

Cohomology and L -values

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Abstract In a paper published in 1959, Shimura presented an elegant calculation of the critical values of L -functions attached to elliptic modular forms using the first cohomology group. We will show that a similar calculation is possible for Hilbert modular forms over real quadratic fields using the second cohomology group. We present explicit numerical examples calculated by this method.

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0. Introduction

In a celebrated paper [Sh1] published in 1959, Shimura showed that ratios of critical values of the L -function attached to an elliptic modular form can be calculated explicitly using the cohomology group. This method was developed into the theory of modular symbols by Manin [Man]. Though there were great advances during the next half-century in understanding the relationship of automorphic forms and group cohomologies, it seems that no explicit calculations of L -values using cohomology groups were performed beyond the one-dimensional case. The purpose of this paper is to show that we can use cohomology groups effectively for calculations of L -values even in higher-dimensional cases.

To explain our ideas and results, it is best to review first the calculation in [Sh1]. Let \mathfrak{H} be the complex upper half-plane. Let Γ be a Fuchsian group, and let Ω be a cusp form of weight $k \geq 2$ with respect to Γ . Put $l = k - 2$, and let ρ_l be

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the symmetric tensor representation of $\mathrm{GL}(2, \mathbf{C})$ of degree l on a vector space V . We regard V as a Γ -module. Put $\rho = \rho_l$. We consider a V -valued differential form on \mathfrak{H} :

$$\mathfrak{d}(\Omega) = \Omega(z) \begin{bmatrix} z \\ 1 \end{bmatrix}^l dz.$$

Here $\begin{bmatrix} z \\ 1 \end{bmatrix}^l$ denotes the column vector of dimension $l+1$ whose components are $z^l, z^{l-1}, \dots, 1$. We have $\mathfrak{d}(\Omega) \circ \gamma = \rho(\gamma)\mathfrak{d}(\Omega)$ for every $\gamma \in \Gamma$. Here $\mathfrak{d}(\Omega) \circ \gamma$ denotes the transform of $\mathfrak{d}(\Omega)$ by γ . Take a point of the complex upper half-plane or a cusp of Γ , and denote it by z_0 . For $\gamma \in \Gamma$, we consider the integral

$$(1) \quad f(\gamma) = \int_{z_0}^{\gamma z_0} \mathfrak{d}(\Omega).$$

Then f satisfies the 1-cocycle condition:

$$f(\gamma_1 \gamma_2) = f(\gamma_1) + \rho(\gamma_1)f(\gamma_2).$$

The cohomology class of f in $H^1(\Gamma, V)$ does not depend on the choice of z_0 . Let $p \in \Gamma$ be a parabolic element, and let z'_0 be the cusp fixed by p . Then we have

$$f(p) = (\rho(p) - 1) \int_{z'_0}^{z_0} \mathfrak{d}(\Omega).$$

Thus $f(p)$ looks like a coboundary, which is the parabolic condition on f .

Now let $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ and $z_0 = i\infty$. Put

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then we find

$$(2) \quad f(\sigma\tau) = - \left(\int_0^{i\infty} \Omega(z) z^t dz \right)_{0 \leq t \leq l} = - (i^{t+1} R(t+1, \Omega))_{0 \leq t \leq l},$$

where $R(s, \Omega) = (2\pi)^{-s} \Gamma(s) L(s, \Omega)$ with the L -function $L(s, \Omega)$ of Ω . Since $(\sigma\tau)^3 = 1$ and $\sigma^2 = 1$, the 1-cocycle condition gives

$$(3) \quad \begin{aligned} [1 + \rho(\sigma\tau) + \rho((\sigma\tau)^2)] f(\sigma\tau) &= 0, \\ [1 + \rho(\sigma)] f(\sigma) &= [1 + \rho(\sigma)] f(\sigma\tau) = 0. \end{aligned}$$

In other words, $f(\sigma\tau)$ is annihilated by the elements $1 + \sigma\tau + (\sigma\tau)^2$ and $1 + \sigma$ of the group ring $\mathbf{Z}[\mathrm{SL}(2, \mathbf{Z})]$. This gives constraints on the critical values of $L(s, \Omega)$. For $k = 12$ and $\Omega = \Delta$, Shimura obtained

$$R(8, \Delta) = \frac{5}{4} R(6, \Delta), \quad R(10, \Delta) = \frac{12}{5} R(6, \Delta), \quad \text{and so on.}$$

In this paper, we will treat the case of Hilbert modular forms over a real quadratic field F . Let \mathcal{O}_F be the ring of integers of F , and let Γ be a congruence subgroup of $\mathrm{SL}(2, \mathcal{O}_F)$. Let Ω be a Hilbert modular cusp form of weight (k_1, k_2) with respect to Γ . We assume $2 \leq k_2 \leq k_1$ and put $l_i = k_i - 2$, $i = 1, 2$. The first step is to attach an explicitly given 2-cocycle of Γ to Ω . This is given in the

author's book [Y3] as follows. Let $\rho = \rho_{l_1} \otimes \rho_{l_2}$, and let V be the representation space of ρ . We consider a V -valued differential form on \mathfrak{H}^2 :

$$\mathfrak{d}(\Omega) = \Omega(z) \begin{bmatrix} z_1 \\ 1 \end{bmatrix}^{l_1} \otimes \begin{bmatrix} z_2 \\ 1 \end{bmatrix}^{l_2} dz_1 dz_2, \quad z = (z_1, z_2) \in \mathfrak{H}^2.$$

We have

$$(4) \quad \mathfrak{d}(\Omega) \circ \gamma = \rho(\gamma) \mathfrak{d}(\Omega), \quad \gamma \in \Gamma.$$

Take a point $w = (w_1, w_2)$ on \mathfrak{H}^2 . For $\gamma_1, \gamma_2 \in \Gamma$, we consider the integral

$$(5) \quad f(\gamma_1, \gamma_2) = \int_{\gamma_1 \gamma_2 w_1}^{\gamma_1 w_1} \int_{w_2}^{\gamma_1' w_2} \mathfrak{d}(\Omega).$$

Here γ_1' denotes the conjugate of γ_1 by $\text{Gal}(F/\mathbf{Q})$. Then f is a 2-cocycle of Γ taking values in V . The cohomology class of $f \in H^2(\Gamma, V)$ does not depend on the choice of w . Let $p \in \Gamma$ be a parabolic element, and let (w_1^*, w_2^*) be the cusp fixed by p . Since Ω is a cusp form, we may replace w_2 by w_2^* . By this procedure, we find the parabolic condition satisfied by f .

Next let $\Gamma = \text{SL}(2, \mathcal{O}_F)$, and let ϵ be the fundamental unit of F . We assume that $l_1 \equiv l_2 \pmod{2}$ and replace Γ by $\text{PSL}(2, \mathcal{O}_F)$. Put

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}.$$

We choose $w_1 = i\epsilon^{-1}$, $w_2 = i\infty$. Then we have

$$f(\sigma, \mu) = f(\sigma, \sigma) = - \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \mathfrak{d}(\Omega).$$

For $0 \leq s \leq l_1$, $0 \leq t \leq l_2$, we put

$$P_{s,t} = \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \Omega(z) z_1^s z_2^t dz_1 dz_2.$$

The $((l_1 + 1)(l_2 + 1))$ -components of $f(\sigma, \mu)$ are given by $-P_{s,t}$. We have

$$(6) \quad P_{m, m-(k_1-k_2)/2} = (-1)^{m+1} i^{-(k_1-k_2)/2} (2\pi)^{(k_1-k_2)/2} R(m+1, \Omega)$$

where $R(s, \Omega) = (2\pi)^{-2s} \Gamma(s) \Gamma(s - (k_1 - k_2)/2) L(s, \Omega)$ with the L -function $L(s, \Omega)$ of Ω . The formula (6) gives a generalization of (2); (5) and (6) were known to the author nine years ago.

The L -value $L(m, \Omega)$, $m \in \mathbf{Z}$, is a critical value if and only if

$$\frac{l_1 - l_2}{2} + 1 \leq m \leq \frac{l_1 + l_2}{2} + 1.$$

Since all of them appear as components of $f(\sigma, \mu)$, we expect that we can deduce information on critical values once we know the second cohomology group $H^2(\Gamma, V)$ well. Before we are able to materialize this hope, we need to answer the following conceptual question: Can we annihilate the effect of adding a coboundary to f ? We can give an affirmative answer to this question by using the para-

bolic condition. Put

$$P = \left\{ \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} \mid u \in \mathcal{O}_F^\times, v \in \mathcal{O}_F \right\} / \{\pm 1_2\} \subset \Gamma.$$

Then we have

$$(7) \quad f(p\gamma_1, \gamma_2) = pf(\gamma_1, \gamma_2) \quad \text{for every } p \in P, \gamma_1, \gamma_2 \in \Gamma.$$

This is the parabolic condition on f when $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$. A 2-cocycle which satisfies (7) is called a *parabolic 2-cocycle*. In Section 3, we will prove the following.

THEOREM

Let $i = 1$ or 2 . Then

$$\dim H^i(P, V) = \begin{cases} 0 & \text{if } l_1 \neq l_2 \text{ or } N(\epsilon)^{l_1} = -1, \\ 1 & \text{if } l_1 = l_2 \text{ and } N(\epsilon)^{l_1} = 1. \end{cases}$$

Now, suppose that we add a coboundary to f , keeping the parabolic condition (7). In Section 4, using this theorem for the case $i = 1$, we will show the following. If $l_1 \neq l_2$, the components of $f(\sigma, \mu)$ related to the critical values do not change. If $l_1 = l_2$, the same assertion holds except for the critical values on the edges: $L(1, \Omega)$ and $L(l_1 + 1, \Omega)$. Therefore we can deduce information on critical values $L(m, \Omega)$ once we know a parabolic 2-cocycle corresponding to Ω .

The final step is to find constraints on $f(\sigma, \mu)$ which generalize (3). This is technically the most difficult step. Let $\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\omega$, and put

$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

It is known (see [V]) that Γ is generated by σ , μ , τ , and η . Let \mathcal{F} be the free group on four letters $\tilde{\sigma}$, $\tilde{\mu}$, $\tilde{\tau}$, $\tilde{\eta}$. Define a surjective homomorphism $\pi: \mathcal{F} \rightarrow \Gamma$ by $\pi(\tilde{\sigma}) = \sigma$, $\pi(\tilde{\mu}) = \mu$, $\pi(\tilde{\tau}) = \tau$, $\pi(\tilde{\eta}) = \eta$, and let R be the kernel of π . Then we have $\Gamma = \mathcal{F}/R$ and (cf. Section 1.4)

$$(8) \quad H^2(\Gamma, V) \cong H^1(R, V)^\Gamma / \mathrm{Im}(H^1(\mathcal{F}, V)).$$

Here we have

$$H^1(R, V)^\Gamma = \{ \varphi \in \mathrm{Hom}(R, V) \mid \varphi(grg^{-1}) = g\varphi(r), g \in \mathcal{F}, r \in R \}.$$

We write

$$\epsilon^2 = A + B\omega, \quad \epsilon^2\omega = C + D\omega.$$

We have relations:

- (i) $\sigma^2 = 1$,
- (ii) $(\sigma\tau)^3 = 1$,
- (iii) $(\sigma\mu)^2 = 1$,
- (iv) $\tau\eta = \eta\tau$,
- (v) $\mu\tau\mu^{-1} = \tau^A\eta^B$,
- (vi) $\mu\eta\mu^{-1} = \tau^C\eta^D$.

For $t \in \mathcal{O}_F^\times$, we have

$$(vii) \quad \sigma \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} -t & 1 \\ 0 & -t^{-1} \end{pmatrix}.$$

The relation (ii) follows from (vii) by taking $t = 1$. We call the relation group R *minimal* if it is generated by the elements corresponding to (i) \sim (vii) and their conjugates. We see that μ , τ , and η generate P and that (iv) \sim (vi) are their fundamental relations.

Now, let $\varphi \in H^1(R, V)^\Gamma$ be a corresponding element to f . Adding an element of $\text{Im}(H^1(\mathcal{F}, V))$, we may assume that $\varphi(\tilde{\sigma}^2) = 0$. Then we find (cf. (5.3))

$$f(\sigma, \mu) = -\varphi((\tilde{\sigma}\tilde{\mu})^2).$$

Our problem is reduced to finding constraints on $\varphi((\tilde{\sigma}\tilde{\mu})^2)$. We have an obvious constraint $\sigma\mu\varphi((\tilde{\sigma}\tilde{\mu})^2) = \varphi((\tilde{\sigma}\tilde{\mu})^2)$, but of course it is not enough.

To proceed further, we assume that l_1 and l_2 are even and change ρ to $\rho' = \rho'_{l_1} \otimes \rho'_{l_2}$, where $\rho'_i(g) = \rho_i(g) \det(g)^{-l_i/2}$, and regard V as a $\text{PGL}(2, \mathcal{O}_F)$ -module. The Γ -module structure does not change. We put

$$\nu = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These two elements act on Γ as outer automorphisms and induce automorphisms of $H^2(\Gamma, V)$ of order 2. Hence $H^2(\Gamma, V)$ decomposes into four pieces under their actions. Let Γ^* be the subgroup of $\text{PGL}(2, \mathcal{O}_F)$ generated by Γ and ν . The transfer map gives an isomorphism of the plus part of $H^2(\Gamma, V)$ under the action of ν onto $H^2(\Gamma^*, V)$. For simplicity suppose that we can take $\omega = \epsilon$. Then σ , ν , and τ generate Γ^* . Let \mathcal{F}^* be the free group on three letters $\tilde{\sigma}$, $\tilde{\nu}$, $\tilde{\tau}$. Define a surjective homomorphism $\pi^*: \mathcal{F}^* \rightarrow \Gamma^*$ by $\pi^*(\tilde{\sigma}) = \sigma$, $\pi^*(\tilde{\nu}) = \nu$, $\pi^*(\tilde{\tau}) = \tau$, and let R^* be the kernel of π^* . Then we have $\Gamma^* = \mathcal{F}^*/R^*$ and

$$(8^*) \quad H^2(\Gamma^*, V) \cong H^1(R^*, V)^{\Gamma^*} / \text{Im}(H^1(\mathcal{F}^*, V)).$$

Let f^* be the transfer of f to Γ^* , and let f^+ be the restriction of f^* to Γ . Then f^+ is the projection of f to the plus part. (We perform this procedure on the cocycle level.) We have

$$f^*(\sigma, \mu) = f^+(\sigma, \mu) = (1 + \nu)f(\sigma, \mu).$$

In Γ^* , σ , ν , and τ satisfy the relations (i), (ii), and (iii*): $(\sigma\nu)^2 = 1$; (iv*): $\tau\nu\tau\nu^{-1} = \nu\tau\nu^{-1}\tau$; (v*): $\nu^2\tau\nu^{-2} = \tau^A(\nu\tau\nu^{-1})^B$. Let P^* be the subgroup of Γ^* generated by P and ν . We see that P^* is generated by ν and τ and that (iv*) and (v*) are the fundamental relations between generators ν and τ . Let $\varphi^* \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to f^* . By the parabolic condition on f , we may assume that φ^* vanishes on the elements of R^* corresponding to (iv*) and (v*). Adding an element of $\text{Im}(H^1(\mathcal{F}^*, V))$, we may also assume that $\varphi^*(\tilde{\sigma}^2) = 0$. Then we have (cf. (6.6))

$$f^*(\sigma, \mu) = -(1 + \nu^{-1})\varphi^*((\tilde{\sigma}\tilde{\nu})^2),$$

and two quantities

$$A = \varphi^*((\tilde{\sigma}\tilde{\nu})^2), \quad B = \varphi^*((\tilde{\sigma}\tilde{\tau})^3)$$

remain to be determined. The Hecke operators act on $H^2(\Gamma^*, V)$. We can analyze its action on the right-hand side of (8*) and will give a simple formula for it. The quantity A is related to the critical values of $L(s, \Omega)$. We may assume that the class of f^* is in the plus space of $H^2(\Gamma^*, V)$ under the action of δ . Then A must satisfy the constraints

$$(9) \quad (\sigma\nu - 1)A = 0, \quad (\delta - 1)A = 0.$$

We will execute the determination of A for $F = \mathbf{Q}(\sqrt{5})$ and $F = \mathbf{Q}(\sqrt{13})$. First assume $F = \mathbf{Q}(\sqrt{5})$. In this case, we can show that R is minimal and that R^* is generated by the elements corresponding to the relations (i), (ii), (iii*), (iv*), (v*), and their conjugates. Calculating the action of the Hecke operator $T(2)$ on the right-hand side of (8*), we find a certain element $x \in \mathcal{F}^*$ such that $\pi^*(x)^3 = 1$. We can give an explicit formula expressing $\varphi^*(x^3)$ in terms of A and B . In every case examined, we find by numerical computations that we may assume that $B = 0$ by adding an element of $\text{Im}(H^1(\mathcal{F}^*, V))$. Therefore

$$(10) \quad (x - 1)\varphi^*(x^3) = 0$$

gives a new constraint on A . Let Z_A^+ be the subspace of V consisting of all A which satisfy (9) and (10), and let B_A^+ be the subspace of Z_A^+ which represents the contribution from $\text{Im}(H^1(\mathcal{F}^*, V))$. Again in every case examined, we find by numerical computations that $\dim S_{l_1+2, l_2+2}(\Gamma) = \dim Z_A^+/B_A^+$. If this is one-dimensional, we can deduce information on L -values by calculating Z_A^+ . In general, calculating the action of $T(2)$ on Z_A^+/B_A^+ and taking eigenvectors, we can obtain many examples on L -values. Actually, by considering f^+ , we are losing half of the information on critical values (cf. Section 5.6). To treat all critical values, we need to consider f^- , the projection of f to the minus part of $H^2(\Gamma, V)$ under the action of ν . To handle f^- is a somewhat more complicated task, and we leave the explanation of it to the text. Next let $F = \mathbf{Q}(\sqrt{13})$. The procedure is almost the same. Let \mathfrak{p} be the prime ideal generated by $4 - \sqrt{13}$. Calculating the action of the Hecke operator $T(\mathfrak{p})$, we obtain a certain element $x \in \mathcal{F}^*$ such that $\pi^*(x)$ is of order 3. Then the constraint $(x - 1)\varphi^*(x^3) = 0$ obtained from x is sufficient. Here, remarkably, we can perform rigorous calculations without proving that R is minimal. (This is actually true also for the case $F = \mathbf{Q}(\sqrt{5})$.) We have used Pari [PARI] for the numerical calculations in Sections 6 and 7.

To calculate the ratios of critical values of L -functions, there is another method initiated by Shimura [Sh2] which employs the Rankin-Selberg convolution and differential operators. A comparison of this method and the cohomological method will be discussed in Section 8.

Now let us explain the organization of this paper briefly. In Section 1, we review several facts on cohomology of a group which will be repeatedly used in later sections. In Section 2, we will review Hilbert modular forms. We prove (5)

and (6). In Section 3, we will study cohomology groups of P and will prove the theorem stated above. In Section 4, we will examine the parabolic condition on a cocycle applying results in Section 3. We will prove the nonvanishing of the cohomology class of f under mild conditions. In Section 5, we will study the decomposition of $H^2(\Gamma, V)$ under the action of outer automorphisms of Γ . It decomposes into four pieces under this action. In Section 6, we will study the case $F = \mathbf{Q}(\sqrt{5})$ in detail and will give many examples. In Section 7, we will study the case $F = \mathbf{Q}(\sqrt{13})$. We devote Section 8 to the comparison of the two methods mentioned above. In Section 9, we will show that it is possible to deduce some information on the components of the cocycle $f(\Omega)$ which are not related to critical values in certain cases.

A more detailed version of this paper (except for Section 9) is available (see [Y4]).

NOTATION

For an associative ring A with identity element, A^\times denotes the group of all invertible elements of A . Let R be a commutative ring with identity element. We denote by $M(n, R)$ the ring of all $(n \times n)$ -matrices with entries in R . We define $\mathrm{GL}(n, R) = M(n, R)^\times$, $\mathrm{SL}(n, R) = \{g \in \mathrm{GL}(n, R) \mid \det g = 1\}$. The quotient group of $\mathrm{GL}(n, R)$ (resp., $\mathrm{SL}(n, R)$) by its center is denoted by $\mathrm{PGL}(n, R)$ (resp., $\mathrm{PSL}(n, R)$). Let G be a group. The subgroup of G generated by $g_1, \dots, g_n \in G$ is denoted by $\langle g_1, \dots, g_n \rangle$. When G acts on a module M , M^G denotes the submodule of M consisting of all elements fixed by G . For an algebraic number field F , \mathcal{O}_F denotes the ring of integers of F . For $a \in \mathcal{O}_F$, the ideal $a\mathcal{O}_F$ generated by a is denoted by (a) . We denote by E_F the group of units of F ; that is, $E_F = \mathcal{O}_F^\times$. When F is totally real and $\alpha \in F$, $\alpha \gg 0$ means that α is totally positive. We denote by \mathfrak{H} the complex upper half-plane. The set of all positive real numbers is denoted by \mathbf{R}_+ .

1. Preparations on cohomology groups

In this section, we will review group cohomology. Most of the results, except for the results presented in Section 1.5, can be found in standard textbooks such as Cartan and Eilenberg [CE], Serre [Se1], and Suzuki [Su].

1.1

Let G be a group, and let M be a left G -module. We set $C^0(G, M) = M$, and for $0 < n \in \mathbf{Z}$, let $C^n(G, M)$ be the abelian group consisting of all mappings of G^n into M . We define the coboundary operator $d_n : C^n(G, M) \longrightarrow C^{n+1}(G, M)$ by the usual formula

$$(1.1) \quad \begin{aligned} (d_n f)(g_1, \dots, g_{n+1}) &= g_1 f(g_2, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n) \\ &\quad + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}). \end{aligned}$$

We set

$$Z^n(G, M) = \text{Ker}(d_n), \quad B^n(G, M) = \text{Im}(d_{n-1}).$$

Here we understand $B^0(G, M) = \{0\}$. An element of $C^n(G, M)$ (resp., $Z^n(G, M)$, resp., $B^n(G, M)$) is called an n -cochain (resp., n -cocycle, resp., n -coboundary). The cohomology group $H^n(G, M)$ is that of the complex $\{C^n(G, M), d_n\}$; that is, $H^n(G, M) = Z^n(G, M)/B^n(G, M)$.

Let G' be a group, and let M' be a left G' -module. Let $\varphi : G \rightarrow G'$ be a group homomorphism, and let $\psi : M' \rightarrow M$ be a homomorphism of abelian groups. We assume that φ and ψ are compatible; that is,

$$\psi(\varphi(g)m') = g(\psi(m')), \quad m' \in M', g \in G.$$

For $f \in C^n(G', M')$, define $\omega_n f \in C^n(G, M)$ by the formula

$$(1.2) \quad (\omega_n f)(g_1, g_2, \dots, g_n) = \psi(f(\varphi(g_1), \varphi(g_2), \dots, \varphi(g_n))).$$

Then ω_n sends $Z^n(G', M')$ (resp., $B^n(G', M')$) into $Z^n(G, M)$ (resp., $B^n(G, M)$) and induces a homomorphism $H^n(G', M') \rightarrow H^n(G, M)$.

Now let N be a subgroup of G . Let $g \in G$. We define

$$\varphi(n) = gng^{-1}, \quad n \in g^{-1}Ng, \quad \psi(m) = g^{-1}m, \quad m \in M.$$

Then φ is an isomorphism of $g^{-1}Ng$ onto N ; φ and ψ are compatible. Hence we obtain an isomorphism of $H^p(N, M)$ onto $H^p(g^{-1}Ng, M)$, which is induced by sending $f \in Z^p(N, M)$ to $f' \in Z^p(g^{-1}Ng, M)$:

$$(1.3) \quad f'(n_1, n_2, \dots, n_p) = g^{-1}f(gn_1g^{-1}, gn_2g^{-1}, \dots, gn_pg^{-1}).$$

1.2

Let H be a subgroup of G of finite index. An explicit form of the transfer map $T : H^n(H, M) \rightarrow H^n(G, M)$ is given as follows (cf. Eckmann [E], [Y4]).

PROPOSITION 1.1

Let G be a group, let H be a subgroup of finite index, and let M be a left G -module. Let $G = \bigsqcup_{i=1}^r x_i H$ be a coset decomposition. Let $f \in Z^n(H, M)$ be an n -cocycle representing $c \in H^n(H, M)$. Then an n -cocycle $\tilde{f} \in Z^n(G, M)$ which represents $T(c) \in H^n(G, M)$ is given by

$$\tilde{f}(g_1, g_2, \dots, g_n) = \sum_{i=1}^r x_i f(x_i^{-1}g_1x_{p_i(1)}, x_{p_i(1)}^{-1}g_2x_{p_i(2)}, \dots, x_{p_i(n-1)}^{-1}g_nx_{p_i(n)}).$$

Here $x_{p_i(l)}$ is chosen so that

$$x_i^{-1}g_1x_{p_i(1)} \in H, \quad x_{p_i(l-1)}^{-1}g_lx_{p_i(l)} \in H, \quad 2 \leq l \leq n.$$

Let $\text{Res} : H^n(G, M) \rightarrow H^n(H, M)$ be the restriction homomorphism. Then we have the well-known result:

$$(1.4) \quad T \circ \text{Res}(c) = [G : H]c, \quad c \in H^n(G, M).$$

1.3

We consider the action of Hecke operators on cohomology groups. Let \tilde{G} be a group, and let G be a subgroup. Let M be a \tilde{G} -module. We assume that G and tGt^{-1} are commensurable for every $t \in \tilde{G}$. For $t \in \tilde{G}$, we put

$$G_t = G \cap t^{-1}Gt.$$

Let

$$\text{conj} : H^n(G, M) \longrightarrow H^n(t^{-1}Gt, M)$$

be the isomorphism induced by (1.3). Let Res be the restriction map from $H^n(t^{-1}Gt, M)$ to $H^n(G_t, M)$, and let $T : H^n(G_t, M) \longrightarrow H^n(G, M)$ be the transfer map. Then we define

$$(1.5) \quad [GtG] = T \circ \text{Res} \circ \text{conj}.$$

(It is not difficult to check that the right-hand side of (1.5) depends only on the double coset GtG and that (1.5) defines a homomorphism of the Hecke ring $H(G, \tilde{G})^*$ into $\text{End}(H^n(G, M))$.) An explicit form of this operator when $n = 2$ is given as follows (cf. [Y4]).

PROPOSITION 1.2

Let $c \in H^2(G, M)$, and let $f \in Z^2(G, M)$ be a 2-cocycle representing c . Let $GtG = \bigsqcup_{i=1}^d G\beta_i$ be a coset decomposition. Then a 2-cocycle $h \in Z^2(G, M)$ representing $[GtG](c)$ is given by

$$h(g_1, g_2) = \sum_{i=1}^d \beta_i^{-1} f(\beta_i g_1 \beta_{j(i)}^{-1}, \beta_{j(i)} g_2 \beta_{k(i)}^{-1}).$$

Here, for $1 \leq i \leq d$, we choose $j(i)$ and $k(i)$ such that

$$\beta_i g_1 \beta_{j(i)}^{-1} \in G, \quad \beta_i g_2 \beta_{k(i)}^{-1} \in G.$$

1.4

Let G be a group, and let M be a left G -module. Let N be a normal subgroup of G . Then we have the Hochschild–Serre spectral sequence

$$(1.6) \quad E_2^{p,q} = H^p(G/N, H^q(N, M)) \implies H^n(G, M).$$

In low dimensions, this gives an exact sequence

$$(1.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^1(G/N, M^N) & \longrightarrow & H^1(G, M) & \longrightarrow & H^1(N, M)^{G/N} \\ & & \longrightarrow & H^2(G/N, M^N) & \longrightarrow & H^2(G, M). \end{array}$$

Now we describe a method to calculate $H^2(G, M)$, which is originally due to MacLane (cf. [K, Section 50]). Taking a free group \mathcal{F} , we write $G = \mathcal{F}/R$. Let $\pi : \mathcal{F} \longrightarrow G$ be the canonical homomorphism such that $\text{Ker}(\pi) = R$. We regard

*For the definition of the Hecke ring, see [Sh5, Chapter 3, Section 1].

M as an \mathcal{F} -module by $gm = \pi(g)m$, $g \in \mathcal{F}$, $m \in M$. Since

$$(1.8) \quad H^i(\mathcal{F}, M) = 0, \quad i \geq 2,$$

(1.7) yields an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1(G, M) \longrightarrow H^1(\mathcal{F}, M) \longrightarrow H^1(R, M)^G \\ &\longrightarrow H^2(G, M) \longrightarrow 0. \end{aligned}$$

Therefore we have

$$(1.9) \quad H^2(G, M) \cong H^1(R, M)^G / \text{Im}(H^1(\mathcal{F}, M)).$$

Since R acts on M trivially, we have $B^1(R, M) = 0$ and $H^1(R, M) = \text{Hom}(R, M)$. Therefore we have

$$H^1(R, M)^G = \{ \varphi \in \text{Hom}(R, M) \mid \varphi(grg^{-1}) = g\varphi(r), \quad g \in \mathcal{F}, r \in R \}.$$

The isomorphism (1.9) is explicitly given as follows. For $g \in \mathcal{F}$, we put $\pi(g) = \bar{g}$. Take a 2-cocycle $f \in Z^2(G, M)$. The mapping $(g_1, g_2) \longrightarrow f(\bar{g}_1, \bar{g}_2)$ is an M -valued 2-cocycle of \mathcal{F} . By (1.8), there exists a 1-cochain $a \in C^1(\mathcal{F}, M)$ such that

$$(1.10) \quad f(\bar{g}_1, \bar{g}_2) = g_1 a(g_2) + a(g_1) - a(g_1 g_2), \quad g_1, g_2 \in \mathcal{F}.$$

Let $\varphi = a|_R$, the restriction of a to R . We may assume that f is normalized; that is,

$$f(1, g) = f(g, 1) = 0 \quad \text{for all } g \in G.$$

If $r_1, r_2 \in R$, then, by (1.10), we have

$$a(r_2) + a(r_1) - a(r_1 r_2) = 0.$$

Therefore we get $\varphi \in Z^1(R, M) = \text{Hom}(R, M)$. By (1.10), we have

$$(1.11) \quad a(gr) = ga(r) + a(g), \quad g \in \mathcal{F}, r \in R.$$

Again by (1.10), we have

$$\begin{aligned} a(grg^{-1}) &= gra(g^{-1}) + a(gr) - f(\bar{g}, \bar{g}^{-1}) \\ &= ga(g^{-1}) + ga(r) + a(g) - f(\bar{g}, \bar{g}^{-1}) \end{aligned}$$

for $g \in \mathcal{F}$, $r \in R$. Using (1.10) with $g_1 = g$, $g_2 = g^{-1}$ and noting $a(1) = 0$, we obtain

$$(1.12) \quad \varphi(grg^{-1}) = g\varphi(r), \quad g \in \mathcal{F}, r \in R.$$

This formula shows that φ belongs to $H^1(R, M)^G$. Suppose that a' is another 1-cochain satisfying (1.10). Put $\varphi' = a'|_R$, $a' = a + b$. Then $b \in Z^1(\mathcal{F}, M)$. Hence the classes of φ and φ' in $H^1(R, M)^G / \text{Im}(H^1(\mathcal{F}, M))$ are the same. Suppose that we add the coboundary of a 1-cochain c to f . Then (1.10) holds when we replace $a(g)$ by $a(g) + c(\bar{g})$. Then $a|_R$ does not change. Thus we have defined a homomorphism

$$\omega : H^2(G, M) \longrightarrow H^1(R, M)^G / \text{Im}(H^1(\mathcal{F}, M)).$$

We can verify without difficulty that ω is a surjective isomorphism.

1.5

Let $f \in Z^2(G, M)$ be a normalized cocycle. Take $a \in C^1(\mathcal{F}, M)$ which satisfies (1.10), and put $\varphi = a \mid R \in H^1(R, M)^G$. For every $g \in G$, we choose $\tilde{g} \in \mathcal{F}$ such that $\pi(\tilde{g}) = g$. The formula (1.10) can be written as

$$f(g_1, g_2) = g_1 a(\tilde{g}_2) + a(\tilde{g}_1) - a(\tilde{g}_1 \tilde{g}_2), \quad g_1, g_2 \in G.$$

By (1.11), we have

$$a(\widetilde{g_1 g_2} (\widetilde{g_1 g_2})^{-1} \tilde{g}_1 \tilde{g}_2) = g_1 g_2 \varphi((\widetilde{g_1 g_2})^{-1} \tilde{g}_1 \tilde{g}_2) + a(\widetilde{g_1 g_2}).$$

Then, using (1.12), we have

$$a(\tilde{g}_1 \tilde{g}_2) = a(\widetilde{g_1 g_2}) + \varphi(\tilde{g}_1 \tilde{g}_2 (\widetilde{g_1 g_2})^{-1}).$$

Therefore we obtain

$$(1.13) \quad f(g_1, g_2) = g_1 a(\tilde{g}_2) + a(\tilde{g}_1) - a(\widetilde{g_1 g_2}) - \varphi(\tilde{g}_1 \tilde{g}_2 (\widetilde{g_1 g_2})^{-1}), \quad g_1, g_2 \in G.$$

This formula shows that, adding a coboundary to f , we may assume that

$$(1.14) \quad f(g_1, g_2) = -\varphi(\tilde{g}_1 \tilde{g}_2 (\widetilde{g_1 g_2})^{-1}).$$

Conversely we note the following lemma.

LEMMA 1.3

Let $\varphi \in H^1(R, M)^G$. For $g_1, g_2 \in G$, define $f(g_1, g_2)$ by (1.14). Then $f \in Z^2(G, M)$. If $\tilde{1} = 1$, f is normalized.

Proof

The cocycle condition is

$$g_1 f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = 0.$$

We have

$$\begin{aligned} & g_1 \varphi(\tilde{g}_2 \tilde{g}_3 (\widetilde{g_2 g_3})^{-1}) - \varphi(\widetilde{g_1 g_2} \tilde{g}_3 (\widetilde{g_1 g_2 g_3})^{-1}) + \varphi(\tilde{g}_1 \widetilde{g_2 g_3} (\widetilde{g_1 g_2 g_3})^{-1}) \\ & \quad - \varphi(\tilde{g}_1 \tilde{g}_2 (\widetilde{g_1 g_2})^{-1}) \\ & = \varphi(\tilde{g}_1 \tilde{g}_2 \tilde{g}_3 (\widetilde{g_2 g_3})^{-1} \tilde{g}_1^{-1}) + \varphi(\tilde{g}_1 \widetilde{g_2 g_3} (\widetilde{g_1 g_2 g_3})^{-1}) + \varphi(\widetilde{g_1 g_2} \tilde{g}_3 \tilde{g}_3^{-1} (\widetilde{g_1 g_2})^{-1}) \\ & \quad + \varphi(\widetilde{g_1 g_2} \tilde{g}_2^{-1} \tilde{g}_1^{-1}) \\ & = \varphi(\tilde{g}_1 \tilde{g}_2 \tilde{g}_3 (\widetilde{g_2 g_3})^{-1} \tilde{g}_1^{-1}) + \varphi(\tilde{g}_1 \widetilde{g_2 g_3} \tilde{g}_3^{-1} (\widetilde{g_1 g_2})^{-1}) + \varphi(\widetilde{g_1 g_2} \tilde{g}_2^{-1} \tilde{g}_1^{-1}) \\ & = \varphi(\tilde{g}_1 \tilde{g}_2 (\widetilde{g_1 g_2})^{-1}) + \varphi(\widetilde{g_1 g_2} \tilde{g}_2^{-1} \tilde{g}_1^{-1}) = 0. \end{aligned}$$

Hence the cocycle condition holds. The latter assertion is obvious. This completes the proof. \square

We write the action of Hecke operators on the right-hand side of (1.9) explicitly. Let the notation be the same as in Sections 1.3 and 1.4. Let $f \in Z^2(G, M)$ be a normalized 2-cocycle of the cohomology class c . Let h be the 2-cocycle given

by Proposition 1.2 which represents the class $[GtG](c)$. Clearly h is normalized. There exists a 1-cochain $b \in C^1(\mathcal{F}, M)$ such that

$$h(\bar{g}_1, \bar{g}_2) = g_1 b(g_2) + b(g_1) - b(g_1 g_2), \quad g_1, g_2 \in \mathcal{F}.$$

PROPOSITION 1.4

Let $\varphi \in H^1(R, M)^G$, and let a normalized 2-cocycle $f \in Z^2(G, M)$ be given by (1.14). Suppose that $g_j \in G$ are given for $1 \leq j \leq m$. For every j , we define a permutation on d letters $p_j \in S_d$ by

$$\beta_i g_j \beta_{p_j(i)}^{-1} \in G, \quad 1 \leq i \leq d.$$

We define $q_j \in S_d$ inductively by

$$q_1 = p_1, \quad q_k = p_k q_{k-1}, \quad 2 \leq k \leq m.$$

We assume that $b(\tilde{g}_j) = 0$ for $1 \leq j \leq m$. Then we have

$$\begin{aligned} (1.15) \quad & b(\tilde{g}_1 \tilde{g}_2 \cdots \tilde{g}_m) \\ &= \sum_{i=1}^d \beta_i^{-1} \varphi(\widetilde{\beta_i g_1 \beta_{q_1(i)}^{-1}} \widetilde{\beta_{q_1(i)} g_2 \beta_{q_2(i)}^{-1}} \cdots \\ & \quad \times \widetilde{\beta_{q_{m-1}(i)} g_m \beta_{q_m(i)}^{-1}} (\beta_i g_1 g_2 \cdots g_m \beta_{q_m(i)}^{-1})^{-1}). \end{aligned}$$

Proof

If $m = 1$, the left-hand side of (1.15) is zero and the right-hand side is zero since $\varphi(1) = 0$. We assume that $m \geq 2$ and the formula is valid for $m - 1$. Then, by Proposition 1.2 and (1.14), we have

$$\begin{aligned} & b(\tilde{g}_1 \tilde{g}_2 \cdots \tilde{g}_{m-1} \tilde{g}_m) \\ &= g_1 g_2 \cdots g_{m-1} b(\tilde{g}_m) + b(\tilde{g}_1 \tilde{g}_2 \cdots \tilde{g}_{m-1}) - h(g_1 \cdots g_{m-1}, g_m) \\ &= \sum_{i=1}^d \beta_i^{-1} \varphi(\widetilde{\beta_i g_1 \beta_{q_1(i)}^{-1}} \cdots \widetilde{\beta_{q_{m-2}(i)} g_{m-1} \beta_{q_{m-1}(i)}^{-1}} (\beta_i g_1 g_2 \cdots g_{m-1} \beta_{q_{m-1}(i)}^{-1})^{-1}) \\ & \quad + \sum_{i=1}^d \beta_i^{-1} \varphi(\widetilde{\beta_i g_1 g_2 \cdots g_{m-1} \beta_{q_{m-1}(i)}^{-1}} \\ & \quad \times \widetilde{\beta_{q_{m-1}(i)} g_m \beta_{q_m(i)}^{-1}} (\beta_i g_1 g_2 \cdots g_m \beta_{q_m(i)}^{-1})^{-1}) \\ &= \sum_{i=1}^d \beta_i^{-1} \varphi(\widetilde{\beta_i g_1 \beta_{q_1(i)}^{-1}} \cdots \widetilde{\beta_{q_{m-1}(i)} g_m \beta_{q_m(i)}^{-1}} (\beta_i g_1 g_2 \cdots g_m \beta_{q_m(i)}^{-1})^{-1}) \end{aligned}$$

since $b(\tilde{g}_m) = 0$. This completes the proof. \square

We have

$$b(g_1 g_2) = g_1 b(g_2) + b(g_1) - h(\bar{g}_1, \bar{g}_2), \quad g_1, g_2 \in \mathcal{F}.$$

We may take $b(g) = 0$ for a fixed set of generators of \mathcal{F} , and we can apply the above formula to determine the value of $b(g)$ according to the length of $g \in \mathcal{F}$. But Proposition 1.5 is useful beyond this case, as will be seen after Section 5.

2. Hilbert modular forms

2.1

In this subsection, we follow the exposition given in Shimura [Sh3]. Let F be a totally real algebraic number field of degree n . Let \mathfrak{d}_F denote the difference of F over \mathbf{Q} , and let $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ be the set of all isomorphisms of F into \mathbf{R} . For $\xi \in F$, we put $\xi^{(\nu)} = \xi^{\sigma_\nu}$. For $z = (z_1, z_2, \dots, z_n) \in \mathfrak{H}^n$, we put

$$\mathbf{e}_F(\xi z) = \exp\left(2\pi i \sum_{\nu=1}^n \xi^{(\nu)} z_\nu\right).$$

Let $k = (k_1, k_2, \dots, k_n) \in \mathbf{Z}^n$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{R})_+$ and $z \in \mathfrak{H}$, we put $gz = (az + b)/(cz + d)$, $j(g, z) = cz + d$, where $\mathrm{GL}(2, \mathbf{R})_+ = \{g \in \mathrm{GL}(2, \mathbf{R}) \mid \det g > 0\}$; $\mathrm{GL}(2, \mathbf{R})_+^n$ acts on \mathfrak{H}^n . For a function Ω on \mathfrak{H}^n , $g = (g_1, \dots, g_n) \in \mathrm{GL}(2, \mathbf{R})_+^n$ and $z = (z_1, \dots, z_n) \in \mathfrak{H}^n$, we define a function $\Omega|_k g$ on \mathfrak{H}^n by the formula

$$(\Omega|_k g)(z) = \prod_{\nu=1}^n \det(g_\nu)^{k_\nu/2} j(g_\nu, z_\nu)^{-k_\nu} \Omega(gz).$$

We embed $\mathrm{GL}(2, F)$ into $\mathrm{GL}(2, \mathbf{R})^n$ by

$$\mathrm{GL}(2, F) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix} \right) \in \mathrm{GL}(2, \mathbf{R})^n.$$

Let Γ be a congruence subgroup of $\mathrm{SL}(2, \mathcal{O}_F)$. A holomorphic function Ω on \mathfrak{H}^n is called a Hilbert modular form of weight k with respect to Γ if

$$\Omega|_k \gamma = \Omega$$

holds for every $\gamma \in \Gamma$, and the usual conditions hold at cusps when $F = \mathbf{Q}$. For every $g \in \mathrm{SL}(2, F)$, $\Omega|_k g$ has a Fourier expansion of the form $(\Omega|_k g)(z) = \sum_{\xi \in L} a_g(\xi) \mathbf{e}_F(\xi z)$, where L is a lattice in F . We have $a_g(\xi) = 0$ if $\xi \neq 0$ is not totally positive. We call Ω a *cusp form* if the constant term $a_g(0)$ vanishes for every $g \in \mathrm{SL}(2, F)$. We denote the space of Hilbert modular forms (resp., cusp forms) of weight k with respect to Γ by $M_k(\Gamma) = M_{k_1, k_2, \dots, k_n}(\Gamma)$ (resp., $S_k(\Gamma) = S_{k_1, k_2, \dots, k_n}(\Gamma)$).

Hereafter until the end of this subsection, we assume that $\Gamma = \mathrm{SL}(2, \mathcal{O}_F)$ and $0 \neq \Omega \in S_k(\Gamma)$. The Fourier expansion of Ω takes the form

$$(2.1) \quad \Omega(z) = \sum_{0 \ll \xi \in \mathfrak{o}_F^{-1}} a(\xi) \mathbf{e}_F(\xi z).$$

Since $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \Gamma$ for $u \in E_F$, we have

$$u^k \sum_{0 \ll \xi \in \mathfrak{d}_F^{-1}} a(\xi) \mathbf{e}_F(\xi u^2 z) = \sum_{0 \ll \xi \in \mathfrak{d}_F^{-1}} a(\xi) \mathbf{e}_F(\xi z),$$

where we put $u^k = \prod_{\nu=1}^n (u^{(\nu)})^{k_\nu}$. Therefore we have

$$(2.2) \quad a(u^2 \xi) = u^k a(\xi), \quad u \in E_F.$$

In particular, taking $u = -1$, we have

$$(2.3) \quad \sum_{\nu=1}^n k_\nu \equiv 0 \pmod{2}.$$

For the sake of simplicity, we assume that

$$(A) \quad u^k > 0 \quad \text{for every } u \in E_F.$$

Put

$$k_0 = \max(k_1, k_2, \dots, k_n), \quad k'_\nu = k_0 - k_\nu, \quad k' = (k'_1, k'_2, \dots, k'_n).$$

We define the L -function of Ω by

$$(2.4) \quad L(s, \Omega) = \sum_{\xi \in E_F^2} a(\xi) \xi^{k'/2} N(\xi)^{-s}, \quad \xi^{k'/2} = \prod_{\nu=1}^n (\xi^{(\nu)})^{k'_\nu/2}.$$

Here the summation extends over all cosets ξE_F^2 with ξ satisfying $0 \ll \xi \in \mathfrak{d}_F^{-1}$. By (2.2) and (A), we see that the sum is well defined. The series (2.4) converges when $\Re(s)$ is sufficiently large. We put

$$(2.5) \quad R(s, \Omega) = (2\pi)^{-ns} \prod_{\nu=1}^n \Gamma\left(s - \frac{k'_\nu}{2}\right) L(s, \Omega).$$

By the standard calculation, we obtain the integral representation

$$(2.6) \quad \int_{\mathbf{R}_+^n / E_F^2} \Omega(iy_1, iy_2, \dots, iy_n) \prod_{\nu=1}^n y_\nu^{s-k'_\nu/2-1} dy_\nu = (2\pi)^{\sum_{\nu=1}^n k'_\nu/2} R(s, \Omega)$$

when $\Re(s)$ is sufficiently large. By a suitable transformation of this integral, we can show that $R(s, \Omega)$ is an entire function of s and satisfies the functional equation

$$(2.7) \quad R(s, \Omega) = (-1)^{\sum_{\nu=1}^n k_\nu/2} R(k_0 - s, \Omega).$$

2.2

In [Y3, Chapter V, Section 5], we gave an explicit method to attach a cohomology class to a Hilbert modular form. We will review it in this subsection. For $0 \leq l \in \mathbf{Z}$ and $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathbf{C}^2$, put

$$\begin{bmatrix} u \\ v \end{bmatrix}^l = {}^t(u^l u^{l-1} v \dots uv^{l-1} v^l).$$

Define a representation $\rho_l : \mathrm{GL}(2, \mathbf{C}) \longrightarrow \mathrm{GL}(l+1, \mathbf{C})$ by

$$\rho_l(g) \begin{bmatrix} u \\ v \end{bmatrix}^l = \left(g \begin{bmatrix} u \\ v \end{bmatrix} \right)^l.$$

Let Γ be a congruence subgroup of $\mathrm{SL}(2, \mathcal{O}_F)$. Let l_1, l_2, \dots, l_n be nonnegative integers. Let V be the representation space of $\rho_{l_1} \otimes \rho_{l_2} \otimes \dots \otimes \rho_{l_n}$. Let $\Omega \in M_{l_1+2, l_2+2, \dots, l_n+2}(\Gamma)$ be a Hilbert modular form of weight $(l_1+2, l_2+2, \dots, l_n+2)$. Define a holomorphic V -valued n -form $\mathfrak{d}(\Omega)$ on \mathfrak{H}^n by

$$(2.8) \quad \mathfrak{d}(\Omega) = \Omega(z) \begin{bmatrix} z_1 \\ 1 \end{bmatrix}^{l_1} \otimes \begin{bmatrix} z_2 \\ 1 \end{bmatrix}^{l_2} \otimes \dots \otimes \begin{bmatrix} z_n \\ 1 \end{bmatrix}^{l_n} dz_1 dz_2 \dots dz_n.$$

We put $\rho = \rho_{l_1} \otimes \rho_{l_2} \otimes \dots \otimes \rho_{l_n}$.

Let $g = (g_1, \dots, g_n) \in \mathrm{GL}(2, \mathbf{R})_+^n$. Under the action of g on \mathfrak{H}^n , $\mathfrak{d}(\Omega)$ transforms to $\mathfrak{d}(\Omega) \circ g$, where

$$\mathfrak{d}(\Omega) \circ g = \Omega(g(z)) \begin{bmatrix} g_1 z_1 \\ 1 \end{bmatrix}^{l_1} \otimes \dots \otimes \begin{bmatrix} g_n z_n \\ 1 \end{bmatrix}^{l_n} (dz_1 \circ g_1) \dots (dz_n \circ g_n).$$

By an easy calculation, we obtain

$$(2.9_a) \quad \mathfrak{d}(\Omega) \circ g = \prod_{\nu=1}^n (\det g_\nu)^{-l_\nu/2} \rho(g) \mathfrak{d}(\Omega|_k g), \quad g \in \mathrm{GL}(2, \mathbf{R})_+^n \cap \mathrm{GL}(2, F).$$

In particular, we have

$$(2.9_b) \quad \mathfrak{d}(\Omega) \circ \gamma = \rho(\gamma) \mathfrak{d}(\Omega), \quad \gamma \in \Gamma.$$

We discuss the case $n = 2$ in detail. Take $w = (w_1, w_2) \in \mathfrak{H}^2$. For $z = (z_1, z_2) \in \mathfrak{H}^2$, we put

$$(2.10) \quad F(z) = \int_{w_1}^{z_1} \int_{w_2}^{z_2} \mathfrak{d}(\Omega),$$

a period integral of Eichler–Shimura type. Let \mathcal{H} denote the vector space of all V -valued holomorphic functions on \mathfrak{H}^2 . For $\varphi \in \mathcal{H}$ and $\gamma \in \Gamma$, we define a function $\gamma\varphi$ on \mathfrak{H}^2 by

$$(2.11) \quad (\gamma\varphi)(z) = \rho(\gamma)\varphi(\gamma^{-1}z).$$

Then \mathcal{H} becomes a left Γ -module. Since

$$\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} (\gamma F - F) = 0,$$

we can write

$$\gamma F - F = g(\gamma; z_1) + h(\gamma; z_2),$$

where $g(\gamma; z_1) \in \mathcal{H}$ (resp., $h(\gamma; z_2) \in \mathcal{H}$) is a function which depends only on z_1 (resp., z_2) (cf. [Y3, Lemma 5.1, p. 208]). We regard g and h as 1-cochains in $C^1(\Gamma, \mathcal{H})$. Then clearly we have (d_1 in Section 1.1 is abbreviated to d)

$$dg(\gamma_1, \gamma_2; z_1) + dh(\gamma_1, \gamma_2; z_2) = 0.$$

Put

$$f(\Omega)(\gamma_1, \gamma_2) = dg(\gamma_1, \gamma_2; z_1).$$

We abbreviate $f(\Omega)$ to f . We see that $f(\gamma_1, \gamma_2) \in V$ is a constant. Furthermore, in \mathcal{H} , f is a coboundary. Hence f satisfies the cocycle condition

$$(2.12) \quad \gamma_1 f(\gamma_2, \gamma_3) - f(\gamma_1 \gamma_2, \gamma_3) + f(\gamma_1, \gamma_2 \gamma_3) - f(\gamma_1, \gamma_2) = 0.$$

The 2-cocycle f determines a cohomology class in $H^2(\Gamma, V)$.

Let us give an explicit formula for f . For $x \in F$, let x' denote the conjugate of x over \mathbf{Q} . For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, let $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. We regard $\gamma, \gamma' \in \mathrm{SL}(2, \mathbf{R})$. Then, for $\gamma \in \Gamma$, we have

$$\begin{aligned} F(\gamma(z)) &= F(\gamma z_1, \gamma' z_2) = \int_{w_1}^{\gamma z_1} \int_{w_2}^{\gamma' z_2} \mathfrak{d}(\Omega) \\ &= \int_{\gamma w_1}^{\gamma z_1} \int_{\gamma' w_2}^{\gamma' z_2} \mathfrak{d}(\Omega) + \int_{\gamma w_1}^{\gamma z_1} \int_{w_2}^{\gamma' w_2} \mathfrak{d}(\Omega) + \int_{w_1}^{\gamma w_1} \int_{w_2}^{\gamma' z_2} \mathfrak{d}(\Omega) \\ &= (\rho_{l_1}(\gamma) \otimes \rho_{l_2}(\gamma')) F(z) + \int_{\gamma w_1}^{\gamma z_1} \int_{w_2}^{\gamma' w_2} \mathfrak{d}(\Omega) + \int_{w_1}^{\gamma w_1} \int_{w_2}^{\gamma' z_2} \mathfrak{d}(\Omega). \end{aligned}$$

Substituting z for $\gamma^{-1}z$ in this formula, we get

$$(2.13) \quad (\rho_{l_1}(\gamma) \otimes \rho_{l_2}(\gamma')) F(\gamma^{-1}z) - F(z) = - \int_{\gamma w_1}^{z_1} \int_{w_2}^{\gamma' w_2} \mathfrak{d}(\Omega) - \int_{w_1}^{\gamma w_1} \int_{w_2}^{z_2} \mathfrak{d}(\Omega).$$

We may take

$$(2.14) \quad g(\gamma; z_1) = - \int_{\gamma w_1}^{z_1} \int_{w_2}^{\gamma' w_2} \mathfrak{d}(\Omega),$$

$$(2.15) \quad h(\gamma; z_2) = - \int_{w_1}^{\gamma w_1} \int_{w_2}^{z_2} \mathfrak{d}(\Omega).$$

For $\gamma_1, \gamma_2 \in \Gamma$, we have

$$(2.16) \quad f(\gamma_1, \gamma_2) = (\gamma_1 g)(\gamma_2; z_1) - g(\gamma_1 \gamma_2; z_1) + g(\gamma_1; z_1),$$

$$(2.17) \quad f(\gamma_1, \gamma_2) = - \{ (\gamma_1 h)(\gamma_2; z_2) - h(\gamma_1 \gamma_2; z_2) + h(\gamma_1; z_2) \}.$$

By (2.14) and (2.16), we have

$$\begin{aligned} f(\gamma_1, \gamma_2) &= (\rho_{l_1}(\gamma_1) \otimes \rho_{l_2}(\gamma'_1)) g(\gamma_2; \gamma_1^{-1} z_1) - g(\gamma_1 \gamma_2; z_1) + g(\gamma_1; z_1) \\ &= - (\rho_{l_1}(\gamma_1) \otimes \rho_{l_2}(\gamma'_1)) \int_{\gamma_2 w_1}^{\gamma_1^{-1} z_1} \int_{w_2}^{\gamma'_2 w_2} \mathfrak{d}(\Omega) \\ &\quad + \int_{\gamma_1 \gamma_2 w_1}^{z_1} \int_{w_2}^{\gamma'_1 \gamma'_2 w_2} \mathfrak{d}(\Omega) - \int_{\gamma_1 w_1}^{z_1} \int_{w_2}^{\gamma'_1 w_2} \mathfrak{d}(\Omega) \\ &= - \int_{\gamma_1 \gamma_2 w_1}^{z_1} \int_{\gamma'_1 w_2}^{\gamma'_1 \gamma'_2 w_2} \mathfrak{d}(\Omega) + \int_{\gamma_1 \gamma_2 w_1}^{z_1} \int_{w_2}^{\gamma'_1 \gamma'_2 w_2} \mathfrak{d}(\Omega) - \int_{\gamma_1 w_1}^{z_1} \int_{w_2}^{\gamma'_1 w_2} \mathfrak{d}(\Omega) \end{aligned}$$

$$\begin{aligned}
&= \int_{\gamma_1 \gamma_2 w_1}^{z_1} \int_{w_2}^{\gamma_1' w_2} \mathfrak{d}(\Omega) - \int_{\gamma_1 w_1}^{z_1} \int_{w_2}^{\gamma_1' w_2} \mathfrak{d}(\Omega) \\
&= \int_{\gamma_1 \gamma_2 w_1}^{\gamma_1 w_1} \int_{w_2}^{\gamma_1' w_2} \mathfrak{d}(\Omega)
\end{aligned}$$

using (2.9_b). Thus we obtain an explicit formula

$$(2.18) \quad f(\gamma_1, \gamma_2) = \int_{\gamma_1 \gamma_2 w_1}^{\gamma_1 w_1} \int_{w_2}^{\gamma_1' w_2} \mathfrak{d}(\Omega).$$

By (2.9_b), (2.18) can be written as

$$(2.19) \quad f(\gamma_1, \gamma_2) = (\rho_{l_1}(\gamma_1) \otimes \rho_{l_2}(\gamma_1')) \int_{\gamma_2 w_1}^{w_1} \int_{\gamma_1'^{-1} w_2}^{w_2} \mathfrak{d}(\Omega).$$

Suppose that w_1 is replaced by w_1^* , w_2 remaining the same. Then $g(\gamma; z_1)$ changes to $g(\gamma, z_1) + a(\gamma)$, where

$$a(\gamma) = \int_{\gamma w_1}^{\gamma w_1^*} \int_{w_2}^{\gamma' w_2} \mathfrak{d}(\Omega).$$

Hence $f(\gamma_1, \gamma_2)$ changes to $f(\gamma_1, \gamma_2) + \gamma_1 a(\gamma_2) - a(\gamma_1 \gamma_2) + a(\gamma_1)$. Suppose that w_2 is replaced by w_2^* , w_1 remaining the same. Then $h(\gamma; z_2)$ changes to $h(\gamma, z_2) + b(\gamma)$, where

$$b(\gamma) = \int_{w_1}^{\gamma w_1} \int_{w_2}^{w_2^*} \mathfrak{d}(\Omega).$$

By (2.17), $f(\gamma_1, \gamma_2)$ changes to $f(\gamma_1, \gamma_2) - \gamma_1 b(\gamma_2) + b(\gamma_1 \gamma_2) - b(\gamma_1)$. Therefore the cohomology class of f does not depend on the choice of the “base points” w_1, w_2 .

Put $\bar{\Gamma} = \Gamma / (\{\pm 1_2\} \cap \Gamma)$. By (2.18), we see that f can be regarded as a 2-cocycle of $\bar{\Gamma}$ taking values in V . Depending on the context, we consider f as a 2-cocycle on $\bar{\Gamma}$. We see that the cocycle f is normalized; that is,

$$(2.20) \quad f(1, \gamma) = f(\gamma, 1) = 0 \quad \text{for every } \gamma \in \bar{\Gamma}.$$

Now assume that Ω is a cusp form. Then the cocycle $f = f(\Omega)$ satisfies the “parabolic condition.” Namely, let $q \in \Gamma$ be a parabolic element, and let $w^* = (w_1^*, w_2^*)$ be the fixed point of q' . Since Ω is a cusp form, we may replace w_2 by w_2^* .[†] Let f^* be the cocycle obtained from (w_1, w_2^*) . We have

$$f^*(\gamma_1, \gamma_2) = f(\gamma_1, \gamma_2) - \gamma_1 b(\gamma_2) + b(\gamma_1 \gamma_2) - b(\gamma_1)$$

with a 1-cochain b and $f^*(q, \gamma) = 0$. Therefore

$$f(q, \gamma) = qb(\gamma) - b(q\gamma) + b(q), \quad \gamma \in \Gamma;$$

[†]For every $g \in \mathrm{SL}(2, F)$, we have the Fourier expansion $(\Omega|_k g)(z) = \sum_{0 \ll \xi \in L} a_g(\xi) \mathbf{e}_F(\xi z)$ where L is a lattice in F . We have the estimate $|a_g(\xi)| \leq M \xi^{k_1/2} \xi'^{k_2/2}$ with a positive constant M depending on Ω and g (cf. [Sh7, Proposition A6.4, p. 280]). Using this estimate, it is not difficult to check the absolute convergence of the integral (2.10) defining $F(z)$ when w_2 is replaced by w_2^* .

that is, $f(q, \gamma)$ is of the form of a coboundary whenever q is parabolic. A similar argument applies to $f(\gamma, q)$.

2.3

We investigate closely the relation between the critical values of the L -function $L(s, \Omega)$ and the cocycle $f(\Omega)$. Until the end of this subsection, we assume $\Gamma = \mathrm{SL}(2, \mathcal{O}_F)$. Let ϵ be the fundamental unit of F . We put

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}.$$

We regard σ and μ as elements of $\bar{\Gamma}$. Taking $\gamma_1 = \gamma_2 = \gamma_3 = \sigma$ in (2.12), we obtain

$$(2.21) \quad \sigma f(\sigma, \sigma) = f(\sigma, \sigma)$$

in view of (2.20). As the base points, we choose

$$w_1 = i\epsilon^{-1}, \quad w_2 = i\infty.$$

By (2.18), we get

$$(2.22) \quad f(\sigma, \mu) = f(\sigma, \sigma) = - \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \mathfrak{d}(\Omega).$$

Put

$$P = \left\{ \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} \mid u \in E_F, v \in \mathcal{O}_F \right\} \subset \Gamma.$$

By (2.18), we get

$$(2.23) \quad f(p, \gamma) = 0 \quad \text{for every } p \in P, \gamma \in \Gamma$$

since we have $pw_2 = w_2$ for $p \in P$. Taking $\gamma_1 = p \in P$ in (2.12), we obtain

$$(2.24) \quad f(p\gamma_1, \gamma_2) = pf(\gamma_1, \gamma_2) \quad \text{for every } p \in P, \gamma_1, \gamma_2 \in \Gamma.$$

This is the parabolic condition for $\Gamma = \mathrm{SL}(2, \mathcal{O}_F)$ and will play a crucial role in the succeeding sections.

For $0 \leq s \leq l_1$, $0 \leq t \leq l_2$, we put

$$(2.25) \quad P_{s,t} = \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \Omega(z) z_1^s z_2^t dz_1 dz_2.$$

The components of $f(\sigma, \sigma)$ are given by $-P_{s,t}$. The condition $\sigma f(\sigma, \sigma) = f(\sigma, \sigma)$ is equivalent to

$$(2.26) \quad P_{s,t} = (-1)^{l_1+l_2-s-t} P_{l_1-s, l_2-t}.$$

Put $k_1 = l_1 + 2$, $k_2 = l_2 + 2$. By (2.3), we have

$$(2.27) \quad l_1 \equiv l_2 \pmod{2}.$$

We assume that $l_1 \geq l_2$. Then we have

$$k_0 = k_1, \quad k'_1 = 0, \quad k'_2 = k_1 - k_2.$$

Since $E_F^2 = \langle \epsilon^2 \rangle$, a fundamental domain of \mathbf{R}_+^2 / E_F^2 is given by $[\epsilon^{-1}, \epsilon] \times \mathbf{R}_+$. By (2.6), we obtain

$$(2.28) \quad \int_{\epsilon^{-1}}^{\epsilon} \int_0^{\infty} \Omega(iy_1, iy_2) y_1^{s-1} y_2^{s-(k_1-k_2)/2-1} dy_1 dy_2 = (2\pi)^{(k_1-k_2)/2} R(s, \Omega)$$

when $\Re(s)$ is sufficiently large. We can verify that the integral converges locally uniformly for $s \in \mathbf{C}$. Take $m \in \mathbf{Z}$, and put $s = m$, $t = m - (k_1 - k_2)/2$. Then $0 \leq s \leq l_1$, $0 \leq t \leq l_2$ hold if and only if

$$(2.29) \quad \frac{k_1 - k_2}{2} \leq m \leq \frac{k_1 + k_2}{2} - 2.$$

For an integer m in this range, we have

$$\begin{aligned} P_{m, m-(k_1-k_2)/2} &= \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{\infty} \Omega(z) z_1^m z_2^{m-(k_1-k_2)/2} dz_1 dz_2 \\ &= i^{2m-(k_1-k_2)/2+2} \int_{\epsilon^{-1}}^{\epsilon} \int_0^{\infty} \Omega(iy_1, iy_2) y_1^m y_2^{m-(k_1-k_2)/2} dy_1 dy_2. \end{aligned}$$

Therefore we obtain

$$(2.30) \quad P_{m, m-(k_1-k_2)/2} = (-1)^{m+1} i^{-(k_1-k_2)/2} (2\pi)^{(k_1-k_2)/2} R(m+1, \Omega)$$

by (2.28). By the functional equation (2.7), this is equal to

$$(-1)^{m+1} i^{-(k_1-k_2)/2} (2\pi)^{(k_1-k_2)/2} (-1)^{(k_1+k_2)/2} R(k_1 - m - 1, \Omega).$$

Since $k_1 - m - 2$ satisfies (2.29), we obtain

$$(2.31) \quad P_{m, m-(k_1-k_2)/2} = (-1)^{(k_1-k_2)/2} P_{k_1-m-2, (k_1+k_2)/2-m-2}$$

using (2.30). We see that (2.31) is consistent with (2.26). Note that (2.29) is the condition for $L(m+1, \Omega)$ to be a critical value (cf. [Sh3, (4.14)]).

2.4

Let $\Omega \in M_k(\Gamma)$, and let $f = f(\Omega) \in Z^2(\Gamma, V)$ be the 2-cocycle attached to Ω defined by (2.18). In this subsection, we will write the action of Hecke operators on the cohomology class of $f(\Omega)$ explicitly. We denote $f(\Omega)$ also by f_Ω .

Let F be a totally real number field of degree n , and let Γ be a congruence subgroup of $\mathrm{SL}(2, \mathcal{O}_F)$. Let ϖ be a totally positive element of F , and let

$$(2.32) \quad \Gamma \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} \Gamma = \bigsqcup_{i=1}^d \Gamma \beta_i$$

be a coset decomposition. Let $\Omega \in M_k(\Gamma)$. We define the Hecke operator $T(\varpi)$ by

$$(2.33) \quad \Omega | T(\varpi) = N(\varpi)^{k_0/2-1} \sum_{i=1}^d \Omega|_k \beta_i.$$

Clearly $T(\varpi)$ does not depend on the choice of the coset decomposition (2.32). We have $\Omega | T(\varpi) \in M_k(\Gamma)$; it is a cusp form if Ω is. By (2.9_a), we have

$$(2.34) \quad \mathfrak{d}(\Omega | T(\varpi)) = \prod_{\nu=1}^n (\varpi^{(\nu)})^{(k_0+k_\nu)/2-2} \sum_{i=1}^d \rho(\beta_i)^{-1} (\mathfrak{d}(\Omega) \circ \beta_i).$$

Put

$$(2.35) \quad c = \prod_{\nu=1}^n (\varpi^{(\nu)})^{(k_0+k_\nu)/2-2}.$$

Until the end of this subsection, we assume that $n = 2$. We define (cf. (2.10))

$$(2.36) \quad F_{\Omega|T(\varpi)}(z) = \int_{w_1}^{z_1} \int_{w_2}^{z_2} \mathfrak{d}(\Omega | T(\varpi)), \quad z = (z_1, z_2).$$

By the procedure given in Section 2.2, we can calculate the 2-cocycle attached to $\Omega | T(\varpi)$. We omit the details (cf. [Y4]). The result is as follows. For $\gamma_1, \gamma_2 \in \Gamma$, we put

$$(2.37) \quad \beta_i \gamma_1 = \delta_i^{(1)} \beta_{j(i)}, \quad \delta_i^{(1)} \in \Gamma, \quad \beta_i \gamma_2 = \delta_i^{(2)} \beta_{k(i)}, \quad \delta_i^{(2)} \in \Gamma, \quad 1 \leq i \leq d.$$

Then, modulo coboundary, we have

$$(2.38) \quad f_{\Omega|T(\varpi)}(\gamma_1, \gamma_2) = c \sum_{i=1}^d \beta_i^{-1} f_{\Omega}(\beta_i \gamma_1 \beta_{j(i)}^{-1}, \beta_{j(i)} \gamma_2 \beta_{k(j(i))}^{-1}).$$

This formula is consistent with Proposition 1.2.

2.5

Assume that the class number of F in the narrow sense is 1. Suppose that Ω is a Hecke eigenform. Then the L -function $L(s, \Omega)$ defined by (2.4) essentially coincides with the Euler product given in [Sh3] or in Jacquet and Langlands [JL], but there is a subtle difference; we explain it briefly for the reader's convenience.

We write $\mathfrak{d}_F = (\delta)$ with $\delta \gg 0$. Let $\Omega \in S_{k_1, k_2}(\Gamma)$, $\Gamma = \mathrm{SL}(2, \mathcal{O}_F)$, and let

$$\Omega(z) = \sum_{0 \ll \alpha \in \mathcal{O}_F} c(\alpha) \mathbf{e}_F\left(\frac{\alpha}{\delta} z\right)$$

be the Fourier expansion. We have $a(\alpha/\delta) = c(\alpha)$ (cf. (2.1)). We set

$$\Delta = \{\alpha \in M(2, \mathcal{O}_F) \mid \det \alpha \gg 0\}.$$

Let \mathfrak{m} be an integral ideal of F , and take $m \gg 0$ so that $\mathfrak{m} = (m)$. Then we define

$$T(\mathfrak{m}) = \sum_{\alpha \in \Delta, \det \alpha = m} \Gamma \alpha \Gamma,$$

which is an element of the abstract Hecke ring $\mathcal{H}(\Gamma, \Delta)$ (cf. [Sh5, p. 54]). Let $T(\mathfrak{m}) = \bigsqcup_{i=1}^e \Gamma \beta_i$ be a coset decomposition. Assume that $k_1 \geq k_2$. We define the action of $T(\mathfrak{m})$ on Ω by

$$\Omega | T(\mathfrak{m}) = N(\mathfrak{m})^{k_1/2-1} \sum_{i=1}^e \Omega|_{\beta_i}.$$

Then $\Omega | T(\mathfrak{m}) \in S_k(\Gamma)$; we can verify easily that it does not depend on the choices of m and β_i .

Assume that Ω is a nonzero common eigenfunction for all Hecke operators $T(\mathfrak{m})$. We put

$$\Omega | T(\mathfrak{m}) = \lambda(\mathfrak{m})\Omega.$$

We assume that Ω is normalized so that $c(1) = 1$. Then calculating similarly to [Sh5, pp. 79–80], we have[†]

$$\lambda(\mathfrak{m}) = c(m)(m^{(2)})^{(k_1-k_2)/2}.$$

Then we obtain

$$(2.39) \quad L(s, \Omega) = (\delta^{(2)})^{(k_1-k_2)/2} D_F^s \prod_{\mathfrak{p}} (1 - \lambda(\mathfrak{p})N(\mathfrak{p})^{-s} + N(\mathfrak{p})^{k_1-1-2s})^{-1}.$$

Here \mathfrak{p} extends over all prime ideals of F , and $D_F = N(\delta)$ is the discriminant of F .

When $0 \ll \varpi \in \mathcal{O}_F$ generates a prime ideal \mathfrak{p} , we denote $T(\varpi)$ defined by (2.33) also by $T(\mathfrak{p})$.

3. Cohomology of P

In this section, we will study cohomology groups of P . Main results are Theorems 3.7 and 3.9, which give the vanishing of $H^1(P, V)$ and $H^2(P, V)$ when $l_1 \neq l_2$. Hereafter in this paper, we assume that $[F : \mathbf{Q}] = 2$. We also assume $l_1 \equiv l_2 \pmod{2}$ and $l_2 \leq l_1$.

3.1

Put $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$. In this section, we define subgroups P and U of Γ by

$$P = \left\{ \begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix} \mid t \in E_F, u \in \mathcal{O}_F \right\} / \{\pm 1_2\},$$

$$U = \left\{ \begin{pmatrix} \pm 1 & 0 \\ u & \pm 1 \end{pmatrix} \mid u \in \mathcal{O}_F \right\} / \{\pm 1_2\}.$$

We write $\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\omega$. Let ϵ be the fundamental unit of F , and let

$$\epsilon^2 = A + B\omega, \quad \epsilon^2\omega = C + D\omega.$$

Then we see that ϵ^2 is an eigenvalue of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$. We put

$$u_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \in U, \quad t = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix}.$$

We have

$$(3.1) \quad tu_1t^{-1} = u_1^A u_2^B, \quad tu_2t^{-1} = u_1^C u_2^D.$$

We put

$$(3.2) \quad \mathcal{Z} = \{(U_1, U_2) \in V \times V \mid (u_1 - 1)U_2 = (u_2 - 1)U_1\}.$$

[†]Correction to [Y4]: In [Y4, (2.46)], $(m^{(2)})^{(k_1-k_2)}$ should read $(m^{(2)})^{(k_1-k_2)/2}$.

It is easy to see that by the mapping

$$Z^1(U, V) \ni f \longrightarrow (f(u_1), f(u_2)) \in \mathcal{Z},$$

we have an isomorphism $Z^1(U, V) \cong \mathcal{Z}$. Put

$$(3.3) \quad \mathcal{B} = \{((u_1 - 1)\mathbf{b}, (u_2 - 1)\mathbf{b}) \mid \mathbf{b} \in V\}.$$

Then we have $B^1(U, V) \cong \mathcal{B} \subset \mathcal{Z}$.

We have $V = V_1 \otimes V_2$, $V_1 = \mathbf{C}^{l_1+1}$, $V_2 = \mathbf{C}^{l_2+1}$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{l_1+1}\}$ (resp., $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_{l_2+1}\}$) be the standard basis of V_1 (resp., V_2). The following four lemmas deal with linear algebra. The proofs are not difficult and can be found in [Y4]. We omit them.

LEMMA 3.1

We have $\dim V^U = 1$, and V^U is spanned by $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_{l_2+1}$.

LEMMA 3.2

Let $g = \left(\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c_2 & 1 \end{pmatrix}\right) \in \mathrm{SL}(2, \mathbf{C})^2$. We assume that $c_1 \neq 0$, $c_2 \neq 0$. Then the dimension of the subspace of V consisting of all vectors fixed by g is $l_2 + 1$. (Note that we have assumed $l_1 \geq l_2$.)

LEMMA 3.3

Let $u = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, $0 \neq c \in F$. Then we have

$$\mathbf{e}_i \otimes \mathbf{e}'_j \in \mathrm{Im}(u - 1) \quad \text{for } 1 \leq j \leq l_2 + 1 \text{ if } i \geq l_2 + 3 - j.$$

Here $\mathrm{Im}(u - 1)$ denotes the image of the linear mapping $V \ni \mathbf{v} \mapsto (\rho(u) - \rho(1))\mathbf{v} \in V$.

LEMMA 3.4

We have

$$\mathrm{Im}(u_1 - 1) + \mathrm{Im}(u_2 - 1) = \left(\bigoplus_{j=2}^{l_2+1} \mathbf{C}(\mathbf{e}_1 \otimes \mathbf{e}'_j) \right) \oplus \left(\bigoplus_{i=2}^{l_1+1} \bigoplus_{j=1}^{l_2+1} \mathbf{C}(\mathbf{e}_i \otimes \mathbf{e}'_j) \right).$$

In particular, $\dim(\mathrm{Im}(u_1 - 1) + \mathrm{Im}(u_2 - 1)) = \dim V - 1$.

By Lemma 3.1, we have

$$(3.4) \quad \dim \mathcal{B} = \dim V - 1.$$

Consider the surjective linear mapping

$$\mathcal{Z} \ni (U_1, U_2) \mapsto (u_2 - 1)U_1 \in \mathrm{Im}(u_1 - 1) \cap \mathrm{Im}(u_2 - 1).$$

The kernel of this mapping consists of (U_1, U_2) such that $U_1 \in \mathrm{Ker}(u_2 - 1)$, $U_2 \in \mathrm{Ker}(u_1 - 1)$. Hence by Lemma 3.2, we have

$$(3.5) \quad \dim \mathcal{Z} = \dim(\mathrm{Im}(u_1 - 1) \cap \mathrm{Im}(u_2 - 1)) + 2l_2 + 2.$$

By (3.4) and (3.5), we obtain

$$(3.6) \quad \dim H^1(U, V) = \dim(\operatorname{Im}(u_1 - 1) \cap \operatorname{Im}(u_2 - 1)) + 2l_2 + 3 - \dim V.$$

LEMMA 3.5

We have $\dim H^1(U, V) = 2$.

Proof

We have

$$\begin{aligned} & \dim(\operatorname{Im}(u_1 - 1) \cap \operatorname{Im}(u_2 - 1)) \\ &= \dim(\operatorname{Im}(u_1 - 1)) + \dim(\operatorname{Im}(u_2 - 1)) - \dim(\operatorname{Im}(u_1 - 1) + \operatorname{Im}(u_2 - 1)). \end{aligned}$$

By Lemma 3.2, we have $\dim(\operatorname{Im}(u_i - 1)) = \dim V - (l_2 + 1)$, $i = 1, 2$. Then by Lemma 3.4, we get

$$\dim(\operatorname{Im}(u_1 - 1) \cap \operatorname{Im}(u_2 - 1)) = \dim V - 2l_2 - 1.$$

The assertion follows from (3.6). \square

3.2

In this subsection, we will prove the following two theorems.

THEOREM 3.6

The eigenvalues of the action of t on $H^1(U, V)$ are $\epsilon^{l_1+2}(\epsilon')^{-l_2}$ and $\epsilon^{-l_1-2}(\epsilon')^{l_2}$. In particular, $H^1(U, V)^{P/U} = 0$.

THEOREM 3.7

We have

$$\dim H^1(P, V) = \begin{cases} 0 & \text{if } l_1 \neq l_2 \text{ or } N(\epsilon)^{l_1} = -1, \\ 1 & \text{if } l_1 = l_2 \text{ and } N(\epsilon)^{l_1} = 1. \end{cases}$$

Here $N(\epsilon)$ denotes the norm of ϵ .

Taking $G = P$, $N = U$, $M = V$ in (1.7), we obtain the exact sequence

$$0 \longrightarrow H^1(P/U, V^U) \longrightarrow H^1(P, V) \longrightarrow H^1(U, V)^{P/U} \longrightarrow 0,$$

since $P/U \cong \mathbf{Z}$. Therefore Theorem 3.7 follows immediately from Theorem 3.6, since $\dim H^1(P/U, V^U)$ is easily seen to be equal to zero (resp., 1) if $l_1 \neq l_2$ or $N(\epsilon)^{l_1} = -1$ (resp., if $l_1 = l_2$ and $N(\epsilon)^{l_1} = 1$), in view of Lemma 3.1.

Proof of Theorem 3.6

First we recall the following fact on the action of t on $H^q(U, V)$ (cf. (1.3)). Let $f \in Z^q(U, V)$, and let $\bar{f} \in H^q(U, V)$ be the cohomology class represented by f . Put

$$g(n_1, n_2, \dots, n_q) = t^{-1}f(tn_1t^{-1}, tn_2t^{-1}, \dots, tn_qt^{-1}), \quad n_i \in U, 1 \leq i \leq q.$$

Then $g \in Z^q(U, V)$ and $\bar{f} \mapsto \bar{g}$ is the action of t .

As in Lemma 3.4, let

$$W = \text{Im}(u_1 - 1) + \text{Im}(u_2 - 1) = \left(\bigoplus_{j=2}^{l_2+1} \mathbf{C}(\mathbf{e}_1 \otimes \mathbf{e}'_j) \right) \oplus \left(\bigoplus_{i=2}^{l_1+1} \bigoplus_{j=1}^{l_2+1} \mathbf{C}(\mathbf{e}_i \otimes \mathbf{e}'_j) \right).$$

We have

$$V = \mathbf{C}(\mathbf{e}_1 \otimes \mathbf{e}'_1) \oplus W.$$

We may assume that $l_1 > 0$ since our assertion is clearly true if $l_1 = l_2 = 0$.

Put $\mathbf{t}_1 = \mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$. Let us show that for

$$\mathbf{t}_2 = \omega(\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}) + \sum_{i=2}^{l_1+1} x_i (\mathbf{e}_i \otimes \mathbf{e}'_{l_2+1})$$

with suitably chosen $x_i \in \mathbf{C}$, we have

$$(3.7) \quad (u_2 - 1)\mathbf{t}_1 = (u_1 - 1)\mathbf{t}_2.$$

To this end, for $i \geq 1$, put

$$W_i = \bigoplus_{k=i}^{l_1+1} \mathbf{C}(\mathbf{e}_k \otimes \mathbf{e}'_{l_2+1}).$$

We have

$$\begin{aligned} (u_2 - 1)(\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}) &= (\omega \mathbf{e}_2 + \omega^2 \mathbf{e}_3 + \cdots) \otimes \mathbf{e}'_{l_2+1}, \\ (u_1 - 1)(\mathbf{e}_i \otimes \mathbf{e}'_{l_2+1}) &= \left(i \mathbf{e}_{i+1} + \binom{i+1}{i-1} \mathbf{e}_{i+2} + \cdots \right) \otimes \mathbf{e}'_{l_2+1}. \end{aligned}$$

We see that

$$(u_2 - 1)\mathbf{t}_1 \equiv (u_1 - 1)\mathbf{t}_2 \pmod{W_3}.$$

For $x_2 = (\omega^2 - \omega)/2$, we have

$$(u_2 - 1)\mathbf{t}_1 \equiv (u_1 - 1)\mathbf{t}_2 \pmod{W_4}.$$

In this way, we can determine x_i successively so that (3.7) holds. Let $f_1 \in Z^1(U, V)$ be the 1-cocycle which corresponds to the point $(\mathbf{t}_1, \mathbf{t}_2) \in \mathcal{Z}$.

Put $\mathbf{t}_3 = \mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$. Similarly to the above, we can show that for

$$\mathbf{t}_4 = \omega'(\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1) + \sum_{j=2}^{l_2+1} y_j (\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_j),$$

the relation

$$(3.8) \quad (u_2 - 1)\mathbf{t}_3 = (u_1 - 1)\mathbf{t}_4$$

holds when y_j are suitably chosen. Let $f_2 \in Z^1(U, V)$ be the 1-cocycle which corresponds to the point $(\mathbf{t}_3, \mathbf{t}_4) \in \mathcal{Z}$.

Let \bar{f}_i be the class of f_i in $H^1(U, V)$, $i = 1, 2$. Let us show that $\{\bar{f}_1, \bar{f}_2\}$ gives a basis of $H^1(U, V)$. To this end, assume that $\alpha f_1 + \beta f_2 \in B^1(U, V)$ for $\alpha, \beta \in \mathbf{C}$. Then there exists $\mathbf{b} \in V$ such that the following hold:

- (i) $\alpha \mathbf{t}_1 + \beta \mathbf{t}_3 = (u_1 - 1)\mathbf{b}$,
- (ii) $\alpha \mathbf{t}_2 + \beta \mathbf{t}_4 = (u_2 - 1)\mathbf{b}$.

Put

$$\mathbf{b} = \sum_{i=1}^{l_1+1} \sum_{j=1}^{l_2+1} x_{ij}(\mathbf{e}_i \otimes \mathbf{e}'_j).$$

On the left-hand side of (i), the coefficient of the tensor $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ is α and the coefficients of $\mathbf{e}_1 \otimes \mathbf{e}'_j$ are 0 for $1 \leq j \leq l_2$. We have

$$(u_1 - 1)(\mathbf{e}_1 \otimes \mathbf{e}'_j) = j(\mathbf{e}_1 \otimes \mathbf{e}'_{j+1}) + \sum_{l=j+2}^{l_2+1} z_l(\mathbf{e}_1 \otimes \mathbf{e}'_l) + A,$$

where $z_l \in \mathbf{Z}$ and A is a term which does not contain $\mathbf{e}_1 \otimes \mathbf{e}'_l$. Therefore we have $x_{11} = \cdots = x_{1l_2-1} = 0$. By comparing the coefficients of the tensor $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ on both sides of (i), we obtain

$$\alpha = l_2 x_{1l_2}.$$

By comparing the coefficients of the tensor $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ on both sides of (ii), we get

$$\alpha \omega = l_2 \omega' x_{1l_2}.$$

Hence we obtain $x_{1l_2} = 0$, $\alpha = 0$. Similarly, by comparing the coefficients of the tensor $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$ for both sides of (i) and (ii), we obtain $\beta = 0$.

Let f'_1 be the image of f_1 under the action of t , and let $(U'_1, U'_2) \in \mathcal{Z}$ be the point corresponding to f'_1 . Then we have

$$\begin{aligned} U'_1 &= f'_1(u_1) = t^{-1} f_1(tu_1 t^{-1}) = t^{-1} f_1(u_1^A u_2^B) = t^{-1} [u_1^A f_1(u_2^B) + f_1(u_1^A)], \\ U'_2 &= f'_1(u_2) = t^{-1} f_1(tu_2 t^{-1}) = t^{-1} f_1(u_1^C u_2^D) = t^{-1} [u_1^C f_1(u_2^D) + f_1(u_1^C)]. \end{aligned}$$

For $i = 1, 2$, we have $f_1(u_i) = \mathbf{t}_i$ and

$$(3.9) \quad f_1(u_i^n) = (1 + u_i + \cdots + u_i^{n-1})\mathbf{t}_i \quad \text{if } n > 0,$$

$$(3.10) \quad f_1(u_i^{-n}) = -(u_i^{-1} + \cdots + u_i^{-n})\mathbf{t}_i \quad \text{if } n > 0.$$

From these formulas, we see easily that the coefficient of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ in tU'_1 is $A + B\omega$. Hence the coefficient of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ in U'_1 is $\epsilon^{l_1}(\epsilon')^{-l_2}(A + B\omega) = \epsilon^{l_1+2}(\epsilon')^{-l_2}$. Similarly we see that the coefficient of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ in U'_2 is $\omega \epsilon^{l_1+2}(\epsilon')^{-l_2}$.

Now let

$$f'_1 \equiv \gamma f_1 + \delta f_2 \pmod{B^1(U, V)}$$

with $\gamma, \delta \in \mathbf{C}$. Then there exists $\mathbf{c} \in V$ such that

- (iii) $\gamma \mathbf{t}_1 + \delta \mathbf{t}_3 - U'_1 = (u_1 - 1)\mathbf{c}$,
- (iv) $\gamma \mathbf{t}_2 + \delta \mathbf{t}_4 - U'_2 = (u_2 - 1)\mathbf{c}$.

Put $\mathbf{c} = \sum_{i=1}^{l_1+1} \sum_{j=1}^{l_2+1} y_{ij}(\mathbf{e}_i \otimes \mathbf{e}'_j)$. Comparing the coefficients of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ on both sides of (iii), we obtain

$$\gamma - \epsilon^{l_1+2}(\epsilon')^{-l_2} = l_2 y_{1l_2}.$$

Comparing the coefficients of $\mathbf{e}_1 \otimes \mathbf{e}'_{l_2+1}$ on both sides of (iv), we obtain

$$(\gamma - \epsilon^{l_1+2}(\epsilon')^{-l_2})\omega = l_2 y_{1l_2} \omega'.$$

From these two formulas, we obtain $y_{1l_2} = 0$, $\gamma = \epsilon^{l_1+2}(\epsilon')^{-l_2}$. Similarly, comparing the coefficients of $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$ on both sides of (iii) and (iv), we obtain $\delta = 0$. Thus we have shown

$$(3.11) \quad f'_1 \equiv \epsilon^{l_1+2}(\epsilon')^{-l_2} f_1 \pmod{B^1(U, V)}.$$

Next let f'_2 be the image of f_2 under the action of t , and let (U'_3, U'_4) be the point of \mathcal{Z} corresponding to f'_2 . Here $U'_3 = f'_2(u_1)$, $U'_4 = f'_2(u_2)$. Then we have

$$U'_3 = f'_2(u_1) = t^{-1} f_2(tu_1 t^{-1}) = t^{-1} f_2(u_1^A u_2^B) = t^{-1} [u_1^A f_2(u_2^B) + f_2(u_1^A)],$$

$$U'_4 = f'_2(u_2) = t^{-1} f_2(tu_2 t^{-1}) = t^{-1} f_2(u_1^C u_2^D) = t^{-1} [u_1^C f_2(u_2^D) + f_2(u_1^C)].$$

The coefficient of $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$ in tU'_3 is $A + B\omega' = \epsilon^{-2}$. The coefficient of $\mathbf{e}_{l_1+1} \otimes \mathbf{e}'_1$ in tU'_4 is $C + D\omega' = \epsilon^{-2}\omega'$. By an argument similar to the above, we obtain

$$(3.12) \quad f'_2 \equiv \epsilon^{-l_1-2}(\epsilon')^{l_2} f_2 \pmod{B^1(U, V)}.$$

This completes the proof of Theorem 3.6. \square

3.3

In this subsection, we will prove the following two theorems.

THEOREM 3.8

We have $\dim H^2(U, V) = 1$, and t acts on it as the multiplication by $\epsilon^{l_1}(\epsilon')^{l_2}$.

THEOREM 3.9

We have $H^2(P, V) = 0$ except for the case when $l_1 = l_2$ and $N(\epsilon)^{l_1} = 1$. If $l_1 = l_2$ and $N(\epsilon)^{l_1} = 1$, then we have $\dim H^2(P, V) = 1$.

First we will prove the part of Theorem 3.8 concerning the dimension.

LEMMA 3.10

We have $\dim H^2(U, V) = 1$.

Proof

Let U_1 be the subgroup of U generated by u_1 . We have the exact sequence

$$(3.13) \quad 0 \longrightarrow U_1 \longrightarrow U \longrightarrow U_2 \longrightarrow 0$$

and the associated spectral sequence (cf. (1.6))

$$(3.14) \quad E_2^{p,q} = H^p(U_2, H^q(U_1, V)) \implies H^n(U, V).$$

Let $E^n = H^n(U, V)$, and let $\{F^i\}$ denote the filtration on E^n induced by (3.14). We have $F^p(E^n)/F^{p+1}(E^n) \cong E_\infty^{p,n-p}$. Since $U_2 \cong \mathbf{Z}$, we have $E_2^{2,q} = E_\infty^{2,q} = 0$.

Since $F^3(E^2) = 0$, we get $F^2(E^2) = 0$. Since $U_1 \cong \mathbf{Z}$, we have $E_2^{p,2} = E_\infty^{p,2} = 0$. Hence we get $E^2/F^1(E^2) = 0$. We have $F^1(E^2)/F^2(E^2) \cong E_\infty^{1,1}$. Therefore it is sufficient to show that $\dim E_\infty^{1,1} = 1$.

We consider

$$E_2^{1,1} = H^1(U_2, H^1(U_1, V)).$$

The map $Z^1(U_1, V) \ni f \mapsto f(u_1) \in V$ induces the isomorphism

$$(3.15) \quad H^1(U_1, V) \cong V / \text{Im}(u_1 - 1).$$

The action of $u \in U_2$ on the right-hand side of (3.15) is given by

$$V / \text{Im}(u_1 - 1) \ni v \bmod \text{Im}(u_1 - 1) \longrightarrow u^{-1}v \bmod \text{Im}(u_1 - 1) \in V / \text{Im}(u_1 - 1).$$

Since $\bar{u}_2 = u_2 \bmod U_1$ is a generator of U_2 , we have

$$H^1(U_2, H^1(U_1, V)) \cong (V / \text{Im}(u_1 - 1)) / \text{Im}(\bar{u}_2 - 1) \cong V / (\text{Im}(u_1 - 1) + \text{Im}(u_2 - 1)).$$

By Lemma 3.4, we obtain

$$\dim H^1(U_2, H^1(U_1, V)) = \dim E_2^{1,1} = 1.$$

Since $E_2^{3,0} = 0$, we have $E_\infty^{1,1} = E_2^{1,1}$. This completes the proof. \square

Proof of Theorem 3.8

We set $\tau = u_1$, $\eta = u_2$. Let \mathcal{F} be the free group on two free generators $\tilde{\tau}$ and $\tilde{\eta}$, and let $\pi : \mathcal{F} \longrightarrow U$ be the surjective homomorphism such that

$$\pi(\tilde{\tau}) = \tau, \quad \pi(\tilde{\eta}) = \eta.$$

Let R be the kernel of π . For $a, b \in \mathcal{F}$, let $[a, b] = aba^{-1}b^{-1}$ be the commutator of a and b . We see easily that

$$R = \langle x[\tilde{\tau}, \tilde{\eta}]x^{-1} \mid x \in \mathcal{F} \rangle, \quad R = [\mathcal{F}, \mathcal{F}].$$

We have the isomorphism (cf. (1.9))

$$(3.16) \quad H^2(U, V) \cong H^1(R, V)^U / \text{Im}(H^1(\mathcal{F}, V)).$$

We have

$$(3.17) \quad H^1(R, V)^U = \{ \varphi \in \text{Hom}(R, V) \mid \varphi(grg^{-1}) = g\varphi(r), g \in \mathcal{F}, r \in R \}.$$

Hence $\varphi \in H^1(R, V)^U$ is completely determined by $\varphi([\tilde{\tau}, \tilde{\eta}])$. For $b \in H^1(\mathcal{F}, V)$, we have

$$b([\tilde{\tau}, \tilde{\eta}]) = (1 - \eta)b(\tilde{\tau}) + (\tau - 1)b(\tilde{\eta}).$$

Let W be the subspace $\text{Im}(u_1 - 1) + \text{Im}(u_2 - 1)$ of V (cf. Lemma 3.4). For $\varphi \in \text{Im}(H^1(\mathcal{F}, V))$, the formula above shows that $\varphi([\tilde{\tau}, \tilde{\eta}])$ can take an arbitrary vector in W . In particular, it follows that $\dim H^2(U, V) \leq 1$. Since $\dim H^2(U, V) = 1$ by Lemma 3.10, we see that there exists $\varphi_1 \in H^1(R, V)^U$ such that $\varphi_1([\tilde{\tau}, \tilde{\eta}]) = \mathbf{e}_1 \otimes \mathbf{e}'_1$. This φ_1 corresponds to a generator of $H^2(U, V)$.

Let $f \in Z^2(U, V)$. For $g \in \mathcal{F}$, we put $\bar{g} = \pi(g)$. There exists $a \in C^1(\mathcal{F}, V)$ such that (cf. (1.10))

$$(3.18) \quad f(\bar{g}_1, \bar{g}_2) = g_1 a(g_2) + a(g_1) - a(g_1 g_2), \quad g_1, g_2 \in \mathcal{F}.$$

The corresponding element $\varphi \in H^1(R, V)^U$ to f is obtained as the restriction of a to R . Now let ξ be an automorphism of \mathcal{F} . Since ξ stabilizes $R = [\mathcal{F}, \mathcal{F}]$, ξ induces an automorphism of $U = \mathcal{F}/R$, which we denote by $\bar{\xi}$. We have

$$\bar{\xi}(\bar{g}) = \overline{\xi(g)}, \quad g \in \mathcal{F}.$$

From (3.18), we obtain

$$(3.19) \quad f(\bar{\xi}(\bar{g}_1), \bar{\xi}(\bar{g}_2)) = \xi(g_1) a(\xi(g_2)) + a(\xi(g_1)) - a(\xi(g_1) \xi(g_2)), \quad g_1, g_2 \in \mathcal{F}.$$

LEMMA 3.11

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbf{Z})$, let $\xi(\gamma)$ be the automorphism of U defined by $\xi(\gamma)(\tau) = \tau^a \eta^c$, $\xi(\gamma)(\eta) = \tau^b \eta^d$. Then there exists an automorphism $\widetilde{\xi(\gamma)}$ of \mathcal{F} such that $\xi(\gamma) = \widetilde{\xi(\gamma)}$. Moreover, $\widetilde{\xi(\gamma)}$ can be taken so that

$$(3.20) \quad \varphi(\widetilde{\xi(\gamma)}(g)) \equiv \varphi(g) \pmod{W}$$

holds for every $\varphi \in H^1(R, V)^U$ and every $g \in [\mathcal{F}, \mathcal{F}]$.

Proof

For $\gamma_1, \gamma_2 \in \text{SL}(2, \mathbf{Z})$, we have $\xi(\gamma_1 \gamma_2) = \xi(\gamma_1) \xi(\gamma_2)$. For two automorphisms ξ_1, ξ_2 of \mathcal{F} , we have $\xi_1 \xi_2 = \bar{\xi}_1 \bar{\xi}_2$. Therefore to show the first assertion, it is sufficient to verify it for generators $\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\text{SL}(2, \mathbf{Z})$. Clearly the formulas $\widetilde{\xi(\gamma_1)}(\tau) = \tilde{\tau}$, $\widetilde{\xi(\gamma_1)}(\eta) = \tilde{\tau} \tilde{\eta}$, $\widetilde{\xi(\gamma_2)}(\tau) = \tilde{\eta}^{-1}$, $\widetilde{\xi(\gamma_2)}(\eta) = \tilde{\tau}$ define automorphisms $\widetilde{\xi(\gamma_1)}$ and $\widetilde{\xi(\gamma_2)}$ of \mathcal{F} satisfying the requirements.

To show the latter assertion, we first note that

$$(3.21) \quad u \mathbf{v} \equiv \mathbf{v} \pmod{W} \quad \text{for every } u \in U \text{ and every } \mathbf{v} \in V.$$

Let $\varphi \in H^1(R, V)^U$. Since $\widetilde{\xi(\gamma)}$ can be taken from the subgroup of $\text{Aut}(\mathcal{F})$ generated by $\widetilde{\xi(\gamma_1)}$ and $\widetilde{\xi(\gamma_2)}$, it is sufficient to show (3.20) for these generators. Moreover, since $\varphi(x[\tilde{\tau}, \tilde{\eta}]x^{-1}) = x\varphi([\tilde{\tau}, \tilde{\eta}])$ for $x \in \mathcal{F}$, it is enough to verify (3.20) for $g = [\tilde{\tau}, \tilde{\eta}]$ in view of (3.21). For $\xi(\gamma_1)$, we have

$$\varphi(\widetilde{\xi(\gamma_1)}([\tilde{\tau}, \tilde{\eta}])) = \varphi(\tilde{\tau}[\tilde{\tau}, \tilde{\eta}]\tilde{\tau}^{-1}) = \tau\varphi([\tilde{\tau}, \tilde{\eta}]) \equiv \varphi([\tilde{\tau}, \tilde{\eta}]) \pmod{W}$$

by (3.21). For $\xi(\gamma_2)$, we can check (3.20) similarly since $\widetilde{\xi(\gamma_2)}([\tilde{\tau}, \tilde{\eta}]) = \tilde{\eta}^{-1}[\tilde{\tau}, \tilde{\eta}]\tilde{\eta}$. This completes the proof of Lemma 3.11. \square

Applying Lemma 3.11 to $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we see that there exists an automorphism ξ_t of \mathcal{F} such that (cf. (3.1))

$$\bar{\xi}_t(u) = tut^{-1}, \quad u \in U.$$

Under the action of t , f is transformed to the 2-cocycle $f' \in Z^2(U, V)$ where

$$f'(h_1, h_2) = t^{-1}f(th_1t^{-1}, th_2t^{-1}), \quad h_1, h_2 \in U.$$

By (3.19), we obtain

$$(3.22) \quad \begin{aligned} & t^{-1}f(t\bar{g}_1t^{-1}, t\bar{g}_2t^{-1}) \\ &= g_1t^{-1}a(\xi_t(g_2)) + t^{-1}a(\xi_t(g_1)) - t^{-1}a(\xi_t(g_1)\xi_t(g_2)), \quad g_1, g_2 \in \mathcal{F}. \end{aligned}$$

This formula shows that a 1-cochain $a' \in C^1(\mathcal{F}, V)$ which splits f' is given by

$$a'(g) = t^{-1}a(\xi_t(g)), \quad g \in \mathcal{F}.$$

Now suppose that f (resp., f') $\in Z^2(U, V)$ corresponds to φ (resp., φ') $\in H^1(R, V)^U$. We have

$$(3.23) \quad \varphi'([\tilde{\tau}, \tilde{\eta}]) = t^{-1}\varphi(\xi_t([\tilde{\tau}, \tilde{\eta}])).$$

We may assume that $\varphi = \varphi_1$; that is, $\varphi([\tilde{\tau}, \tilde{\eta}]) = \mathbf{e}_1 \otimes \mathbf{e}'_1$. Then by (3.20), we obtain

$$\varphi'([\tilde{\tau}, \tilde{\eta}]) \equiv t^{-1}\varphi([\tilde{\tau}, \tilde{\eta}]) \equiv \epsilon^{l_1}(\epsilon')^{l_2}\varphi([\tilde{\tau}, \tilde{\eta}]) \pmod{W}.$$

This completes the proof of Theorem 3.8. \square

Proof of Theorem 3.9

Set $T = P/U$. Then T is generated by $t \pmod{U}$. We consider the spectral sequence

$$(3.24) \quad E_2^{p,q} = H^p(T, H^q(U, V)) \implies H^n(P, V).$$

Let $E^n = H^n(P, V)$, and let $\{F^i\}$ denote the filtration induced by (3.24). Since $T \cong \mathbf{Z}$, we have $E_2^{p,q} = 0$ for $p \geq 2$, $q \geq 0$. Hence $F^2(E^2)/F^3(E^2) \cong E_\infty^{2,0} = 0$. Since $F^3(E^2) = 0$, we obtain $F^2(E^2) = 0$. By Theorem 3.6, we have $E_2^{1,1} = H^1(T, H^1(U, V)) = 0$. Hence we have $F^1(E^2)/F^2(E^2) \cong E_\infty^{1,1} = 0$. Therefore we obtain

$$(3.25) \quad \dim H^2(P, V) = \dim E^2/F^1(E^2) = \dim E_\infty^{0,2}.$$

Now assume $l_1 \neq l_2$ or $N(\epsilon)^{l_1} \neq 1$. By Theorem 3.8, we have $H^2(U, V)^T = 0$. Hence we get $E_2^{0,2} = E_\infty^{0,2} = 0$. Next assume that $l_1 = l_2$ and $N(\epsilon)^{l_1} = 1$. By Theorem 3.8, we have $\dim E_2^{0,2} = \dim H^2(U, V)^T = 1$. We clearly have $E_2^{0,2} \cong E_\infty^{0,2}$. This completes the proof. \square

4. On the parabolic condition

In this section (in particular, Section 4.1), we will show that it is possible to deduce information on critical values of L -functions once we know a corresponding 2-cocycle which satisfies the parabolic condition.

From this section until the end of the paper, we define subgroups of Γ by

$$\begin{aligned} P &= \left\{ \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} \middle| u \in E_F, v \in \mathcal{O}_F \right\} / \{\pm 1_2\}, \\ U &= \left\{ \begin{pmatrix} \pm 1 & v \\ 0 & \pm 1 \end{pmatrix} \middle| v \in \mathcal{O}_F \right\} / \{\pm 1_2\} \end{aligned}$$

restoring the notation to that of Section 2. We see that Theorems 3.7 and 3.9 and the fact that $H^1(P, V)^{P/U} = 0$ stated in Theorem 3.6 are valid, considering the isomorphism $P \ni p \mapsto {}^t p^{-1} \in {}^t P$ and noting that $g \mapsto \rho(g)$ and $g \mapsto \rho({}^t g^{-1})$ are equivalent as representations of $\mathrm{SL}(2, \mathbf{C})^2$.

4.1

Let V_1 (resp., V_2) be the representation space of ρ_{l_1} (resp., ρ_{l_2}). We take a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{l_1+1}\}$ of V_1 so that $\rho_{l_1}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{e}_i = a^{l_1+1-i}\mathbf{e}_i$. Similarly we take a basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_{l_2+1}\}$ of V_2 so that $\rho_{l_2}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{e}'_i = a^{l_2+1-i}\mathbf{e}'_i$. We assume that $l_1 \geq l_2$, $l_1 \equiv l_2 \pmod{2}$. We put $k_1 = l_1 + 2$, $k_2 = l_2 + 2$, $k = (k_1, k_2)$. Let $\Omega \in S_k(\Gamma)$. We assume that l_1 is even if $N(\epsilon) = -1$. (This assumption is (A) in Section 1.)

We recall the formulas

$$(4.1) \quad f(\gamma_1, \gamma_2) = \int_{\gamma_1 \gamma_2 w_1}^{\gamma_1 w_1} \int_{w_2}^{\gamma'_1 w_2} \mathfrak{d}(\Omega), \quad w_1 = i\epsilon^{-1}, w_2 = i\infty,$$

$$(4.2) \quad f(\sigma, \mu) = - \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \mathfrak{d}(\Omega).$$

The formula (2.30) shows that the coefficients of $\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}$ in $f(\sigma, \mu)$, $(l_1 - l_2)/2 + 1 \leq i \leq (l_1 + l_2)/2 + 1$, are related to the critical values of $L(s, \Omega)$.

The parabolic condition on the cocycle f is

$$(4.3) \quad f(p\gamma_1, \gamma_2) = pf(\gamma_1, \gamma_2) \quad \text{for every } p \in P, \gamma_1, \gamma_2 \in \Gamma.$$

Now suppose that we add the coboundary of $b \in C^1(\Gamma, V)$,

$$b(\gamma_1 \gamma_2) - \gamma_1 b(\gamma_2) - b(\gamma_1),$$

to f . We assume that the resulting 2-cocycle is normalized and still satisfies the parabolic condition (4.3). Then $b(1) = 0$, and using the parabolic condition, we obtain

$$p\gamma_1 b(\gamma_2) + b(p\gamma_1) - b(p\gamma_1 \gamma_2) = p\gamma_1 b(\gamma_2) + pb(\gamma_1) - pb(\gamma_1 \gamma_2)$$

for $p \in P$. Taking $\gamma_2 = \gamma_1^{-1}$ and writing γ_1 as γ , we find that b must satisfy the condition

$$(4.4) \quad b(p\gamma) = pb(\gamma) + b(p), \quad p \in P, \gamma \in \Gamma.$$

Put $A = f(\sigma, \mu)$. After adding the coboundary of b , A changes to $A + b(\sigma\mu) - \sigma b(\mu) - b(\sigma)$. By (4.4), we have

$$b(\sigma\mu) = b(\mu^{-1}\sigma) = \mu^{-1}b(\sigma) + b(\mu^{-1}), \quad b(\mu^{-1}) = -\mu^{-1}b(\mu).$$

Therefore A changes to

$$A + (\mu^{-1} - 1)b(\sigma) - (\sigma + \mu^{-1})b(\mu).$$

By (4.4), we have $b|P \in Z^1(P, V)$. Suppose that $l_1 \neq l_2$. By Theorem 3.7, we have

$$b(\mu) = (\mu - 1)\mathbf{b}, \quad \mathbf{b} \in V.$$

Since $(\sigma + \mu^{-1})(\mu - 1) = (\mu^{-1} - 1)(\sigma - 1)$, we see that A changes to

$$A + (\mu^{-1} - 1)[b(\sigma) + (1 - \sigma)\mathbf{b}].$$

This formula shows that the components of A related to the critical values do not change by adding a coboundary, since $\mu^{-1}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}) = N(\epsilon)^{l_1}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2})$. Next suppose that $l_1 = l_2$. By Theorem 3.7 and by the exact sequence below it, we have

$$b(\mu) = (\mu - 1)\mathbf{b} + \mathbf{b}_0, \quad \mathbf{b} \in V, \mathbf{b}_0 \in V^U.$$

Hence A changes to

$$A + (\mu^{-1} - 1)[b(\sigma) + (1 - \sigma)\mathbf{b}] - (\sigma + \mu^{-1})\mathbf{b}_0.$$

Since $\mathbf{b}_0 \in V^U$, this formula shows that the components of A related to the critical values do not change except for two critical values $L(1, \Omega)$ and $L(l_1 + 1, \Omega)$ at the edges.

4.2

Let $\bar{Z}^2(\Gamma, V)$ be the subgroup of $Z^2(\Gamma, V)$ consisting of normalized 2-cocycles. Put

$$\bar{B}^2(\Gamma, V) = \{f = db \mid b \in C^1(\Gamma, V), b(1) = 0\}.$$

Then we have

$$\bar{Z}^2(\Gamma, V) \cap B^2(\Gamma, V) = \bar{B}^2(\Gamma, V),$$

and therefore

$$\bar{Z}^2(\Gamma, V)/\bar{B}^2(\Gamma, V) \subset Z^2(\Gamma, V)/B^2(\Gamma, V).$$

Since every 2-cocycle can be normalized by adding a coboundary, we have

$$H^2(\Gamma, V) = \bar{Z}^2(\Gamma, V)/\bar{B}^2(\Gamma, V).$$

Put

$$(4.5) \quad Z_P^2(\Gamma, V) = \{f \in \bar{Z}^2(\Gamma, V) \mid f \text{ satisfies the parabolic condition (4.3)}\},$$

$$(4.6) \quad B_P^2(\Gamma, V) = \{f \in \bar{B}^2(\Gamma, V) \mid f = db, b \in C^1(\Gamma, V), \\ b(p\gamma) = pb(\gamma) + b(p), \quad p \in P, \gamma \in \Gamma\}.$$

An element of $Z_P^2(\Gamma, V)$ is called a *normalized parabolic 2-cocycle*. The next lemma can easily be verified.

LEMMA 4.1

We have

$$Z_P^2(\Gamma, V) \cap \bar{B}^2(\Gamma, V) = B_P^2(\Gamma, V).$$

By Lemma 4.1, we have

$$Z_P^2(\Gamma, V)/B_P^2(\Gamma, V) \subset \bar{Z}^2(\Gamma, V)/\bar{B}^2(\Gamma, V) = H^2(\Gamma, V).$$

We define the parabolic part $H_P^2(\Gamma, V)$ of $H^2(\Gamma, V)$ by

$$(4.7) \quad H_P^2(\Gamma, V) = Z_P^2(\Gamma, V) / B_P^2(\Gamma, V).$$

4.3

As another application of Theorem 3.7, we show the nonvanishing of the cohomology class attached to a Hecke eigenform.

LEMMA 4.2

Assume that l_1 is even if $N(\epsilon) = -1$. Let $f \in Z_P^2(\Gamma, V)$ be a normalized parabolic 2-cocycle. For $(l_1 - l_2)/2 + 1 \leq i \leq (l_1 + l_2)/2 + 1$, let c_i be the coefficient of $\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}$ in $f(\sigma, \mu)$. Assume that $c_i \neq 0$ for some i if $l_1 \neq l_2$ and that $c_i \neq 0$ for some $i \neq 1, l_1 + 1$ if $l_1 = l_2$. Then the cohomology class of f is nontrivial.

Proof

Suppose that the cohomology class of f is trivial. Then there exists $b \in C^1(\Gamma, V)$ such that

$$f(\gamma_1, \gamma_2) = \gamma_1 b(\gamma_2) + b(\gamma_1) - b(\gamma_1 \gamma_2), \quad \gamma_1, \gamma_2 \in \Gamma.$$

By a computation similar to that given in Section 4.1, we obtain

$$(4.8) \quad f(\sigma, \mu) = (1 - \mu^{-1})b(\sigma) + (\sigma + \mu^{-1})b(\mu).$$

First we consider the case $l_1 \neq l_2$. Since $b \mid P \in Z^1(P, V)$ and $H^1(P, V) = 0$ (see Theorem 3.7), there exists $\mathbf{b} \in V$ such that $b(\mu) = (\mu - 1)\mathbf{b}$. Then we have

$$f(\sigma, \mu) = (1 - \mu^{-1})[b(\sigma) + (1 - \sigma)\mathbf{b}].$$

We have $\mu^{-1}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}) = N(\epsilon)^{l_1}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2})$. Hence c_i vanishes for all i . This is a contradiction, and the proof is complete in this case.

Next we consider the case $l_1 = l_2$. By Theorem 3.7, there exist $\mathbf{b} \in V$ and $\mathbf{b}_0 \in V^U$ such that

$$b(\mu) = (\mu - 1)\mathbf{b} + \mathbf{b}_0.$$

Then we have

$$f(\sigma, \mu) = (1 - \mu^{-1})[b(\sigma) + (1 - \sigma)\mathbf{b}] + (\sigma + \mu^{-1})\mathbf{b}_0.$$

Since $\mathbf{b}_0 \in V^U$, this formula shows that $c_i = 0$ if $i \neq 1, l_1 + 1$. This is a contradiction and completes the proof. \square

PROPOSITION 4.3

Let $k = (k_1, k_2)$, $k_1 \geq k_2$, $k_1 \equiv k_2 \equiv 0 \pmod{2}$. Let $\Omega \in S_k(\Gamma)$, and let $f = f(\Omega)$ be the normalized parabolic 2-cocycle attached to Ω (cf. (4.1)). We assume that the class number of F in the narrow sense is 1 and that Ω is a nonzero Hecke eigenform. If $k_1 \neq k_2$, we assume $k_2 \geq 4$. If $k_1 = k_2$, we assume $k_2 \geq 6$. Then the cohomology class of f in $H^2(\Gamma, V)$ is nontrivial.

Proof

Let $k_1 = l_1 + 2$, $k_2 = l_2 + 2$. By (2.30), we see that the coefficient c_i of $\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}$ in $f(\sigma, \mu)$ is $L(l_1 + 2 - i, \Omega)$ times a nonzero constant for $(l_1 - l_2)/2 + 1 \leq i \leq (l_1 + l_2)/2 + 1$. It is well known that $L(s, \Omega) \neq 0$ for $\Re(s) \geq (k_1 + 1)/2$ (cf. [Sh3, Proposition 4.16]). For $i = (l_1 - l_2)/2 + 1$, c_i is nonzero times $L((k_1 + k_2)/2 - 1, \Omega)$. Since $(k_1 + k_2)/2 - 1 \geq (k_1 + 1)/2$ if $k_2 \geq 3$, our assertion follows from Lemma 4.2 if $k_1 \neq k_2$. Assume $k_1 = k_2$. For $i = 2$, c_i is nonzero times $L(k_1 - 2, \Omega)$. Since $k_1 - 2 \geq (k_1 + 1)/2$ if $k_1 \geq 5$, our assertion in this case also follows from Lemma 4.2. \square

4.4

With a free group \mathcal{F} , we write $\Gamma = \mathcal{F}/R$. Let $\pi: \mathcal{F} \rightarrow \Gamma$ be the canonical homomorphism with $\text{Ker}(\pi) = R$. For $g \in \mathcal{F}$, we put $\pi(g) = \bar{g}$. We regard V as an \mathcal{F} -module by $gv = \bar{g}v$, $g \in \mathcal{F}$, $v \in V$. By (1.9), we have

$$(4.9) \quad H^2(\Gamma, V) \cong H^1(R, V)^\Gamma / \text{Im}(H^1(\mathcal{F}, V)).$$

We examine the part of the right-hand side of (4.9) which corresponds to $H^2_P(\Gamma, V)$. Put $\mathcal{P} = \pi^{-1}(P)$. Let $f \in Z^2_P(\Gamma, V)$. Take a 1-cochain $a \in C^1(\mathcal{F}, V)$ which satisfies (1.10). Then we have

$$f(\bar{p}\bar{g}_1, \bar{g}_2) = pg_1a(g_2) + a(pg_1) - a(pg_1g_2), \quad p \in \mathcal{P}, g_1, g_2 \in \mathcal{F}.$$

By the parabolic condition on f , this is equal to

$$p(g_1a(g_2) + a(g_1) - a(g_1g_2)).$$

Hence we have

$$a(pg_1g_2) - a(pg_1) = pa(g_1g_2) - pa(g_1), \quad p \in \mathcal{P}, g_1, g_2 \in \mathcal{F}.$$

Taking $g_1 = g_2^{-1} = g$, we obtain

$$(4.10) \quad a(pg) = pa(g) + a(p), \quad p \in \mathcal{P}, g \in \mathcal{F}.$$

Conversely if a satisfies (4.10), then f satisfies the parabolic condition.

Let $\varphi = a|_R$. We note that a satisfies (1.11) and $\varphi \in H^1(R, V)^\Gamma$. For every $s \in P$, we take an element $\tilde{s} \in \mathcal{P}$ such that $\pi(\tilde{s}) = s$. We fix the choice of \tilde{s} . Then we write $a(\tilde{s})$ as $\tilde{a}(s)$. By (1.11), we have

$$(4.11) \quad a(\tilde{s}r) = s\varphi(r) + \tilde{a}(s), \quad s \in P, r \in R.$$

Now for $s_1, s_2 \in P$ and $r_1, r_2 \in R$, we have

$$\begin{aligned} a(\tilde{s}_1r_1\tilde{s}_2r_2) &= a((\widetilde{s_1s_2})(\widetilde{s_1s_2})^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_2^{-1}r_1\tilde{s}_2r_2) \\ &= s_1s_2\varphi((\widetilde{s_1s_2})^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_2^{-1}r_1\tilde{s}_2r_2) + \tilde{a}(s_1s_2) \\ &= s_1s_2[\varphi(\tilde{s}_2^{-1}r_1\tilde{s}_2) + \varphi(r_2) + \varphi((\widetilde{s_1s_2})^{-1}\tilde{s}_1\tilde{s}_2)] + \tilde{a}(s_1s_2) \\ &= s_1\varphi(r_1) + s_1s_2\varphi(r_2) + \varphi(\tilde{s}_1\tilde{s}_2(\widetilde{s_1s_2})^{-1}) + \tilde{a}(s_1s_2), \end{aligned}$$

using (1.11), (1.12), and (4.11). On the other hand, by (4.10), we have

$$\begin{aligned} a(\tilde{s}_1 r_1 \tilde{s}_2 r_2) &= s_1 a(\tilde{s}_2 r_2) + a(\tilde{s}_1 r_1) \\ &= s_1 (s_2 \varphi(r_2) + \tilde{a}(s_2)) + s_1 \varphi(r_1) + \tilde{a}(s_1). \end{aligned}$$

Comparing two results, we obtain

$$(4.12) \quad \varphi(\tilde{s}_1 \tilde{s}_2 (\widetilde{s_1 s_2})^{-1}) = s_1 \tilde{a}(s_2) + \tilde{a}(s_1) - \tilde{a}(s_1 s_2).$$

The condition (4.12) can be interpreted as follows. The group extension

$$1 \longrightarrow R \longrightarrow \mathcal{P} \longrightarrow P \longrightarrow 0.$$

defines the factor set

$$(4.13) \quad (s_1, s_2) \longrightarrow \tilde{s}_1 \tilde{s}_2 (\widetilde{s_1 s_2})^{-1}$$

of P taking values in R . Mapping this factor set by φ , we obtain a 2-cocycle of P taking values in V (cf. Lemma 1.3). Then (4.12) means that this 2-cocycle splits. The converse holds (see [Y4] for a proof), and we have the following proposition.

PROPOSITION 4.4

On the right-hand side of (4.9), the subgroup which corresponds to $H_P^2(\Gamma, V)$ consists of the class of $\varphi \in H^1(R, V)^\Gamma$ for which the 2-cocycle $(s_1, s_2) \mapsto \varphi(\tilde{s}_1 \tilde{s}_2 (\widetilde{s_1 s_2})^{-1})$ of P taking values in V splits.

By Theorem 3.9, we have $H^2(P, V) = 0$ if $l_1 \neq l_2$. Hence the next proposition follows.

PROPOSITION 4.5

If $l_1 \neq l_2$, then we have $H^2(\Gamma, V) = H_P^2(\Gamma, V)$.

It is known that there are no holomorphic Eisenstein series of weight (k_1, k_2) if $k_1 \neq k_2$ (see [Sh6, Proposition 2.1]). We can interpret this proposition as the cohomological counterpart of this fact.

REMARK 4.6

In view of the results of Matsushima and Shimura [MS], Hida [Hi1], [Hi2], and Harder [Ha], we should be able to prove that $\dim H_P^2(\Gamma, V) = 4 \dim S_{l_1+2, l_2+2}(\Gamma)$. The author has not worked out the details yet. The parabolic cohomology group is also discussed in [Hi2].

5. Decompositions of $H^2(\Gamma, V)$

5.1

Let F be a real quadratic field, and let $\Gamma = \mathrm{PSL}(2, \mathcal{O}_F)$. We define elements σ , μ , τ , and η of Γ by

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Here we choose an ω such that $\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\omega$. Let \mathcal{F} be the free group on four letters $\tilde{\sigma}, \tilde{\mu}, \tilde{\tau}, \tilde{\eta}$. Let $\pi : \mathcal{F} \longrightarrow \Gamma$ be the homomorphism such that

$$\pi(\tilde{\sigma}) = \sigma, \quad \pi(\tilde{\mu}) = \mu, \quad \pi(\tilde{\tau}) = \tau, \quad \pi(\tilde{\eta}) = \eta.$$

By Vaserstein [V], π is surjective. Let R be the kernel of π . For $\gamma \in \Gamma$, we choose a $\tilde{\gamma} \in \mathcal{F}$ so that $\pi(\tilde{\gamma}) = \gamma$. For $\gamma = \sigma, \mu, \tau$, and η , we choose $\tilde{\gamma}$ so that the notation will be consistent. We choose $\tilde{1} = 1$. For other γ , we will specify the choice of $\tilde{\gamma}$ later (cf. (5.2) and Section 6.2).

Let $f \in Z^2(\Gamma, V)$ be a normalized 2-cocycle. There exists $a \in C^1(\mathcal{F}, V)$ which satisfies

$$f(\gamma_1, \gamma_2) = \gamma_1 a(\tilde{\gamma}_2) + a(\tilde{\gamma}_1) - a(\tilde{\gamma}_1 \tilde{\gamma}_2).$$

A corresponding element $\varphi \in H^1(R, V)^\Gamma$ to f is given by $\varphi = a|_R$. As was shown in Section 1.5, adding a coboundary to f , we may assume that $f \in Z^2(\Gamma, V)$ is given by

$$(5.1) \quad f(\gamma_1, \gamma_2) = -\varphi(\tilde{\gamma}_1 \tilde{\gamma}_2 (\widetilde{\gamma_1 \gamma_2})^{-1}), \quad \gamma_1, \gamma_2 \in \Gamma.$$

Let \mathcal{F}_P be the subgroup of \mathcal{F} generated by $\tilde{\mu}, \tilde{\tau}$, and $\tilde{\eta}$. Let π_P be the restriction of π to \mathcal{F}_P , and let R_P be the kernel of π_P . We see that R_P is generated by the elements corresponding to the relations (iv), (v), (vi) given in the introduction and their conjugates. Suppose that f satisfies the parabolic condition (4.3). Then, by (4.12), we see that we may assume that $\varphi|_{R_P} = 0$ in addition to (5.1), adding a coboundary to f if necessary.

Conversely assume that $\varphi|_{R_P} = 0$. Take a complete set of representatives Δ for $P \backslash \Gamma$, and fix it. We have

$$\Gamma = \bigsqcup_{\delta \in \Delta} P\delta.$$

For $\gamma = p\delta$, $p \in P$, $\delta \in \Delta$, we define

$$(5.2) \quad \tilde{\gamma} = \tilde{p}\tilde{\delta}.$$

In (5.1), write $\gamma_1 = p_1\delta_1$, $p_1 \in P$, $\delta_1 \in \Delta$, $\gamma_1\gamma_2 = p_2\delta_2$, $p_2 \in P$, $\delta_2 \in \Delta$. Let $p \in P$. Then we have

$$\widetilde{p\gamma_1} = \widetilde{pp_1\delta_1} = \widetilde{pp_1}(\tilde{p}\tilde{p_1})^{-1}\tilde{p}\tilde{\gamma_1}, \quad \widetilde{p\gamma_1\gamma_2} = \widetilde{pp_2\delta_2} = \widetilde{pp_2}\tilde{p_2}^{-1}\tilde{\gamma_1}\tilde{\gamma_2}.$$

Hence, by (5.1), we have

$$\begin{aligned} f(p\gamma_1, \gamma_2) &= -\varphi(\widetilde{pp_1}(\tilde{p}\tilde{p_1})^{-1}\tilde{p}\tilde{\gamma_1}\tilde{\gamma_2}\{\widetilde{pp_2}(\tilde{p}\tilde{p_2})^{-1}\tilde{p}\tilde{\gamma_1}\tilde{\gamma_2}\}^{-1}) \\ &= -\varphi(\tilde{p}\tilde{\gamma_1}\tilde{\gamma_2}(\widetilde{\gamma_1\gamma_2})^{-1}\tilde{p}^{-1}) = -p\varphi(\tilde{\gamma_1}\tilde{\gamma_2}(\widetilde{\gamma_1\gamma_2})^{-1}) = pf(\gamma_1, \gamma_2). \end{aligned}$$

Therefore f satisfies the parabolic condition (4.3).

The value $f(\sigma, \mu)$ of the cocycle is related to the critical values of the L -function. By (5.1), we have

$$f(\sigma, \mu) = -\varphi(\tilde{\sigma}\tilde{\mu}(\widetilde{\sigma\mu})^{-1}) = -\varphi(\tilde{\sigma}\tilde{\mu}(\widetilde{\mu^{-1}\sigma})^{-1}).$$

We assume that $\sigma \in \Delta$. Then we have

$$f(\sigma, \mu) = -\varphi(\widetilde{\sigma\tilde{\mu}\tilde{\sigma}^{-1}(\mu^{-1})^{-1}}),$$

since $\widetilde{\mu^{-1}\sigma} = \widetilde{\mu^{-1}\tilde{\sigma}}$. As $\widetilde{\mu^{-1}\tilde{\mu}} \in R_P$, we have

$$\begin{aligned} f(\sigma, \mu) &= -\varphi(\widetilde{\sigma\tilde{\mu}\tilde{\sigma}^{-1}\tilde{\mu}}) = -\varphi(\widetilde{\sigma\tilde{\mu}\tilde{\sigma}^{-2}\tilde{\sigma}\tilde{\mu}}) = -\varphi(\widetilde{\sigma\tilde{\mu}\tilde{\sigma}^{-2}(\tilde{\sigma}\tilde{\mu})^{-1}\tilde{\sigma}\tilde{\mu}\tilde{\sigma}\tilde{\mu}}) \\ &= -\sigma\mu\varphi(\tilde{\sigma}^{-2}) - \varphi(\tilde{\sigma}\tilde{\mu}\tilde{\sigma}\tilde{\mu}). \end{aligned}$$

Therefore we obtain

$$(5.3) \quad f(\sigma, \mu) = -\varphi((\tilde{\sigma}\tilde{\mu})^2) + \sigma\mu\varphi(\tilde{\sigma}^2).$$

5.2

Let us consider the action of Hecke operators. Let ϖ be a totally positive element of F . Let

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} \Gamma = \bigsqcup_{i=1}^d \Gamma\beta_i$$

be a coset decomposition. We put (cf. (2.35))

$$c = \prod_{\nu=1}^2 (\varpi^{(\nu)})^{(k_0+k_\nu)/2-2}.$$

Let $f \in Z^2(\Gamma, V)$, and put $g = cT(\varpi)f$. The explicit form of g is given as follows (cf. Proposition 1.2 and (2.38)). Let

$$\beta_i\gamma_1 = \delta_i^{(1)}\beta_{j(i)}, \quad \delta_i^{(1)} \in \Gamma, \quad \beta_i\gamma_2 = \delta_i^{(2)}\beta_{k(i)}, \quad \delta_i^{(2)} \in \Gamma,$$

for $1 \leq i \leq d$. Here j and k are permutations on d letters. Then

$$(5.4) \quad g(\gamma_1, \gamma_2) = c \sum_{i=1}^d \beta_i^{-1} f(\beta_i\gamma_1\beta_{j(i)}^{-1}, \beta_{j(i)}\gamma_2\beta_{k(j(i))}^{-1}).$$

We assume that $f \in Z_P^2(\Gamma, V)$ and that it is given by (5.1) with $\varphi \in H^1(R, V)^\Gamma$ satisfying $\varphi|_{R_P} = 0$. Then we have

$$(5.5) \quad g(\gamma_1, \gamma_2) = -c \sum_{i=1}^d \beta_i^{-1} \varphi(\widetilde{\beta_i\gamma_1\beta_{j(i)}^{-1}} \widetilde{\beta_{j(i)}\gamma_2\beta_{k(j(i))}^{-1}} (\beta_i\gamma_1\gamma_2\beta_{k(j(i))}^{-1})^{-1}).$$

Let $\psi \in H^1(R, V)^\Gamma$ be a corresponding element to g . We give an explicit form of ψ . There exists $b \in C^1(\mathcal{F}, V)$ such that

$$g(\bar{x}_1, \bar{x}_2) = x_1b(x_2) + b(x_1) - b(x_1x_2), \quad x_1, x_2 \in \mathcal{F},$$

and ψ is given as the restriction of b to R . Here $\bar{x} = \pi(x)$, $x \in \mathcal{F}$. We assume that (ϖ) is a prime ideal. Then $d = N(\varpi) + 1$, and $\{\beta_i\}$ can be taken as

$$\left\{ \begin{pmatrix} 1 & u \\ 0 & \varpi \end{pmatrix}, u \pmod{\varpi}, \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Take $p \in P$, and let $\beta_i p \beta_{j(i)}^{-1} \in \Gamma$ for $1 \leq i \leq d$. Then we see easily that

$$(5.6) \quad \beta_i p \beta_{j(i)}^{-1} \in P, \quad 1 \leq i \leq d.$$

By (5.5), (5.6), and $\varphi|_{R_P} = 0$, we find

$$(5.7) \quad g(p_1, p_2) = 0, \quad p_1, p_2 \in P.$$

We have

$$b(x_1 x_2) = x_1 b(x_2) + b(x_1) - g(\bar{x}_1, \bar{x}_2), \quad x_1, x_2 \in \mathcal{F},$$

and we can use this formula to determine the value $b(x)$, $x \in \mathcal{F}$, by the induction on the length of the element x . As the initial conditions, we may assume that

$$b(\tilde{\mu}) = 0, \quad b(\tilde{\tau}) = 0, \quad b(\tilde{\eta}) = 0, \quad b(\tilde{\sigma}) = 0.$$

Then, by (5.7), we see that

$$(5.8) \quad b|_{\mathcal{F}_P} = 0.$$

The next Proposition is a special case of Proposition 1.4.

PROPOSITION 5.1

Suppose that $\gamma_j \in \Gamma$ are given for $1 \leq j \leq m$. For every j , we define $p_j \in S_d$ by

$$\beta_i \gamma_j \beta_{p_j(i)}^{-1} \in \Gamma, \quad 1 \leq i \leq d.$$

We define $q_j \in S_d$ inductively by

$$q_1 = p_1, \quad q_k = p_k q_{k-1}, \quad 2 \leq k \leq m.$$

We assume that $\gamma_j \in P$ or $\gamma_j = \sigma$ for every j . Then we have

$$(5.9) \quad \begin{aligned} & b(\tilde{\gamma}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_m) \\ &= c \sum_{i=1}^d \beta_i^{-1} \varphi \left(\widetilde{\beta_i \gamma_1 \beta_{q_1(i)}^{-1}} \widetilde{\beta_{q_1(i)} \gamma_2 \beta_{q_2(i)}^{-1}} \cdots \right. \\ & \quad \left. \times \widetilde{\beta_{q_{m-1}(i)} \gamma_m \beta_{q_m(i)}^{-1}} (\beta_i \gamma_1 \gamma_2 \cdots \gamma_m \beta_{q_m(i)}^{-1})^{-1} \right). \end{aligned}$$

5.3

For the practical computation, it is convenient to decompose $H^2(\Gamma, V)$ into a direct sum of subspaces under the action of the automorphisms of Γ . We put

$$Z = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in E_F \right\},$$

which is the center of $\mathrm{GL}(2, \mathcal{O}_F)$. Then we have

$$Z \cdot \mathrm{SL}(2, \mathcal{O}_F) / Z \cong \mathrm{SL}(2, \mathcal{O}_F) / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \mathrm{PSL}(2, \mathcal{O}_F) = \Gamma.$$

By this isomorphism, we regard Γ as a subgroup of $\mathrm{PGL}(2, \mathcal{O}_F) = \mathrm{GL}(2, \mathcal{O}_F)/Z$. Hereafter we assume that l_1 and l_2 are even. When l is even, we define a representation ρ'_l of $\mathrm{GL}(2, \mathbf{C})$ by

$$\rho'_l(g) = \rho_l(g) \det(g)^{-l/2}, \quad g \in \mathrm{GL}(2, \mathbf{C}).$$

Then ρ'_l is trivial on the center. We put $\rho' = \rho'_{l_1} \otimes \rho'_{l_2}$. By $gv = \rho'(g)v$, $g \in \mathrm{GL}(2, \mathcal{O}_F)$, $v \in V$, we regard V as a left $\mathrm{GL}(2, \mathcal{O}_F)$ -module. Since $\rho'(z) = \mathrm{id}$, $z \in Z$, we can regard V as a $\mathrm{PGL}(2, \mathcal{O}_F)$ -module. Since $\rho' \mid \Gamma = \rho \mid \Gamma$, the Γ -module structure of V is the same as before.

We have

$$\mathrm{PGL}(2, \mathcal{O}_F) / \mathrm{PSL}(2, \mathcal{O}_F) \cong E_F / E_F^2 \cong \mathbf{Z} / 2\mathbf{Z} \oplus \mathbf{Z} / 2\mathbf{Z}.$$

By conjugation, $\mathrm{PGL}(2, \mathcal{O}_F)$ acts on $H^2(\Gamma, V)$, and it decomposes into a direct sum of four subspaces. We put

$$\nu = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that $\mathrm{PGL}(2, \mathcal{O}_F)$ is generated by ν and δ over $\mathrm{PSL}(2, \mathcal{O}_F)$. We first examine the action of ν . For $f \in Z^2(\Gamma, V)$, define $\tilde{e}f \in Z^2(\Gamma, V)$ by (cf. (1.3))

$$(5.10) \quad \tilde{e}f(\gamma_1, \gamma_2) = \nu^{-1}f(\nu\gamma_1\nu^{-1}, \nu\gamma_2\nu^{-1}), \quad \gamma_1, \gamma_2 \in \Gamma.$$

Then \tilde{e} induces an automorphism e of $H^2(\Gamma, V)$. Since $\nu^2 = \mu$, \tilde{e}^2 is obtained from the inner automorphism by μ . Hence $e^2 = 1$. By (5.10), we see that $\tilde{e}f$ is a parabolic cocycle if f is parabolic. Therefore, by the action of e , we have the decompositions

$$H^2(\Gamma, V) = H^2(\Gamma, V)^+ \oplus H^2(\Gamma, V)^-, \quad H_P^2(\Gamma, V) = H_P^2(\Gamma, V)^+ \oplus H_P^2(\Gamma, V)^-.$$

Here we put

$$\begin{aligned} H^2(\Gamma, V)^\pm &= \{c \in H^2(\Gamma, V) \mid ec = \pm c\}, \\ H_P^2(\Gamma, V)^\pm &= \{c \in H_P^2(\Gamma, V) \mid ec = \pm c\}. \end{aligned}$$

Explicitly the decomposition is given by

$$f = \frac{1}{2}[(1 + \tilde{e})f + (1 - \tilde{e})f], \quad f \in Z^2(\Gamma, V).$$

PROPOSITION 5.2

Let $k = (k_1, k_2)$, $k_1 \geq k_2$, k_1 and k_2 are even. Let $\Omega \in S_k(\Gamma)$, and let $f = f(\Omega)$ be the normalized parabolic 2-cocycle attached to Ω by (4.1). We assume that the class number of F in the narrow sense is 1 and that Ω is a nonzero Hecke eigenform.

(1) If $k_1 \neq k_2$, we assume $k_2 \geq 6$. If $k_1 = k_2$, we assume $k_2 \geq 8$. Then the cohomology class of $(1 + \tilde{e})f$ in $H^2(\Gamma, V)$ is nontrivial.

(2) If $k_1 \neq k_2$, we assume $k_2 \geq 4$. If $k_1 = k_2$, we assume $k_2 \geq 6$. Then the cohomology class of $(1 - \tilde{e})f$ in $H^2(\Gamma, V)$ is nontrivial.

Proof

We apply Lemma 4.2 in a way similar to the proof of Proposition 4.3. We use the same notation as there. By (5.10), we have

$$(\tilde{e}f)(\sigma, \mu) = \nu^{-1}f(\nu\sigma\nu^{-1}, \nu\mu\nu^{-1}) = \nu^{-1}f(\mu\sigma, \mu) = \nu^{-1}\mu f(\sigma, \mu) = \nu f(\sigma, \mu).$$

We have

$$\begin{aligned} \nu(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}) &= N(\epsilon)^{l_1/2+1-i}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}) \\ &= N(\epsilon)^{k_1/2-i}(\mathbf{e}_i \otimes \mathbf{e}'_{i-(l_1-l_2)/2}). \end{aligned}$$

By the assumption, we have $N(\epsilon) = -1$. The range of i is $k_1/2 - l_2/2 \leq i \leq k_1/2 + l_2/2$. We see that $L(l_1 + 2 - i, \Omega)$ is nonvanishing if $i \neq k_1/2$. To conclude the nonvanishing of the cohomology class of $(1 + \tilde{e})f$, it suffices to find an even integer j such that $0 < j \leq l_2/2$ if $k_1 \neq k_2$ and $0 < j \leq l_2/2 - 1$ if $k_1 = k_2$. Such a j exists under the condition stated in (1). To conclude the nonvanishing of the cohomology class of $(1 - \tilde{e})f$, it suffices to find an odd integer j such that $0 < j \leq l_2/2$ if $k_1 \neq k_2$ and $0 < j \leq l_2/2 - 1$ if $k_1 = k_2$. Such a j exists under the condition stated in (2). This completes the proof. \square

We put

$$\bar{\Gamma}^* = \{\gamma \in \mathrm{GL}(2, \mathcal{O}_F) \mid \det(\gamma) = \epsilon^n, n \in \mathbf{Z}\}, \quad \Gamma^* = Z\bar{\Gamma}^*/Z.$$

Then Γ^* is generated by ν over Γ , and we have $[\Gamma^* : \Gamma] = 2$. Let

$$\mathrm{Res} : H^2(\Gamma^*, V) \longrightarrow H^2(\Gamma, V), \quad T : H^2(\Gamma, V) \longrightarrow H^2(\Gamma^*, V)$$

be the restriction map and the transfer map, respectively.

PROPOSITION 5.3

We have

- (1) $\mathrm{Res}(H^2(\Gamma^*, V)) = H^2(\Gamma, V)^+$,
- (2) $T(H^2(\Gamma, V)^+) = H^2(\Gamma^*, V)$,
- (3) $\mathrm{Ker}(T) = H^2(\Gamma, V)^-$.

We omit the proof since it is easy.

5.4

We have

$$H^2(\Gamma, V) \cong H^1(R, V)^\Gamma / \mathrm{Im}(H^1(\mathcal{F}, V)).$$

Let us consider the action of e on the right-hand side under this isomorphism. We use the same notation as in Section 5.1. Let ξ be the automorphism of Γ defined by $\xi(\gamma) = \nu\gamma\nu^{-1}$, $\gamma \in \Gamma$. Put

$$\epsilon = A + B\omega, \quad \epsilon\omega = C + D\omega.$$

Then we have $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$. We have

$$\nu\sigma\nu^{-1} = \sigma\mu^{-1}, \quad \nu\mu\nu^{-1} = \mu, \quad \nu\tau\nu^{-1} = \tau^A\eta^B, \quad \nu\eta\nu^{-1} = \tau^C\eta^D.$$

Using Lemma 3.11, we can check that there exists an automorphism $\tilde{\xi}$ of \mathcal{F} which satisfies

$$(5.11) \quad \pi(\tilde{\xi}(g)) = \xi(\pi(g)), \quad g \in \mathcal{F}.$$

Now let $f \in Z^2(\Gamma, V)$, and take $a \in C^1(\mathcal{F}, V)$ so that

$$f(\pi(g_1), \pi(g_2)) = g_1 a(g_2) + a(g_1) - a(g_1 g_2), \quad g_1, g_2 \in \mathcal{F}.$$

Then we have

$$\begin{aligned} (\tilde{e}f)(\pi(g_1), \pi(g_2)) &= \nu^{-1} f(\xi(\pi(g_1)), \xi(\pi(g_2))) = \nu^{-1} f(\pi(\tilde{\xi}(g_1)), \pi(\tilde{\xi}(g_2))) \\ &= g_1 \nu^{-1} a(\tilde{\xi}(g_2)) + \nu^{-1} a(\tilde{\xi}(g_1)) - \nu^{-1} a(\tilde{\xi}(g_1 g_2)) \end{aligned}$$

for $g_1, g_2 \in \mathcal{F}$. Put

$$a'(g) = \nu^{-1} a(\tilde{\xi}(g)), \quad g \in \mathcal{F}.$$

Then we have

$$(\tilde{e}f)(\pi(g_1), \pi(g_2)) = g_1 a'(g_2) + a'(g_1) - a'(g_1 g_2), \quad g_1, g_2 \in \mathcal{F}.$$

Thus we obtain the following proposition.

PROPOSITION 5.4

Let $f \in Z^2(\Gamma, V)$, and let $\varphi \in H^1(R, V)^\Gamma$ be a corresponding element. Then a corresponding element ψ of $H^1(R, V)^\Gamma$ to $\tilde{e}f$ is given by

$$\psi(r) = \nu^{-1} \varphi(\tilde{\xi}(r)), \quad r \in R.$$

We can check easily that the map $\varphi \longrightarrow \psi$ induces a map from $H^1(R, V)^\Gamma / \mathrm{Im}(H^1(\mathcal{F}, V))$ to itself and gives an automorphism of order 2.

5.5

For the actual computation, the cohomology group $H^2(\Gamma^*, V)$ is easier to handle than $H^2(\Gamma, V)$. By the action of δ , we can further decompose $H^2(\Gamma^*, V)$ so that

$$H^2(\Gamma^*, V) = H^2(\Gamma^*, V)^+ \oplus H^2(\Gamma^*, V)^-.$$

Let \tilde{d} (resp., d) denote the action of δ on $Z^2(\Gamma^*, V)$ (resp., $H^2(\Gamma^*, V)$).

PROPOSITION 5.5

Let $k = (k_1, k_2)$, $k_1 \geq k_2$, k_1 and k_2 are even. Let $\Omega \in S_k(\Gamma)$, and let $f = f(\Omega)$ be the normalized parabolic 2-cocycle attached to Ω by (4.1). We assume that the class number of F in the narrow sense is 1 and that Ω is a nonzero Hecke eigenform. Take $f^* \in Z^2(\Gamma^*, V)$ so that $f^*|_\Gamma = (1 + \tilde{e})f$.[†] If $k_1 \neq k_2$, we assume

[†]The cocycle $f^* = \tilde{T}(f)$ satisfies this condition.

$k_2 \geq 6$. If $k_1 = k_2$, we assume $k_2 \geq 8$. Then the cohomology class of $(1 + \tilde{d})f^*$ in $H^2(\Gamma^*, V)$ is nontrivial.

We omit the proof since it is similar to that of Proposition 5.2.

Until the end of this subsection, we assume that σ , ν , and τ generate Γ^* . (This assumption is satisfied if $\mathcal{O}_F = \mathbf{Z} + \mathbf{Z}\epsilon$.) Let \mathcal{F}^* be the free group on three letters $\tilde{\sigma}$, $\tilde{\nu}$, and $\tilde{\tau}$. We define a surjective homomorphism π^* of \mathcal{F}^* onto Γ^* by

$$\pi^*(\tilde{\sigma}) = \sigma, \quad \pi^*(\tilde{\nu}) = \nu, \quad \pi^*(\tilde{\tau}) = \tau$$

and let R^* be the kernel of π^* . We see that δ commutes with σ and ν and $\delta\tau\delta^{-1} = \tau^{-1}$. We can define an automorphism $x \mapsto x_\delta$ of \mathcal{F}^* by $(\tilde{\sigma})_\delta = \tilde{\sigma}$, $(\tilde{\nu})_\delta = \tilde{\nu}$, $(\tilde{\tau})_\delta = \tilde{\tau}^{-1}$. Then we have

$$\pi^*(x_\delta) = \delta\pi^*(x)\delta^{-1}, \quad x \in \mathcal{F}^*.$$

The following proposition can be shown in a way similar to Proposition 5.4.

PROPOSITION 5.6

Let $f \in Z^2(\Gamma^*, V)$, and let $\varphi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element. Then a corresponding element ψ of $H^1(R^*, V)^{\Gamma^*}$ to $\tilde{d}f$ is given by

$$\psi(r) = \delta^{-1}\varphi(r_\delta), \quad r \in R^*.$$

Let $\varphi \in H^1(R^*, V)^{\Gamma^*}$. We define $\varphi_\delta \in H^1(R^*, V)^{\Gamma^*}$ by the formula

$$(5.12) \quad \varphi_\delta(r) = \delta^{-1}\varphi(r_\delta).$$

Then we can check easily that $(\varphi_\delta)_\delta = \varphi$ and $H^1(R^*, V)^{\Gamma^*}$ decomposes into a direct sum of ± 1 eigenspaces under the action of δ :

$$(5.13) \quad H^1(R^*, V)^{\Gamma^*} = H^1(R^*, V)^{\Gamma^*, +} \oplus H^1(R^*, V)^{\Gamma^*, -}.$$

5.6

Let l_1 and l_2 be nonnegative even integers. We assume that $l_1 \geq l_2$. Let $\Omega \in S_{l_1+2, l_2+2}(\Gamma)$. Define $L(s, \Omega)$ and $R(s, \Omega)$ by (2.4) and (2.5), respectively. The functional equation is (cf. (2.7))

$$R(s, \Omega) = (-1)^{(l_1+l_2)/2} R(l_1+2-s, \Omega).$$

For an integer m , $L(m, \Omega)$ is a critical value if and only if

$$(5.14) \quad \frac{l_1 - l_2}{2} + 1 \leq m \leq \frac{l_1 + l_2}{2} + 1.$$

The central critical value is $L(l_1/2 + 1, \Omega)$ which vanishes if $(l_1 + l_2)/2$ is odd. By (2.30), we have

$$(5.15) \quad R(m, \Omega) = (-1)^{m_i(l_1-l_2)/2} (2\pi)^{(l_2-l_1)/2} P_{m-1, m-1-(l_1-l_2)/2}.$$

Here $P_{s,t}$ denotes the period integral given by (2.25). Let $f = f(\Omega) \in Z_P^2(\Gamma, V)$

be the parabolic 2-cocycle defined by (4.1). Then we have

$$f(\sigma, \mu) = - \int_{i\epsilon^{-1}}^{i\epsilon} \int_0^{i\infty} \mathfrak{d}(\Omega),$$

and $-P_{m-1, m-1-(l_1-l_2)/2}$ is equal to the coefficient of $\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}$ in $f(\sigma, \mu)$.

Using the operator \tilde{e} (cf. (5.10)), we define

$$f^+ = (1 + \tilde{e})f, \quad f^- = (1 - \tilde{e})f.$$

We have $f^\pm \in Z_P^2(\Gamma, V)$. As was shown in the proof of Proposition 5.2, we have

$$(5.16) \quad f^+(\sigma, \mu) = (1 + \nu)f(\sigma, \mu), \quad f^-(\sigma, \mu) = (1 - \nu)f(\sigma, \mu).$$

We have

$$(5.17) \quad \nu(\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}) = N(\epsilon)^{m-1-l_1/2} \mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}.$$

Assume $N(\epsilon) = -1$. Suppose that $l_1/2$ is even. By (5.17), we see that $f^+(\sigma, \mu)$ contains information on $R(m, \Omega)$ for odd m and $f^-(\sigma, \mu)$ contains information on $R(m, \Omega)$ for even m . If $l_1/2$ is odd, then $f^+(\sigma, \mu)$ contains information on $R(m, \Omega)$ for even m and $f^-(\sigma, \mu)$ contains information on $R(m, \Omega)$ for odd m .

To treat f^- efficiently, we will need more techniques which will be explained in the next section.

6. Numerical examples, I

6.1

In this section, we assume that $F = \mathbf{Q}(\sqrt{5})$. (The formulas (6.1) \sim (6.6) and those given in Section 6.5 are valid for any real quadratic field.) We use the notation of Section 5. The elements σ , μ , τ , and η generate $\Gamma = \text{PSL}(2, \mathcal{O}_F)$ (cf. Vaserštejn [V]). We take $\omega = \epsilon$. Then they satisfy the relations

- (i) $\sigma^2 = 1$,
- (ii) $(\sigma\tau)^3 = 1$,
- (iii) $(\sigma\mu)^2 = 1$,
- (iv) $\tau\eta = \eta\tau$,
- (v) $\mu\tau\mu^{-1} = \tau\eta$,
- (vi) $\mu\eta\mu^{-1} = \tau\eta^2$,
- (vii) $\sigma\eta\sigma = \tau\eta^{-1}\sigma\eta^{-1}\mu$.

THEOREM 6.1

Let $F = \mathbf{Q}(\sqrt{5})$ and $\Gamma = \text{PSL}(2, \mathcal{O}_F)$. We take $\omega = \epsilon$. The fundamental relations satisfied by the generators σ , μ , τ , and η are (i) \sim (vii).

A proof is given in the appendix of [Y4]. This theorem is not necessary for the calculations in this section, but it clarifies the exposition.

Now the elements σ , ν , and τ of Γ^* satisfy the relations

$$\begin{aligned} \text{(i')} \quad & \sigma^2 = 1, \\ \text{(ii')} \quad & (\sigma\tau)^3 = 1, \\ \text{(iii')} \quad & (\sigma\nu)^2 = 1, \\ \text{(iv')} \quad & \tau\nu\tau\nu^{-1} = \nu\tau\nu^{-1}\tau, \\ \text{(v')} \quad & \nu^2\tau\nu^{-2} = \tau\nu\tau\nu^{-1}. \end{aligned}$$

THEOREM 6.2

The fundamental relations satisfied by the generators σ , ν , τ of Γ^ are (i') \sim (v').*

This theorem follows from Theorem 6.1. We sketch a proof. We have $\mu = \nu^2$, $\eta = \nu\tau\nu^{-1}$. Then we can check easily that the relations (i) \sim (vii) in Theorem 6.1 follow from (i') \sim (v'). Suppose that

$$(*) \quad u_1 u_2 \cdots u_m = 1$$

is a relation. Here u_i is one of σ , ν , ν^{-1} , τ , τ^{-1} . In (*), we substitute ν^{-1} by $\mu^{-1}\nu$. Then we obtain a relation

$$(**) \quad v_1 v_2 \cdots v_n = 1.$$

Here v_i is one of σ , ν , μ^{-1} , τ , τ^{-1} . The number of v_i such that $v_i = \nu$ is even. If this number is zero, then (**) is the relation among the elements σ , μ , and τ . If this number is positive, then in (**), a term of the form $\nu X \nu$ is contained, where X is an expression which contains only σ , τ , and μ . We may replace $\nu X \nu$ by $\nu X \nu^{-1} \mu$. By the relations

$$\nu\sigma\nu^{-1} = \sigma\nu^{-2} = \sigma\mu^{-1}, \quad \nu\tau\nu^{-1} = \eta,$$

$\nu X \nu^{-1}$ is transformed to an expression which contains only σ , μ , τ , η , and their inverses. Repeating this procedure, (**) can be reduced to a relation among the elements σ , μ , τ , and η . By Theorem 6.1, this relation follows from the fundamental relations (i) \sim (vii). Since (i) \sim (vii) follow from (i') \sim (v'), our assertion is proved.

Let \mathcal{F}^* be the free group on three letters $\tilde{\sigma}$, $\tilde{\nu}$, $\tilde{\tau}$. We define a surjective homomorphism $\pi^* : \mathcal{F}^* \rightarrow \Gamma^*$ by $\pi^*(\tilde{\sigma}) = \sigma$, $\pi^*(\tilde{\nu}) = \nu$, $\pi^*(\tilde{\tau}) = \tau$. Let R^* be the kernel of π^* . We have $\Gamma^* = \mathcal{F}^*/R^*$. By Theorem 6.2, R^* is generated by the elements

$$\begin{aligned} \text{(i}^*) \quad & \tilde{\sigma}^2, \\ \text{(ii}^*) \quad & (\tilde{\sigma}\tilde{\tau})^3, \\ \text{(iii}^*) \quad & (\tilde{\sigma}\tilde{\nu})^2, \\ \text{(iv}^*) \quad & \tilde{\tau}\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1}(\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1}\tilde{\tau})^{-1}, \\ \text{(v}^*) \quad & \tilde{\nu}^2\tilde{\tau}\tilde{\nu}^{-2}(\tilde{\tau}\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1})^{-1} \end{aligned}$$

and their conjugates.

Let P^* be the subgroup of Γ^* consisting of elements which can be represented by upper-triangular matrices. Let \mathcal{F}_{P^*} be the subgroup of \mathcal{F}^* generated

by $\tilde{\nu}$ and $\tilde{\tau}$. Then $\pi^* | \mathcal{F}_{P^*} : \mathcal{F}_{P^*} \longrightarrow P^*$ is surjective. Let R_{P^*} be the kernel of this homomorphism. We see that R_{P^*} is generated by (iv^*) and (v^*) and their conjugates.

We have $[\mathcal{F}^* : (\pi^*)^{-1}(\Gamma)] = 2$. The following lemma can be proved easily by applying the method of Reidemeister and Schreier (cf. Schreier [Sc], Suzuki [Su, Section 6]).

LEMMA 6.3

The group $(\pi^)^{-1}(\Gamma)$ is the free group on five elements $\tilde{\sigma}$, $\tilde{\nu}^2$, $\tilde{\tau}$, $\tilde{\nu}\tilde{\sigma}\tilde{\nu}^{-1}$, and $\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1}$.*

We put $\tilde{\nu}^2 = \tilde{\mu}$, $\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1} = \tilde{\eta}$. Let \mathcal{F} be the free group on four elements $\tilde{\sigma}$, $\tilde{\mu}$, $\tilde{\tau}$, and $\tilde{\eta}$. Then our notation becomes consistent with that given in the beginning of Section 5. We have $\mathcal{F}R^* = (\pi^*)^{-1}(\Gamma)$.

6.2

For every $\gamma \in \Gamma^*$, we choose $\tilde{\gamma} \in \mathcal{F}^*$ so that $\pi^*(\tilde{\gamma}) = \gamma$. For explicit calculations, it is necessary to specify the choice of $\tilde{\gamma}$. First let $p \in P$. We can write $p = \mu^a \tau^b \eta^c$, and this expression is unique. We put $\tilde{p} = \tilde{\mu}^a \tilde{\tau}^b \tilde{\eta}^c$. Next let $p \in P^*$. We have $p \in P$ or $p = \nu p_1$ with $p_1 \in P$. In the latter case, we put $\tilde{p} = \tilde{\nu} \tilde{p}_1$.

Let Δ be a complete set of representatives for $P \backslash \Gamma$ as in Section 5.1. Then Δ is also a complete set of representatives for $P^* \backslash \Gamma^*$. For $\gamma \in \Gamma^*$, we write $\gamma = p\delta$ with $p \in P^*$, $\delta \in \Delta$ and put $\tilde{\gamma} = \tilde{p}\tilde{\delta}$. Our task is to specify the choice of Δ and define $\tilde{\delta}$ for $\delta \in \Delta$. To specify Δ is equivalent to choosing one element from every coset $P\gamma$, $\gamma \in \Gamma$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(1) In the case where $P\gamma = P$, we take 1 as the representative. We take the identity element of \mathcal{F} as $\tilde{1}$.

(2) In the case where $c \in E_F$, we can take an element of the form $\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}$ as the representative. We define

$$\widetilde{\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}} = \tilde{\sigma} \widetilde{\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}}.$$

(3) In the case where $c \neq 0$ and $c \notin E_F$, we note that \mathcal{O}_F is a Euclidean ring with respect to the absolute value of the norm (cf. [HW, Theorem 247, p. 213]).[†] For every $x, y \in \mathcal{O}_F$, $x \neq 0$, there exist $q, r \in \mathcal{O}_F$ such that

$$y = qx + r, \quad |N(r)| < |N(x)|.$$

We have

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ua & ub \\ u^{-1}c & u^{-1}d \end{pmatrix}, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}.$$

[†]In this paper, this step will be used for the actual calculations only in the case $a \in E_F$. Since it will become necessary in future calculations, we write one (tentative) algorithm explicitly.

First multiplying γ on the left by $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$, $u \in E_F$, we normalize c so that

$$c \gg 0, \quad 1 \leq c'/c < \epsilon^4.$$

Next multiplying γ on the left by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \in \mathcal{O}_F$, we may assume that $|N(a)| < |N(c)|$ by the Euclidean algorithm. However, to specify the choice of t is not necessarily easy. In other words, there can be many choices of such a 's. We make the preference order of the choice of a as follows. Put $a = \alpha + \beta\epsilon$, $\alpha, \beta \in \mathbf{Z}$.

- (1) $|\alpha| + |\beta|$ is minimum.
- (2) $|\alpha|$ is minimum.
- (3) $|\beta|$ is minimum.
- (4) We have $\alpha \geq 0$.
- (5) We have $\beta \geq 0$.

We define $\tilde{\delta}$ for $\delta \in \Delta$ as follows. We put $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and proceed by induction on $|N(c)|$. The case $|N(c)| = 0$ or 1 is settled by (1) and (2). By our choice of Δ , we have $|N(a)| < |N(c)|$. Put $\sigma^{-1}\delta = p_1\delta_1$, $p_1 \in P$, $\delta_1 \in \Delta$, $\delta_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. We have $|N(c_1)| = |N(a)| < |N(c)|$. We define $\tilde{\delta} = \tilde{\sigma}p_1\tilde{\delta}_1$.

6.3

Let $f \in Z_P^2(\Gamma, V)$ be a normalized parabolic 2-cocycle. We first consider f^+ (cf. Section 5.6). We put $f^* = \tilde{T}(f)$. Then $f^* \in Z^2(\Gamma^*, V)$ and $f^*|_{\Gamma} = f^+$ (cf. Section 5.3). We can verify easily the parabolic condition

$$(6.1) \quad f^*(p\gamma_1, \gamma_2) = pf^*(\gamma_1, \gamma_2), \quad p \in P^*, \gamma_1, \gamma_2 \in \Gamma^*.$$

We have

$$(6.2) \quad H^2(\Gamma^*, V) \cong H^1(R^*, V)^{\Gamma^*} / \text{Im}(H^1(\mathcal{F}^*, V)).$$

Let $\varphi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to f^* . We recall that φ is obtained in the following way. There exists $a \in C^1(\mathcal{F}^*, V)$ such that

$$(6.3) \quad a(g_1g_2) = g_1a(g_2) + a(g_1) - f^*(\pi^*(g_1), \pi^*(g_2)), \quad g_1, g_2 \in \mathcal{F}^*.$$

Then $\varphi = a|_{R^*}$. We may regard (6.3) as a rule for determining the value $a(g)$ according to the length of a word $g \in \mathcal{F}^*$. We can take $a(\tilde{\sigma}) = a(\tilde{\nu}) = a(\tilde{\tau}) = 0$. Then we have $a|_{\mathcal{F}P^*} = 0$, since (6.1) yields $f^*(p, \gamma) = 0$, $p \in P^*$, $\gamma \in \Gamma^*$. In particular, we have

$$(6.4) \quad \varphi|_{R_{P^*}} = 0.$$

As shown in Section 1.5, we may assume that

$$(6.5) \quad f^*(\gamma_1, \gamma_2) = -\varphi(\tilde{\gamma}_1\tilde{\gamma}_2(\widetilde{\gamma_1\gamma_2})^{-1})$$

adding a coboundary to f^* . By (6.4), we can check that f^* satisfies (6.1) in the same way as in Section 5.1. We have (cf. (5.3))

$$f^*(\sigma, \mu) = -\varphi((\tilde{\sigma}\tilde{\mu})^2) + \sigma\mu\varphi(\tilde{\sigma}^2).$$

We have

$$\begin{aligned}
 \varphi((\tilde{\sigma}\tilde{\mu})^2) &= \varphi(\tilde{\sigma}\tilde{\nu}^2\tilde{\sigma}\tilde{\nu}^2) = \varphi(\tilde{\sigma}\tilde{\nu}\tilde{\sigma}\tilde{\nu}\tilde{\nu}^{-1}\tilde{\sigma}^{-1}\tilde{\nu}\tilde{\sigma}\tilde{\nu}^2) \\
 &= \varphi(\tilde{\sigma}\tilde{\nu}\tilde{\sigma}\tilde{\nu}) + \varphi(\tilde{\nu}^{-1}\tilde{\sigma}^{-2}\tilde{\nu}) + \varphi(\tilde{\nu}^{-1}\tilde{\sigma}\tilde{\nu}\tilde{\sigma}\tilde{\nu}^2) \\
 &= (1 + \nu^{-1})\varphi((\tilde{\sigma}\tilde{\nu})^2) - \nu^{-1}\varphi(\tilde{\sigma}^2).
 \end{aligned}$$

Therefore we obtain

$$(6.6) \quad f^*(\sigma, \mu) = -(1 + \nu^{-1})\varphi((\tilde{\sigma}\tilde{\nu})^2) + (\sigma\mu + \nu^{-1})\varphi(\tilde{\sigma}^2).$$

Clearly φ is determined by its values on the elements $(i^*) \sim (v^*)$. By (6.4), φ takes the value zero on the elements (iv^*) and (v^*) . We have $\sigma\varphi(\tilde{\sigma}^2) = \varphi(\tilde{\sigma}^2)$. Take $h \in H^1(\mathcal{F}^*, V)$ so that $h(\tilde{\sigma}) = -\varphi(\tilde{\sigma}^2)/2$, $h(\tilde{\nu}) = 0$, $h(\tilde{\tau}) = 0$. Adding $h \mid R^*$ to φ , we may assume that $\varphi(\tilde{\sigma}^2) = 0$; φ still satisfies (6.4).

We analyze the process of adding $h \mid R^*$ to φ in more detail. For $S, T, U \in V$, we can find $h \in H^1(\mathcal{F}^*, V)$ such that

$$h(\tilde{\sigma}) = S, \quad h(\tilde{\tau}) = T, \quad h(\tilde{\nu}) = U.$$

We find easily that the conditions when h vanishes on the elements (iv^*) and (v^*) are

$$(6.7) \quad (1 + \tau\nu - \nu - \nu\tau\nu^{-1})T + (\tau - 1)(1 - \nu\tau\nu^{-1})U = 0,$$

$$(6.8) \quad (\nu^2 - 1 - \tau\nu)T + (1 + \nu - \nu^2\tau\nu^{-1} - \tau)U = 0,$$

respectively. We have

$$(6.9) \quad h(\tilde{\sigma}^2) = (1 + \sigma)S.$$

We put

$$A = \varphi((\tilde{\sigma}\tilde{\nu})^2), \quad B = \varphi((\tilde{\sigma}\tilde{\tau})^3).$$

We note that

$$(6.10) \quad \sigma\nu A = A, \quad \sigma\tau B = B.$$

Our objective is to determine A explicitly.

6.4

Let us consider the Hecke operators. We put $g^* = T(\varpi)f^*$ where g^* is defined by (5.4) with Γ^* in place of Γ . Let $\psi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to g^* . We see that Proposition 5.1 remains valid with Γ^* and P^* in place of Γ and P . In particular, we may assume that ψ is given by the formula

$$\begin{aligned}
 (6.11) \quad &\psi(\tilde{\gamma}_1\tilde{\gamma}_2\cdots\tilde{\gamma}_m) \\
 &= c \sum_{i=1}^d \beta_i^{-1} \varphi(\widetilde{\beta_i\gamma_1\beta_{q_1(i)}^{-1}} \widetilde{\beta_{q_1(i)}\gamma_2\beta_{q_2(i)}^{-1}} \cdots \widetilde{\beta_{q_{m-1}(i)}\gamma_m\beta_{q_m(i)}^{-1}}).
 \end{aligned}$$

Here $\gamma_j = \sigma$ or $\gamma_j \in P^*$ and $\gamma_1\gamma_2\cdots\gamma_m = 1$.

EXAMPLE 6.4

Let us consider $T(2)$. We may take

$$\begin{aligned}\beta_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, & \beta_2 &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, & \beta_3 &= \begin{pmatrix} 1 & \epsilon \\ 0 & 2 \end{pmatrix}, \\ \beta_4 &= \begin{pmatrix} 1 & \epsilon^2 \\ 0 & 2 \end{pmatrix}, & \beta_5 &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

By (6.11), we find

$$\psi((\tilde{\sigma}\tilde{\tau})^3) = c(\beta_3^{-1}Z_3 + \beta_4^{-1}Z_4),$$

where

$$(6.12) \quad Z_3 = \varphi\left(\left(\widetilde{\begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix}}\tilde{\tau}\right)^3\right), \quad Z_4 = \varphi\left(\left(\widetilde{\begin{pmatrix} \epsilon^2 & -\epsilon^2 \\ 2 & -\epsilon \end{pmatrix}}\right)^3\right).$$

We have

$$\widetilde{\begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix}} = \tilde{\sigma}\left(\widetilde{\begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix}}\right)\widetilde{\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}}^{-2}\tilde{\sigma}\left(\widetilde{\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}}\right)^{-1}.$$

Hence, using (6.4), we have

$$Z_3 = \varphi\left(\left(\tilde{\sigma}\left(\widetilde{\begin{pmatrix} \epsilon^{-1} & -2 \\ 0 & \epsilon \end{pmatrix}}\right)\tilde{\sigma}\left(\widetilde{\begin{pmatrix} 1 & -\epsilon^{-1} \\ 0 & 1 \end{pmatrix}}\right)\right)^3\right).$$

Similarly we obtain

$$Z_4 = \varphi\left(\left(\tilde{\sigma}\left(\widetilde{\begin{pmatrix} \epsilon^{-2} & -2 \\ 0 & \epsilon^2 \end{pmatrix}}\right)\tilde{\sigma}\left(\widetilde{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}\right)\right)^3\right).$$

6.5

In general, every element r of R^* can be written as (by using (i^*) , (iv^*) , (v^*) , and taking conjugation by $\tilde{\sigma}$ if necessary)

$$r = \tilde{\sigma}\tilde{p}_1\tilde{\sigma}\tilde{p}_2\cdots\tilde{\sigma}\tilde{p}_m$$

with $p_i \in P^*$, $1 \leq i \leq m$, such that $\sigma p_1 \sigma p_2 \cdots \sigma p_m = 1$. We call such an element an m -terms relation. Theorem 6.2 assures us that $\varphi(r)$ can be expressed by A and B . The following formulas can be proved easily.

$$(6.13a) \quad \varphi((\tilde{\sigma}\tilde{\nu}^n)^2) = (1 + \nu^{-1} + \cdots + \nu^{1-n})A, \quad n \geq 1,$$

$$(6.13b) \quad \varphi((\tilde{\sigma}\tilde{\nu}^{-n})^2) = -(\nu + \nu^2 + \cdots + \nu^n)A, \quad n \geq 1.$$

For $t \in E_F$, we put

$$B(t) = \varphi\left(\tilde{\sigma}\left(\widetilde{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}\right)\tilde{\sigma}\left(\widetilde{\begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}}\right)\tilde{\sigma}\left(\widetilde{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}\right)\left(\widetilde{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}}\right)\right).$$

Then we have $B(1) = B$,

$$(6.14) \quad B(-t) = -\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} B(t) - \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \varphi \left(\left(\widetilde{\sigma \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}} \right)^2 \right),$$

$$(6.15) \quad \begin{aligned} B(\epsilon t) &= \nu^{-1} B(t) \\ &+ \left[1 + \sigma \begin{pmatrix} 1 & \epsilon t \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & \epsilon^{-1} t^{-1} \\ 0 & 1 \end{pmatrix} - \sigma \begin{pmatrix} 1 & \epsilon t \\ 0 & 1 \end{pmatrix} \sigma \right] A, \end{aligned}$$

$$(6.16) \quad B(t) = \sigma \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} B(t^{-1}) + \varphi \left(\left(\widetilde{\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}} \right)^2 \right).$$

By these formulas, we can express $B(t)$ in terms of A and B explicitly. Using $B(t)$, we have an explicit formula for $\varphi(r)$ for a three-term relation r :

$$(6.17) \quad \begin{aligned} &\varphi \left(\widetilde{\sigma \begin{pmatrix} u_1 & x_1 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma \begin{pmatrix} u_2 & x_2 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma \begin{pmatrix} u_3 & x_3 \\ 0 & 1 \end{pmatrix}} \right) \\ &= \begin{pmatrix} u_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} B(u_1^{-1} x_1) + \varphi \left(\left(\widetilde{\sigma \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}} \right)^2 \right) \\ &\quad + \begin{pmatrix} u_3^{-1} & -u_3^{-1} x_3 \\ 0 & 1 \end{pmatrix} \sigma \varphi \left(\left(\widetilde{\sigma \begin{pmatrix} u_2 & 0 \\ 0 & 1 \end{pmatrix}} \right)^2 \right). \end{aligned}$$

For an m -term relation $r \in R^*$, $m \geq 4$, we may write $p_i = \begin{pmatrix} u_i & x_i \\ 0 & 1 \end{pmatrix}$, $u_i \in E_F$, $x_i \in \mathcal{O}_F$, $1 \leq i \leq m$. We see that $\varphi(r)$ reduces to an $(m-2)$ -term relation if $x_i = 0$ for some i . If $x_i \in E_F$ for some i , $\varphi(r)$ reduces to an $(m-1)$ -term relation. For example, if $x_1 \in E_F$ and $m \geq 4$, we have

$$(6.18) \quad \begin{aligned} &\varphi \left(\widetilde{\sigma \begin{pmatrix} u_1 & x_1 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma \begin{pmatrix} u_2 & x_2 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma \begin{pmatrix} u_3 & x_3 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma \dots \sigma \begin{pmatrix} u_m & x_m \\ 0 & 1 \end{pmatrix}} \right) \\ &= \begin{pmatrix} u_1^{-1} u^{-1} & -u_1^{-1} \\ 0 & u \end{pmatrix} \varphi \left(\widetilde{\sigma \begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma \begin{pmatrix} u_2 & x_2 \\ 0 & 1 \end{pmatrix}} \widetilde{\sigma \begin{pmatrix} u_3 & x_3 \\ 0 & 1 \end{pmatrix}} \right) \\ &\quad \dots \widetilde{\sigma \begin{pmatrix} u_m & x_m \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} u_1^{-1} u^{-1} & -u_1^{-1} \\ 0 & u \end{pmatrix} \\ &\quad + \begin{pmatrix} u_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} B(u) + \varphi \left(\left(\widetilde{\sigma \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}} \right)^2 \right). \end{aligned}$$

Here $u = u_1^{-1} x_1$. For a general m -term relation r , the explicit reduction of $\varphi(r)$ to A and B is a highly nontrivial problem. The author has an idea on a heuristic algorithm to solve this problem, but it will not be discussed in this paper. For our present purposes, the formulas (6.13a) \sim (6.18) are sufficient.

6.6

For actual computations, it is convenient to use the decomposition (5.13). Proposition 5.5 shows that we will lose little information by assuming $\varphi \in H^1(R^*, V)^{\Gamma^*, +}$, so we do assume this. Then we have

$$\begin{aligned} -\varphi((\tilde{\sigma}\tilde{\tau})^3) &= \varphi(\tilde{\tau}^{-1}\tilde{\sigma}\tilde{\tau}^{-1}\tilde{\sigma}\tilde{\tau}^{-1}\tilde{\sigma}) = \tau^{-1}\varphi((\tilde{\sigma}\tilde{\tau}^{-1})^3) \\ &= \tau^{-1}\varphi(((\tilde{\sigma}\tilde{\tau})^3)_\delta) = \tau^{-1}\delta\varphi((\tilde{\sigma}\tilde{\tau})^3). \end{aligned}$$

Hence

$$(\delta\tau + 1)B = 0.$$

Similarly we obtain

$$(\delta - 1)A = 0.$$

Now we are ready to state explicit numerical examples. First by numerical computations, we have verified the following.

FACT 6.1

Suppose $0 \leq l_2 \leq l_1 \leq 20$. Then adding $h \mid R^*$, $h \in H^1(\mathcal{F}^*, V)$ to φ (keeping φ in the plus space under the action of δ and the condition (6.4)), we may assume $B = 0$.

Therefore our task is to find constraints on $A = \varphi((\tilde{\sigma}\tilde{\nu})^2)$. Note that $(\sigma\nu - 1)A = 0$. We put $x = \begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix} \tau$ and

$$(6.19) \quad Z_A^+ = \{\mathbf{v} \in V \mid (\sigma\nu - 1)\mathbf{v} = 0, (\delta - 1)\mathbf{v} = 0, xZ_3 = Z_3\}.$$

Here some explanation is called for on the meaning of $xZ_3 = Z_3$. First note that Z_3 is defined by (6.12); clearly we must have $xZ_3 = Z_3$. Using the formulas (6.13a) \sim (6.18), we see that Z_3 can be expressed by A . Therefore $xZ_3 = Z_3$ gives a constraint on A . We define a linear mapping

$$(6.20) \quad \zeta^+ : Z_A^+ \longrightarrow \mathbf{C}^{l_2+1}$$

as follows. Let $\mathbf{v} \in Z_A^+$. We let the coefficient of $\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}$ in $(1 + \nu^{-1})\mathbf{v}$ be equal to the $((l_1 + l_2)/2 + 2 - m)$ th coefficient of $\zeta^+(\mathbf{v})$, for $(l_1 - l_2)/2 + 1 \leq m \leq (l_1 + l_2)/2 + 1$ (cf. (6.6)).

Suppose that φ as above corresponds to a (nonzero) Hecke eigenform $\Omega \in S_{l_1+2, l_2+2}(\Gamma)$. Suppose that l_1 and l_2 are in the range of Fact 6.1. Then $\zeta^+(A) \neq 0$ if $l_2 \geq 4$ in the case $l_1 \neq l_2$, if $l_2 \geq 6$ in the case $l_1 = l_2$ by Proposition 5.5.

EXAMPLE 6.5

We take $l_1 = 8$, $l_2 = 4$. Then $\dim S_{10,6}(\Gamma) = 1$. We find that $\zeta^+(Z_A^+)$ is one-dimensional and consists of scalar multiples of ${}^t(4, 0, 1, 0, 4)$. Hence we obtain

$$R(7, \Omega)/R(5, \Omega) = 4, \quad \Omega \in S_{10,6}(\Gamma).$$

My computer calculates this example in six seconds.

EXAMPLE 6.6

In the same way as in Example 6.5, we obtain the following numerical values:

$$\begin{aligned} R(9, \Omega)/R(7, \Omega) &= 6, & \Omega \in S_{14,6}(\Gamma), \\ R(6, \Omega)/R(4, \Omega) &= \frac{25}{6}, & \Omega \in S_{8,8}(\Gamma), \\ R(8, \Omega)/R(6, \Omega) &= 7, & \Omega \in S_{12,8}(\Gamma), \\ R(10, \Omega)/R(8, \Omega) &= \frac{720}{11}, & \Omega \in S_{12,10}(\Gamma). \end{aligned}$$

The spaces of cusp forms appearing in this example are all one-dimensional.

6.7

To deal with the case where $\dim S_{l_1+2, l_2+2}(\Gamma) > 1$, it is necessary to use the action of Hecke operators. To this end, we consider the contribution of $H^1(\mathcal{F}^*, V)$ to Z_A^+ . Take $h \in H^1(\mathcal{F}^*, V)$, and put

$$h(\tilde{\sigma}) = S, \quad h(\tilde{\nu}) = U, \quad h(\tilde{\tau}) = T.$$

We require that $h \mid R^*$ vanishes on the elements (i^*) , (ii^*) , (iv^*) , (v^*) . These conditions are equivalent to

$$(6.21) \quad (\sigma + 1)S = 0,$$

$$(6.22) \quad \{(\sigma\tau)^2 + \sigma\tau + 1\}(\sigma T + S) = 0,$$

and (6.7), (6.8). We have

$$h((\tilde{\sigma}\tilde{\nu})^2) = (\sigma\nu + 1)(\sigma U + S).$$

We also require that

$$(6.23) \quad (\delta - 1)(\sigma\nu + 1)(\sigma U + S) = 0.$$

Let B_A^+ be the subspace of V generated by $(\sigma\nu + 1)(\sigma U + S)$ when S, T, U extend over vectors of V satisfying the relations (6.7), (6.8), (6.21), (6.22), and (6.23). We have $B_A^+ \subset Z_A^+$. As shown in Section 4.1, we have

$$(6.24) \quad \zeta^+(B_A^+) = \{0\} \quad \text{if } l_1 \neq l_2, \quad \dim \zeta^+(B_A^+) \leq 1 \quad \text{if } l_1 = l_2.$$

By Proposition 5.5, we have

$$\dim Z_A^+/B_A^+ \geq \dim S_{l_1+2, l_2+2}(\Gamma) \quad \text{if } l_2 \geq 4, l_1 \neq l_2, \text{ or if } l_1 = l_2, l_2 \geq 6.$$

Now by numerical computations, we have verified the following.

FACT 6.2

Suppose $0 \leq l_2 \leq l_1 \leq 20$. Then $\dim S_{l_1+2, l_2+1}(\Gamma) = \dim Z_A^+/B_A^+$.

This fact means that the constraints posed on $A = \varphi((\tilde{\sigma}\tilde{\nu})^2)$ are enough.

EXAMPLE 6.7

We take $l_1 = 12, l_2 = 8$. We have $\dim S_{14,10}(\Gamma) = 2$. Moreover, we have $\zeta^+(Z_A^+) =$

2 in this case. Hence ζ^+ gives an isomorphism of Z_A^+/B_A^+ into \mathbf{C}^{l_2+1} . Calculating the action of $T(2)$ on Z_A^+/B_A^+ using (6.11), we find that the eigenvalues are $-2560 \pm 960\sqrt{106}$. Take an eigenvector in Z_A^+/B_A^+ , and map it by ζ^+ . Then we find

$$R(11, \Omega)/R(7, \Omega) = 1616 - 76\sqrt{106}, \quad R(9, \Omega)/R(7, \Omega) = \frac{58}{3} - \frac{5}{6}\sqrt{106}$$

if $0 \neq \Omega \in S_{14,10}(\Gamma)$ satisfies $\Omega | T(2) = (-2560 + 960\sqrt{106})\Omega$. If $0 \neq \Omega \in S_{14,10}(\Gamma)$ satisfies $\Omega | T(2) = (-2560 - 960\sqrt{106})\Omega$, then we have

$$R(11, \Omega)/R(7, \Omega) = 1616 + 76\sqrt{106}, \quad R(9, \Omega)/R(7, \Omega) = \frac{58}{3} + \frac{5}{6}\sqrt{106}.$$

REMARK 6.8

The relation $\dim \zeta^+(Z_A^+) = \dim S_{l_1+2, l_2+2}(\Gamma)$ is rather accidental in the above example. It holds in many cases, but we have $\dim S_{l_1+2, l_2+2}(\Gamma) > \dim \zeta^+(Z_A^+)$ in general. Even in the general case, we can obtain ratios of L -values by finding an eigenvector of Hecke operators in Z_A^+/B_A^+ and mapping it by ζ^+ .

6.8

We next consider the 2-cocycle f^- (cf. Section 5.6). The technique of calculation is basically the same as for f^+ , but this case is somewhat more complicated. Put

$$(6.25) \quad H^1(R^*, V)^\Gamma = \{\varphi \in \text{Hom}(R^*, V) \mid \varphi(grg^{-1}) = g\varphi(r), g \in \mathcal{F}, r \in R^*\}.$$

Let $\varphi \in H^1(R^*, V)^\Gamma$. We put

$$(e\varphi)(r) = \nu^{-1}\varphi(\tilde{\nu}r\tilde{\nu}^{-1}), \quad r \in R^*.$$

Then we can verify easily that

$$e\varphi \in H^1(R^*, V)^\Gamma, \quad e^2\varphi = \varphi.$$

Therefore $H^1(R^*, V)^\Gamma$ decomposes as

$$(6.26) \quad H^1(R^*, V)^\Gamma = H^1(R^*, V)^{\Gamma, +} \oplus H^1(R^*, V)^{\Gamma, -},$$

where, for $\epsilon = \pm 1$,

$$H^1(R^*, V)^{\Gamma, \epsilon} = \{\varphi \in \text{Hom}(R^*, V)^\Gamma \mid \varphi(\tilde{\nu}r\tilde{\nu}^{-1}) = \epsilon\nu\varphi(r), r \in R^*\}.$$

First we take an arbitrary normalized 2-cocycle $f \in Z^2(\Gamma, V)$. Since $\mathcal{F}R^*$ is a free group, there exists $a \in C^1(\mathcal{F}R^*, V)$ such that

$$(6.27) \quad f(\pi^*(g_1), \pi^*(g_2)) = g_1a(g_2) + a(g_1) - a(g_1g_2), \quad g_1, g_2 \in \mathcal{F}R^*.$$

As shown in Section 1.4, we have

$$a(gr) = ga(r) + a(g), \quad a(grg^{-1}) = ga(r), \quad g \in \mathcal{F}R^*, r \in R^*.$$

Put $\varphi = a | R^*$. Then the above formulas imply $\varphi \in H^1(R^*, V)^\Gamma$. From the isomorphism $\Gamma \cong \mathcal{F}R^*/R^* (\cong \mathcal{F}/\mathcal{F} \cap R^* = \mathcal{F}/R)$, we obtain

$$(6.28) \quad H^2(\Gamma, V) \cong H^1(R^*, V)^\Gamma / \text{Im}(H^1(\mathcal{F}R^*, V)),$$

and the procedure $f \mapsto \varphi$ described above gives an explicit form of the isomorphism (6.28). We consider the decomposition of $H^2(\Gamma, V)$ under the action of ν (cf. the formula below (5.10)). Then we have

$$(6.29) \quad H^2(\Gamma, V)^\pm \cong H^1(R^*, V)^{\Gamma, \pm} / (\text{Im}(H^1(\mathcal{F}R^*, V)) \cap H^1(R^*, V)^{\Gamma, \pm}).$$

6.9

Now we consider the 2-cocycle f^- . Let $\varphi \in H^1(R^*, V)^{\Gamma, -}$ be a corresponding element. As for f^+ , we may assume that

$$(6.30) \quad \varphi \mid R_{P^*} = 0,$$

$$(6.31) \quad f^-(\gamma_1, \gamma_2) = -\varphi(\widetilde{\gamma}_1 \widetilde{\gamma}_2 (\widetilde{\gamma_1 \gamma_2})^{-1})$$

adding a coboundary to f^- . We put

$$A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2), \quad B = \varphi((\widetilde{\sigma}\widetilde{\tau})^3).$$

The formulas (6.13a) \sim (6.18) hold with the following modifications:

$$(6.13a^-) \quad \varphi((\widetilde{\sigma}\widetilde{\nu}^n)^2) = (1 - \nu^{-1} + \nu^{-2} + \cdots + (-1)^{1-n}\nu^{1-n})A, \quad n \geq 1,$$

$$(6.13b^-) \quad \varphi((\widetilde{\sigma}\widetilde{\nu}^{-n})^2) = (\nu - \nu^2 + \nu^3 - \cdots + (-1)^{1-n}\nu^n)A, \quad n \geq 1.$$

We define $B(t)$, $t \in E_F$, by the same formula as before. In (6.15), the term $\nu^{-1}B(t)$ should be replaced by $-\nu^{-1}B(t)$; (6.14) and (6.16) hold without any change. For $u = \pm\epsilon^n \in E_F$, we define $\epsilon_0(u) = (-1)^n$. On the right-hand side of (6.17), the first term should be multiplied by $\epsilon_0(u_1)$, and the third term should be multiplied by $\epsilon_0(u_3)$. On the right-hand side of (6.18), both the first and the second terms should be multiplied by $\epsilon_0(u_1)$.

We may and do assume that f^- belongs to the plus subspace of $H^2(\Gamma, V)^-$ under the action of δ . Then we have

$$(\delta - 1)A = 0, \quad (\delta\tau + 1)B = 0.$$

By numerical computations, we have verified the following.

FACT 6.3

Suppose $0 \leq l_2 \leq l_1 \leq 20$. Then adding $h \mid R^*$ for $h \in H^1(\mathcal{F}R^*, V)$ such that $h \mid R^* \in H^1(R^*, V)^{\Gamma, -}$ to φ (keeping φ in the plus space under the action of δ and the condition (6.30)), we may assume $B = 0$.

Therefore our task is to find constraints on $A = \varphi((\widetilde{\sigma}\widetilde{\nu})^2)$. Note that $(\sigma\nu + 1) \cdot A = 0$. We put $x = \begin{pmatrix} \epsilon & -\epsilon^2 \\ 2 & -\epsilon^2 \end{pmatrix} \tau$ and

$$(6.32) \quad Z_A^- = \{ \mathbf{v} \in V \mid (\sigma\nu + 1)\mathbf{v} = 0, (\delta - 1)\mathbf{v} = 0, xZ_3 = Z_3 \}.$$

Here the meaning of the constraint $xZ_3 = Z_3$ is the same as for Z_A^+ . We define a linear mapping

$$\zeta^- : Z_A^- \longrightarrow \mathbf{C}^{l_2+1}$$

as follows. Let $\mathbf{v} \in Z_A^-$. We let the coefficient of $\mathbf{e}_{l_1+2-m} \otimes \mathbf{e}'_{(l_1+l_2)/2+2-m}$ in $(1-\nu^{-1})\mathbf{v}$ be equal to the $((l_1+l_2)/2+2-m)$ th coefficient of $\zeta^-(\mathbf{v})$, for $(l_1-l_2)/2+1 \leq m \leq (l_1+l_2)/2+1$ (cf. (6.6)).

EXAMPLE 6.9

We take $l_1 = 8$, $l_2 = 6$. Then $\dim S_{10,8}(\Gamma) = 1$. We find that $\zeta^-(Z_A^-)$ is one-dimensional and consists of scalar multiples of ${}^t(2, 0, 7/90, 0, -7/90, 0, -2)$. Hence we obtain

$$R(8, \Omega)/R(6, \Omega) = \frac{180}{7}, \quad \Omega \in S_{10,8}(\Gamma).$$

EXAMPLE 6.10

In the same way as in Example 6.9, we obtain the following numerical values:

$$R(9, \Omega)/R(7, \Omega) = \frac{70}{3}, \quad \Omega \in S_{12,8}(\Gamma),$$

$$R(9, \Omega)/R(7, \Omega) = 42, \quad \Omega \in S_{12,10}(\Gamma).$$

The spaces of cusp forms appearing in this example are all one-dimensional.

6.10

To treat the case where $\dim S_{l_1+2, l_2+2}(\Gamma) > 1$, it is necessary to consider Hecke operators.

First let us write down $\text{Im}(H^1(\mathcal{F}R^*, V)) \cap H^1(R^*, V)^{\Gamma, \pm}$, which appears on the right-hand side of (6.29), explicitly. Take $h \in Z^1(\mathcal{F}R^*, V)$. We put

$$(6.33) \quad (e_0 h)(x) = \nu^{-1} h(\tilde{\nu} x \tilde{\nu}^{-1}), \quad x \in \mathcal{F}R^*.$$

We can check easily that $e_0 h \in Z^1(\mathcal{F}R^*, V)$ and that

$$(e_0^2 h)(x) = h(x) + (\nu^{-2} - x\nu^{-2})h(\tilde{\nu}^2), \quad x \in \mathcal{F}R^*.$$

If we restrict h to R^* , then the action e_0 coincides with the action of e defined in Section 6.8. We have $(e_0^2 h) \mid R^* = h \mid R^*$. We put

$$h^\pm = h \pm e_0 h.$$

A general element of $\text{Im}(H^1(\mathcal{F}R^*, V)) \cap H^1(R^*, V)^{\Gamma, \pm}$ can be obtained as $h^\pm \mid R^*$ from a general element $h \in Z^1(\mathcal{F}R^*, V)$.

Let $Z^1(\mathcal{F}R^*, V)^\pm$ be the subgroup of $Z^1(\mathcal{F}R^*, V)$ consisting of all elements whose restrictions to R^* belong to $H^1(R^*, V)^{\Gamma, \pm}$. Take $\epsilon_1 = \pm 1$, and put $h^\pm = h + \epsilon_1 e_0 h$. For the free generators $\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{\nu}^2$, $\tilde{\nu}\tilde{\sigma}\tilde{\nu}^{-1}$, $\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1}$ of $\mathcal{F}R^*$, we put

$$\begin{aligned} h(\tilde{\sigma}) &= S_1, & h(\tilde{\tau}) &= T_1, & h(\tilde{\nu}^2) &= U, \\ h(\tilde{\nu}\tilde{\sigma}\tilde{\nu}^{-1}) &= V_1, & h(\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1}) &= W_1. \end{aligned}$$

Then we find

$$\begin{aligned} h^\pm(\tilde{\sigma}) &= S_1 + \epsilon_1 \nu^{-1} V_1, \\ h^\pm(\tilde{\tau}) &= T_1 + \epsilon_1 \nu^{-1} W_1, \end{aligned}$$

$$\begin{aligned}
h^\pm(\tilde{\nu}^2) &= (1 + \epsilon_1 \nu^{-1})U, \\
h^\pm(\tilde{\nu}\tilde{\sigma}\tilde{\nu}^{-1}) &= V_1 + \epsilon_1 \nu S_1 + \epsilon_1 \nu^{-1}(1 - \nu^4 \sigma)U, \\
h^\pm(\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1}) &= W_1 + \epsilon_1 \nu T_1 + \epsilon_1(\nu^{-1} - \nu\tau\nu^{-2})U.
\end{aligned}$$

Fix $\epsilon_1 = \pm 1$, and put

$$(6.34) \quad h^\pm(\tilde{\sigma}) = S, \quad h^\pm(\tilde{\tau}) = T.$$

Then V_1 and W_1 are eliminated, and we obtain

$$(6.35) \quad h^\pm(\tilde{\nu}^2) = (1 + \epsilon_1 \nu^{-1})U,$$

$$(6.36) \quad h^\pm(\tilde{\nu}\tilde{\sigma}\tilde{\nu}^{-1}) = \epsilon_1 \nu S + \epsilon_1 \nu^{-1}(1 - \nu^4 \sigma)U,$$

$$(6.37) \quad h^\pm(\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1}) = \epsilon_1 \nu T + \epsilon_1(\nu^{-1} - \nu\tau\nu^{-2})U.$$

Clearly S , T , and U can take the arbitrary three vectors of V . The formulas (6.34) ~ (6.37) describe a general element of $Z^1(\mathcal{F}R^*, V)^\pm$. The conditions for h^\pm to vanish on the elements (iv^*) and (v^*) are

$$(6.38) \quad \{\nu\tau\nu^{-1} - 1 + \epsilon_1(1 - \tau)\nu\}T + \epsilon_1(1 - \tau)(\nu^{-1} - \nu^{-1}\tau\nu^{-2})U = 0,$$

$$(6.39) \quad (1 + \epsilon_1\tau\nu - \nu^2)T + \{\epsilon_1\tau(\nu^{-1} - \nu\tau\nu^{-2}) - (1 - \nu^2\tau\nu^{-2})(1 + \epsilon_1\nu^{-1})\}U = 0,$$

respectively. For $h^\pm \in Z^1(\mathcal{F}R^*, V)^\pm$ as above, we have

$$(6.40) \quad h^\pm((\tilde{\sigma}\tilde{\nu})^2) = (1 + \epsilon_1\sigma\nu)S + (\nu^{-2} + \epsilon_1\sigma\nu^{-1})U.$$

Now we consider the case $\epsilon_1 = -1$. Let B_A^- be the subspace of V generated by $(1 - \sigma\nu)S + (\nu^{-2} - \sigma\nu^{-1})U$ when S , T , U extend over vectors of V satisfying the relations (6.21), (6.22), (6.38), (6.39), and

$$(6.41) \quad (\delta - 1)\{(1 - \sigma\nu)S + (\nu^{-2} - \sigma\nu^{-1})U\} = 0.$$

We have $B_A^- \subset Z_A^-$. As shown in Section 4.1, we have

$$\zeta^-(B_A^-) = \{0\} \quad \text{if } l_1 \neq l_2, \quad \dim \zeta^-(B_A^-) \leq 1 \quad \text{if } l_1 = l_2.$$

Using Proposition 5.2(2), we can show that

$$\dim Z_A^-/B_A^- \geq \dim S_{l_1+2, l_2+2}(\Gamma) \quad \text{if } l_2 \geq 2, l_1 \neq l_2 \text{ or if } l_1 = l_2, l_2 \geq 4.$$

Now by numerical computations, we have verified the following.

FACT 6.4

Suppose $0 \leq l_2 \leq l_1 \leq 20$. Then $\dim S_{l_1+2, l_2+1}(\Gamma) = \dim Z_A^-/B_A^-$.

The formula (6.11) can be generalized in the following way. We put $g^- = T(\varpi)f^-$ where g^- is defined by (5.4). Let $\varphi \in H^1(R^*, V)^{\Gamma, -}$ be a corresponding element to f^- . We may assume that (6.31) holds. There exists a 1-cochain $b \in C^1(\mathcal{F}R^*, V)$ such that

$$(6.42) \quad f^-(\pi^*(x_1), \pi^*(x_2)) = x_1 b(x_2) + b(x_1) - b(x_1 x_2), \quad x_1, x_2 \in \mathcal{F}R^*.$$

As the initial conditions, we may assume that

$$b(\tilde{\sigma}) = 0, \quad b(\tilde{\nu}^2) = 0, \quad b(\tilde{\tau}) = 0, \quad b(\tilde{\nu}\tilde{\sigma}\tilde{\nu}^{-1}) = 0, \quad b(\tilde{\nu}\tilde{\tau}\tilde{\nu}^{-1}) = 0$$

for the free generators of $\mathcal{F}R^*$. Then the formula (5.9) holds when $b(\tilde{\gamma}_j) = 0$, $1 \leq j \leq m$. This condition holds if $\tilde{\gamma}_j$ is equal to one of the five free generators as above or their inverses. In particular, $\psi = b \mid R^*$ is given by

$$\begin{aligned} & \psi(\tilde{\gamma}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_m) \\ &= c \sum_{i=1}^d \beta_i^{-1} \varphi(\widetilde{\beta_i \gamma_1 \beta_{q_1(i)}^{-1}} \widetilde{\beta_{q_1(i)} \gamma_2 \beta_{q_2(i)}^{-1}} \cdots \widetilde{\beta_{q_{m-1}(i)} \gamma_m \beta_{q_m(i)}^{-1}}) \end{aligned}$$

provided $\tilde{\gamma}_j$ is equal to one of the five free generators of $\mathcal{F}R^*$ or their inverses and $\gamma_1 \gamma_2 \cdots \gamma_m = 1$. The above formula is the same as (6.11), but there is one important point about which we must be careful. This ψ belongs to $H^1(R^*, V)^\Gamma$ and gives a corresponding element to g^- , but it does not necessarily belong to $H^1(R^*, V)^{\Gamma,-}$. We obtain $\psi^- \in H^1(R^*, V)^{\Gamma,-}$ corresponding to g^- by $\psi^- = (1 - e)\psi/2$ (cf. Section 6.8).

EXAMPLE 6.11

We take $l_1 = 12$, $l_2 = 8$. We have $\dim S_{14,10}(\Gamma) = 2$. Moreover, we have $\zeta^-(Z_A^-) = 2$ in this case. Hence ζ^- gives an isomorphism of Z_A^-/B_A^- into \mathbf{C}^{l_2+1} . Take an eigenvector of $T(2)$ in Z_A^-/B_A^- , and map it by ζ^- . Then we find

$$R(10, \Omega)/R(8, \Omega) = 50 - \sqrt{106}$$

if $0 \neq \Omega \in S_{14,10}(\Gamma)$ satisfies $\Omega \mid T(2) = (-2560 + 960\sqrt{106})\Omega$. If $0 \neq \Omega \in S_{14,10}(\Gamma)$ satisfies $\Omega \mid T(2) = (-2560 - 960\sqrt{106})\Omega$, then we have

$$R(10, \Omega)/R(8, \Omega) = 50 + \sqrt{106}.$$

Let Ω be a Hecke eigenform of $S_{14,10}(\Gamma)$. Then $L(m, \Omega)$ is a critical value for integers in the range $3 \leq m \leq 11$ (cf. (5.14)). We have $L(s, \Omega) = L(14 - s, \Omega)$ (cf. (2.7)). By Examples 6.7 and 6.11, we have treated all critical values on the right of the critical line.

EXAMPLE 6.12

We take $l_1 = l_2 = 18$. We have $\dim S_{20,20}(\Gamma) = 7$. Calculating the action of $T(2)$ on Z_A^+/B_A^+ using (6.11), we find that the characteristic polynomial of $T(2)$ is (we can use Z_A^-/B_A^- , which gives the same result)

$$\begin{aligned} & (X - 97280)^2(X + 840640)(X^4 - 1286780X^3 + 19006483200X^2 \\ & + 27181090390835200X - 22979876427231395840000). \end{aligned}$$

The irreducible factor of degree four corresponds to the base-change part from $S_{20}(\Gamma_0(5), (\frac{5}{5}))$; $X + 840640$ corresponds to the base-change part from $S_{20}(\mathrm{SL}_2(\mathbf{Z}))$; the factor $(X - 97280)^2$ corresponds to the non-base-change part.

Let $\Omega \in \dim S_{20,20}(\Gamma)$ be a Hecke eigenform in the non-base-change part. A calculation for the plus part yields the result

$$\begin{aligned} R(18, \Omega)/R(10, \Omega) &= 39355680000, & R(16, \Omega)/R(10, \Omega) &= 33163650, \\ R(14, \Omega)/R(10, \Omega) &= \frac{1266460}{27}, & R(12, \Omega)/R(10, \Omega) &= \frac{26075}{216}. \end{aligned}$$

A calculation for the minus part yields the result

$$\begin{aligned} R(17, \Omega)/R(11, \Omega) &= \frac{111006792000}{803}, & R(15, \Omega)/R(11, \Omega) &= \frac{54618434}{365}, \\ R(13, \Omega)/R(11, \Omega) &= \frac{453159}{1606}. \end{aligned}$$

We note that though there are two Hecke eigenforms in the non-base-change part, these ratios are the same for them.[†]

7. Numerical examples, II

7.1

In this section, we treat the case $F = \mathbf{Q}(\sqrt{13})$. We use the same notation as in Section 6. Many results there remain valid in the present case, so we will be brief.

The fundamental unit of F is $\epsilon = (3 + \sqrt{13})/2$. The elements σ , ν , and τ of Γ^* satisfy the relations $(i') \sim (iv')$ in Section 6.1 and

$$(v') \quad \nu^2 \tau \nu^{-2} = \tau(\nu \tau \nu^{-1})^3.$$

Though we do not know that $(i') \sim (v')$ are the fundamental relations, we will show that it is possible to calculate ratios of critical values of L -functions rigorously.

Let \mathcal{F}^* be the free group on three letters $\tilde{\sigma}$, $\tilde{\nu}$, $\tilde{\tau}$. We define a surjective homomorphism $\pi^* : \mathcal{F}^* \rightarrow \Gamma^*$ by $\pi^*(\tilde{\sigma}) = \sigma$, $\pi^*(\tilde{\nu}) = \nu$, $\pi^*(\tilde{\tau}) = \tau$. Let R^* be the kernel of π^* . Then R^* contains the elements $(i^*) \sim (iv^*)$ in Section 6.1 and

$$(v^*) \quad \tilde{\nu}^2 \tilde{\tau} \tilde{\nu}^{-2} \{ \tilde{\tau}(\tilde{\nu} \tilde{\tau} \tilde{\nu}^{-1})^3 \}^{-1}.$$

For every $\gamma \in \Gamma^*$, we choose $\tilde{\gamma} \in \mathcal{F}^*$ so that $\pi^*(\tilde{\gamma}) = \gamma$. We use the same algorithm as in Section 6.

We consider f^+ (cf. Section 5.6). We put $f^* = \tilde{T}(f)$. Then $f^* \in Z^2(\Gamma^*, V)$ and $f^* | \Gamma = f^+$ (cf. Section 5.3). Let $\varphi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to f^* . We may assume that (6.4) and (6.5) hold. We may also assume that $\varphi(\tilde{\sigma}^2) = 0$. We need to analyze the process of adding $h | R^*$ to φ . For S , T , $U \in V$, there exists $h \in H^1(\mathcal{F}^*, V)$ such that

$$h(\tilde{\sigma}) = S, \quad h(\tilde{\tau}) = T, \quad h(\tilde{\nu}) = U.$$

[†]We can show that the L -functions (2.39) are the same for two Hecke eigenforms in the non-base-change part. In fact, let $\Omega \neq 0$ be a Hecke eigenform in the non-base-change part, and let $\lambda(\mathfrak{m})$ be the eigenvalue of $T(\mathfrak{m})$ for Ω . For the nontrivial automorphism σ of F , there exists a Hecke eigenform $\Omega_\sigma \neq 0$ such that $\Omega_\sigma | T(\mathfrak{m}) = \lambda(\mathfrak{m}^\sigma) \Omega_\sigma$ (cf. [Y2, Remark, p. 1035]). Since Ω is not a base change, we have $\lambda(\mathfrak{m}) \neq \lambda(\mathfrak{m}^\sigma)$ for some \mathfrak{m} . Hence Ω_σ is not a constant multiple of Ω . On the other hand, $L(s, \Omega_\sigma)$ is equal to $L(s, \Omega)$.

We find that the conditions for h to vanish on the elements (iv^*) and (v^*) are (6.7) and

$$(7.1) \quad \begin{aligned} & [\nu^2 - \tau\{1 + \nu\tau\nu^{-1} + (\nu\tau\nu^{-1})^2\}\nu - 1]T \\ & + [(1 - \nu^2\tau\nu^{-2})(1 + \nu) - \tau\{1 + \nu\tau\nu^{-1} + (\nu\tau\nu^{-1})^2\} \\ & \times (1 - \nu\tau\nu^{-1})]U = 0, \end{aligned}$$

respectively. We put

$$A = \varphi((\tilde{\sigma}\tilde{\nu})^2), \quad B = \varphi((\tilde{\sigma}\tilde{\tau})^3).$$

Then (6.10) holds. As in Section 6, our objective is to determine A explicitly.

7.2

Let us consider the Hecke operators. We put $g^* = T(\varpi)f^*$ where g^* is defined by (5.4) with Γ^* in place of Γ . Let $\psi \in H^1(R^*, V)^{\Gamma^*}$ be a corresponding element to g^* . We may assume that ψ is given by (6.11).

We have $3 = (4 + \sqrt{13})(4 - \sqrt{13})$ in F . Put $\varpi = 4 - \sqrt{13} = -2\epsilon + 7$, $\mathfrak{p} = (\varpi)$, and consider the Hecke operator $T(\mathfrak{p}) = T(\varpi)$. We may take

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 1 & 1 \\ 0 & \varpi \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 1 & \epsilon \\ 0 & \varpi \end{pmatrix}, \quad \beta_4 = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}.$$

Using (6.11), we can compute $\psi(\tilde{\sigma}^2)$, $\psi((\tilde{\sigma}\tilde{\nu})^2)$, and $\psi((\tilde{\sigma}\tilde{\tau})^3)$. Remarkably it turns out that these quantities can be expressed by A and B . Since this is technically the essential part of calculation, we are going to explain the computation of $\psi((\tilde{\sigma}\tilde{\tau})^3)$ in some detail. By (6.11), we have

$$\psi((\tilde{\sigma}\tilde{\tau})^3) = c\beta_3^{-1}Z_3,$$

where

$$(7.2) \quad Z_3 = \varphi\left(\left(\tilde{\sigma}\begin{pmatrix} \epsilon^{-1} & 2\epsilon - 7 \\ 0 & \epsilon \end{pmatrix}\tilde{\sigma}\begin{pmatrix} 1 & -2\epsilon \\ 0 & 1 \end{pmatrix}\right)^3\right).$$

For $x \in \mathcal{O}_F$ and $u \in E_F$ such that x divides $u - 1$, we put

$\{x, u\}_4$

$$= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tilde{\sigma} \begin{pmatrix} 1 & (1-u)/x \\ 0 & 1 \end{pmatrix} \tilde{\sigma} \begin{pmatrix} 1 & -x/u \\ 0 & 1 \end{pmatrix} \tilde{\sigma} \begin{pmatrix} 1 & -u(1-u)/x \\ 0 & 1 \end{pmatrix} \tilde{\sigma} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}.$$

Then $\{x, u\}_4 \in R^*$. As a quantitative version of [Y4, Lemma A.6(3)], we can show that

$$(7.3) \quad \begin{aligned} \varphi(\{x, u^e\}_4) &= \varphi(\{x, u\}_4) \\ &+ \sigma \begin{pmatrix} u^{-1} & u^{-e+1}(1-u^e)/x \\ 0 & u \end{pmatrix} \sigma \begin{pmatrix} 1 & u^{e-2}x \\ 0 & 1 \end{pmatrix} \\ &\cdot \varphi(\{-u^{e-2}x, u^{e-1}\}_4) \end{aligned}$$

$$\begin{aligned}
& -\sigma \begin{pmatrix} u^{-1} & u^{-e+1}(1-u^e)/x \\ 0 & u \end{pmatrix} \sigma \varphi \left(\left(\widetilde{\sigma} \begin{pmatrix} u^{1-e} & 0 \\ 0 & u^{e-1} \end{pmatrix} \right)^2 \right) \\
& + \sigma \begin{pmatrix} u^{-e} & 0 \\ 0 & u^e \end{pmatrix} \varphi \left(\left(\widetilde{\sigma} \begin{pmatrix} u^{-e} & 0 \\ 0 & u^e \end{pmatrix} \right)^2 \right) \\
& - \sigma \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \varphi \left(\left(\widetilde{\sigma} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \right)^2 \right)
\end{aligned}$$

for $e \in \mathbf{Z}$. (This formula holds for any real quadratic field F .) By (7.3) and using the formulas given in Section 6.5, we can express $\psi((\tilde{\sigma}\tilde{\tau})^3)$ in terms of A and B .

7.3

We assume $\varphi \in H^1(R^*, V)^{\Gamma^*, +}$ (cf. Section 5.5). Then, as in Section 6.6, we have

$$(\delta\tau + 1)B = 0, \quad (\delta - 1)A = 0.$$

FACT 7.1

Suppose $0 \leq l_2 \leq l_1 \leq 20$. Then adding $h \mid R^*$, $h \in H^1(\mathcal{F}^*, V)$ to φ (keeping φ in the plus space under the action of δ and the condition $\varphi \mid R_{P^*} = 0$), we may assume $B = 0$.

Therefore our task is to find constraints on $A = \varphi((\tilde{\sigma}\tilde{\nu})^2)$. We put $x = \sigma \begin{pmatrix} \epsilon^{-1} & 2\epsilon-7 \\ 0 & \epsilon \end{pmatrix} \sigma \begin{pmatrix} 1 & -2\epsilon \\ 0 & 1 \end{pmatrix}$ and let

$$(7.4) \quad Z_A^+ = \{ \mathbf{v} \in V \mid (\sigma\nu - 1)\mathbf{v} = 0, (\delta - 1)\mathbf{v} = 0, xZ_3 = Z_3 \}.$$

Here Z_3 is defined by (7.2), and the meaning of $xZ_3 = Z_3$ is the same as in Section 6.6. Namely, $xZ_3 = Z_3$ must hold because $x^3 = 1$; since Z_3 can be expressed by A , $xZ_3 = Z_3$ gives a constraint on A .

We consider the contribution of $H^1(\mathcal{F}^*, V)$ to Z_A^+ . Take $h \in H^1(\mathcal{F}^*, V)$, and put

$$h(\tilde{\sigma}) = S, \quad h(\tilde{\nu}) = U, \quad h(\tilde{\tau}) = T.$$

We require that $h \mid R^*$ vanishes on the elements (i*), (ii*), (iv*), (v*). These conditions are equivalent to (6.21), (6.22), (6.7), and (7.1). We have

$$h((\tilde{\sigma}\tilde{\nu})^2) = (\sigma\nu + 1)(\sigma U + S).$$

We also require that (6.23) holds. Let B_A^+ be the subspace of V generated by $(\sigma\nu + 1)(\sigma U + S)$ when S, T, U extend over vectors of V satisfying the relations (6.7), (6.21), (6.22), (6.23), and (7.1). We have $B_A^+ \subset Z_A^+$. As shown in Section 4.1, (6.24) holds. By Proposition 5.5, we have

$$\dim Z_A^+ / B_A^+ \geq \dim S_{l_1+2, l_2+2}(\Gamma) \quad \text{if } l_2 \geq 4, l_1 \neq l_2 \text{ or if } l_1 = l_2, l_2 \geq 6.$$

Now by numerical computations, we have verified the following.

FACT 7.2

Suppose $0 \leq l_2 \leq l_1 \leq 20$. Then $\dim S_{l_1+2, l_2+1}(\Gamma) = \dim Z_A^+/B_A^+$.

This fact means that the constraints posed on $A = \varphi((\tilde{\sigma}\tilde{\nu})^2)$ are enough.

EXAMPLE 7.1

We take $l_1 = l_2 = 6$. We have $\dim S_{8,8}(\Gamma) = 5$. Calculating the action of $T(\mathfrak{p})$ on Z_A^+/B_A^+ using (6.11), we find that the characteristic polynomial of $T(\mathfrak{p})$ is

$$(X^2 - 40X - 3957)(X^3 + 28X^2 - 2601X - 71748).$$

The quadratic factor corresponds to the non-base-change part; the irreducible factor of degree three corresponds to the base-change part from $S_8(\Gamma_0(13), (\frac{1}{13}))$. Let $\Omega \in S_{8,8}(\Gamma)$ be the Hecke eigenform such that $\Omega | T(\mathfrak{p}) = (20 + \sqrt{4357})\Omega$. Then we find

$$R(6, \Omega)/R(4, \Omega) = 70/3.$$

EXAMPLE 7.2

We take $l_1 = l_2 = 8$. We have $\dim S_{10,10}(\Gamma) = 7$. We find that the characteristic polynomial of $T(\mathfrak{p})$ is

$$(X^2 - 16X - 42789) \\ \times (X^5 + X^4 - 66033X^3 + 1260423X^2 + 530326440X + 14266185264).$$

The quadratic factor corresponds to the non-base-change part. Let $\Omega \in S_{10,10}(\Gamma)$ be the Hecke eigenform such that $\Omega | T(\mathfrak{p}) = (8 + \sqrt{42853})\Omega$. Then we find

$$R(7, \Omega)/R(5, \Omega) = 50.$$

EXAMPLE 7.3

We take $l_1 = l_2 = 10$. We have $\dim S_{12,12}(\Gamma) = 11$. We find that the characteristic polynomial of $T(\mathfrak{p})$ is

$$(X - 252)(X^4 + 252X^3 - 496198X^2 - 116604684X + 25202349477) \\ \times (X^6 + 244X^5 - 665334X^4 - 129598956X^3 + 109163403621X^2 \\ + 14522233287672X - 255121008509808).$$

The irreducible factor of degree four corresponds to the non-base-change part; $X - 252$ corresponds to the base-change part from $S_{12}(\mathrm{SL}_2(\mathbf{Z}))$; and the irreducible factor of degree six corresponds to the base-change part from $S_{12}(\Gamma_0(13), (\frac{1}{13}))$. Put

$$f(X) = X^4 + 252X^3 - 496198X^2 - 116604684X + 25202349477.$$

Let θ be a root of $f(X)$, and put $K = \mathbf{Q}(\theta)$. We find that K contains a quadratic subfield $F = \mathbf{Q}(\sqrt{7 \cdot 5167})$. Put $d = 7 \cdot 5167$. Then a root of $f(X)$ is given by

$$\psi = -(63 + \sqrt{d}) + \sqrt{223837 - 360\sqrt{d}}.$$

We have

$$N(223837 - 360\sqrt{d}) = 13 \cdot 563 \cdot 6205151.$$

This number and the quadratic fields in Examples 7.1 and 7.2 are consistent with the table given in Doi, Hida, and Ishii [DHI].

For the Hecke eigenform $\Omega \in S_{12,12}(\Gamma)$ such that $\Omega | T(\varpi) = \psi\Omega$, we find

$$R(10, \Omega)/R(6, \Omega) = \frac{3732099 + 18663\sqrt{d}}{5},$$

$$R(8, \Omega)/R(6, \Omega) = \frac{24367 + 121\sqrt{d}}{20}.$$

8. A comparison of two methods

In [Sh2], Shimura gave a method to calculate critical values of $D(s, f, g)$ for two elliptic modular forms f and g . Here $D(s, f, g)$ is the Rankin–Selberg convolution of f and g . Shortly later he gave a generalization to the case of Hilbert modular forms (see [Sh3]). Taking one argument in the convoluted L -function as a suitable Eisenstein series, this method enables us to calculate the ratios of critical values of $L(s, \Omega)$ for a Hilbert modular form Ω . We call this technique method A. We call the cohomological technique method B, which was initiated in [Sh1] and studied in this paper when $[F : \mathbf{Q}] = 2$. It is interesting to compare A and B.

(0) Method A is more general and conceptually simpler. It has the advantage of giving the relation of the product of the plus and minus periods to the Petersson norm. It is applicable also to modular forms of half-integral weights.

(1) If $n = [F : \mathbf{Q}] > 2$, method B has to calculate $H^n(\Gamma, V)$, which is beyond reach at present. Therefore when $[F : \mathbf{Q}] > 2$, A is definitively superior than B.

(2) Suppose that $[F : \mathbf{Q}] = 2$. Method B is still incomplete. But in the cases well worked out, $F = \mathbf{Q}(\sqrt{5})$, for example, B has the advantage that we can write a program which calculates everything by machine. It can also be used to calculate the characteristic polynomials of Hecke operators. (In this respect, it is desirable to solve the problem mentioned at the end of Section 6.5.) We employed essentially a single program to obtain examples in Section 7. Therefore in some cases at least, B will have the advantage over A. But in general the method A is conceptually simpler.

In Doi and Goto [DG] and Doi and Ishii [DI], the authors gave interesting examples of critical values of $D(s, f, g)$ for Hilbert modular forms f and g . Their interest was the relation of this value to the congruences between Hilbert modular forms. However, they did not give examples of critical values of $L(s, \Omega)$. Recently Dr. K. Okada calculated the ratios of critical values of $L(s, \Omega)$ and confirmed the numerical value of Example 7.1 by method A. He obtained one more example for $F = \mathbf{Q}(\sqrt{17})$.

(3) Suppose that $F = \mathbf{Q}$. Method B is developed into the theory of modular symbols which is presently used to calculate characteristic polynomials of Hecke operators. For the L -values, the author does not know which is faster. But the

calculation of [Sh1] reviewed in the introduction suggests that B would not be more complex than A .

9. A remark on periods unrelated to critical values

In this section, we will show that we can deduce some information on the components of the cocycle $f(\Omega)$ which are not related to critical values in certain cases. We use the notation of Section 6 assuming $F = \mathbf{Q}(\sqrt{5})$. For simplicity, we consider the plus space assuming $l_1 \neq l_2$.

To explain our ideas, let us recall that

$$(9.1) \quad H^2(\Gamma^*, V)^+ \cong Z_A^+/B_A^+,$$

which is verified for $4 \leq l_2 < l_1 \leq 20$ (cf. Facts 6.1, 6.2, and Proposition 5.5). We assume that (9.1) always holds. Let ζ^+ be the linear mapping of Z_A^+ into \mathbf{C}^{l_2+1} (see (6.20)) which picks up information on critical values. A crucial point of our calculation of L -values is the fact that $\zeta^+(B_A^+) = 0$. By the functional equation of $L(s, \Omega)$ (or by (2.21)) and by Section 5.6, we see easily that the components of ζ^+ consist of (at most) $[l_2/4] + 1$ linearly independent linear forms on Z_A^+ . We have

$$Z_A^+ \supset \text{Ker}(\zeta^+) \supset B_A^+, \quad \dim \zeta^+(Z_A^+) = \dim Z_A^+ / \text{Ker}(\zeta^+).$$

Put $g^+ = \dim \text{Ker}(\zeta^+)/B_A^+$ and $L = \text{Hom}(Z_A^+/B_A^+, \mathbf{C})$. We regard an element of L as a linear form on Z_A^+ which is trivial on B_A^+ . Let L_0 be the subspace of L spanned by the components of ζ^+ .

Now our idea is very simple: By the dimensionality reason, we have $g^+ > 0$ when l_1 is sufficiently large for a fixed l_2 (e.g., $g^+ = 1$ when $(l_1, l_2) = (12, 6), (18, 6), (18, 8)$). Hence there exists $l \in L$ which does not belong to L_0 . In view of (9.1), l defines the linear form of $Z^2(\Gamma^*, V)^+$ which is trivial on the coboundary space. Considering the image under l of the cocycle obtained from Ω , we can deduce information on periods which are not related to critical values.

More concretely, let χ be the system of eigenvalues of Hecke operators attached to Ω . Let $(Z_A^+/B_A^+)(\chi)$ be the χ -isotypic component of Z_A^+/B_A^+ , and let $Z_A^+(\chi)$ be its pullback under the canonical homomorphism $Z_A^+ \rightarrow Z_A^+/B_A^+$. By the method of Section 6, we can calculate $(Z_A^+/B_A^+)(\chi)$ algebraically. Take $\varphi \in Z_A^+(\chi)$ whose components are in $\overline{\mathbf{Q}}$. On the other hand, we can calculate the corresponding element $\psi \in Z_A^+(\chi)$ from values of the cocycle $f(\Omega)$. We have $\psi \equiv c\varphi \pmod{B_A^+}$ with $c \in \mathbf{C}^\times$, and therefore

$$(9.2) \quad l(\psi) = cl(\varphi).$$

Equation (9.2) contains information on the values of $f(\Omega)$ unrelated to the critical values.

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