On the complement of effective divisors with semipositive normal bundle

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Abstract A Hartogs-type extension theorem is proved for the complement of certain effective divisors on compact Kähler manifolds. A method employed in the proof yields a result supplementing some solutions of the Levi problem on two-dimensional spaces.

0. Introduction

Given a complex manifold M, locally pseudoconvex domains over M are basic objects naturally arising in complex analysis of several variables. In fact, it is essentially contained in Oka's [O] theory that, for any positive line bundle L over a projective algebraic manifold X, there exists a positive integer m_0 such that every locally pseudoconvex domain over X is equivalent, as a Riemann domain over X, to a connected component of the sheaf of holomorphic sections of L^m for any $m \geq m_0$. More specifically, it is known that locally pseudoconvex domains over $\mathbb{C}P^n$ are nothing but connected components of the structure sheaf of $\mathbb{C}P^n$ (cf. [F], [T]).

Topologically there are three main types of locally pseudoconvex domains: covering spaces, bounded (i.e., relatively compact) domains with C^2-smooth pseudoconvex boundary, and complements of (the support of) effective divisors. An index theorem was established for covering spaces of compact manifolds and applied to the existence problem (cf. [A]), boundary value problems are solved for smoothly bounded domains (cf. [Kn]), and concavity properties are studied for effective divisors (cf. [U], [Oh3]).

Among the questions on covering spaces, a well-known open problem is whether or not the universal covering space of a compact Kähler manifold is holomorphically convex (cf. [Sh]). The answer is negative without the Kählerianity assumption because Hopf manifolds are obvious counterexamples. Another longstanding question is whether every bounded domain with C^2-smooth pseudoconvex boundary in a Kähler manifold admits a plurisubharmonic exhaustion function. This is nontrivial only when the boundary is not strongly pseudoconvex. The answer is no if M is not Kähler (cf. [DF]).

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Putting such existence questions aside, the author has studied in [Oh4] the Hartogs-type extension property for the smoothly bounded domains. As a result, it turned out that a vanishing theorem for the L^2 $\overline{\partial}$-cohomology leads to the following.

THEOREM 0.1 (SEE [Oh4])
Let Ω be a locally pseudoconvex bounded domain in a Kähler manifold M. Assume that the boundary of Ω is a real hypersurface of class C^2 and not everywhere Levi flat. Then, for any compact set $K \subset \Omega$ and for any holomorphic function f on $\Omega - K$, there exists a holomorphic function on Ω which coincides with f outside a compact subset of Ω.

COROLLARY
The boundary of Ω as above is connected.

Let us call Ω *a domain of Hartogs type* if the conclusion of Theorem 0.1 holds true for Ω. The purpose of the present article is to establish a result analogous to Theorem 0.1 for the complements of effective divisors. Namely, we shall prove the following.

THEOREM 0.2
Let M be a connected compact Kähler manifold, let D be an effective divisor on M, let $[D]$ be the line bundle associated to D, and let $|D|$ be the support of D. Assume that $[D]$ has a fiber metric whose curvature form is semipositive on the Zariski tangent spaces of $|D|$ everywhere and not identically zero at some point of $|D|$. Then $M - |D|$ is a domain of Hartogs type. In particular, $|D|$ is connected.

In short, for any compact Kähler manifold, the complement of an effective divisor with semipositive normal bundle has the Hartogs extendibility property unless the curvature form is identically zero.

As well as Theorem 0.1, the proof of Theorem 0.2 is based on the solvability of a $\overline{\partial}$-equation. However, the geometric reasoning for reducing the problem there is characteristic to the case of divisorial boundaries and quite different from the smoothly bounded case (cf. Proposition 1.5). As we shall show in Section 1 (cf. Theorem 1.2), one can combine such a geometry with Demailly and Peternell's vanishing theorem (cf. [DP]), which is a strengthened version of the vanishing theorem of Kawamata and Viehweg (cf. [Dm2], [Ka], [V]), to give a proof of Theorem 0.2. Nevertheless, for the sake of completeness, we shall also give an alternate proof of Theorem 0.2 just by applying a basic L^2 vanishing theorem on complete Kähler manifolds (cf. Theorem 2.1). For that, it is crucial to have an elementary construction of exhaustion functions employed in [Oh1], [FOh], and [Oh4, Lemma 4.1] (see Section 1, Propositions 1.2, 1.4). The vanishing theorem itself is not new, but the choice of a weight function in the application seems to be new. It might be worthwhile to note that the argument of the latter proof can be applied to extend the result to the locally pseudoconvex domains whose boundaries are partially smooth and partially divisorial (see Remark 3.1). We would like to note that the part of constructing an exhaustion function on $M - |D|$ resembles a sheaf theoretic argument of Simha [S], who proved that the complement of a curve on a Stein surface is Stein. Accordingly, our method can be applied to give an alternate proof of Simha's theorem and to prove the following.

THEOREM 0.3
Let M be a connected compact complex manifold of dimension 2, and let D be an effective divisor on M such that $[D]$ has a fiber metric whose curvature form is semipositive on $|D|$ and positive at some point of $|D|$. Then $M - |D|$ is holomorphically convex and properly bimeromorphic to a Stein space.

It may be worthwhile to add a remark that Theorem 0.1 supplements the author's previous work [Oh2] on Levi flat hypersurfaces, Theorem 0.2 the result in [Oh3] on the divisors with topologically trivial normal bundles, and Theorem 0.3 the solution [DOh] of a two-dimensional Levi problem.

1. q-convex and q-concave

For the proof of Theorem 0.2, we shall recall some generalized convexity notions.

Let Ω be a bounded domain with C^2-smooth boundary $\partial\Omega$ in a connected complex manifold M of dimension n, and let ρ be a defining function of Ω; that is, ρ is a real-valued C^2-function defined on a neighborhood, say, U of the closure $\overline{\Omega}$ of Ω such that $\Omega = \{x \in U; \rho(x) < 0\}$ holds and $d\rho$ vanishes nowhere on $\partial\Omega$. Let us denote by $T(\partial\Omega)$ the tangent bundle of $\partial\Omega$ which is naturally embedded in the tangent bundle of M. We put

$$T^{1,0}(\partial\Omega) = v \in T^{1,0}M \cap \big(T(\partial\Omega) \otimes \mathbb{C}\big); \qquad \partial\rho(v) = 0, \tag{1.1}$$

where $T^{1,0}M$ stands for the holomorphic tangent bundle of M and $\partial\rho$ the (1, 0)-part of $d\rho$.

Let $x \in \partial\Omega$. By the Levi signature of $\rho\Omega$ at x, we shall mean the signature of the Hermitian form

$$\begin{array}{ccc} T^{1,0}(\partial\Omega) \times T^{1,0}(\partial\Omega) & \longrightarrow & \mathbb{C} \\ \Cup & & \Cup \\ (v, w) & \longmapsto & \partial\overline{\partial}\rho(v \wedge \overline{w}). \end{array}$$

Here $\partial\overline{\partial}\rho$ denotes the (1, 1)-part of $-d\partial\rho$. It is clear that the Levi signature does not depend on the choices of defining functions of Ω. If a Hermitian metric, say g is given on M, the eigenvalues of $\partial\overline{\partial}\rho(v \wedge \overline{w})$ with respect to g depend on the choice of ρ, but only up to multiplication of a positive function on $\partial\Omega$.

If the Levi signature (s, t) of $\partial\Omega$ everywhere satisfies $s \geq n - q$ (resp., $t \geq n - q$), we say that $\partial\Omega$ is *q-convex* (resp., *q-concave*). If there exists a Kähler metric g on a neighborhood of Ω such that the sums of q eigenvalues of $\partial\overline{\partial}\rho(v \wedge \overline{w})$

with respect to g are everywhere positive (resp., negative) on $\partial\Omega$, then we say $\partial\Omega$ is *hyper-q-convex* (resp., *hyper-q-concave*). We say that Ω is q-convex, q-concave, and so on if $\partial\Omega$ is. In [GR] hyper-q-convex domains in Kähler manifolds are defined, a cohomology vanishing theorem is proved on hyper-q-convex domains, and an example of a q-convex domain is given which is not hyper-q-convex in our sense.

Recall that M is called a *q-complete manifold* if there exists a C^2 exhaustion function φ on M such that φ is everywhere *q-convex* in the sense that the Hermitian form

$$\begin{array}{ccc} T^{1,0}M \times T^{1,0}M & \longrightarrow & \mathbb{C} \\ \Downarrow & & \Downarrow \\ (v,w) & \longmapsto & \partial\overline{\partial}\varphi(v \wedge \overline{w}). \end{array}$$

which is called the *Levi form* of φ, has everywhere at least $n-q+1$ positive eigenvalues. The Levi form of φ will be denoted simply by $\partial\overline{\partial}\varphi$.

It was first proved by Greene and Wu [GW] that M is n-complete if and only if M is noncompact. An extension of Greene and Wu's theorem to complex spaces with arbitrary singularities can be found in [Oh1] (see also [Dm1]). Here the notion of q-convex functions on complex spaces is defined in such a way that it is inherited by restrictions to closed complex subspaces (cf. [AG]). The latter method will be applied afterward.

The following is an immediate consequence of the maximum principle.

PROPOSITION 1.1
If $\partial\Omega$ is compact and $(n-1)$-concave, then there exist no nonconstant holomorphic functions on the connected neighborhoods of $\overline{\Omega}$.

Recall also the following.

THEOREM 1.1 (SEE [G])
For any 1-convex domain $\Omega \subset M$, there exist a Stein space $\hat{\Omega}$ and a proper bimeromorphic morphism from Ω to $\hat{\Omega}$ which is one-to-one outside a compact analytic subset of Ω. M is a Stein manifold if and only if M is 1-complete.

Given a compact complex submanifold $S \subset M$, the above-mentioned convexity properties of neighborhoods of S are derived from the curvature properties of the normal bundle of S. This relation naturally extends to the case of effective divisors.

Let D be an effective divisor on M such that $[D]$ has a fiber metric h whose curvature form satisfies the assumption of Theorem 0.2. (Compactness and Kählerianity of M is not assumed here.)

We fix a canonical section s of $[D]$ and denote by $|s|$ the length of s with respect to h. Let $h^\wedge$ be a fiber metric of $[|D|]$, let $s^\wedge$ be a canonical section of $[|D|]$, and let $|s^\wedge|$ be the length of $s^\wedge$ with respect to $h^\wedge$.

If $|D|$ is compact, then replacing h by $h\exp(-A|s^\wedge|^2)$ for sufficiently large $A > 0$, we may assume in advance that the curvature form of h is semipositive at every point of $|D|$ and of rank ≥ 2 at some regular point x_0 of $|D|$.

The following is crucial for our purpose.

PROPOSITION 1.2

If $|D|$ is compact and x_0 is as above, then the connected component of $|D|$ containing x_0 has an $(n-1)$-concave neighborhood system.

Proof

In view of Hironaka's desingularization theorem and the fact that the pullbacks of semipositive bundles are semipositive, without losing the generality we may assume that $|D|$ is a divisor of simple normal crossing.

Let D_k $(k \in 0 \cup N)$ be defined inductively as follows; $D_0 = \emptyset, D_1 =$ the irreducible component of $|D|$ containing x_0, and, for $k \in N$, $D_{k+1} =$ the union of those irreducible components of $|D|$ which do not intersect with D_{k-1} but intersect with D_k transversally. We put $D(k) = \bigcup_{j \leq k} D_j$ and $m = \sup\{k; D_k \neq \emptyset\}$.

Let s_k $(1 \leq k \leq m)$ be canonical sections of $[D_k]$. Then, since D_k intersects with every irreducible component of D_{k+1} and is disjoint from D_{k+2}, $[D_k]$ admits a fiber metric h_k which satisfies the following conditions.

(i) The curvature form of h_k has at least one positive eigenvalue on the Zariski tangent spaces of D_{k+1} except at the points of $D_{k+1} \cap D_{k+2}$.

(ii) On D_{k+1}, the length $|s_k|$ of s_k with respect to h_k takes its maximum along $D_{k+1} \cap D_{k+2}$, and that $-\log|s_k| = |s_{k+2}|^2$ holds on a neighborhood of $D_{k+1} \cap D_{k+2}$ in D_{k+1}.

(iii) $|s_k| = 1$ on $|D| - D(k+1)$.

A canonical way of constructing h_k is in terms of a Morse function on $D_{k+1} - D_k$ which has no local maximum. In fact, it was shown in the proof of [FOh, Theorem 1] by such a method, basically modifying functions by composing diffeomorphisms and convex increasing functions that, for any fiber metric h_{k_0} of $[D_k]$, there exists a $C^\infty(n-1)$-convex exhaustion function on $D_{k+1} - D_k$, say, Φ_k, such that the length $|s_k|_0$ of s_k with respect to h_{k_0} satisfies that $\Phi_k + \log|s_k|_0$ is extendible as a C^∞-function on D_{k+1}. Letting h_k^1 be a fiber metric of $[D_k]$ such that Φ_k equals $-\log|s_k|_1$, where $|s_k|_1$ denotes the length of s_k with respect to h_k^1, it is easy to modify h_k^1 on a neighborhood of D_{k+2} to obtain h_k fulfilling the additional requirements (ii) and (iii), by adding $\varepsilon\chi\log|s_{k+2}|$ $(0 < \varepsilon \ll 1)$ to Φ_k for some C^∞-cutoff function χ with support in a neighborhood of $D_{k+1} \cap D_{k+2}$ and then composing to a resulting sum a convex increasing function $F(x)$ of the form $F(x) = \exp(\varepsilon^{-1}x)$ for $x \ll 0$ and $F(x) = x$ for $x \gg 1$.

Moreover, it is easy to see that, if a Hermitian metric on M is given in advance, we may assume that the sums of $n-1$ eigenvalues of the curvature form of h_k are positive on the Zariski tangents of D_{k+1} as above except along $D_{k+1} \cap D_{k+2}$.

Let ν_0 be a nonnegative C^2 exhaustion function on $D_1 - x_0$ such that ν_0 is $(n-1)$-convex outside $D_1 \cap D_2$ and that $\nu_0/|s_2|^2$ is constant on a neighborhood of $D_1 \cap D_2$. Let χ be a nonnegative C^2-function on D_1 satisfying the following:

(a) $\chi(x) = 1$ holds on a neighborhood of x_0;
(b) $\operatorname{supp}\chi$ is disjoint from $D_1 \cap D_2$;
(c) the rank of the curvature form of h is at least 2 on $\operatorname{supp}\chi$.

Then we put

$$\varphi(x) = \begin{cases} \varepsilon(1-\chi)\nu_0 & \text{on } D_1 - x_0 \text{ for sufficiently small } \varepsilon > 0, \\ 0 & \text{if } x = x_0 \text{ or } x \in D \end{cases}$$

and extend $h\exp(-\varphi)$ to a fiber metric h^* of $[D]$ in such a way that, for some neighborhood $U \supset D_1 \cap D_2$, the length with respect to h^*, say, $|s|^*$, satisfies

$$-\log|s|^* = -\log|s| + c|s_2|^2 \quad \text{on } U - (D_1 \cup D_2)$$

for some positive number c.

Let D_{kj} $(1 \le j \le m_k)$ be the irreducible components of D_k, let s_{kj} be canonical sections of $[D_{kj}]$, let h_{kj} be fiber metrics of $[D_{kj}]$, and let $|s_{kj}|$ be the length of s_{kj} with respect to h_{kj}. We put

$$\begin{aligned} \sigma_{k_1} &= \sum_{j=1}^{m_k} |s_{kj}|^2, \\ \sigma_{k_2} &= \sum_{1 \le i \le j \le m_k} |s_{ki}|^2 |s_{kj}|^2, \\ &\;\;\vdots \\ \sigma_{k_{m_k}} &= |s_{k1}|^2 \cdots |s_{k_{m_k}}|^2, \end{aligned}$$

and put

$$h = h^* \prod_{k=1}^{m} h_k^{1/R^{m_k}} \exp\Big(-\sum_{j=1}^{m_k} R^{j-m_k} \sigma_{kj}\Big)$$

for $R > 0$, as a fiber metric of the fractional bundle $[D] + \sum_{k=1}^{m} R^{-m_k}[D_k]$.

Then it is clear that the function

$$\Psi = -\log|s|^* - \sum_{k=1}^{m} (R^{-m_k} \log|s_k|^2) + \sum_{j=1}^{m_k} R^{j-m_k} \sigma_{kj} + R|s^\wedge|^2$$

becomes $(n-1)$-convex near $|D|$ for sufficiently large R. Note that σ_{kj} are needed to keep at least 2 positive eigenvalues of $\partial\bar{\partial}\Psi$ near the singular points of D. Near the set $s_{k1} = \cdots = s_{km_k} = 0$, σ_{k1} works, so does σ_{k2} near $s_{k1} = \cdots = \check{s_{kj}} = \cdots = s_{km_k} = 0$ with $s_{kj} \ne 0$ $(1 \le j \le m_k)$, and so on. Since Ψ is exhaustive we are done. $\square$

Combining Proposition 1.2 with Theorem 1.1 we obtain Theorem 0.3. Since $[D]||D|$ is semipositive and $M - D(m)$ is a proper modification of a Stein space, $|D|$ is connected.

The above proof of Proposition 1.2 can be easily generalized to prove the following.

PROPOSITION 1.3

Let M be a complex manifold of dimension n, and let D be an effective divisor on M such that $|D|$ is connected. If $|D|$ has a noncompact irreducible component, then there exist a neighborhood $U \supset |D|$ and a C^2-function φ on $M - |D|$ such that φ is $(n-1)$-convex on $U - |D|$ and satisfies $\lim_{x \to y} \varphi(x) = \infty$ for any $y \in |D|$.

In virtue of desingularization and Theorem 1.1, we have the following.

COROLLARY (CF. [S])

The complement of a curve in a Stein space of dimension 2 is Stein.

In the situation of Proposition 1.2, if a Hermitian metric is given on M, it is clear from the method of finding h_k as above that one may assume moreover that there exists a neighborhood $U \supset D(m)$ such that the sums of $n-1$ eigenvalues of the Levi form of the function Ψ are positive on $U - D(m)$. In particular we have also the following.

PROPOSITION 1.4

The complement of $D(m)$ is exhausted by hyper-$(n-1)$-convex domains if M is compact and Kählerian.

Let us observe also the following consequence of Proposition 1.2 which holds under the assumption of Theorem 0.2.

PROPOSITION 1.5

There exist no nonconstant holomorphic functions on any connected neighborhood of any connected component of $|D|$.

Proof

Let D_0 be a connected component of $|D|$. If D_0 admits an $(n-1)$-concave neighborhood system, then there exist no nonconstant holomorphic functions on the connected neighborhoods of D_0 by Proposition 1.1. In general, let $U \supset D_0$ be a connected neighborhood, and let f be a holomorphic function on U such that $f^{-1}(0) \supseteq D_0$. It suffices to show that $f = 0$. Supposing on the contrary that $f \neq 0$, let us examine two alternate cases.

Case 1. Assume that $f^{-1}(0)$ has no irreducible component intersecting with D_0. Then there exists a neighborhood $U' \supset D_0$ such that $f|U'$ is a proper holomorphic map onto the unit disc $\Delta = z \in \mathbb{C}; |z| < 1$. Since M is a compact Kähler manifold, $f|U'$ extends to a holomorphic map, say, f from M onto a compact Riemann surface C containing Δ (cf. [Fk]).

Clearly, every connected component of $|D|$ is contained in the preimage of a point in C by f. But this is absurd because f must be constant on a neighborhood of D_1 (cf. Propositions 1.1, 1.2).

Case 2. Assume that there exists an irreducible component Z of $f^{-1}(0)$ such that $Z \cap D_0$ is $(n-2)$-dimensional. Let $D[k]$ $(1 \leq k \leq \mu)$ be the irreducible components of $|D|$ such that $Z \cap D[1] \neq \emptyset$ and $|D_0| = \bigcap_{k \leq \mu_0} D[k]$ $(\mu_0 \leq \mu)$, let $D = \sum \beta_k D[k]$ $(\beta_k \in N)$, and let α_k be the order of multiplicity of the zeros of f along $D[k]$ $(1 \leq k \leq \mu_0)$.

We fix any m satisfying

$$\beta_m/\alpha_m = \max\{\beta_k/\alpha_k; 1 \leq k \leq \mu_0\}.$$

Then the function $-\alpha_m \log|s| + \beta_m \log|f|$ extends to a plurisubharmonic function, say, φ on $D[m]$ because of its boundedness near $D[m]$ and the semipositivity of the curvature of h along $|D|$.

Since $D[m]$ is compact, φ becomes constant. Therefore it follows that $\beta_m/\alpha_m = \beta_k/\alpha_k$ must hold as long as $D[m] \cap D[k] \neq \emptyset$. Hence one sees by induction that

$$\beta_1/\alpha_1 = \beta_2/\alpha_2 = \cdots = \beta_{\mu_0}/\alpha_{\mu_0}$$

holds true. This means that $-\alpha_m \log|s| + \beta_m \log|f|$ extends to a plurisubharmonic function on $D[1]$, which is absurd in the presence of Z. □

REMARK 1.1

According to the referees, case 1 may be treated without the assumption that M is Kähler. In fact, since M is compact, the sheaf cohomology group $H^1(M, \mathcal{O})$ is finite-dimensional. Here $\mathcal{O}$ denotes the structure sheaf of M. Hence there exist holomorphic functions g_0 on $M - D_0$, g_1 on U' and complex numbers $a_1, \ldots, a_d$ with $a_d \neq 0$, such that $g_0 - g_1 = \sum_{j=1}^{d} a_j f^{-j}$ on $U' - D_0$. Then g is nonconstant, which is an absurdity.

Proof of Theorem 0.2.

If D is as in Theorem 0.2, then clearly $\dim M \geq 2$, D is nef, that is, D belongs to the closure of the Kähler cone, and $D^2 \neq 0$. Hence, $H^1(M, I) = 0$ holds for the ideal sheaf I of D by Demailly and Peternell's vanishing theorem (cf. [DP]). Therefore, for any neighborhood $U \supset |D|$, for any holomorphic function f on $U - |D|$, and for any C^2-extension $f^\wedge$ of f to $M - |D|$, there exists a C^2-section u of I such that $f^\wedge - u$ is holomorphic on $M - |D|$. Since u is then holomorphic on a neighborhood of $|D|$, by Proposition 1.5 it follows that u is zero on a neighborhood of $|D|$. Hence $f^\wedge - u$ extends $f|(V - |D|)$ for some neighborhood $V \supset |D|$. □

2. An L^2 vanishing theorem

We recall here a basic result on the L^2-cohomology of complete Kähler manifolds. It is a variant of Kodaira's vanishing theorem in [K] in the spirit of Andreotti and Vesentini [AV1], [AV2] and Grauert and Riemenschneider [GR]. Theorem 2.1 stated below is essentially well known and contained in the union of [AV1], [AV2] and [GR] up to the method of Hörmander [H].

To state the theorem with an outline of the proof, we start by recalling basic formulas. Let (N, g) be a connected complete Kähler manifold of dimension n, and let E be a holomorphic vector bundle over N equipped with a fiber metric, say, h. Let $\overline{\partial}$ (resp., ∂) denote the complex exterior derivative of type $(0,1)$ (resp., $(1,0)$), and let $\partial_h = h^{-1} \cdot \partial \cdot h$, regarding h as a smooth section of $\mathrm{Hom}(E, \overline{E}^*)$. Let ϑ_h (resp., $\overline{\vartheta}$) be the formal adjoint of $\overline{\partial}$ (resp., ∂) with respect to g and h.

Let ω be the fundamental form of g, and let Λ be the adjoint of exterior multiplication by ω.

Recall that

$$\vartheta_h u = \sqrt{-1}[\partial_h, \Lambda]u \tag{2.1}$$

and

$$\overline{\vartheta} u = -\sqrt{-1}[\overline{\partial}, \Lambda]u \tag{2.2}$$

hold for any C^2 E-valued (p,q)-form u on N.

Let Θ_h be the curvature form of h, identified with its exterior multiplication from the left-hand side. As a straightforward consequence of (2.1) and (2.2) one has

$$\overline{\partial}\vartheta_h + \vartheta_h\overline{\partial} - \partial_h\overline{\vartheta} - \overline{\vartheta}\partial_h = \sqrt{-1}[\Theta_h, \Lambda] \text{ (Nakano's identity).} \tag{2.3}$$

Let (u, v) denote the inner product of C^2 E-valued compactly supported forms u and v, and let $\|u\|^2 = (u, u)$.

Then (2.3) implies that

$$\|\overline{\partial}u\|^2 + \|\vartheta_h u\|^2 \geq (-\sqrt{-1}\Lambda\Theta_h u, u) \tag{2.4}$$

holds if u is a compactly supported E-valued C^2 $(0,q)$-form. Note that the right-hand side of (2.4) is nonnegative if the dual of (E, h) is Nakano semipositive.

For any C^2 real-valued function φ and any E-valued $(0,q)$-form u, let us recall a local expression of $\sqrt{-1}\Lambda\partial\overline{\partial}\varphi \wedge u$.

Let $x \in N$, and let $\sigma_1, \ldots, \sigma_n$ be a basis of the holomorphic cotangent space of N at x such that $\omega = \sqrt{-1}\sum \sigma_k \wedge \overline{\sigma}_k$ and $\partial\overline{\partial}\varphi = \sum \lambda_k \sigma_k \wedge \overline{\sigma}_k (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n)$ hold at x.

Then, by letting

$$u = \sum u_K \overline{\sigma}_K \quad \text{at } x,$$

where $\overline{\sigma}_K = \overline{\sigma}_{k_1} \wedge \cdots \wedge \overline{\sigma}_{k_q}$ for $K = (k_1, \ldots, k_q)$, we have

$$\sqrt{-1}\Lambda\partial\overline{\partial}\psi \wedge u = \sum (\lambda^* - \lambda_K) u_K \overline{\sigma}_K \quad \text{at } x, \tag{2.5}$$

where $\lambda^* = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and $\lambda_K = \lambda_{k_1} + \cdots + \lambda_{k_q}$. Note that λ_k are continuous.

Let $H^{p,q}_\psi(N,E)$ denote the $L^2\overline{\partial}$-cohomology group of type (p,q) with respect to g and $h \exp\psi$. Since (N,g) is complete, combining (2.4) and (2.5) with $\Theta_{he^\psi} = \Theta_h - \mathrm{Id}_E \otimes \partial\overline{\partial}\psi$, one obtains the following by applying a routine argument of functional analysis and the unique continuation theorem of Aronszajn (cf. [AV1], [AV2], [GR], [H]).

THEOREM 2.1

Assume that the dual of the Hermitian vector bundle (E,h) is Nakano semipositive, $\inf_{N-B}\lambda^*$ is positive for some compact set $B \subset N$, and $\lambda^* - \lambda_n$ is everywhere nonnegative and somewhere positive. Then $H^{0,0}_\psi(N,E) = 0$ and $H^{0,1}_\psi(N,E) = 0$.

3. Alternate proof of Theorem 0.2

Let M and D be as in Theorem 0.2. Let D' be the union of connected components of $|D|$ along which the curvature form of h is tangentially identically zero. Then, with respect to a Kähler metric g on M, for any connected component D_0 of $|D| - D'$ one can find a function Ψ on $M - D_0$ as in the remark after the corollary of Proposition 1.3. By an abuse of notation, we denote the sum of these functions for all such D_0's by Ψ, too. Let ω denote the fundamental form of g.

We fix $c \in \mathbb{R}$ in such a way that the sums of $n-1$ eigenvalues of $\partial\overline{\partial}\Psi$ are positive on the set $V(c) = \{x; \Psi(x) > c\}$. We may assume that $V(c-1) \cup D' = \emptyset$. Then we put

$$b = \inf\{\log|s(x)|; x \in \partial V(c)\}$$

and

$$\omega_\varepsilon = \begin{cases} \omega - \varepsilon\sqrt{-1}\partial\overline{\partial}\lambda(\Psi - c) & \text{on } V(c), \\ \omega - \varepsilon\sqrt{-1}\partial\overline{\partial}\xi(-\log|s| + b) & \text{on } M - V(c) - D' \end{cases}$$

where ε is a positive number, λ is a real-valued C^∞-function on $\mathbb{R}$ satisfying $\lambda(t) = 0$ if $t < 1$ and $\lambda(t) = \log t$ if $t \geq 2$, and $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^∞-function with $\operatorname{supp}\xi \subset (1,\infty)$ such that $\xi(t) = t^2$ if $t \geq 2$. It is easy to see that ω_ε is the fundamental form of a complete Kähler metric, say, g_ε on $M - |D|$ if ε is sufficiently small.

Denoting by $\chi(x)$ the minimum of the sums of $n-1$ eigenvalues of $\partial\overline{\partial}(\Psi - c)^2$ at $x \in M - |D|$ with respect to g_ε, it is easy to see that $\inf \chi(x); \Psi(x) > c$ diverges to ∞ as $c \longrightarrow \infty$. Hence, by composing a convex increasing function to $(\Psi - c)^2$ we obtain a function, say, φ, satisfying the conditions of φ in Theorem 2.1 with respect to g_ε.

Then, by Theorem 2.1, for any neighborhood $V \supset |D|$, for any holomorphic function f on $V - |D|$, and for any C^2-function $\hat{f}$ which coincides with f outside a compact subset of $M - |D|$, the equation $\overline{\partial}u = \overline{\partial}\hat{f}$ has a solution u which is square integrable on $M - |D|$ with respect to the measure defined as the product of the volume form of g_ε and $\exp\varphi$. In particular, u is extendible holomorphically across D', so that u is constant near D'.

Since $|s(x)|^\nu \exp\varphi(x)$ diverges as $x \longrightarrow |D| - D'$ for any $\nu \in \mathbb{N}$, u must vanish outside a compact subset of $M - |D| \cup D'$. On the other hand, by Proposition 1.5 we know that u is locally constant on a neighborhood of D'. Combining this constancy with the infiniteness of the volume of g_ϵ around D', we conclude that u is zero on a neighborhood of D'. Thus $\hat{f} - u$ is the desired extension of f. □

REMARK 3.1

In view of the above proof, it is easy to see that, given a bounded domain Ω with C^2-smooth pseudoconvex boundary in a Kähler manifold, Ω admits an effective divisor D with compact support such that $[D]||D|$ is semipositive only if $\partial\Omega$ is Levi flat and $D^2 = 0$. This fact seems to suggest the validity of a vanishing theorem of Demailly–Peternell type on the domains with Levi flat boundary.

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