Then the second theorem is stated as follows.

THEOREM 1.2

Let the notation be as above. Assume that the magnetic flux α *is a half integer and that the four backward amplitudes* $f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0)$ *and* $f_0(\pm\hat{d} \to \mp\hat{d}; E_0)$ *at energy* $E_0 > 0$ *do not vanish. Then we have the following statements.*

(1) *Assume that* $0 < \kappa < 1/2$. *Then we can take* $\delta_0 > 0$ *so small that the neighborhood* D_{+d} *defined by (1.15) has the following property:* H_d *has the resonances*

$$\{\zeta_{\mathrm{res},j}^{(+)}(d)\}, \quad \zeta_{\mathrm{res},j}^{(+)}(d) \in D_{+d}, 1 \le j \le N_{+d},$$

in a neighborhood of $\zeta_j^{(+)}(d)$ *for* $|d| \gg 1$ *as in Theorem 1.1, and* $R(\zeta; H_d)$ *depends analytically on* ζ *over* $D_{+d} \setminus \bigcup_{1 \le j \le N_{+d}} \{\zeta_{\mathrm{res},j}^{(+)}(d)\}$.

(2) *Assume that $1/2 < \kappa < 1$. Then we can take $\delta_0 > 0$ so small that the neighborhood D_{-d} defined by (1.15) has the following property: H_d has the resonances*

$$\{\zeta^{(-)}_{\mathrm{res},j}(d)\}, \quad \zeta^{(-)}_{\mathrm{res},j}(d) \in D_{-d}, 1 \le j \le N_{-d},$$

in a neighborhood of $\zeta^{(-)}_j(d)$ for $|d| \gg 1$ as in Theorem 1.1, and $R(\zeta; H_d)$ depends analytically on ζ over $D_{-d} \setminus \bigcup_{1\le j\le N_{-d}} \{\zeta^{(-)}_{\mathrm{res},j}(d)\}$.

(3) *Assume that $\kappa = 1/2$. Then, for any $\varepsilon > 0$ small enough, there exists $d_\varepsilon \gg 1$ such that $\zeta \in \boldsymbol{C}$ with $|\mathrm{Re}\,\zeta - E_0| < E_0/2$ and with*

$$0 \ge \mathrm{Im}\,\zeta > -(2-\varepsilon)(\mathrm{Re}\,\zeta)^{1/2}\big((\log|d|)/|d|\big)$$

is not a resonance of H_d for $|d| > d_\varepsilon$.

We remark that the third statement does not require the assumption that the backward amplitudes do not vanish. The two theorems above are proved in Section 4 after stating some preliminary propositions and lemmas on the scattering theory and on the asymptotic properties of the Green function for the magnetic Schrödinger operator H defined by (1.7) in Sections 2 and 3.

The resonance problem is one of the most active subjects in scattering theory at present. There are a large number of works devoted to the problem of resonances near the real axis generated by closed classical trajectories. In particular, the semiclassical problem of shape resonances has been studied in detail, and upper or lower bounds on the resonance width (the imaginary part of the resonance) and its asymptotic expansion have been obtained by many authors under various kind of assumptions (see, e.g., [5], [6], [8]–[11], [14], [17]). We refer to the book [11] for an extensive list of references and to [9] for the recent development. However, it seems that there are few works which discuss the resonances of magnetic Schrödinger operators in connection with the AB quantum effect. In [4], we have studied how the AB effect is reflected in the lower bound on the resonance widths. Roughly speaking, the bound has been determined from the relation $|h(\zeta; d)| < 1$ strictly. In other words, H_d does not have any resonances in the region where $|h(\zeta; d)| < 1$ is fulfilled. However, the neighborhood D_d defined by (1.8) contains points at which $|h(\zeta; d)| > 1$. Thus we can improve considerably the result obtained by the previous work [4] by showing the actual existence in D_d of resonances.

We end the section by explaining from a physical point of view how reasonable (1.14) is as an approximate relation to determine the location of the resonances. We denote by

$$\varphi_0(x; \omega, E) = \exp(iE^{1/2} x \cdot \omega) \tag{1.17}$$

the plane-wave incident from the direction ω at energy $E > 0$, where the notation $\cdot$ denotes the scalar product in $\boldsymbol{R}^2$. We write $x_\pm$ for $x_\pm = x - d_\pm$. The incident plane wave $\varphi_0(x_-; -\hat{d}, E)$ takes the form $f_-(-\hat{d} \to \hat{d}; E)(e^{iE^{1/2}|x_-|}/|x_-|^{1/2})$ after it is scattered into the direction $\hat{d} = d/|d|$ by the potential V_{-d}, and the scattered wave hits the support of the other potential V_{+d}. Since $|x_-|$ behaves like

$$|x_-| = |x - d_-| = |d + x_+| = |d| + \hat{d} \cdot x_+ + O(|d|^{-1})$$

for $x \in B_{+d}$, $B_{\pm d}$ being as in (1.5), the scattered wave behaves like the plane wave

$$(e^{iE^{1/2}|d|}/|d|^{1/2}) f_-(-\hat{d} \to \hat{d}; E) \varphi_0(x_+; \hat{d}, E)$$

when it arrives at the support of V_{+d}, provided that the vector potential $A(x)$ vanishes identically. If, however, $A(x)$ does not necessarily vanish, then the wave function undergoes a change of the phase factor by the AB quantum effect. We consider the particle moving from d_- to d_+ under the assumption that the center $d_\pm$ of $\operatorname{supp} V_{\pm d}$ is located on the x_1-axis. We distinguish between the trajectories passing over $x_2 > 0$ and $x_2 < 0$ to denote the former and latter trajectories by τ_+ and τ_-, respectively. The vector potential $A(x)$ defined by (1.4) satisfies the relation

$$A(x) = \alpha \nabla \gamma(x), \tag{1.18}$$

where $\gamma(x)$ denotes the azimuth angle from the positive x_1-axis. Then the AB effect causes the change in the phase factor of the wave function, which is given by the line integral

$$\int_{\tau_\pm} A(x) \cdot dx = \mp \alpha\pi$$

along $\tau_\pm$. The factor $\cos(\alpha\pi)$ is generated from the sum of $e^{i\alpha\pi}$ and $e^{-i\alpha\pi}$. Thus the scattered wave takes

$$(e^{iE^{1/2}|d|}/|d|^{1/2}) f_-(-\hat{d} \to \hat{d}; E) \cos(\alpha\pi) \varphi_0(x_+; \hat{d}, E)$$

as an approximate form, when it hits the support of V_{+d}. A similar argument applies to the plane wave $\varphi_0(x_+; \hat{d}, E)$ after it is scattered into the direction $-\hat{d}$ by the potential V_{+d}, so that it again returns to the support of V_{-d}, taking the approximate form $h(E; d)\varphi_0(x_-; -\hat{d}, E)$. Hence the contribution from the trapping effect between $\operatorname{supp} V_{-d}$ and $\operatorname{supp} V_{+d}$ is described by the series

$$\Big(\sum_{n=1}^{\infty} h(E; d)^n\Big) \varphi_0(x_-; -\hat{d}, E).$$

For example, the term with $h(E; d)^n$ describes the contribution from the trajectory oscillating n times. Thus the location of the resonances is approximately determined by relation (1.14), and we see that the resonances in Theorem 1.1 are just generated near the real axis through a combination of the trapping effect from classical mechanics and the AB effect from quantum mechanics. If α is a half integer, then $\cos(\alpha\pi)$ vanishes by cancellation, and Theorem 1.2 asserts that the second longest oscillating trajectory determines the location of the resonances.

2. The AB Hamiltonian

In this section we give a brief review of the scattering by one solenoidal field. Such a system is known as one of the exactly solvable models in quantum mechanics. We refer to [1]–[3], [7], and [16] for more detailed expositions. To avoid confusing the notation, we often write

$$A_\alpha(x) = \alpha(-x_2/|x|^2, x_1/|x|^2) = \alpha(-\partial_2 \log|x|, \partial_1 \log|x|)$$

for the AB potential A defined by (1.4), when considered as a vector potential over the whole space $\boldsymbol{R}^2$.

We now consider the energy operator

$$P_\alpha = H(A_\alpha, 0) = (-i\nabla - A_\alpha)^2 \tag{2.1}$$

on $L^2(\boldsymbol{R}^2)$. This operator governs the quantum particle moving in the solenoidal field $2\pi\alpha\delta(x)$ and is often called the AB Hamiltonian in physics literature. The operator P_α is symmetric over $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$, but it is not necessarily essentially self-adjoint in $L^2(\boldsymbol{R}^2)$ because of the strong singularity at the origin of A_α. We know (see [1], [7]) that it is a symmetric operator with type $(2,2)$ deficiency indices. The self-adjoint extension is realized by imposing a boundary condition at the origin. Its Friedrichs extension denoted by the same notation P_α is obtained by imposing the boundary condition $\lim_{|x|\to 0} |u(x)| < \infty$ at the center of the solenoidal field.

We calculate the generalized eigenfunction of the eigenvalue problem

$$P_\alpha \varphi = E\varphi, \quad \lim_{|x|\to 0} |\varphi(x)| < \infty, \tag{2.2}$$

with energy $E > 0$ as an eigenvalue. Since P_α is rotationally invariant, we work in the polar coordinate system (r, θ). Let U be the unitary mapping defined by

$$(Uu)(r,\theta) = r^{1/2} u(r\theta) : L^2 \to L^2\big((0,\infty); dr\big) \otimes L^2(S^1).$$

We write $\sum_l$ for the summation ranging over all integers l. Then U allows us to decompose P_α into the partial wave expansion

$$P_\alpha \simeq U P_\alpha U^* = \sum_l \oplus (P_{l\alpha} \otimes \mathrm{Id}), \tag{2.3}$$

where $P_{l\alpha} = -\partial_r^2 + (\nu^2 - 1/4) r^{-2}$ with $\nu = |l - \alpha|$ is self-adjoint in $L^2((0,\infty); dr)$ under the boundary condition $\lim_{r\to 0} r^{-1/2} |u(r)| < \infty$ at $r = 0$. Let $\varphi_0(x;\omega,E)$ be defined by (1.17). We denote by $\gamma(x;\omega)$ the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$. Then the outgoing eigenfunction $\varphi_{\alpha+}(x;\omega,E)$ with ω as an incident direction is calculated as

$$\varphi_{\alpha+}(x;\omega,E) = \sum_l \exp(-i\nu\pi/2) \exp\big(il\gamma(x;-\omega)\big) J_\nu(E^{1/2}|x|) \tag{2.4}$$

with $\nu = |l - \alpha|$, where $J_\mu(z)$ denotes the Bessel function of order μ. The eigenfunction $\varphi_{\alpha+}$ behaves like $\varphi_{\alpha+}(x;\omega,E) \sim \varphi_0(x;\omega,E)$ as $|x| \to \infty$ in the direction $-\omega$ $(x = -|x|\omega)$, and the difference $\varphi_{\alpha+} - \varphi_0$ satisfies the outgoing radiation condition at infinity. On the other hand, the incoming eigenfunction $\varphi_{\alpha-}(x;\omega,E)$ is given by

$$\varphi_{\alpha-}(x;\omega,E)=\sum_l \exp(i\nu\pi/2)\exp\big(il\gamma(x;\omega)\big)J_\nu(E^{1/2}|x|), \tag{2.5}$$

which behaves like $\varphi_{\alpha-}\sim\varphi_0(x;\omega,E)$ as $|x|\to\infty$ in the direction ω. The eigenfunctions $\varphi_{\alpha\pm}(x;\omega,E)$ admit the analytic extension

$$\varphi_{\alpha\pm}(x;\omega,\zeta)=\sum_l \exp(\mp i\nu\pi/2)\exp\big(il\gamma(x;\mp\omega)\big)J_\nu(k|x|),\quad k=\zeta^{1/2},$$

over the complex plane.

We decompose $\varphi_{\alpha+}(x;\omega,E)$ into the sum $\varphi_{\alpha+}=\varphi_{\mathrm{in}}+\varphi_{\mathrm{sc}}$ of incident and scattering waves to calculate the scattering amplitude through the asymptotic behavior at infinity of the scattering wave $\varphi_{\mathrm{sc}}(x;\omega,E)$. If we set $\sigma=\sigma(x;\omega)=\gamma(x;\omega)-\pi$, then

$$\varphi_{\alpha+}=\sum_l e^{-i\nu\pi/2}e^{il\sigma}J_\nu(E^{1/2}|x|),\quad \nu=|l-\alpha|.$$

If we further make use of the formula $e^{-i\mu\pi/2}J_\mu(iw)=I_\mu(w)$ for the Bessel function

$$I_\mu(w)=(1/\pi)\Big(\int_0^\pi e^{w\cos\rho}\cos(\mu\rho)\,d\rho-\sin(\mu\pi)\int_0^\infty e^{-w\cosh p-\mu p}\,dp\Big) \tag{2.6}$$

with $\mathrm{Re}\,w\ge 0$ (see [18, p. 181]), then $\varphi_{\alpha+}(x;\omega,E)$ takes the form

$$\begin{aligned}\varphi_{\alpha+}&=(1/\pi)\sum_l e^{il\sigma}\int_0^\pi e^{-iE^{1/2}|x|\cos\rho}\cos(\nu\rho)\,d\rho\\&\quad-(1/\pi)\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^\infty e^{iE^{1/2}|x|\cosh p}e^{-\nu p}\,dp.\end{aligned} \tag{2.7}$$

We take the incident wave $\varphi_{\mathrm{in}}(x;\omega,E)$ as

$$\varphi_{\mathrm{in}}=e^{i\alpha\sigma}\varphi_0(x;\omega,E)=e^{i\alpha\sigma}e^{iE^{1/2}|x|\cos\gamma(x;\omega)}=e^{i\alpha\sigma}e^{-iE^{1/2}|x|\cos\sigma},$$

which is different from the usual plane wave $\varphi_0(x;\omega,E)$. Since the vector potential $A_\alpha(x)$ has the representation $A_\alpha(x)=\alpha\nabla\gamma(x;\omega)$ by (1.18), the modified factor $e^{i\alpha\sigma}$ may be interpreted as the change of the phase factor

$$\int_{l_x}A_\alpha(y)\cdot dy=\alpha\int_{-\infty}^0(d/ds)\gamma(x+s\omega;\omega)\,ds=\alpha\big(\gamma(x;\omega)-\pi\big)=\alpha\sigma(x;\omega)$$

which the vector potential A_α causes to the wave function of the quantum particle moving in the direction ω by the AB effect, where $l_x=\{y=x+s\omega: s\le 0\}$.

The incident wave admits the Fourier expansion

$$\varphi_{\mathrm{in}}(x;\omega,E)=(1/\pi)\sum_l\Big(\int_0^\pi e^{-iE^{1/2}|x|\cos\rho}\cos(\nu\rho)\,d\rho\Big)e^{il\sigma(x;\omega)}$$

for $|\sigma|<\pi$. This, together with (2.7), yields

$$\varphi_{\mathrm{sc}}(x;\omega,E)=-(1/\pi)\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^\infty e^{iE^{1/2}|x|\cosh p}e^{-\nu p}\,dp.$$

We compute the series

$$\sum_{l} e^{il\sigma} e^{-\nu p} \sin(\nu\pi) = \Big\{ \sum_{l\le[\alpha]} + \sum_{l\ge[\alpha]+1} \Big\} e^{il\sigma} e^{-\nu p} \sin(\nu\pi)$$
$$= \sin(\alpha\pi)(-1)^{[\alpha]} \Big\{ \frac{e^{-\alpha p}(e^{i\sigma}e^{p})^{[\alpha]}}{1+e^{-i\sigma}e^{-p}} + \frac{e^{\alpha p}(e^{i\sigma}e^{-p})^{[\alpha]}}{1+e^{-i\sigma}e^{p}} \Big\}$$

for $|\sigma| < \pi$, where the Gauss notation $[\alpha]$ denotes the greatest integer not exceeding α. Thus we have

$$\varphi_{\mathrm{sc}} = -\frac{\sin(\alpha\pi)}{\pi}(-1)^{[\alpha]} e^{i[\alpha]\sigma(x;\omega)} \int_{-\infty}^{\infty} e^{iE^{1/2}|x|\cosh p} \frac{e^{-\beta p}}{1+e^{-i\sigma}e^{-p}}\, dp$$

with $\beta = \alpha - [\alpha]$. We apply the stationary phase method to the integral on the right side. Since $e^{i\sigma(x;\omega)} = e^{i(\gamma(x;\omega)-\pi)} = -e^{i(\theta-\omega)}$ by identifying $\theta = x/|x| = \hat{x} \in S^1$ with the azimuth angle θ, $\varphi_{\mathrm{sc}}(x;\omega,E)$ obeys

$$\varphi_{\mathrm{sc}} = f(\omega\to\hat{x};E) e^{iE^{1/2}|x|}|x|^{-1/2} + o(|x|^{-1/2}), \quad |x|\to\infty,$$

where $f(\omega\to\theta;E)$ is defined as

$$(2.8) \qquad f(\omega\to\theta;E) = \Big(\frac{2}{\pi}\Big)^{1/2} e^{i\pi/4} E^{-1/4} \sin(\alpha\pi) e^{i[\alpha](\theta-\omega)} \frac{e^{i(\theta-\omega)}}{1-e^{i(\theta-\omega)}}$$

for $\omega \ne \theta$ by identifying $\omega, \theta \in S^1$ with the azimuth angles from the positive x_1-axis. The quantity $f(\omega\to\theta;E)$ is called the amplitude for scattering from the initial direction $\omega \in S^1$ to the final one θ at energy $E > 0$. By definition, the amplitude admits the analytic extension $f(\omega\to\theta;\zeta)$ over the complex plane.

We calculate the Green function of the resolvent $R_\alpha(\zeta) = R(\zeta;P_\alpha)$ with $\mathrm{Im}\,\zeta > 0$. Let $P_{l\alpha}$ be as in (2.3), and let $k = \zeta^{1/2}$ with $\mathrm{Im}\,k > 0$. Then the equation $(P_{l\alpha}-\zeta)u = 0$ has $\{r^{1/2}J_\nu(kr), r^{1/2}H_\nu(kr)\}$ with Wronskian $2i/\pi$ as a pair of linearly independent solutions, where $H_\mu(z) = H_\mu^{(1)}(z)$ denotes the Hankel function of the first kind. Thus $(P_{l\alpha}-\zeta)^{-1}$ has the integral kernel

$$R_{l\alpha}(r,\rho;\zeta) = (i\pi/2) r^{1/2}\rho^{1/2} J_\nu\big(k(r\wedge\rho)\big) H_\nu\big(k(r\vee\rho)\big), \quad \nu = |l-\alpha|,$$

where $r\wedge\rho = \min(r,\rho)$ and $r\vee\rho = \max(r,\rho)$. Hence the Green function $R_\alpha(x,y;\zeta)$ of $R_\alpha(\zeta)$ is given by

$$(2.9) \qquad R_\alpha(x,y;\zeta) = (i/4)\sum_{l} e^{il(\theta-\omega)} J_\nu\big(k(|x|\wedge|y|)\big) H_\nu\big(k(|x|\vee|y|)\big),$$

where $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ in the polar coordinates. This makes sense even for ζ in the lower half plane of the complex plane by analytic continuation. Then $R_\alpha(\zeta)$ with $\mathrm{Im}\,\zeta \le 0$ is well defined as an operator from $L^2_{\mathrm{comp}}(\boldsymbol{R}^2)$ to $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$. Thus $R_\alpha(\zeta)$ does not have any poles as a function with values in operators from $L_{\mathrm{comp}}(\boldsymbol{R}^2)$ to $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$. We can say that P_α with one solenoidal field $2\pi\alpha\delta(x)$ has no resonances. We do not discuss the possibility of resonances at zero energy.

We end the section by summarizing the asymptotic properties of the Green function $R_\alpha(x,y;\zeta)$ as the three propositions below. We sketch proofs for these propositions in the last section (Section 6). The propositions are used to establish the asymptotic properties of the Green function of the resolvent $R(\zeta; H)$ in Section 3.

PROPOSITION 2.1

Let $E_0 > 0$ and $c_1 > 0$ be fixed. Let $\lambda \gg 1$ be large enough. Assume that $\zeta = E + i\eta$ satisfies $|E - E_0| < E_0/2$ and $|\eta| \le c_1(\log\lambda)/\lambda$. If x and y fulfill

$$\lambda/c \le |x|, |y|, |x-y| \le c\lambda$$

for some $c > 1$ and if $\hat{x}$ and $\hat{y}$ satisfy $|\hat{x}\cdot\hat{y} + 1| < c\lambda^{2(\mu-1)}$ for some $0 \le \mu < 1/2$, then

$$\begin{aligned} R_\alpha(x,y;\zeta) &= (i/4)\cos(\alpha\pi)e^{i\alpha(\gamma(\hat{x};\hat{y})-\pi)}H_0(k|x-y|) \\ &\qquad + e^{ik(|x|+|y|)}(|x|+|y|)^{-1/2}e_{1N}(x,y;\zeta,\lambda) + O(\lambda^{-N}) \end{aligned}$$

for any $N \gg 1$, where e_{1N} is analytic in ζ as above and obeys

$$|\partial_x^n \partial_y^m e_{1N}| = O(\lambda^{\mu-1/2-|n|/2-|m|/2}) \tag{2.10}$$

uniformly in x, y, and ζ.

PROPOSITION 2.2

Let $\zeta = E + i\eta$ be as in Proposition 2.1. If x and y fulfill

$$\lambda/c \le |x|, |y|, |x-y| \le c\lambda$$

for some $c > 1$ and if $\hat{x}$ and $\hat{y}$ satisfy $|\hat{x}\cdot\hat{y}+1| > 1/c$, then

$$\begin{aligned} R_\alpha(x,y;\zeta) &= (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) \\ &\qquad + c_0(\zeta)e^{ik(|x|+|y|)}(|x||y|)^{-1/2}\big(f(-\hat{y}\to\hat{x};\zeta) + e_{2N}(x,y;\zeta,\lambda)\big) \\ &\qquad + O(\lambda^{-N}) \end{aligned}$$

for any $N \gg 1$, where $c_0(\zeta)$ is the constant defined by

$$c_0(\zeta) = (8\pi)^{-1/2}e^{i\pi/4}\zeta^{-1/4}, \tag{2.11}$$

while e_{2N} is analytic in ζ and obeys

$$|\partial_x^n \partial_y^m e_{2N}| = O(\lambda^{-1-|n|-|m|}) \tag{2.12}$$

uniformly in x, y, and ζ.

REMARK 2.1

The first term on the right-hand side describes the free trajectory which goes from y to x directly without being scattered at the origin, while the second term comes from the scattering trajectory which starts from y and arrives at x after it is scattered at the origin.

PROPOSITION 2.3

Let $\zeta = E + i\eta$ be again as in Proposition 2.1. Let $\varphi_{\alpha+}(x;\omega,E)$ and $\varphi_{\alpha-}(x;\omega,E)$

be the outgoing and incoming eigenfunctions of P_α, respectively. Then we have the following statements.

(1) *Denote by $\overline{\varphi}_{\alpha-}(y;\omega,\overline{\zeta})$ the complex conjugate of $\varphi_{\alpha-}(y;\omega,\overline{\zeta})$. If x and y fulfill $\lambda/c \le |x| \le c\lambda$ and $1/c \le |y| \le c$ for some $c > 1$, then*

$$R_\alpha(x,y;\zeta) = c_0(\zeta)e^{ik|x|}|x|^{-1/2}\big(\overline{\varphi}_{\alpha-}(y;\hat{x},\overline{\zeta}) + e_{3N}(x,y;\zeta,\lambda)\big) + O(\lambda^{-N}),$$

where e_{3N} is analytic in ζ and obeys $|\partial_x^n\partial_y^m e_{3N}| = O(\lambda^{-1-|n|})$ uniformly in x, y, and ζ.

(2) *If x and y fulfill $1/c \le |x| \le c$ and $\lambda/c \le |y| \le c\lambda$, then*

$$R_\alpha(x,y;\zeta) = c_0(\zeta)e^{ik|y|}|y|^{-1/2}\big(\varphi_{\alpha+}(x;-\hat{y},\zeta) + e_{4N}(x,y;\zeta,\lambda)\big) + O(\lambda^{-N}),$$

where e_{4N} is analytic in ζ and obeys $|\partial_x^n\partial_y^m e_{4N}| = O(\lambda^{-1-|m|})$ uniformly in x, y, and ζ.

REMARK 2.2

By definition (2.5), $\overline{\varphi}_{\alpha-}(y;\omega,\overline{\zeta})$ is given by

$$\overline{\varphi}_{\alpha-}(y;\omega,\overline{\zeta}) = \sum_l \exp(-i\nu\pi/2)\exp\big(-il\gamma(x;\omega)\big)J_\nu(\zeta^{1/2}|x|),$$

and hence we see that $\overline{\varphi}_{\alpha-}(y;\omega,\overline{\zeta})$ is analytic in ζ.

3. Magnetic Schrödinger operators in exterior domains

We begin by recalling the notation: $H = H(A,0)$ is the self-adjoint operator defined by (1.7) with $\mathcal{D}(H) = H^2(\Omega) \cap H_0^1(\Omega)$, and $f_0(\omega \to \theta; E)$ denotes the amplitude at energy E for the pair (K_0, H), $K_0 = -\Delta$ being the free Hamiltonian acting on $L^2(\boldsymbol{R}^2)$. The aim of the present section is to study the asymptotic behavior of the kernel $R(x,y;\zeta)$ with $|x-y| \gg 1$ for the resolvent $R(\zeta) = R(\zeta;H)$. Here we introduce a smooth nonnegative cutoff function $\chi \in C_0^\infty[0,\infty)$ with the properties

$$0 \le \chi \le 1, \qquad \operatorname{supp}\chi \subset [0,2], \qquad \chi = 1 \quad \text{on } [0,1]. \tag{3.1}$$

This function is often used in the future discussion without further references. We also use the notation $(\ ,\ )$ to denote the L^2 scalar product in $L^2(\boldsymbol{R}^2)$ or $L^2(\Omega)$.

LEMMA 3.1

Let $E > 0$ be fixed. Then there exists a complex neighborhood of E where the resolvent $R(\zeta) = R(\zeta;H)$ is analytic as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$.

Proof

The proof is based on the complex scaling method developed by [5] and [17], and the lemma follows as a particular case of such a general theory. The operator H is a long-range perturbation to the free Hamiltonian K_0, but the coefficients of H are analytic in Ω. The operator P_α defined by (2.1) is transformed into $e^{-2\theta}P_\alpha$

under the group of dilations $x \to e^{\theta}x$. By assumption (1.1), $\mathcal{O} \subset \{|x| < 1\}$. Let $\Sigma = \{|x| < 8\}$. Since H has no positive eigenvalues, we can show by making use of the analytic dilation which leaves Σ invariant that there exists a complex neighborhood of E in which the operator

$$j_\sigma R(\zeta) : L^2_{\mathrm{comp}}(\Sigma_0) \to L^2_{\mathrm{comp}}(\Sigma_0), \quad \Sigma_0 = \Sigma \setminus \mathcal{O},$$

restricted to $L^2_{\mathrm{comp}}(\Sigma_0) \subset L^2(\Omega)$ is analytic as a function with values in bounded operators, where j_σ is the characteristic function of Σ_0, and the multiplication operator j_σ is considered to be the restriction to $L^2_{\mathrm{comp}}(\Sigma_0)$ from $L^2(\Omega)$. We assert that $R(\zeta)$ is analytic over the neighborhood above as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. To see this, we set

$$u_0(x) = \chi(|x|/2), \qquad u_1(x) = \chi(|x|/4), \qquad v_0 = 1 - u_0, \qquad v_1 = 1 - u_1$$

for the cutoff function $\chi \in C_0^\infty[0,\infty)$ with properties (3.1). Recall that

$$R_\alpha(\zeta) = R(\zeta; P_\alpha) : L^2_{\mathrm{comp}}(\boldsymbol{R}^2) \to L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$$

depends analytically on ζ. If we regard the multiplication operator $f \mapsto v_1 f$ as the extension from $L^2(\Omega)$ to $L^2(\boldsymbol{R}^2)$, then $R_\alpha(\zeta)v_1$ makes sense as an operator from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$, and similarly for $R_\alpha(\zeta)v_0$. Since $v_0 v_1 = v_1$ and since $H = P_\alpha$ over Ω, $R(\zeta) = R(\zeta)(u_1 + v_1)$ is decomposed into the sum of three terms

$$R(\zeta) = R(\zeta)u_1 + v_0 R_\alpha(\zeta)v_1 - R(\zeta)[P_\alpha, v_0]R_\alpha(\zeta)v_1$$

at least for ζ with $\mathrm{Im}\,\zeta > 0$, where $[X,Y] = XY - YX$ denotes the commutator between two operators X and Y. The coefficients of $[P_\alpha, v_0]$ have supports in Σ_0. Hence we see that

$$j_\sigma R(\zeta) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{comp}}(\Sigma_0)$$

depends analytically on ζ in the complex neighborhood of E. Similarly we obtain the relation

$$R(\zeta) = u_1 R(\zeta) + v_1 R_\alpha(\zeta)v_0 + v_1 R_\alpha(\zeta)[P_\alpha, v_0]R(\zeta)$$

on $L^2_{\mathrm{comp}}(\Omega)$. This yields the analytic dependence on ζ of $R(\zeta) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega)$, and the proof is complete. □

The lemma above, together with the analytic Fredholm theorem, implies that the resolvent $R(\zeta; H_d)$ in question is meromorphically continued from the upper half plane of the complex plane to a region (independent of d) in the lower half plane across the positive real axis, so that $R(\zeta; H_d)$ is a meromorphic function over D_d for $|d| \gg 1$. In fact, we have the relation $(H_d - \zeta)R(\zeta) = \mathrm{Id} + V_d R(\zeta)$, and

$$V_d R(\zeta) j_d : L^2(B_d) \to L^2(B_d), \quad B_d = B_{-d} \cup B_{+d},$$

acting as a compact operator that is analytic in the neighborhood in the lemma, where j_d denotes the characteristic function of B_d. If the solution w to $(H_d - E)w = 0$ satisfies the outgoing radiation condition at infinity, then it vanishes, so that $\mathrm{Id}+V_dR(\zeta)$ is invertible at $\zeta=E$. Thus $\mathrm{Id}+V_dR(\zeta)j_d : L^2(B_d)\to L^2(B_d)$ is also invertible at $\zeta=E$, and we see by the analytic Fredholm theorem that

$$R(\zeta;H_d)=R(\zeta)-R(\zeta)j_d\big(\mathrm{Id}+V_dR(\zeta)j_d\big)^{-1}V_dR(\zeta), \quad \mathrm{Re}\,\zeta>0,\ \mathrm{Im}\,\zeta\ge 0, \tag{3.2}$$

admits the meromorphic continuation over a region in the lower half plane as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. Here $R(\zeta)j_d$ is considered to be an operator from $L^2(B_d)$ to $L^2_{\mathrm{loc}}(\Omega)$ by regarding the multiplication operator $f\mapsto j_df$ as the extension from $L^2(B_d)$ to $L^2_{\mathrm{comp}}(\Omega)$. If we decompose $L^2(\Omega)$ into

$$L^2(\Omega)=L^2(B_d)\oplus L^2(\Omega\setminus B_d),$$

then the operator $\mathrm{Id}+V_dR(\zeta)$ acts as

$$\begin{pmatrix}\mathrm{Id}+V_dR(\zeta)j_d & V_dR(\zeta)(1-j_d)\\ 0 & \mathrm{Id}\end{pmatrix}$$

on $L^2(B_d)\oplus L^2(\Omega\setminus B_d)$, and its inverse takes the form

$$\begin{pmatrix}(\mathrm{Id}+V_dR(\zeta)j_d)^{-1} & -(\mathrm{Id}+V_dR(\zeta)j_d)^{-1}V_dR(\zeta)(1-j_d)\\ 0 & \mathrm{Id}\end{pmatrix}.$$

Thus $(\mathrm{Id}+V_dR(\zeta)j_d)^{-1}$ naturally appears as the component of the matrix representation for $(\mathrm{Id}+V_dR(\zeta))^{-1}$ and coincides with $(\mathrm{Id}+V_dR(\zeta))^{-1}$ on $L^2(B_d)\subset L^2(\Omega)$.

We now define the scattering amplitude $f_0(\omega\to\theta;E)$ for the pair (K_0,H) with $H=H(A,0)$. Let $\varphi_+(x;\omega,E)$ be the outgoing eigenfunction of H. Then the amplitude is defined through the asymptotic form

$$\varphi_+=e^{i\alpha(\gamma(x;\omega)-\pi)}\varphi_0(x;\omega,E)+f_0(\omega\to\theta;E)e^{iE^{1/2}|x|}|x|^{-1/2}+o(|x|^{-1/2})$$

as $|x|\to\infty$ in the direction θ ($x=|x|\theta$).

LEMMA 3.2

Recall that $\varphi_{\alpha+}(x;\omega,E)$ and $\varphi_{\alpha-}(x;\theta,E)$ denote the outgoing and incoming eigenfunctions of $P_\alpha=H(A_\alpha,0)$, respectively, and that $f(\omega\to\theta;E)$ is the scattering amplitude for the pair (K_0,P_α). Set $u_0=\chi(|x|/2)$, and set $u_1=\chi(|x|/4)$. Then the amplitude $f_0(\omega\to\theta;E)$ for the pair (K_0,H) has the representation

$$f_0=f(\omega\to\theta;E)+c_0(E)\big(R(E)[P_\alpha,u_0]\varphi_{\alpha+}(\cdot;\omega,E),[P_\alpha,u_1]\varphi_{\alpha-}(\cdot;\theta,E)\big),$$

where $R(E)=R(E;H)$ and $c_0(E)$ is defined by (2.11).

REMARK 3

This lemma, together with Lemma 3.1, allows us to extend analytically $f_0(\omega\to\theta;E)$ in a complex neighborhood of E, and its extension $f_0(\omega\to\theta;\zeta)$ takes the form

$$f_0=f(\omega\to\theta;\zeta)+c_0(\zeta)\big(R(\zeta)[P_\alpha,u_0]\varphi_{\alpha+}(\cdot;\omega,\zeta),[P_\alpha,u_1]\varphi_{\alpha-}(\cdot;\theta,\overline{\zeta})\big).$$

Proof
Let $\varphi_+(x;\omega,E)$ be as above. By assumption (1.1), $P_\alpha = H = H(A,0)$ outside the support of u_0, and hence we have

$$\varphi_+ = (1-u_0)\varphi_{\alpha+} + R(E)[P_\alpha, u_0]\varphi_{\alpha+}. \tag{3.3}$$

Similarly

$$\varphi_{\alpha+} = (1-u_1)\varphi_+ + R_\alpha(E)[P_\alpha, u_1]\varphi_+$$

with $R_\alpha(E) = R(E;P_\alpha)$, and hence

$$\varphi_+ = \varphi_{\alpha+} + u_1\varphi_+ - R_\alpha(E)[P_\alpha, u_1]\varphi_+. \tag{3.4}$$

It follows from Proposition 2.3(1) with $\lambda = r = |x|$ that the last term on the right-hand side of (3.4) behaves like

$$c_0(E)\big(\varphi_+(\cdot;\omega,E), [P_\alpha,u_1]\varphi_{\alpha-}(\cdot;\theta,E)\big)|x|^{-1/2}e^{iE^{1/2}|x|} + o(|x|^{-1/2})$$

as $|x|\to\infty$ in the direction θ. We insert (3.3) into φ_+ on the right-hand side. Since

$$\big((1-u_0)\varphi_{\alpha+}, [u_1,P_\alpha]\varphi_{\alpha-}\big) = (\varphi_{\alpha+}, [u_1,P_\alpha]\varphi_{\alpha-}) = 0,$$

we obtain the desired relation. □

The following two propositions correspond to Propositions 2.1 and 2.2 in Section 2. We keep the notation with the same meaning as ascribed there to formulate the propositions.

PROPOSITION 3.1
Let $E_0 > 0$ and $c_1 > 0$ be fixed. Assume that $\zeta = E + i\eta$ satisfies $|E - E_0| < E_0/2$ and $|\eta| \le c_1(\log\lambda)/\lambda$ for $\lambda \gg 1$. If x and y fulfill

$$\lambda/c \le |x|, |y|, |x-y| \le c\lambda$$

for some $c > 1$ and if $\hat{x}$ and $\hat{y}$ satisfy $|\hat{x}\cdot\hat{y} + 1| < c\lambda^{2(\mu-1)}$ for some $0 \le \mu < 1/2$, then

$$\begin{aligned} R(x,y;\zeta) = (i/4)\cos(\alpha\pi)e^{i\alpha(\gamma(\hat{x};\hat{y})-\pi)}H_0(k|x-y|) \\ + e^{ik(|x|+|y|)}(|x|+|y|)^{-1/2}r_{1N}(x,y;\zeta,\lambda) + O(\lambda^{-N}) \end{aligned}$$

for any $N \gg 1$, where r_{1N} is analytic in ζ in a complex neighborhood of E as in Lemma 3.1, and it obeys the same bound as in (2.10).

Proof
We again set

$$u_0(x) = \chi(|x|/2), \qquad u_1(x) = \chi(|x|/4), \qquad v_0 = 1-u_0, \qquad v_1 = 1-u_1$$

and fix $p, q \in \boldsymbol{R}^2$ ($|p|, |q| \gg 1$) as points having the properties in the proposition. If we further set $w_p(x) = \chi(|x-p|)$, then $w_p v_0 = w_p$ and $w_p v_1 = w_p$, and similarly for $w_q = \chi(|x-q|)$. The operator H coincides with P_α on the support of v_1. We compute

$$w_p R(\zeta) w_q = w_p R_\alpha(\zeta) w_q + w_p R_\alpha(\zeta)(P_\alpha v_1 - v_1 H) R(\zeta) w_q$$
$$= w_p R_\alpha(\zeta) w_q + w_p R_\alpha(\zeta)[u_1, P_\alpha] R(\zeta) w_q.$$

Since $v_0 = 1$ on the support of ∇u_1 and since $H = P_\alpha$ on the support of v_0, we repeat the above argument to get

$$w_p R(\zeta) w_q = w_p R_\alpha(\zeta) w_q + w_p R_\alpha(\zeta)[u_1, P_\alpha]\big(R_\alpha(\zeta) + R(\zeta)[P_\alpha, u_0] R_\alpha(\zeta)\big) w_q.$$

Note that

$$w_p R_\alpha(\zeta)[u_1, P_\alpha] R_\alpha(\zeta) w_q = w_p R_\alpha(\zeta) u_1 w_q - w_p u_1 R_\alpha(\zeta) w_q = 0$$

and hence we have

$$w_p R(\zeta) w_q = w_p R_\alpha(\zeta) w_q + w_p R_\alpha(\zeta)[u_1, P_\alpha] R(\zeta)[P_\alpha, u_0] R_\alpha(\zeta) w_q.$$

We apply Proposition 2.3 to the second operator on the right-hand side. Let Σ_0 be again defined by $\Sigma_0 = \Sigma \setminus \mathcal{O}$. Then the coefficients of $[u_1, P_\alpha]$ and $[P_\alpha, u_0]$ have supports in Σ_0. Hence it follows from elliptic estimates that

$$[u_1, P_\alpha] R(\zeta)[P_\alpha, u_0] : L^2_{\mathrm{comp}}(\Sigma_0) \to L^2_{\mathrm{comp}}(\Sigma_0)$$

is bounded uniformly in ζ as in the proposition and is analytic in ζ. According to Proposition 2.3, the integral kernels of $w_p R_\alpha(\zeta)$ and $R_\alpha(\zeta) w_q$ take the forms

$$\big(w_p R_\alpha(\zeta)\big)(x,y) = w_p(x) e^{ik|x|} |x|^{-1/2} \times \{\text{uniformly bounded function}\}$$

for $y \in \Sigma_0$ and

$$\big(R_\alpha(\zeta) w_q\big)(x,y) = \{\text{uniformly bounded function}\} \times e^{ik|y|} |y|^{-1/2} w_q(y)$$

for $x \in \Sigma_0$. Hence the kernel of the second operator obeys the bound

$$e^{ik(|p|+|q|)}(|p||q|)^{-1/2} O(1) = e^{ik(|p|+|q|)}(|p|+|q|)^{-1/2} O(\lambda^{-1/2}).$$

This allows us to deal with this kernel as a remainder term. Thus the proposition follows from Proposition 2.1. □

PROPOSITION 3.2

Let $\zeta = E + i\eta$ be as in Proposition 3.1. If x and y fulfill

$$\lambda/c \le |x|, |y|, |x-y| \le c\lambda, \qquad |\hat{x}\cdot\hat{y} + 1| > 1/c$$

for some $c > 1$, then

$$R(x,y;\zeta) = (i/4) e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)} H_0(k|x-y|)$$
$$+ c_0(\zeta) e^{ik(|x|+|y|)} (|x||y|)^{-1/2} \big(f_0(-\hat{y} \to \hat{x}; \zeta) + r_{2N}(x,y;\zeta,\lambda)\big)$$
$$+ O(\lambda^{-N})$$

for any $N \gg 1$, where r_{2N} is analytic in ζ in a complex neighborhood of E as in Lemma 3.1, and it obeys the same bound as in (2.12).

Proof
We use the same notation and repeat the same argument as in the proof of Proposition 3.1. Then we obtain

$$w_p R(\zeta) w_q = w_p R_\alpha(\zeta) w_q + w_p R_\alpha(\zeta)[u_1, P_\alpha] R(\zeta)[P_\alpha, u_0] R_\alpha(\zeta) w_q.$$

If we apply Propositions 2.2 and 2.3 to the second operator on the right-hand side, then it follows from Lemma 3.2 (see also Remark 3) that the kernel of this operator has the desired asymptotic form at points p and q fixed arbitrarily. This proves the proposition. □

4. Basic lemmas and proof of main theorems

In this section we prove Theorems 1.1 and 1.2, accepting the two basic lemmas (Lemmas 4.1, 4.2) formulated below as proved. These lemmas are proved in Section 5.

In Section 3, we have shown that the amplitude $f_0(\omega \to \theta; E)$ admits the analytic extension in a complex neighborhood of $E > 0$. Before going into the proof, we also make a comment on the analytic extension of the amplitude

$$f_\pm(\omega \to \theta; E) = -c_0(E)\big(V_\pm(\mathrm{Id} - G_\pm(E)V_\pm)\varphi_0(\cdot; \omega, E), \varphi_0(\cdot; \theta, E)\big)$$

for the pair $(K_0, K_\pm)$, where $K_\pm$ is the Schrödinger operator defined by (1.6) and $G_\pm(E) = R(E; K_\pm)$ denotes the resolvent of $K_\pm$. In view of this representation, the analytic extension $f_\pm(\omega \to \theta; \zeta)$ is given by

$$f_\pm(\omega \to \theta; \zeta) = -c_0(\zeta)\big(V_\pm(\mathrm{Id} - G_\pm(\zeta)V_\pm)\varphi_0(\cdot; \omega, \zeta), \varphi_0(\cdot; \theta, \overline{\zeta})\big)$$

in a complex neighborhood of $E > 0$. In fact, the resolvent $G_\pm(\zeta) = R(\zeta; K_\pm)$ admits the analytic extension

$$G_\pm(\zeta) = G_0(\zeta) - G_0(\zeta) j_0 \big(\mathrm{Id} + V_\pm G_0(\zeta) j_0\big)^{-1} V_\pm G_0(\zeta) : L^2_{\mathrm{comp}}(\boldsymbol{R}^2) \to L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$$

over a complex neighborhood of $E > 0$, where $G_0(\zeta)$ denotes the resolvent $R(\zeta; K_0)$ of the free Hamiltonian $K_0 = -\Delta$ and j_0 is the characteristic function of the unit disk $B = \{|x| < 1\}$. (Note that $\operatorname{supp} V_\pm \subset B$.) The derivation of the relation above is done in the same way used to derive (3.2).

We begin by fixing new notation to formulate the lemmas. Let $G_0(\zeta) = R(\zeta; K_0)$ be as above. We again denote by j_0 the characteristic function of the unit disk B. Then

$$j_{\pm d}(x) = j_0(x - d_\pm) = j_0(x_\pm)$$

defines the characteristic function of $B_{\pm d} = \{|x - d_\pm| < 1\}$, where $d_+ = (1-\kappa)d$ and $d_- = -\kappa d$. We construct a function $g_\pm \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ such that $g_\pm$ equals

$$g_\pm(x) = \alpha\gamma(x; \mp\hat{d}) = \alpha\gamma(\hat{x}; \mp\hat{d}) \tag{4.1}$$

on $\{|x - d_\pm| < |d_\pm|/2\}$ and obeys $\partial_x^n g_\pm = O(|x|^{-|n|})$ as $|x| \to \infty$, where $\gamma(\hat{x}; \omega)$ denotes the azimuth angle from ω to $\hat{x} = x/|x|$. By construction, $g_\pm(x)$ satisfies

$$\nabla g_\pm = \alpha(-x_2/|x|^2, x_1/|x|^2) = A(x) \tag{4.2}$$

on $\{|x-d_\pm|<|d_\pm|/2\}$. We also introduce the auxiliary operators

$$K_{\pm d} = K_0 + V_{\pm d}, \qquad \mathcal{D}(K_{\pm d}) = H^2(\boldsymbol{R}^2), \tag{4.3}$$
$$H_{\pm d} = H(A, V_{\pm d}), \qquad \mathcal{D}(H_{\pm d}) = H^2(\Omega) \cap H_0^1(\Omega)$$

and write $G_{\pm d}(\zeta)$ and $R_{\pm d}(\zeta)$ for the resolvents $R(\zeta; K_{\pm d})$ and $R(\zeta; H_{\pm d})$, respectively. We further define

$$\psi_\pm(x;\omega,\overline{\zeta}) = \big[\big(\mathrm{Id} - G_\pm(\zeta)^* V_\pm\big)\varphi_0(\cdot;\omega,\overline{\zeta})\big](x). \tag{4.4}$$

The function $\psi_\pm(x;\omega,\overline{\zeta})$ solves the equation $(K_\pm - \overline{\zeta})\psi_\pm(x;\omega,\overline{\zeta}) = 0$. If, in particular, $\zeta = E > 0$, $\psi_\pm(x;\omega,E)$ turns out to be the incoming eigenfunction of $K_\pm$, and the conjugate function $\overline{\psi}_\pm(x;\omega,\overline{\zeta})$ of $\psi_\pm(x;\omega,\overline{\zeta})$ is analytic in ζ. It should be noted that $\psi_\pm(x;\omega,\overline{\zeta})$ itself is not analytic. We note that $\psi_+(x;\omega,E)$ does not denote the outgoing eigenfunction at energy $E>0$.

4.1. Two basic lemmas

With the notation above, we are now in a position to formulate the two basic lemmas. In the lemmas, we use the multiplication $f \mapsto j_{\pm d} f$ by the characteristic function $j_{\pm d}$ as the extension from $L^2(B_{\pm d})$ to $L^2_{\mathrm{comp}}(\Omega)$ or $L^2_{\mathrm{comp}}(\boldsymbol{R}^2)$.

LEMMA 4.1

(1) *Let ζ be in D_{+d} defined by (1.15), and let $X_+(\zeta;d)$ be the operator defined by*

$$X_+(\zeta;d) = V_{+d} R(\zeta) j_{+d} : L^2(B_{+d}) \to L^2(B_{+d}).$$

Then $\mathrm{Id} + X_+(\zeta;d)$ takes the form

$$\mathrm{Id} + X_+(\zeta;d) = e^{ig_+}\big(\mathrm{Id} + X_{+0}(\zeta;d) + X_{+1}(\zeta;d)\big)\big(\mathrm{Id} + V_{+d}G_0(\zeta)j_{+d}\big)e^{-ig_+},$$

where $X_{+0}(\zeta;d)$ is the integral operator with the kernel $X_{+0}(x,y;\zeta,d)$ defined by

$$X_{+0} = c_0(\zeta)(e^{2ik|d_+|}/|d_+|) f_0(-\hat{d} \to \hat{d};\zeta) V_+(x_+)\varphi_0(x_+;\hat{d},\zeta)\overline{\psi}_+(y_+;-\hat{d},\overline{\zeta}) j_0(y_+)$$

and $X_{+1}(\zeta;d)$ is analytic in $\zeta \in D_{+d}$ with values in bounded operators acting on $L^2(B_{+d})$ and obeys $\|X_{+1}(\zeta;d)\| = O(|d|^{-\mu})$ uniformly in ζ for some $\mu>0$.

(2) *Let ζ be in D_{-d} defined by (1.15), and let $X_-(\zeta;d)$ be the operator defined by*

$$X_-(\zeta;d) = V_{-d} R(\zeta) j_{-d} : L^2(B_{-d}) \to L^2(B_{-d}).$$

Then $\mathrm{Id} + X_-(\zeta;d)$ takes the form

$$\mathrm{Id} + X_-(\zeta;d) = e^{ig_-}\big(\mathrm{Id} + X_{-0}(\zeta;d) + X_{-1}(\zeta;d)\big)\big(\mathrm{Id} + V_{-d}G_0(\zeta)j_{-d}\big)e^{-ig_-},$$

where $X_{-0}(\zeta;d)$ is the integral operator with the kernel $X_{-0}(x,y;\zeta,d)$ defined by

$$X_{-0} = c_0(\zeta)(e^{2ik|d_-|}/|d_-|) f_0(\hat{d} \to -\hat{d};\zeta) V_-(x_-)\varphi_0(x_-;-\hat{d},\zeta)\overline{\psi}_-(y_-;\hat{d},\overline{\zeta}) j_0(y_-)$$

and $X_{-1}(\zeta;d)$ is analytic in $\zeta \in D_{-d}$ with values in bounded operators acting on $L^2(B_{-d})$ and obeys $\|X_{-1}(\zeta;d)\| = O(|d|^{-\mu})$ uniformly in ζ for some $\mu>0$.

Before formulating the second lemma, we make a brief comment on the operators $R_{\pm d}(\zeta) = R(\zeta; H_{\pm d})$. Accepting the above lemma as proved, we can show that these operators are analytic in $\zeta \in D_d$ as functions with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$, where D_d is defined by (1.8) (for details, see the argument after the proof of Lemma 4.3 in this subsection).

LEMMA 4.2

Assume that α is not a half integer. Then we have the following statements.

(1) *Let $Y_+(\zeta; d)$ be the operator defined by*

$$Y_+(\zeta; d) = V_{-d} R_{+d}(\zeta) j_{+d} : L^2(B_{+d}) \to L^2(B_{-d})$$

for $\zeta \in D_d$. Then $Y_+(\zeta; d)$ admits the decomposition

$$Y_+(\zeta; d) = Y_{+0}(\zeta; d) + Y_{+1}(\zeta; d),$$

where $Y_{+0}(\zeta; d)$ is the integral operator with the kernel $Y_{+0}(x, y; \zeta, d)$ defined by

$$Y_{+0} = c_0(\zeta)\cos(\alpha\pi)(e^{ik|d|}/|d|^{1/2}) V_-(x_-)\varphi_0(x_-; -\hat{d}, \zeta)\overline{\psi}_+(y_+; -\hat{d}, \overline{\zeta}) j_0(y_+)$$

and $Y_{+1}(\zeta; d)$ is analytic in $\zeta \in D_d$ with values in bounded operators from $L^2(B_{+d})$ to $L^2(B_{-d})$ and obeys $\|Y_{+1}(\zeta; d)\| = O(|d|^{-\mu})$ uniformly in ζ for some $\mu > 0$.

(2) *Let $Y_-(\zeta; d)$ be the operator defined by*

$$Y_-(\zeta; d) = V_{+d} R_{-d}(\zeta) j_{-d} : L^2(B_{-d}) \to L^2(B_{+d})$$

for $\zeta \in D_d$. Then $Y_-(\zeta; d)$ admits the decomposition

$$Y_-(\zeta; d) = Y_{-0}(\zeta; d) + Y_{-1}(\zeta; d),$$

where $Y_{-0}(\zeta; d)$ is the integral operator with the kernel $Y_{-0}(x, y; \zeta, d)$ defined by

$$Y_{-0} = c_0(\zeta)\cos(\alpha\pi)(e^{ik|d|}/|d|^{1/2}) V_+(x_+)\varphi_0(x_+; \hat{d}, \zeta)\overline{\psi}_-(y_-; \hat{d}, \overline{\zeta}) j_0(y_-)$$

and $Y_{-1}(\zeta; d)$ is analytic in $\zeta \in D_d$ with values in bounded operators from $L^2(B_{-d})$ to $L^2(B_{+d})$ and obeys $\|Y_{-1}(\zeta; d)\| = O(|d|^{-\mu})$ uniformly in ζ for some $\mu > 0$.

The following three lemmas are obtained as simple consequences of Lemmas 4.1 and 4.2.

LEMMA 4.3

(1) *If $\zeta \in D_d$, then*

$$\mathrm{Id} + X_\pm(\zeta; d) : L^2(B_{\pm d}) \to L^2(B_{\pm d})$$

has a bounded inverse and the inverse is bounded uniformly in d and $\zeta \in D_d$. Moreover, we have the relation

$$R_{\pm d}(\zeta) j_{\pm d} = R(\zeta) j_{\pm d}\bigl(\mathrm{Id} + X_\pm(\zeta; d)\bigr)^{-1} : L^2(B_{\pm d}) \to L^2_{\mathrm{loc}}(\Omega)$$

for $\zeta \in D_d$.

(2) *Assume that $0 < \kappa < 1/2$. If $\zeta \in D_{+d}$, then $\mathrm{Id} + X_-(\zeta; d)$ is invertible on $L^2(B_{-d})$ and $R_{-d}(\zeta) j_{-d}$ satisfies the same relation as above.*

LEMMA 4.4

Assume that $0<\kappa<1/2$. *Then*

$$V_{-d}R_{-d}(\zeta)j_{-d}: L^2(B_{-d}) \to L^2(B_{-d})$$

is bounded uniformly in d *and* $\zeta \in D_{+d}$.

LEMMA 4.5

Assume that α *is not a half integer. Let* $\zeta \in D_d$, *and let* $Y_\pm(\zeta;d)$ *be as in Lemma 4.2. Define*

$$Y(\zeta;d) = Y_-(\zeta;d)Y_+(\zeta;d) = V_{+d}R_{-d}(\zeta)V_{-d}R_{+d}(\zeta)j_{+d} \tag{4.5}$$

as an operator acting on $L^2(B_{+d})$. *Then* $Y(\zeta;d)$ *admits the decomposition*

$$Y(\zeta;d) = Y_0(\zeta;d) + Y_1(\zeta;d),$$

where $Y_0(\zeta;d)$ *is the integral operator with the kernel* $Y_0(x,y;\zeta,d)$ *defined by*

$$Y_0 = -c_0(\zeta)\cos^2(\alpha\pi)(e^{2ik|d|}/|d|)f_-(-\hat{d}\to\hat{d};\zeta)$$
$$\times V_+(x_+)\varphi_0(x_+;\hat{d},\zeta)\overline{\psi}_+(y_+;-\hat{d},\overline{\zeta})j_0(y_+)$$

and $Y_1(\zeta;d)$ *is analytic in* $\zeta \in D_d$ *with values in bounded operators from* $L^2(B_{+d})$ *to itself and obeys* $\|Y_1(\zeta;d)\| = O(|d|^{-\mu})$ *uniformly in* ζ *for some* $\mu>0$.

Proof of Lemma 4.3

(1) We recall the notation $G_{\pm d}(\zeta) = R(\zeta;K_{\pm d})$ from (4.3). By the resolvent identity, we have

$$\big(\mathrm{Id} + V_{\pm d}G_0(\zeta)j_{\pm d}\big)\big(\mathrm{Id} - V_{\pm d}G_{\pm d}(\zeta)j_{\pm d}\big) = \mathrm{Id}$$

on $L^2(B_{\pm d})$. This implies that $\mathrm{Id} + V_{\pm d}G_0(\zeta)j_{\pm d}$ is invertible and the inverse

$$\big(\mathrm{Id} + V_{\pm d}G_0(\zeta)j_{\pm d}\big)^{-1} = \mathrm{Id} - V_{\pm d}G_{\pm d}(\zeta)j_{\pm d}: L^2(B_{\pm d}) \to L^2(B_{\pm d}) \tag{4.6}$$

is bounded uniformly in ζ. We consider the operator $X_-(\zeta;d)$ only. If $\zeta \in D_d$, then $D_d \subset D_{-d}$ for $|d| \gg 1$, and it follows from (1.13) that

$$\big|e^{2ik|d_-|}/|d_-|\big| = O(|d|^{-c}), \quad \zeta \in D_d, \tag{4.7}$$

for some $c>0$. Hence Lemma 4.1 shows that

$$\|X_{-0}(\zeta;d)\| + \|X_{-1}(\zeta;d)\| = O(|d|^{-c}) \tag{4.8}$$

as an operator acting on $L^2(B_{-d})$. This, together with (4.6), implies that $\mathrm{Id} + X_-(\zeta;d)$ is invertible on $L^2(B_{-d})$. Since

$$(H_{-d}-\zeta)R(\zeta)j_{-d} = \mathrm{Id} + V_{-d}R(\zeta)j_{-d} = \mathrm{Id} + X_-(\zeta;d)$$

on $L^2(B_{-d})$, the desired relation follows at once.

(2) If $0<\kappa<1/2$, then $D_{+d} \subset D_{-d}$ and (4.7) with another $c>0$ remains true for $\zeta \in D_{+d}$. This follows from (1.16). Hence (2) is obtained in the same way as above. □

We note that Lemma 4.2 is not required in proving Lemma 4.3. By the resolvent identity, it follows from Lemmas 3.1 and 4.3 that

$$R_{\pm d}(\zeta) = R(\zeta) - R_{\pm d}(\zeta)V_{\pm d}R(\zeta) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega)$$

is well defined for $\zeta \in D_d$ and is analytic in ζ. If $0 < \kappa < 1/2$, then this fact remains true for $R_{-d}(\zeta)$ with $\zeta \in D_{+d}$, and if $1/2 < \kappa < 1$, then it holds for $R_{+d}(\zeta)$ with $\zeta \in D_{-d}$.

Proof of Lemma 4.4
By Lemma 4.3, we have

$$V_{-d}R_{-d}(\zeta)j_{-d} = V_{-d}R(\zeta)j_{-d}\big(\mathrm{Id} + X_-(\zeta;d)\big)^{-1} = \mathrm{Id} - \big(\mathrm{Id} + X_-(\zeta;d)\big)^{-1}$$

on $L^2(B_{-d})$. This proves the lemma. □

Proof of Lemma 4.5
We recall that the notation (,) denotes the L^2 scalar product and that $\psi_\pm(x;\omega,\overline{\zeta})$ is defined by (4.4). In view of Lemma 4.2, we compute

$$\begin{aligned} c_0(\zeta)&\int V_-(x_-)\varphi_0(x_-;-\hat{d},\zeta)\overline{\psi}_-(x_-;\hat{d},\overline{\zeta})\,dx \\ &= c_0(\zeta)\big(V_-\varphi_0(\cdot;-\hat{d},\zeta),(\mathrm{Id} - G_-(\zeta)^*V_-)\varphi_0(\cdot;\hat{d},\overline{\zeta})\big) \\ &= c_0(\zeta)\big(V_-(\mathrm{Id} - G_-(\zeta)V_-)\varphi_0(\cdot;-\hat{d},\zeta),\varphi_0(\cdot;\hat{d},\overline{\zeta})\big) \\ &= -f_-(-\hat{d} \to \hat{d};\zeta). \end{aligned}$$

This yields the kernel of $Y_0(\zeta;d)$. Since $|e^{2ik|d|}/|d|| = O(|d|^{c\delta_0})$ over D_d for some $c > 0$ (see (1.12)), we can take δ_0 so small that the remainder operator $Y_1(\zeta;d)$ obeys $\|Y_1(\zeta;d)\| = O(|d|^{-\mu})$ for some $\mu > 0$. Thus the proof is complete. □

4.2. Proof of main theorems

We are now in a position to prove the two main theorems.

Proof of Theorem 1.1
By assumption, α is not a half integer. We can use not only Lemma 4.1 but also Lemma 4.2 to prove the theorem. Note that $D_d \subset D_{+d} \cap D_{-d}$ for $|d| \gg 1$. By the argument after the proof of Lemma 4.3, $R_{\pm d}(\zeta) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega)$ is analytic over D_d. The proof is divided into four steps.

(1) We start with the relation

$$(H_d - \zeta)R_{-d}(\zeta) = \mathrm{Id} + V_{+d}R_{-d}(\zeta). \tag{4.9}$$

We regard the operator on the right-hand side as an operator acting on $L^2(B_{+d})$. By the resolvent identity, the operator on the right-hand side equals

$$\mathrm{Id} + V_{+d}R_{-d}(\zeta)j_{+d} = \mathrm{Id} + V_{+d}R(\zeta)j_{+d} - V_{+d}R_{-d}(\zeta)V_{-d}R(\zeta)j_{+d}.$$

By Lemma 4.3, it is further equal to

$$\mathrm{Id} + V_{+d}R_{-d}(\zeta)j_{+d} = \big(\mathrm{Id} - Y(\zeta;d)\big)\big(\mathrm{Id} + X_+(\zeta;d)\big), \tag{4.10}$$

where $Y(\zeta;d)$ is again defined by

$$Y(\zeta;d) = V_{+d}R_{-d}(\zeta)V_{-d}R_{+d}(\zeta)j_{+d} : L^2(B_{+d}) \to L^2(B_{+d})$$

as in Lemma 4.5. If one is not the eigenvalue of $Y(\zeta;d)$ at $\zeta = \zeta_0(d) \in D_d$, then the resolvent $R_d(\zeta) = R(\zeta;H_d)$ turns out to be analytic in a neighborhood of ζ_0 with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. In fact, $R_d(\zeta)$ is represented as

$$R_d(\zeta) = R_{-d}(\zeta) - R_{-d}(\zeta)j_{+d}\big(\mathrm{Id} + V_{+d}R_{-d}(\zeta)j_{+d}\big)^{-1}V_{+d}R_{-d}(\zeta). \tag{4.11}$$

Thus the problem is reduced to specifying $\zeta \in D_d$ at which $Y(\zeta;d)$ has the eigenvalue one and to showing that this point is really the pole of $R_d(\zeta)$ in D_d. This is done in the following two steps.

(2) We begin by specifying $\zeta \in D_d$ at which $Y(\zeta;d)$ has the eigenvalue one. Lemma 4.5 enables us to write $\mathrm{Id} - Y(\zeta;d)$ as

$$\mathrm{Id} - Y(\zeta;d) = \big(\mathrm{Id} - \tilde{Y}(\zeta;d)\big)\big(\mathrm{Id} - Y_1(\zeta;d)\big) : L^2(B_{+d}) \to L^2(B_{+d}), \tag{4.12}$$

where

$$\tilde{Y}(\zeta;d) = Y_0(\zeta;d)\big(\mathrm{Id} - Y_1(\zeta;d)\big)^{-1} = Y_0(\zeta;d)\big(\mathrm{Id} + \tilde{Y}_1(\zeta;d)\big)$$

with $\tilde{Y}_1(\zeta;d) = Y_1(\zeta;d)(\mathrm{Id} - Y_1(\zeta;d))^{-1}$. We compute the integral

$$c_0(\zeta)\int V_+(x_+)\varphi_0(x_+;\hat{d},\zeta)\overline{\psi}_+(x_+;-\hat{d},\overline{\zeta})\,dx = -f_+(\hat{d} \to -\hat{d};\zeta)$$

as in the proof of Lemma 4.5, and we set

$$\begin{aligned}\tilde{h}(\zeta;d) &= -c_0(\zeta)(e^{2ik|d|}/|d|)\cos^2(\alpha\pi)f_-(-\hat{d} \to \hat{d};\zeta)\\ &\qquad \times \big(\tilde{Y}_1(\zeta;d)V_{+d}\varphi_0(\cdot - d_+;\hat{d},\zeta), j_{+d}\psi_+(\cdot - d_+;-\hat{d},\overline{\zeta})\big).\end{aligned}$$

It follows from Lemma 4.5 that $\tilde{h}(\zeta;d)$ obeys $|\tilde{h}(\zeta;d)| = O(|d|^{-\mu})$ uniformly in $\zeta \in D_d$ for some $\mu > 0$. The operator $\tilde{Y}(\zeta;d)$ of rank one has $h(\zeta;d) + \tilde{h}(\zeta;d)$ as the only nonzero eigenvalue, where $h(\zeta;d) = (e^{2ik|d|}/|d|)e_0(\zeta)$ is defined by (1.9) with

$$e_0(\zeta) = \cos^2(\alpha\pi)f_+(\hat{d} \to -\hat{d};\zeta)f_-(-\hat{d} \to \hat{d};\zeta). \tag{4.13}$$

We accept the lemma below as proved, and its proof is done in the last step.

LEMMA 4.6

Let $h(\zeta;d)$ be as above. Then the equation $h(\zeta;d) = 1$ has a finite number of solutions

$$\{\zeta_j(d)\}_{1\le j\le N_d}, \quad \zeta_j(d) \in D_d, \mathrm{Re}\,\zeta_1(d) < \cdots < \mathrm{Re}\,\zeta_{N_d}(d),$$

and each solution $\zeta_j(d)$ has the properties

$$\big|\mathrm{Im}\,\zeta_j(d) + E_0^{1/2}(\log|d|)/|d|\big| < E_0^{-1/2}\delta_0(\log|d|)/|d|, \tag{4.14}$$

$$\big|\mathrm{Re}\big(\zeta_{j+1}(d) - \zeta_j(d)\big) - 2\pi E_0^{1/2}/|d|\big| < 2\pi E_0^{-1/2}\delta_0/|d| \tag{4.15}$$

for $|d| \gg 1$.

We apply Rouché's theorem to the equation

$$h(\zeta;d)+\tilde{h}(\zeta;d)=1 \tag{4.16}$$

over D_d. Let $\{\zeta_j(d)\}$, $1\le j\le N_d$, be as in Lemma 4.6, and let

$$C_{j\varepsilon}=\big\{|\zeta-\zeta_j(d)|=\varepsilon/|d|\big\},\qquad D_{j\varepsilon}=\big\{|\zeta-\zeta_j(d)|<\varepsilon/|d|\big\},$$

for $\varepsilon>0$ fixed arbitrarily but sufficiently small. We may assume $D_{j\varepsilon}\subset D_d$ for $|d|\gg 1$ by expanding D_d slightly, if necessary. Since $h(\zeta_j(d);d)=1$, we have

$$h'\big(\zeta_j(d);d\big)=i\zeta_j(d)^{-1/2}|d|\big(1+O(|d|^{-1})\big),$$

so that $|h'(\zeta_j(d);d)|\ge c_1|d|$ for some $c_1>0$. Hence it follows that $|h(\zeta;d)-1|\ge c_2\varepsilon$ on $C_{j\varepsilon}$ for some $c_2>0$. Thus equation (4.16) has a unique solution

$$\big\{\zeta_{\mathrm{res},j}(d)\big\}_{1\le j\le N_d},\quad \operatorname{Re}\zeta_{\mathrm{res},1}(d)<\cdots<\operatorname{Re}\zeta_{\mathrm{res},N_d}(d),$$

in $D_{j\varepsilon}$ for $|d|\gg 1$.

(3) We shall show that $\zeta_{\mathrm{res},j}(d)$ becomes the pole of $R_d(\zeta)$. For brevity, we fix one of $\zeta_{\mathrm{res},j}(d)$, $1\le j\le N_d$, and denote it by $\zeta_0(d)$. We restrict ourselves to the neighborhood

$$D_{0\delta}=\big\{|\zeta-\zeta_0(d)|<\delta/|d|\big\}\subset D_d,\quad 0<\delta\ll 1,$$

of $\zeta_0(d)$. If we set $h_0(\zeta;d)=h(\zeta;d)+\tilde{h}(\zeta;d)$, then $1-h_0(\zeta;d)$ admits the decomposition

$$1-h_0(\zeta;d)=\big(\zeta-\zeta_0(d)\big)h_1(\zeta;d), \tag{4.17}$$

where $h_1(\zeta;d)$ is analytic in $D_{0\delta}$ and never vanishes there.

Now we work in the space $L^2(B_{+d})$. We denote by $\langle\,,\,\rangle$ the L^2 scalar product in $L^2(B_{+d})$ and write $u\otimes v$ for the integral operator with the kernel defined by $u(x)\overline{v}(y)$ with u and v in $L^2(B_{+d})$. We combine (4.10) with (4.12) to obtain

$$\mathrm{Id}+V_{+d}R_{-d}(\zeta)j_{+d}=\big(\mathrm{Id}-\tilde{Y}(\zeta;d)\big)\big(\mathrm{Id}-Y_1(\zeta;d)\big)\big(\mathrm{Id}+X_+(\zeta;d)\big),$$

where $\tilde{Y}(\zeta;d)=Y_0(\zeta;d)(\mathrm{Id}+\tilde{Y}_1(\zeta;d))$ is an integral operator of rank one. We consider the inverse $(\mathrm{Id}+V_{+d}R_{-d}(\zeta)j_{+d})^{-1}$ on the right-hand side of relation (4.11). If we define $u_0(x;\zeta,d)$ by

$$u_0=-c_0(\zeta)(e^{2ik|d|}/|d|)\cos^2(\alpha\pi)f_-(-\hat{d}\to\hat{d};\zeta)V_{+d}(x)\varphi_0(x-d_+;\hat{d},\zeta)$$

and $v_0(x;\zeta,d)$ by

$$v_0=\big[\big(\mathrm{Id}+\tilde{Y}_1(\zeta;d)^*\big)j_{+d}(\cdot)\psi_+(\cdot-d_+;-\hat{d},\overline{\zeta})\big](x),$$

then $\tilde{Y}(\zeta;d)=u_0\otimes v_0$ with $\langle u_0,v_0\rangle=h_0(\zeta;d)$, and the inverse $(\mathrm{Id}-\tilde{Y}(\zeta;d))^{-1}$ is represented as

$$\mathrm{Id}+\big(1-h_0(\zeta;d)\big)^{-1}(u_0\otimes v_0)=\mathrm{Id}+\big(p_1(\zeta;d)/(\zeta-\zeta_0(d))\big)(u_0\otimes v_0)$$

by (4.17), where $p_1(\zeta;d)=1/h_1(\zeta;d)$. If we further define $u_1(x;\zeta,d)$ by

$$u_1=\big(\mathrm{Id}+X_+(\zeta;d)\big)^{-1}\big(\mathrm{Id}-Y_1(\zeta;d)\big)^{-1}u_0,$$

and $v_1(x;\zeta,d)$ by $v_1 = V_{+d}v_0(x;\zeta,d)$, then the operator $(\mathrm{Id} + V_{+d}R_{-d}(\zeta)j_{+d})^{-1}$ under consideration takes the form

$$\big(\mathrm{Id} + V_{+d}R_{-d}(\zeta)j_{+d}\big)^{-1}V_{+d} \sim \big(p_1(\zeta;d)/(\zeta-\zeta_0(d))\big)(u_1 \otimes v_1)$$

as a function with values in bounded operators on $L^2(B_{+d})$, where the analytic terms over $D_{0\delta}$ are neglected.

We analyze the behavior as $|d| \to \infty$ of u_1 and v_1 and show that u_1 and v_1 never vanish identically for $|d| \gg 1$. We write $\psi_{\mathrm{in}}(x;\omega,\overline{\zeta})$ for the incoming eigenfunction $\psi_+(x;\omega,\overline{\zeta})$ defined by (4.4) for the operator K_+ and define the outgoing eigenfunction $\psi_{\mathrm{out}}(x;\omega,\zeta)$ by

$$\psi_{\mathrm{out}}(x;\omega,\zeta) = \big[\big(\mathrm{Id} - G_+(\zeta)V_+\big)\varphi_0(\cdot;\omega,\zeta)\big](x).$$

Since $\|Y_1(\zeta;d)\| + \|\tilde{Y}_1(\zeta;d)\| = O(|d|^{-\mu})$ for some $\mu > 0$ as bounded operators on $L^2(B_{+d})$, it is easy to see that $v_1(x;\zeta,d)$ behaves like

$$v_1 = V_+(x_+)\psi_{\mathrm{in}}(x_+;-\hat{d},\overline{\zeta}) + o_2(1), \quad |d| \to \infty,$$

uniformly in $\zeta \in D_{0\delta}$, where $x_+ = x - d_+$ and $o_2(1)$ denotes remainder terms obeying the bound $o(1)$ as $|d| \to \infty$ of their L^2-norms in $L^2(B_{+d})$. We look at the behavior of u_1. Recall the representation for $\mathrm{Id} + X_+(\zeta;d)$ from Lemma 4.1. The function g_+ defined by (4.1) satisfies

$$|e^{\pm ig_+} - e^{\pm i\alpha\pi}| = O(|d|^{-1}) \tag{4.18}$$

on B_{+d}. Hence it follows from (4.6) and (4.8) that $u_1(x;\zeta,d)$ behaves like

$$u_1 = -c_0(\zeta)\Big(\frac{e^{2ik|d|}}{|d|}\Big)\cos^2(\alpha\pi)f_-(-\hat{d} \to \hat{d};\zeta)\big(V_+(x_+)\psi_{\mathrm{out}}(x_+;\hat{d},\zeta) + o_2(1)\big)$$

uniformly in $\zeta \in D_{0\delta}$. By assumption, $f_+(\hat{d} \to -\hat{d};\zeta)$ does not vanish for $\zeta \in D_{0\delta}$ with $|d| \gg 1$. This implies that u_1 and v_1 never vanish identically.

We now define $w_{\mathrm{out}}(x;\hat{d},\zeta,d)$ and $w_{\mathrm{in}}(x;-\hat{d},\overline{\zeta},d)$ by

$$w_{\mathrm{out}} = [R_{-d}(\zeta)j_{+d}u_1(\cdot;\hat{d},\zeta,d)](x), \qquad w_{\mathrm{in}} = [R_{-d}(\zeta)^*j_{+d}v_1(\cdot;-\hat{d},\overline{\zeta},d)](x).$$

Then both w_{out} and w_{in} are in $L^2_{\mathrm{loc}}(\Omega)$. Moreover, these functions fulfill the zero Dirichlet boundary conditions on the boundary $\partial\Omega$ and never vanish identically over Ω for $\zeta \in D_{0\delta}$. By (4.11), we obtain that the resolvent $R_d(\zeta) = R(\zeta;H_d)$ in question behaves like

$$R_d(\zeta) \sim -\big(p_1(\zeta;d)/(\zeta-\zeta_0(d))\big)(w_{\mathrm{out}} \otimes w_{\mathrm{in}}) \tag{4.19}$$

in the neighborhood $D_{0\delta}$ as an operator from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. This shows that $\zeta_0(d)$ becomes the resonance of H_d.

(4) We complete the proof of the theorem by proving Lemma 4.6.

Proof of Lemma 4.6

Let $e_0(\zeta)$ be defined by (4.13), and let

$$\eta_{0d} = E_0^{1/2}(1 + 2\delta_0/E_0)(\log|d|)/|d|$$

be as in (1.8). We write $\zeta = E + i\eta \in D_d$ with $\operatorname{Im}\zeta = \eta \le 0$. Then $|E - E_0| < \delta_0$ and $-\eta_{0d} < \eta \le 0$. We first fix E and consider the function

$$\eta \mapsto |h(E + i\eta; d)| = \Big(\frac{e^{-2|d|\operatorname{Im} k}}{|d|}\Big)|e_0(E + i\eta)|$$

with $k = \zeta^{1/2} = (E + i\eta)^{1/2}$. The function takes the values $|h(E + i\eta; d)| \ll 1$ at $\eta = 0$ and $|h(E + i\eta; d)| \gg 1$ at $\eta = -\eta_{0d}$ (see (1.12)), and it follows from (1.10) that it behaves like

$$|h(E + i\eta; d)| = \Big(\frac{e^{-|d|(\eta/E^{1/2})}}{|d|}\Big)|e_0(E + i\eta)|\big(1 + O(|\eta|^3)\big). \tag{4.20}$$

Hence the function is strictly decreasing over the interval $[-\eta_{0d}, 0]$. Thus there exists a unique value $\eta = \eta(E; d)$ at which $|h(E + i\eta(E; d); d)| = 1$.

Next we write $h(E + i\eta(E; d); d) = \exp(i\theta(E; d))$ and consider the function $E \mapsto \theta(E; d)$ over the interval $(E_0 - \delta_0, E_0 + \delta_0)$. This function takes the form

$$\theta(E; d) = 2|d| \operatorname{Re}\big(E + i\eta(E; d)\big)^{1/2} + \theta_0(E; d),$$

where $\theta_0(E; d)$, $|\theta_0| < 2\pi$, is defined through the relation

$$e_0\big(E + i\eta(E; d)\big) = \big|e_0\big(E + i\eta(E; d)\big)\big| \exp\big(i\theta_0(E; d)\big).$$

Since

$$\operatorname{Re} k = \operatorname{Re}(E + i\eta)^{1/2} = E^{1/2} + O(\eta^2), \tag{4.21}$$

the function $\theta(E; d)$ behaves like

$$\theta(E; d) = 2E^{1/2}|d| + \theta_0(E; d) + O\big(\eta(E; d)^2\big)|d|, \tag{4.22}$$

and the last term on the right-hand side obeys $O(\eta(E; d)^2)|d| = O((\log|d|)^2/|d|)$. This implies that the function is strictly increasing over $(E_0 - \delta_0, E_0 + \delta_0)$, and hence there exist a finite number of solutions

$$\big\{E_j(d)\big\}_{1\le j\le N_d}, \quad E_1(d) < \cdots < E_{N_d}(d),$$

which satisfy $\theta(E_j(d); d) = 2(m_d + j)\pi$ for each j, where m_d is some integer dependent on d. Thus the solutions to the equation $h(\zeta; d) = 1$ are determined as

$$\zeta_j(d) = E_j(d) + i\eta_j(d), \quad \eta_j(d) = \eta\big(E_j(d); d\big), 1 \le j \le N_d.$$

We shall show that $\zeta_j(d)$ has properties (4.14) and (4.15). For brevity, we skip the reference to d to write $\zeta_j = E_j + i\eta_j$, if there is no ambiguity. Since $|h(E_j + i\eta_j; d)| = 1$ and $\eta_j = O((\log|d|)/|d|)$, we have

$$-\eta_j/E_j^{1/2} = |d|^{-1}\big(\log|d| - \log|e_0(\zeta_j)|\big)\big(1 + O(((\log|d|)/|d|)^3)\big)$$

by (4.20). This, together with (1.11), implies that $\eta_j = \operatorname{Im}\zeta_j$ has property (4.14). By (4.22), we have $E_{j+1} - E_j = O(|d|^{-1})$, so that $|\zeta_{j+1} - \zeta_j| = O((\log|d|)/|d|)$ and

$$|e_0(\zeta_{j+1}) - e_0(\zeta_j)| = O\big((\log|d|)/|d|\big).$$

If we set $k_j = \zeta_j^{1/2}$, then it follows that

$$\exp\bigl(2i|d|\,\mathrm{Re}(k_{j+1}-k_j)\bigr) = \Bigl(\frac{|e_0(\zeta_{j+1})|}{e_0(\zeta_{j+1})}\Bigr)\Bigl(\frac{e_0(\zeta_j)}{|e_0(\zeta_j)|}\Bigr) = \bigl(1+O((\log|d|)/|d|)\bigr).$$

This yields

$$\mathrm{Re}(k_{j+1}-k_j) = |d|^{-1}\bigl(\pi + O((\log|d|)/|d|)\bigr).$$

If we write

$$\mathrm{Re}(\zeta_{j+1}-\zeta_j) = E_{j+1}-E_j = (E_{j+1}^{1/2}+E_j^{1/2})(E_{j+1}^{1/2}-E_j^{1/2}),$$

then (4.15) is obtained as a consequence of (1.11) and (4.21). □

The proof of the theorem is now complete. □

Here we make a comment on relation (4.19). We write $w_{\mathrm{out},j}$ and $w_{\mathrm{in},j}$ for w_{out} and w_{in} with $\zeta = \zeta_{\mathrm{res},j}(d)$. By construction, it is not difficult to see that $w_{\mathrm{out},j}$ is the outgoing resonant state associated with the resonance $\zeta_{\mathrm{res},j}(d)$, which solves the equation $(H_d - \zeta_{\mathrm{res},j}(d))w_{\mathrm{out},j} = 0$. However, it is not easy to see directly from the above definition of w_{in} that $w_{\mathrm{in},j}$ is the incoming resonant state which solves $(H_d - \overline{\zeta}_{\mathrm{res},j}(d))w_{\mathrm{in},j} = 0$. To see this, we take the adjoint of the both sides of (4.19) to obtain

$$R_d(\zeta)^* \sim -\bigl(\overline{p}_1(\zeta;d)/(\overline{\zeta}-\overline{\zeta}_0(d))\bigr)(w_{\mathrm{in}}\otimes w_{\mathrm{out}}).$$

On the other hand, we can construct a relation similar to (4.11) for $R_d(\zeta)^*$ and get the relation

$$R_d(\zeta)^* \sim -\bigl(q_1(\overline{\zeta};d)/(\overline{\zeta}-\overline{\zeta}_0(d))\bigr)(\tilde{w}_{\mathrm{in}}\otimes \tilde{w}_{\mathrm{out}}),$$

where $\tilde{w}_{\mathrm{in}}$ solves $(H_d - \overline{\zeta})\tilde{w}_{\mathrm{in}} = 0$ at $\zeta = \zeta_0(d)$. Thus this shows that $w_{\mathrm{in},j}$ becomes the incoming resonant state associated with $\zeta = \zeta_{\mathrm{res},j}(d)$.

Proof of Theorem 1.2
(1) Since $0<\kappa<1/2$ by assumption, $R_{-d}(\zeta) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega)$ is well defined for $\zeta \in D_{+d} \subset D_{-d}$ (see the argument after the proof of Lemma 4.3). We again start with relation (4.9). By the resolvent identity, the operator $\mathrm{Id} + V_{+d}R_{-d}(\zeta)$ on the right-hand side of (4.9) fulfills the relation

$$\mathrm{Id} + V_{+d}R_{-d}(\zeta)j_{+d} = \mathrm{Id} + V_{+d}R(\zeta)j_{+d} - Z(\zeta;d) = \mathrm{Id} + X_+(\zeta;d) - Z(\zeta;d)$$

as an operator acting on $L^2(B_{+d})$, where $Z(\zeta;d) = V_{+d}R_{-d}(\zeta)V_{-d}R(\zeta)j_{+d}$. We again use the resolvent identity to represent $Z(\zeta;d)$ as

$$Z(\zeta;d) = V_{+d}R(\zeta)V_{-d}R(\zeta)j_{+d} - V_{+d}R(\zeta)V_{-d}R_{-d}(\zeta)V_{-d}R(\zeta)j_{+d}.$$

Since α is a half integer, Lemma 4.4 and Proposition 3.1 with $\mu = 0$ and $\lambda = |d|$ enable us to take $\delta_0 > 0$ so small that

$$\|Z(\zeta;d)\| = O\bigl(\bigl|e^{ik|d|}/|d|\bigr|^2\bigr) = O\bigl(\bigl|e^{2ik|d_+|}/|d_+|\bigr|^2\bigr)O(|e^{2ik(2\kappa-1)|d|}|) = O(|d|^{-c})$$

for some $c > 0$ as an operator on $L^2(B_{+d})$. Lemma 4.1 gives the approximate form of $\mathrm{Id} + X_+(\zeta; d)$. We compute the integral

$$c_0(\zeta) \int V_+(x_+)\varphi_0(x_+; \hat{d}, \zeta)\overline{\psi}_+(x_+; -\hat{d}, \overline{\zeta})\, dx = -f_+(\hat{d} \to -\hat{d}; \zeta).$$

This implies that the resonance of H_d is approximately determined as the solution to the equation

$$h_+(\zeta; d) = (e^{2ik|d_+|}/|d_+|) f_0(-\hat{d} \to \hat{d}; \zeta) f_+(\hat{d} \to -\hat{d}; \zeta) = 1.$$

Thus statement (1) is verified by repeating almost the same argument as that in the proof of Theorem 1.1.

(2) The second statement is verified in exactly the same way as the first one. If we start by the relation

$$(H_d - \zeta)R_{+d}(\zeta) = \mathrm{Id} + V_{-d}R_{+d}(\zeta)$$

instead of (4.9), then the argument proceeds in the same way as above. We skip the details for the proof of statement (2).

(3) We deal with the third case, $\kappa = 1/2$ (and hence $|d_\pm| = |d|/2$). A similar result has been already established by [4, Theorem 1.3(3)]. It suffices to prove the statement for ζ such that $|\operatorname{Re}\zeta - E| < \delta_0$ for some $E \in (E_0/2, 3E_0/2)$. We denote by $D_{\pm d}(E)$ the neighborhood defined by (1.15) with E_0 replaced by E. Note that $D_{-d}(E) = D_{+d}(E)$ for $\kappa = 1/2$. Then Lemma 4.1 remains true for $D_{\pm d}(E)$, and the constant $\mu > 0$ in the lemma can be taken independently of E. If $\zeta = \operatorname{Re}\zeta + i \operatorname{Im}\zeta$ satisfies the assumption, then it follows from (1.10) that (4.7) and (4.8) hold true with $c = \varepsilon/3 > 0$ for $|d| \gg 1$. Hence the operator $\mathrm{Id} + X_-(\zeta; d)$ has the inverse

$$\big(\mathrm{Id} + X_-(\zeta; d)\big)^{-1} : L^2(B_{-d}) \to L^2(B_{-d})$$

bounded uniformly in ζ, and $R_{-d}(\zeta)$ is analytic in ζ as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$ (see the argument after the proof of Lemma 4.3). Lemma 4.4 remains true for ζ as in statement (3) even in the case where $\kappa = 1/2$. This allows us to repeat the same argument as in the proof of statement (1) to obtain

$$\mathrm{Id} + V_{+d}R_{-d}(\zeta)j_{+d} = \mathrm{Id} + X_+(\zeta; d) - Z(\zeta; d)$$

on $L^2(B_{+d})$, where $Z(\zeta; d)$ is again defined by

$$Z(\zeta; d) = V_{+d}R(\zeta)V_{-d}R(\zeta)j_{+d} - V_{+d}R(\zeta)V_{-d}R_{-d}(\zeta)V_{-d}R(\zeta)j_{+d}.$$

When $\kappa = 1/2$, $\mathrm{Id} + X_+(\zeta; d)$ is also invertible with the inverse bounded uniformly in ζ, and $Z(\zeta; d)$ obeys

$$\|Z(\zeta; d)\| = O\big(\big|e^{ik|d|}/|d|\big|^2\big) = O(|d|^{-\varepsilon/2})$$

by (1.10) and by assumption. Thus there exists the inverse

$$\big(\mathrm{Id} + V_{+d}R_{-d}(\zeta)j_{+d}\big)^{-1} : L^2(B_{+d}) \to L^2(B_{+d})$$

bounded uniformly in ζ, and we see that

$$R_d(\zeta) = R_{-d}(\zeta) - R_{-d}(\zeta)j_{+d}\big(\mathrm{Id} + V_{+d}R_{-d}(\zeta)j_{+d}\big)^{-1}V_{+d}R_{-d}(\zeta)$$

is well defined for ζ as in statement (3) as an analytic function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. This proves statement (3), and the proof of the theorem is complete. □

5. Proofs of Lemmas 4.1 and 4.2

The present section is devoted to proving Lemmas 4.1 and 4.2, which have played important roles in proving the main theorems.

Proof of Lemma 4.1
We prove only the first statement, and a similar argument applies to the second one. We first fix the new notation. For notational brevity, we write g for g_+ with property (4.2) and introduce the following auxiliary operators:

$$\tilde{K}_0 = H(\nabla g, 0), \qquad \tilde{K}_+ = H(\nabla g, V_+) = \tilde{K}_0 + V_+, \qquad \tilde{K}_{+d} = \tilde{K}_0 + V_{+d},$$

with the same domain $H^2(\boldsymbol{R}^2)$. We further write $\tilde{G}_0(\zeta), \tilde{G}_+(\zeta)$, and $\tilde{G}_{+d}(\zeta)$ for the resolvents $R(\zeta;\tilde{K}_0), R(\zeta;\tilde{K}_+)$, and $R(\zeta;\tilde{K}_{+d})$, respectively. We also recall the notation $G_+(\zeta) = R(\zeta;K_+)$ and $G_{+d}(\zeta) = R(\zeta;K_{+d})$ from (1.6) and (4.3), respectively. By definition, $\tilde{G}_0(\zeta)$ satisfies the relation

$$\tilde{G}_0(\zeta) = e^{ig}G_0(\zeta)e^{-ig}, \tag{5.1}$$

and similarly for $\tilde{G}_+(\zeta)$ and $\tilde{G}_{+d}(\zeta)$. The proof is rather long and is divided into six steps.

(1) Let $w_d(x)$ be defined by

$$w_d(x) = \chi(4|x - d_+|/|d_+|) = \chi(4|x_+|/|d_+|),$$

where χ is the smooth cutoff function with properties (3.1). Then $\nabla g = A$ on the support of w_d, so that $\tilde{K}_0 = H = H(A,0)$ there. We compute

$$\begin{aligned}\mathrm{Id} + X_+(\zeta;d) &= \mathrm{Id} + V_{+d}\tilde{G}_0(\zeta)j_{+d} + V_{+d}R(\zeta)j_{+d} - V_{+d}\tilde{G}_0(\zeta)j_{+d}\\ &= \mathrm{Id} + V_{+d}\tilde{G}_0(\zeta)j_{+d} + V_{+d}R(\zeta)(w_d\tilde{K}_0 - Hw_d)\tilde{G}_0(\zeta)j_{+d}\\ &= \mathrm{Id} + V_{+d}\tilde{G}_0(\zeta)j_{+d} + V_{+d}R(\zeta)[w_d,\tilde{K}_0]\tilde{G}_0(\zeta)j_{+d}.\end{aligned}$$

Since

$$\tilde{G}_{+d}(\zeta)j_{+d} = \tilde{G}_0(\zeta)j_{+d}\big(\mathrm{Id} + V_{+d}\tilde{G}_0(\zeta)j_{+d}\big)^{-1}$$

on $L^2(B_{+d})$, we obtain the representation

$$\begin{aligned}\mathrm{Id} + X_+(\zeta;d) &= \big(\mathrm{Id} + V_{+d}R(\zeta)[w_d,\tilde{K}_0]\tilde{G}_{+d}(\zeta)j_{+d}\big)\big(\mathrm{Id} + V_{+d}\tilde{G}_0(\zeta)j_{+d}\big)\\ &= e^{ig}\big(\mathrm{Id} + e^{-ig}V_{+d}R(\zeta)e^{ig}[w_d,K_0]G_{+d}(\zeta)j_{+d}\big)\\ &\qquad\times\big(\mathrm{Id} + V_{+d}G_0(\zeta)j_{+d}\big)e^{-ig}.\end{aligned}$$

Thus the problem is reduced to analyzing the asymptotic behavior as $|d| \to \infty$ of the kernel $\Pi(x, y; \zeta, d)$ of the operator $\Pi(\zeta; d)$ defined by

$$\Pi(\zeta; d) = V_{+d} e^{-ig} R(\zeta) e^{ig} [w_d, K_0] G_{+d}(\zeta) j_{+d}. \tag{5.2}$$

(2) Let $\Sigma_{+d} = \{x : |d_+|/4 < |x_+| < |d_+|/2\}$ with $x_+ = x - d_+$. Then we have

$$\operatorname{supp} \nabla w_d \subset \Sigma_{+d}.$$

We study the behavior of the kernel $G_{+d}(x, y; \zeta)$ of $G_{+d}(\zeta)$. Let $G_+(x, y; \zeta)$ be the kernel of $G_+(\zeta)$. Then $G_{+d}(x, y; \zeta) = G_+(x_+, y_+; \zeta)$. By the resolvent identity, we have

$$G_+(\zeta) = G_0(\zeta)\big(\mathrm{Id} - V_+ G_+(\zeta)\big). \tag{5.3}$$

The kernel $G_0(x, y; \zeta)$ of $G_0(\zeta) = R(\zeta; K_0)$ is given by

$$G_0(x, y; \zeta) = (i/4) H_0(k|x - y|)$$

and behaves like

$$G_0(x, y; \zeta) = c_0(\zeta) e^{ik|x-y|} |x - y|^{-1/2} \big(1 + O(|x - y|^{-1})\big)$$

when $|x - y| \gg 1$. If $\xi \in \Sigma_{+d}$ and $y \in B_{+d}$, then

$$|\xi_+ - y_+| = |\xi_+| - y_+ \cdot \hat{\xi}_+ + O(|d|^{-1}),$$

and hence

$$e^{ik|\xi_+ - y_+|} = e^{ik|\xi_+|} \big(\overline{\varphi}_0(y_+; \hat{\xi}_+, \overline{\zeta}) + O(|d|^{-1})\big).$$

Thus $G_0(\xi_+, y_+; \zeta)$ behaves like

$$G_0(\xi_+, y_+; \zeta) = c_0(\zeta) e^{ik|\xi_+|} |\xi_+|^{-1/2} \big(\overline{\varphi}_0(y_+; \hat{\xi}_+, \overline{\zeta}) + r(\xi_+, y_+; \zeta)\big),$$

where the remainder term $r(\xi_+, y_+; \zeta)$ obeys

$$|\partial_\xi^n r(\xi_+, y_+; \zeta)| = O(|d|^{-1-|n|}) \tag{5.4}$$

uniformly in ξ, y, and $\zeta \in D_d$. Since

$$\psi_+(x; \hat{\xi}_+, \overline{\zeta}) = \big[\big(\mathrm{Id} - G_+(\zeta)^* V_+\big) \varphi_0(\cdot; \hat{\xi}_+, \overline{\zeta})\big](x)$$

by definition, it follows from (5.3) that the kernel $G_{+d}(\xi, y; \zeta)$ under consideration takes the asymptotic form

$$G_{+d}(\xi, y; \zeta) = c_0(\zeta) e^{ik|\xi_+|} |\xi_+|^{-1/2} \big(\overline{\psi}_+(y_+; \hat{\xi}_+, \overline{\zeta}) + r_0(\xi_+, y_+; \zeta)\big), \tag{5.5}$$

where

$$r_0(\xi_+, y_+; \zeta) = r(\xi_+, y_+; \zeta) - \int r(\xi_+, z; \zeta) V_+(z) G_+(z, y_+; \zeta)\, dz$$

is analytic in $\zeta \in D_d$ and obeys the same bound as in (5.4).

Next we look at the behavior of the kernel of $e^{-ig} R(\zeta) e^{ig}$. We note that (4.18) holds on B_{+d}. Hence it follows from Proposition 3.2 with $\lambda = |d|$ that

$$
\begin{aligned}
&e^{-ig(x)}R(x,\xi;\zeta)e^{ig(\xi)} \\
&\quad \sim (i/4)e^{-ig(x)+i\alpha(\gamma(\hat{x};-\hat{\xi})-\pi)}H_0(k|x-\xi|)e^{ig(\xi)} \\
&\qquad + c_0(\zeta)e^{ik(|x|+|\xi|)}(|x||\xi|)^{-1/2}\big(e^{-i\alpha\pi}f_0(-\hat{\xi}\to\hat{x};\zeta)e^{ig(\xi)}+r_1(x,\xi;\zeta,d)\big)
\end{aligned}
$$

for $x\in B_{+d}$ and for $\xi\in\Sigma_{+d}$, where the remainder term of order $O(|d|^{-N})$ is neglected, and $r_1(x,\xi;\zeta,d)$ is analytic in D_d and obeys the same bound as in (2.12).

(3) We consider the function $[w_d,K_0]G_{+d}(\xi,y;\zeta)$. We compute the commutator

$$
[w_d,K_0]=w_dK_0-K_0w_d=2\nabla w_d\cdot\nabla+(\Delta w_d)=2\nabla w_d\cdot\nabla+O(|d|^{-2}).
$$

Since $\nabla_\xi\psi_+=O(|\xi_+|^{-1})=(|d|^{-1})$ on Σ_{+d} and

$$
\nabla w_d(\xi)=(4/|d_+|)\chi'(4|\xi_+|/|d_+|)\hat{\xi}_+,
$$

the function $[w_d,K_0]G_{+d}(\xi,y;\zeta)$ takes the form

$$
c_0(\zeta)e^{ik|\xi_+|}|\xi_+|^{-1/2}|d_+|^{-1}\big(8ik\chi'(4|\xi_+|/|d_+|)\overline{\psi}_+(y_+;\hat{\xi}_+,\overline{\zeta})+\tilde{r}_0(\xi_+,y_+;\zeta)\big)
$$

by (5.5), where $\tilde{r}_0(\xi_+,y_+;\zeta)$ preserves properties similar to those of $r_0(\xi_+,y_+;\zeta)$ in (5.5). If $x\in B_{+d}$ and $\xi\in\Sigma_{+d}$, then

$$
\big|\nabla_\xi(|x-\xi|+|\xi_+|)\big|=\big|\nabla_\xi(|x-\xi|+|\xi-d_+|)\big|\ge c>0
$$

for some c independent of d. We note that $|e^{ik|\xi_+|}|$ and $|e^{ik|x-\xi|}|$ are at most of polynomial growth in $|d|$, because $|\operatorname{Im}k|=|\operatorname{Im}\zeta^{1/2}|=O((\log|d|)/|d|)$ for $\zeta\in D_d$. Hence it follows by repeated use of the integration by parts that

$$
\int e^{i\alpha\gamma(\hat{x};-\hat{\xi})}H_0(k|x-\xi|)e^{ig(\xi)}[w_d,K_0]G_{+d}(\xi,y;\zeta)\,d\xi=O(|d|^{-N}).
$$

Thus the leading term $X_{+0}(x,y;\zeta,d)$ in the lemma comes from the integral

$$
I_0(x,y;\zeta,d)=\nu(d)e^{ik|x|}|x|^{-1/2}\int e^{ik(|\xi|+|\xi_+|)}\chi'(4|\xi_+|/|d_+|)J(\xi,x,y;\zeta,d)\,d\xi
$$

with $\nu=8ikc_0(\zeta)^2e^{-i\alpha\pi}|d_+|^{-1}$, where

$$
J(\xi,x,y;\zeta,d)=|\xi|^{-1/2}|\xi_+|^{-1/2}f_0(-\hat{\xi}\to\hat{x};\zeta)e^{ig(\xi)}\overline{\psi}_+(y_+;\hat{\xi}_+,\overline{\zeta}).
$$

By (2.11), we see that

$$
\nu(d)=8i\zeta^{1/2}(8\pi)^{-1}e^{i\pi/2}\zeta^{-1/2}e^{-i\alpha\pi}|d_+|^{-1}=(-1/\pi)e^{-i\alpha\pi}|d_+|^{-1}
$$

is independent of ζ.

(4) We work in the coordinates

$$
\xi=\xi_++d_+=|d_+|t(\cos\theta,\sin\theta)+d_+,\quad \theta=\gamma(\hat{\xi}_+;-\hat{d}),
$$

with d_+ as the center to see the asymptotic behavior of the integral I_0 above. Then we have

$$
d\xi=|d_+|^2t\,dt\,d\theta,\quad |\xi|=|d_+|(1+t^2-2t\cos\theta)^{1/2}.
$$

Hence $I_0 = I_0(x, y; \zeta, d)$ takes the form

$$I_0 = \nu_0 e^{ik|x|}|x|^{-1/2} \int_0^\infty \chi'(4t)t^{1/2}\Big\{\int e^{i|d_+|k\varphi(t,\theta)} J_0(t,\theta,x,y;\zeta)\,d\theta\Big\}\,dt$$

with $\nu_0 = (-1/\pi)e^{-i\alpha\pi}$, where $\varphi(t,\theta) = t + (1+t^2-2t\cos\theta)^{1/2}$ and

$$J_0 = (1+t^2-2t\cos\theta)^{-1/2} f_0(-\hat{\xi} \to \hat{x}; \zeta) e^{ig(\xi)}\overline{\psi}_+(y_+; \hat{\xi}_+, \overline{\zeta})$$

with $\xi = |d_+|t(\cos\theta, \sin\theta) + d_+$. We note that $\chi'(4t)t^{1/2} \in C_0^\infty(0,\infty)$ has support in the interval $(1/4, 1/2)$. The stationary points with $\partial_\theta\varphi = 0$ are attained at $\theta = 0$ and $\theta = \pi$. The function $\varphi(t,\theta)$ takes $\varphi = 1+2t$ at $\theta = \pi$ and satisfies $\partial_t\varphi(t,\pi) = 2 > 0$. This implies that the stationary point $\theta = \pi$ does not make any contribution to the asymptotic form of I_0.

(5) We consider the contribution from the other stationary point $\theta = 0$. The phase function $k\varphi(t,\theta)$ does not necessarily take real values, which does not allow us to apply directly the stationary phase method to the integral $I_0(x,y;\zeta,d)$ to obtain the leading term $X_{+0}(x,y;\zeta,d)$. We decompose $\varphi(t,\theta)$ into $\varphi(t,\theta) = 1 + \tilde{\varphi}(t,\theta)$, where $\tilde{\varphi}(t,\theta)$ behaves like

$$\tilde{\varphi}(t,\theta) = t(1-t)^{-1}(1-\cos\theta) + O(\theta^4) = \big(t(1-t)^{-1}/2\big)\theta^2 + O(\theta^4) \tag{5.6}$$

as $|\theta| \to 0$ uniformly in $t \in [1/4, 1/2]$. The analyticity in θ enables us to deform the interval $|\theta| < 2\delta$, $0 < \delta \ll 1$, into the smooth contour defined by $z = ue^{iL(\log|d|)/|d|}$, $|u| < \delta$, in a complex neighborhood of $z = 0$, where $L \gg 1$ is taken large enough. The contour is deformed in such a way that $\operatorname{Im} z < 0$ or $\operatorname{Im} z > 0$ according to whether $\operatorname{Re} z < 0$ or $\operatorname{Re} z > 0$ with $|\operatorname{Re} z| < 2\delta$, and $|\operatorname{Im} z| = O((\log|d|)/|d|)$. Then $\operatorname{Im}\tilde{\varphi}(t,z) > 0$ for $z \neq 0$, and the leading term is obtained from the integral

$$\begin{aligned} I_1(x,y;\zeta,d) &= \nu_1(d) e^{ik(|x|+|d_+|)}|x|^{-1/2} \\ &\quad\times \int_0^\infty \chi'(4t)t^{1/2}\Big\{\int e^{i|d_+|k\tilde{\varphi}(t,z)}\chi(2|u|/\delta) J_0(t,z,x,y;\zeta)\,du\Big\}\,dt \end{aligned} \tag{5.7}$$

with $z = ue^{iL(\log|d|)/|d|}$ (and hence $dz = e^{iL(\log|d|)/|d|}\,du$), where

$$\nu_1(d) = \nu_0 e^{iL(\log|d|)/|d|} = (-1/\pi)e^{-i\alpha\pi}e^{iL(\log|d|)/|d|}.$$

Since

$$k = \zeta^{1/2} = (E+i\eta)^{1/2} = E^{1/2} + iE^{-1/2}\eta/2 + O(|d|^{-1}), \quad \eta = O\big((\log|d|)/|d|\big),$$

for $\zeta = E + i\eta \in D_d$ and since

$$\operatorname{Im}(kz^2) \sim u^2\big\{(E^{-1/2}\eta/2)\cos\big(2L(\log|d|)/|d|\big) + E^{1/2}\sin\big(2L(\log|d|)/|d|\big)\big\},$$

we can take $L \gg 1$ so large that $\operatorname{Im}(kz^2) > 0$ for $z \neq 0$. Thus we have $\operatorname{Im}(k\tilde{\varphi}(t,0)) = 0$ and $\operatorname{Im}(k\tilde{\varphi}(t,z)) > 0$ for $z = ue^{iL(\log|d|)/|d|} \neq 0$.

(6) The proof is complete in this step. We are now in a position to apply the stationary phase method (see [12, Theorem 7.7.5]) to the integral in the brackets in (5.7). We see the value at $u = 0$ (or at $\theta = 0$) of the function

$$J_0(t, ue^{iL(\log|d|)/|d|}, x, y; \zeta).$$

If $\theta = 0$, then we have

$$f_0(-\hat{\xi} \to \hat{x};\zeta) = f_0(-\hat{d} \to \hat{x};\zeta), \qquad \psi_+(y_+;\hat{\xi}_+,\overline{\zeta}) = \psi_+(y_+;-\hat{d},\overline{\zeta}),$$

and $e^{ig(\xi)} = e^{i\alpha\gamma(\xi;-\hat{d})} = e^{i\alpha\pi}$. Hence it follows that

$$J_0(t,0,x,y;\zeta) = e^{i\alpha\pi}(1-t)^{-1/2} f_0(-\hat{d} \to \hat{x};\zeta)\psi_+(y_+;-\hat{d},\overline{\zeta}).$$

Since

$$|x| = |d_+ + x_+| = |d_+| + \hat{d}\cdot x_+ + O(|d|^{-1})$$

for $x \in B_{+d}$, $e^{ik|x|}$ behaves like

$$e^{ik|x|} = e^{ik|d_+|}\big(\varphi_0(x_+;\hat{d},\zeta) + O(|d|^{-1})\big),$$

and also $|x|^{-1/2} = |d_+|^{-1/2}(1 + O(|d|^{-1}))$. We further note that

$$f_0(-\hat{d} \to \hat{x};\zeta) = f_0(-\hat{d} \to \hat{d};\zeta) + O(|d|^{-1}).$$

We calculate the Hessian of the phase function

$$k\tilde{\varphi}(t,z) = \zeta^{1/2}\tilde{\varphi}(t, ue^{iL(\log|d|)/|d|})$$

at $u = 0$. By (5.6), it equals

$$\zeta^{1/2}\partial_u^2\tilde{\varphi}(t,0) = \zeta^{1/2}e^{2iL(\log|d|)/|d|}t(1-t)^{-1}.$$

We finally take into account the relations

$$\int_0^\infty \chi'(4t)\,dt = (1/4)\int_0^\infty (d/dt)\chi(4t)\,dt = -1/4$$

and $(-1/\pi)(-1/4)(2\pi)^{1/2}e^{i\pi/4}\zeta^{-1/4} = c_0(\zeta)$. Then we combine all the results obtained to see that the integral $I_1 = I_1(x,y;\zeta,d)$ defined by (5.7) behaves like

$$I_1 = c_0(\zeta)(e^{2ik|d_+|}/|d_+|)\big(f_0(-\hat{d} \to \hat{d};\zeta)\varphi_0(x_+;\hat{d},\zeta)\psi_+(y_+;-\hat{d},\overline{\zeta}) + O(|d|^{-1})\big).$$

This yields the desired form of the leading term, and the proof is complete. □

Proof of Lemma 4.2

We give the proof for statement (1) only, and we use the notation with the same meanings ascribed in the proof of Lemma 4.1 throughout the proof. The operator $Y_+(\zeta;d)$ in question is represented as

$$Y_+(\zeta;d) = V_{-d}R(\zeta)j_{+d}\big(\mathrm{Id} + X_+(\zeta;d)\big)^{-1}.$$

We have established the relation

$$\mathrm{Id} + X_+(\zeta;d) = e^{ig}\big(\mathrm{Id} + \Pi(\zeta;d)\big)\big(\mathrm{Id} + V_{+d}G_0(\zeta)j_{+d}\big)e^{-ig} : L^2(B_{+d}) \to L^2(B_{+d})$$

in the proof of Lemma 4.1 (see step (1)), where $\Pi(\zeta;d)$ is defined by (5.2). We have also shown that

$$\|\Pi(\zeta;d)\| = O\big(\big|e^{2ik|d_+|}/|d_+|\big|\big) = O(|d|^{-c}), \quad \zeta \in D_d,$$

for some $c > 0$ (see (1.13)) as a bounded operator on $L^2(B_{+d})$. If we take (4.6) and (4.18) into account, then $Y_+(\zeta;d)$ admits the decomposition

$$Y_+(\zeta;d) = \tilde{Y}_{+0}(\zeta;d) + \tilde{Y}_{+1}(\zeta;d),$$

where

$$\tilde{Y}_{+0}(\zeta;d) = V_{-d}R(\zeta)j_{+d}\big(\mathrm{Id} - V_{+d}G_{+d}(\zeta)j_{+d}\big)$$

and $\tilde{Y}_{+1}(\zeta;d)$ is analytic in D_d and obeys $\|\tilde{Y}_{+1}(\zeta;d)\| = O(|d|^{-\mu})$ for some $\mu > 0$. We now apply Proposition 3.1 with $\mu = 0$ and $\lambda = |d|$ to the operator $V_{-d}R(\zeta)j_{+d}$. We recall the behavior as $|x-y| \to \infty$ of $H_0(k|x-y|)$ from step (2) in the proof of Lemma 4.1. If $x \in B_{-d}$ and $y \in B_{+d}$, then

$$\gamma(\hat{x};\hat{y}) = \pi + O(|d|^{-1}), \quad |x-y| = |d| - \hat{d}\cdot(x_- - y_+) + O(|d|^{-1}),$$

and hence the kernel $R(x,y;\zeta)$ behaves like

$$R = c_0(\zeta)\cos(\alpha\pi)(e^{ik|d|}/|d|^{1/2})\big(\varphi_0(x_-;-\hat{d},\zeta)\overline{\varphi}_0(y_+;-\hat{d},\overline{\zeta}) + O(|d|^{-1/2})\big),$$

where the remainder term $O(|d|^{-1/2})$ is analytic in $\zeta \in D_d$ and is bounded uniformly in x, y, and ζ. Since

$$\psi_+(y_+;-\hat{d},\overline{\zeta}) = \big[\big(\mathrm{Id} - G_{+d}(\zeta)^* V_{+d}\big)\varphi_0(\cdot - d_+;-\hat{d},\overline{\zeta})\big](y_+)$$

by definition, we see that the kernel $Y_{+0}(x,y;\zeta,d)$ of the leading operator $Y_{+0}(\zeta;d)$ obtained from $\tilde{Y}_{+0}(\zeta;d)$ takes the desired form. This proves the lemma. □

6. Asymptotic properties of Green functions

In this section we prove Propositions 2.1, 2.2, and 2.3, which remain unproved. Similar results with rather rough remainder estimates have been already established as [4, Propositions 3.1–3.3] under slightly different notation. We give only brief sketches and necessary modifications for the proofs of these propositions.

We begin by making a review on the integral representation for the kernel $R_\alpha(x,y;\zeta)$. We consider only the case $\mathrm{Im}\,\zeta \le 0$ and write $\zeta = E - i\eta$ with $0 \le \eta \le c_1(\log\lambda)/\lambda$. The representation is based on the following formula:

$$H_\nu(Z)J_\nu(z) = \frac{1}{i\pi}\int_0^{\kappa+i\infty} \exp\Big(\frac{t}{2} - \frac{Z^2+z^2}{2t}\Big) I_\nu\Big(\frac{Zz}{t}\Big)\,\frac{dt}{t}, \quad |z| \le |Z|,$$

for the product of Bessel functions (see [18, p. 439]), where $I_\nu(w)$ is defined by (2.6) and the contour is taken to be rectilinear with corner at $\kappa + i0$, $\kappa > 0$ being fixed arbitrarily. We use the notation κ with the meaning ascribed above throughout the section. We apply this formula to (2.9) with $Z = k(|x| \vee |y|)$ and $z = k(|x| \wedge |y|)$, where $k = \zeta^{1/2}$ with $\mathrm{Im}\,k \le 0$. If we write $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ in the polar coordinates, then $R_\alpha(x,y;\zeta)$ is represented as

$$\text{(6.1)} \qquad R_\alpha = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty} \exp\Big(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\Big) I_\nu\Big(\frac{\zeta|x||y|}{t}\Big)\,\frac{dt}{t}$$

with $\nu = |l-\alpha|$, where $\psi = \theta - \omega$. If, in particular, $\alpha = 0$, then the resolvent $R(\zeta;K_0)$ of the free Hamiltonian K_0 has the kernel $(i/4)H_0(k|x-y|)$ represented as

$$\frac{i}{4}H_0(k|x-y|) = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty} \exp\Big(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\Big) I_l\Big(\frac{\zeta|x||y|}{t}\Big)\,\frac{dt}{t},$$

where $I_{|l|}(w) = I_l(w) = (1/\pi)\int_0^\pi e^{w\cos\rho}\cos(l\rho)\,d\rho$ (see (2.6)). By the Fourier expansion, the series $\sum_l e^{il\psi}I_l(w)$ converges to $e^{w\cos\psi}$. Since

$$|x-y|^2 = |x|^2 + |y|^2 - 2|x||y|\cos\psi,$$

the kernel $(i/4)H_0(k|x-y|)$ has the representation

$$\frac{i}{4}H_0(k|x-y|) = \frac{1}{4\pi}\int_0^{\kappa+i\infty} \exp\Big(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\Big)\,\frac{dt}{t}. \tag{6.2}$$

We fix $M \gg 1$ large enough and take

$$\kappa = M^2\log\lambda$$

in the contour of integral (6.1). We divide (6.1) into the sum of integrals over the following four intervals by a smooth partition of unity:

(0) $0 < t < \kappa$, (i) $0 < s < 2\lambda/M$, (ii) $\lambda/M < s < 2M\lambda$, (iii) $s > M\lambda$,

for $t = \kappa + is$. We evaluate the integral over each interval. We have shown in [4] that the main contribution comes from the integral over interval (ii). Indeed, we have

$$\big|\exp\big(-\zeta(|x|^2+|y|^2)/(2t)\big)\big| = t^N O(\lambda^{-2N}), \quad 0 < t < \kappa,$$

for any $N \gg 1$, and also the stationary point of the function

$$t \mapsto t/2 - \zeta(|x|^2+|y|^2)/(2t)$$

in integral (6.1) is away from the intervals (i) and (iii) (see also the proof of Proposition 2.2). If we set

$$\chi_M(s) = \chi(s/M)\big(1-\chi(Ms)\big)$$

for the cutoff function $\chi \in C_0^\infty[0,\infty)$ with properties (3.1), then $\chi_M(s/\lambda)$ has support in $(\lambda/M, 2M\lambda)$ and $\chi_M(s/\lambda) = 1$ on $[2\lambda/M, M\lambda]$. We obtain

$$R_\alpha(x,y;\zeta) = \tilde{R}_\alpha(x,y;\zeta) + O(\lambda^{-N})$$

for any $N \gg 1$, where $\tilde{R}_\alpha(x,y;\zeta)$ is defined by

$$\tilde{R}_\alpha = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty} \chi_M\Big(\frac{\operatorname{Im} t}{\lambda}\Big)\exp\Big(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\Big) I_\nu\Big(\frac{\zeta|x||y|}{t}\Big)\,\frac{dt}{t}.$$

We now use formula (2.6) for $I_\nu(w)$ to calculate the series

$$L(w,\psi) = \sum_l e^{il\psi}I_\nu(w), \quad \nu = |l-\alpha|,$$

in the integrand above, where $\psi = \theta - \omega$ and $w = \zeta|x||y|/t$. Then $L(w,\psi)$ is decomposed into the sum

$$L(w,\psi) = L_{\mathrm{fr}}(w,\psi) + L_{\mathrm{sc}}(w,\psi),$$

where

$$L_{\mathrm{fr}}(w,\psi)=(1/\pi)\sum_l e^{il\psi}\int_0^\pi e^{w\cos\rho}\cos(\nu\rho)\,d\rho,$$

$$L_{\mathrm{sc}}(w,\psi)=-(1/\pi)\sum_l e^{il\psi}\sin(\nu\pi)\int_0^\infty e^{-w\cosh p-\nu p}\,dp.$$

We have $L_{\mathrm{fr}}(w,\psi)=e^{i\alpha\psi}e^{w\cos\psi}$ for $|\psi|<\pi$ by the Fourier expansion and

$$L_{\mathrm{sc}}(w,\psi)=-\frac{\sin(\alpha\pi)}{\pi}(-1)^{[\alpha]}e^{i[\alpha]\psi}\int_{-\infty}^\infty e^{-w\cosh p}\frac{e^{(1-\beta)p}}{e^p+e^{-i\psi}}\,dp \tag{6.3}$$

with $0<\beta=\alpha-[\alpha]<1$ by the same argument that was used to calculate the eigenfunction $\varphi_{\alpha+}$ in Section 2 (see (2.7)). Thus the Green function $R_\alpha(x,y;\zeta)$ admits the decomposition

$$R_\alpha(x,y;\zeta)=R_{\mathrm{fr}}(x,y;\zeta)+R_{\mathrm{sc}}(x,y;\zeta)+O(\lambda^{-N}) \tag{6.4}$$

for any $N\gg 1$, where

$$R_{\mathrm{fr}}=\frac{1}{4\pi}e^{i\alpha\psi}\int_0^{\kappa+i\infty}\chi_M\Big(\frac{\operatorname{Im}t}{\lambda}\Big)\exp\Big(\frac{t}{2}-\frac{\zeta|x-y|^2}{2t}\Big)\frac{dt}{t},$$

$$R_{\mathrm{sc}}=\frac{1}{4\pi}\int_0^{\kappa+i\infty}\chi_M\Big(\frac{\operatorname{Im}t}{\lambda}\Big)\exp\Big(\frac{t}{2}-\frac{\zeta(|x|^2+|y|^2)}{2t}\Big)L_{\mathrm{sc}}\Big(\frac{\zeta|x||y|}{t},\psi\Big)\frac{dt}{t}.$$

We should note that (6.4) is true only for $|\psi|<\pi$. If $\psi=\pm\pi$, then the denominator $e^p+e^{-i\psi}$ in (6.3) vanishes at $p=0$. If α is an integer, then $L_{\mathrm{sc}}(\zeta|x||y|/t,\psi)$ vanishes, and hence so does $R_{\mathrm{sc}}(x,y;\zeta)$.

Proof of Proposition 2.1

The proposition has been proved as [4, Proposition 3.1]. As stated above, decomposition (6.4) holds true only for $|\psi|<\pi$. In particular, the denominator in (6.3) vanishes at $p=0$. The behavior along the forward direction of $R_\alpha(x,y;\zeta)$ comes from this singularity. We skip the detailed proof. □

Proof of Proposition 2.2

In [4, Proposition 3.2], we have used the stationary phase method to obtain the asymptotic formula with e_{2N} obeying the rough remainder estimate

$$|\partial_x^n\partial_y^m e_{2N}|=O\big((\log\lambda)^2\lambda^{-1-|n|-|m|}\big).$$

Here we modify the argument there and use the method of steepest descent to make the remainder estimate sharper as in the proposition.

We first note that

$$\psi=\theta-\omega=\gamma(\hat{x};-\hat{y})-\pi \tag{6.5}$$

for $x=(|x|\cos\theta,|x|\sin\theta)$ and $y=(|y|\cos\omega,|y|\sin\omega)$ in the polar coordinates. Since $\lambda/c<|x-y|<c\lambda$ by assumption, we make repeated use of integration by parts and take (6.2) into account to obtain the relation

$$R_{\rm fr}(x,y;\zeta) = (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) + O(\lambda^{-N}).$$

Thus the first term is obtained.

We look at the behavior of $R_{\rm sc}(x,y;\zeta)$. By assumption, the denominator $e^p + e^{-i\psi}$ in (6.3) does not vanish even at $p=0$. We consider the integral

$$S(x,y,t;\zeta) = \int_{-\infty}^{\infty} e^{i(i\zeta|x||y|/t)(\cosh p-1)} \frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\, dp$$

and apply the stationary phase method (see [12, Theorem 7.7.5]) to this integral. We note that $||x||y|/t| \sim \lambda$ and

$$\operatorname{Im}(i\zeta|x||y|/t)(\cosh p - 1) = |x||y|(\kappa^2+s^2)^{-1}(E\kappa - \eta s)(\cosh p - 1) > 0$$

for $p \neq 0$, because $\kappa = M^2 \log\lambda$ with $M \gg 1$ and $\zeta = E - i\eta$ with $0 \le \eta \le c_1(\log\lambda)/\lambda$. Then we have

$$S(x,y,t;\zeta) = (2\pi)^{1/2}t^{1/2}\zeta^{-1/2}(|x||y|)^{-1/2}\big((1+e^{-i\psi})^{-1} + O(\lambda^{-1})\big),$$

and hence we see that $L_{\rm sc} = L_{\rm sc}(\zeta|x||y|/t, \psi)$ behaves like

$$\begin{aligned} L_{\rm sc} = &-(2/\pi)^{1/2}\sin(\alpha\pi)(-1)^{[\alpha]}e^{i[\alpha]\psi}\zeta^{-1/2} \\ &\times (|x||y|)^{-1/2}e^{-\zeta|x||y|/t}t^{1/2}\big((1+e^{-i\psi})^{-1} + O(\lambda^{-1})\big). \end{aligned}$$

By use of (6.5), we compute

$$(1+e^{-i\psi})^{-1} = (1-e^{-i\gamma(\hat{x};-\hat{y})})^{-1} = e^{i\gamma(\hat{x};-\hat{y})}/(e^{i\gamma(\hat{x};-\hat{y})}-1).$$

Recall the representation for the scattering amplitude $f(\omega\to\theta;E)$ from (2.8). Then it follows that $L_{\rm sc} = L_{\rm sc}(\zeta|x||y|/t, \psi)$ takes the asymptotic form

$$L_{\rm sc} = \zeta^{-1/4}e^{-i\pi/4}(|x||y|)^{-1/2}e^{-\zeta|x||y|/t}t^{1/2}\big(f(-\hat{y}\to\hat{x};\zeta) + O(\lambda^{-1})\big).$$

Since

$$t/2 - \zeta(|x|^2+|y|^2)/2t - \zeta|x||y|/t = t/2 - \zeta(|x|+|y|)^2/2t,$$

we have that $R_{\rm sc} = R_{\rm sc}(x,y;\zeta)$ takes the asymptotic form

$$\begin{aligned} R_{\rm sc} = &(4\pi)^{-1}\zeta^{-1/4}e^{-i\pi/4}(|x||y|)^{-1/2} \\ &\times \int_0^{\kappa+i\infty} \chi_M\Big(\frac{\operatorname{Im} t}{\lambda}\Big)\exp\Big(\frac{t}{2} - \frac{\zeta(|x|+|y|)^2}{2t}\Big)\big(f(-\hat{y}\to\hat{x};\zeta) + O(\lambda^{-1})\big)\frac{dt}{t^{1/2}}. \end{aligned}$$

We calculate only the leading term. A similar argument applies to the remainder term of order $O(\lambda^{-1})$.

We consider the integral

$$S_0(x,y;\zeta) = \int_0^{\kappa+i\infty} \chi_M\Big(\frac{\operatorname{Im} t}{\lambda}\Big)\exp\Big(\frac{t}{2} - \frac{\zeta(|x|+|y|)^2}{2t}\Big)\frac{dt}{t^{1/2}}.$$

We make a change of variable

$$t = \kappa + is = ik(|x|+|y|)\tau = i\zeta^{1/2}(|x|+|y|)\tau \tag{6.6}$$

and set $\sigma = \tau - 1$. Then we have $dt = ik(|x|+|y|)\,d\sigma$ and

$$t/2-\zeta(|x|+|y|)^2/2t = i(|x|+|y|)k(\tau/2+1/2\tau)$$
$$= ik(|x|+|y|)+i(|x|+|y|)k\big(\sigma^2/(2(\sigma+1))\big).$$

The line $t=\kappa+is$ with $\lambda/M < s < 2M\lambda$ is transformed into a certain curve in the complex plane. By (6.6), we have the relation

$$\operatorname{Re}\sigma = |k|^{-2}(|x|+|y|)^{-1}(s\operatorname{Re}k-\kappa\operatorname{Im}k)-1,$$
$$\operatorname{Im}\sigma = |k|^{-2}(|x|+|y|)^{-1}(-s\operatorname{Im}k-\kappa\operatorname{Re}k).$$

Since $k=\zeta^{1/2}$ behaves like

$$k = E^{1/2}-iE^{-1/2}\eta/2+O\big(((\log\lambda)/\lambda)^2\big)$$

for $\zeta = E-i\eta$ with $0\le\eta\le c_1(\log\lambda)/\lambda$, there exists $c_2>0$ such that the curve is contained in the region

$$\big\{\sigma\in\boldsymbol{C} : -1\le\operatorname{Re}\sigma < c_2M, 0>\operatorname{Im}\sigma > -c_2M^2(\log\lambda)/\lambda\big\}$$

for $\lambda\gg 1$. The stationary point $\sigma=0$ is not on the curve. For this reason, we deform the curve into a small real interval around $\sigma=0$ by analyticity. We further deform this interval around $\sigma=0$ into a contour in the complex plane as in step (5) in the proof of Lemma 4.1. Then we see that $S_0(x,y;\zeta)$ takes the asymptotic form

$$S_0 = k^{1/2}e^{i\pi/4}(|x|+|y|)^{1/2}\big(k(|x|+|y|)/2\pi i\big)^{-1/2}e^{ik(|x|+|y|)}\big(1+O(\lambda^{-1})\big)$$
$$= (2\pi)^{1/2}ie^{ik(|x|+|y|)}\big(1+O(\lambda^{-1})\big).$$

We compute

$$(1/4\pi)\zeta^{-1/4}e^{-i\pi/4}(2\pi)^{1/2}i = c_0(\zeta)$$

by (2.11), and hence we obtain the desired asymptotic form

$$R_{\mathrm{sc}} = c_0(\zeta)e^{ik(|x|+|y|)}(|x||y|)^{-1/2}\big(f(-\hat{y}\to\hat{x};\zeta)+O(\lambda^{-1})\big).$$

This proves the proposition. □

Proof of Proposition 2.3

In [4], this proposition has also been established as Proposition 3.3 with rough remainder estimates. We give only a sketch for the proof of statement (1). By assumption, $\lambda/c<|x|<c\lambda$ and $1/c<|y|<c$ for some $c>1$. Then we can show that $R_\alpha = R_\alpha(x,y;\zeta)$ behaves like

$$R_\alpha = \tilde{R}_\alpha(x,y;\zeta)+O(\lambda^{-N})$$

for any $N\gg 1$, where

$$\tilde{R}_\alpha = \frac{1}{4\pi}\int_0^{\kappa+i\infty}\chi_M\Big(\frac{\operatorname{Im}t}{\lambda}\Big)\exp\Big(\frac{t}{2}-\frac{\zeta|x|^2}{2t}\Big)\exp\Big(-\frac{\zeta|y|^2}{2t}\Big)I\Big(\frac{\zeta|x||y|}{t},\psi\Big)\frac{dt}{t}$$

and $I(w,\psi)$ is defined by $I(w,\psi)=\sum_l e^{il\psi}I_\nu(w)$ with $w=\zeta|x||y|/t$. We make a change of variable $t=\kappa+is=ik|x|\tau$ and set $\sigma=\tau-1$ as in the proof of

Proposition 2.2. Then we repeat the same argument as above to obtain that $\tilde{R}_\alpha(x, y; \zeta)$ takes the asymptotic form

$$\tilde{R}_\alpha = c_0(\zeta) e^{ik|x|} |x|^{-1/2} \exp(ik|y|^2/2|x|)\big(I(k|y|/i, \psi) + O(\lambda^{-1})\big).$$

We note that $\exp(ik|y|^2/2|x|) = 1 + O(\lambda^{-1})$. Since $I_\nu(z/i) = e^{-i\nu\pi/2} J_\nu(z)$ by formula and since

$$e^{il\psi} = e^{il(\theta-\omega)} = e^{il\gamma(\hat{x};\hat{y})} = e^{-il\gamma(\hat{y};\hat{x})},$$

we have by (2.5) (see also Remark 2.2) that

$$I(k|y|/i, \psi) = \sum_l e^{il\psi} I_\nu(k|y|/i) = \sum_l e^{-il\gamma(\hat{y};\hat{x})} e^{-i\nu\pi/2} J_\nu(k|y|) = \overline{\varphi}_{\alpha-}(y; \hat{x}, \overline{\zeta}).$$

Thus we get the desired asymptotic form. □

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Department of Mathematics, Okayama University, Okayama, 700–8530, Japan;
tamura@math.okayama-u.ac.jp