Examples of groups which are not weakly amenable

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Abstract We prove that weak amenability of a locally compact group imposes a strong condition on its amenable closed normal subgroups. This extends non–weak amenability results of Haagerup (1988) and Ozawa and Popa (2010). A von Neumann algebra analogue is also obtained.

1. Introduction

Let G be a group which is always assumed to be a locally compact topological group. The group G is said to be *weakly amenable* if the Fourier algebra $\mathcal{A}G$ of G has an approximate identity (φ_n) which is uniformly bounded as Herz–Schur multipliers. (If one requires (φ_n) to be bounded as elements in $\mathcal{A}G$, it becomes one of the equivalent definitions of amenability; see Section 2 for the precise definition.) Weak amenability is strictly weaker than amenability and passes to closed subgroups. It was proved by De Cannière and Haagerup [dCH], Cowling [Co], and Cowling and Haagerup [CH] that real simple Lie groups of real rank one are weakly amenable (see also [Oz]) and by Haagerup [Ha] that real simple Lie groups of real rank at least two are not weakly amenable. For the latter fact, Haagerup proves that $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ is not weakly amenable (see also [Do]). More recently, it was proved by Ozawa and Popa [OP] that the wreath product $\Lambda \wr \Gamma$ of a nontrivial group Λ by a nonamenable discrete group Γ is not "weakly amenable with constant 1." In this paper, we generalize these non–weak amenability results as follows.

THEOREM A

Let G be a weakly amenable group, and let N be an amenable closed normal subgroup of G. Then, there is a $(G \ltimes N)$-invariant state on $L^\infty(N)$, where the semidirect product $G \ltimes N$ acts on N by $(g,a)\cdot x = gaxg^{-1}$.

In particular, the wreath product by a nonamenable group is never weakly amenable. The theorem also gives a new proof of Haagerup's result that $\mathrm{SL}(2,\mathbb{Z}) \ltimes$

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$\mathbb{Z}^2$ is not weakly amenable, without appealing to the lattice embedding into $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$. We note for the sake of completeness that there is an even weaker variant of weak amenability called the *approximation property* (see [HK]), and $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ has the approximation property, while $\mathrm{SL}(n \geq 3,\mathbb{R})$ does not (see [LdS]).

As [OP, Theorem 3.5], there is an analogous result for von Neumann algebras. We refer to [OP, Section 3] and Section 4 of this paper for the terminology used in the following theorem.

THEOREM B

Let M be a finite von Neumann algebra with the weak completely bounded approximation property. Then, every amenable von Neumann subalgebra P is weakly compact in M.*

It follows that a type II_1 factor having the weak* completely bounded approximation property and property (T) (e.g., the group von Neumann algebra of a torsion-free lattice in $\mathrm{Sp}(1,n)$) is not isomorphic to a group-measure-space von Neumann algebra.

2. Preliminary on Herz–Schur multipliers

Let G be a group. We denote by λ the left regular representation of G on $L^2(G)$, by $C^*_\lambda G$ the reduced group C*-algebra, and by $\mathcal{L}G$ the group von Neumann algebra of G. The *Fourier algebra* $\mathcal{A}G$ of G consists of all functions φ on G such that there are vectors $\xi,\eta \in L^2(G)$ satisfying $\varphi(x) = \langle \lambda(x)\xi, \eta\rangle$ for every $x \in G$. (In other words, $\mathcal{A}G = L^2(G) * L^2(G)$.) It is a Banach algebra with the norm $\|\varphi\| = \inf\{\|\xi\|\|\eta\|\}$, where the infimum is taken over all $\xi,\eta \in L^2(G)$ as above. The Fourier algebra $\mathcal{A}G$ is naturally identified with the predual of $\mathcal{L}G$ under the duality pairing $\langle \varphi, \lambda(f)\rangle = \int_G \varphi f$ for $\varphi \in \mathcal{A}G$ and $\lambda(f) \in \mathcal{L}G$. If H is a closed subgroup of G, then $\varphi|_H \in \mathcal{A}H$ for every $\varphi \in \mathcal{A}G$. A continuous function φ on G is called a *Herz–Schur multiplier* if there are a Hilbert space $\mathcal{H}$ and bounded continuous functions $\xi,\eta\colon G \to \mathcal{H}$ such that $\varphi(y^{-1}x) = \langle \xi(x), \eta(y)\rangle$ for every $x, y \in G$. The Herz–Schur norm of φ is defined by

$$\|\varphi\|_{\mathrm{cb}} = \inf\bigl\{\|\xi\|_\infty \|\eta\|_\infty\bigr\},$$

where the infimum is taken over all $\xi,\eta \in C(G,\mathcal{H})$ as above. The Banach space of Herz–Schur multipliers is denoted by $B_2(G)$. Clearly, one has a contractive embedding of $\mathcal{A}G$ into $B_2(G)$. The Herz–Schur norm $\|\varphi\|_{\mathrm{cb}}$ coincides with the cb-norm of the corresponding multipliers on $\mathcal{L}G$ or on $C^*_\lambda G$:

$$\|\varphi\|_{\mathrm{cb}} = \|m_\varphi\colon \mathcal{L}G \ni \lambda(f) \mapsto \lambda(\varphi f) \in \mathcal{L}G\|_{\mathrm{cb}} = \|m_\varphi|_{C^*_\lambda G}\|_{\mathrm{cb}}.$$

Indeed, $\|\varphi\|_{\mathrm{cb}} \geq \|m_\varphi\|_{\mathrm{cb}}$ is easy to see: Given a factorization $\varphi(x^{-1}y) = \langle \xi(x), \eta(y)\rangle$ with $\xi,\eta \in C(G,\mathcal{H})$, we define $V_\xi\colon L^2(G) \to L^2(G,\mathcal{H})$ by $(V_\xi f)(x) = f(x)\xi(x^{-1})$, and likewise for V_η. Then, $\lambda(\varphi f) = V_\eta^*(\lambda(f) \otimes 1_{\mathcal{H}})V_\xi$ and $\|m_\varphi\|_{\mathrm{cb}} \leq \|\xi\|_\infty\|\eta\|_\infty$. We will give a proof of the converse inequality in Lemma 1, but we sketch it here in the case of amenable groups. Let N be an amenable group, and let $\varphi \in B_2(N)$. Since the unit character τ_0 is continuous on $C^*_\lambda N$, the linear functional $\omega_\varphi = \tau_0 \circ m_\varphi$ is bounded on $C^*_\lambda N$ and satisfies $\|\omega_\varphi\| \le \|m_\varphi\|_{\mathrm{cb}}$. Let $(\pi, \mathcal{H})$ be the GNS representation for $|\omega_\varphi|$, and view π as a continuous unitary N-representation. Then, there are vectors $\xi, \eta \in \mathcal{H}$ such that $\|\xi\|\|\eta\| = \|\omega_\varphi\|$ and $\varphi(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in N$. (Hence, $\|\omega_\varphi\| = \|\varphi\|_{\mathrm{cb}}$.)

DEFINITION

Let G be a group. By an *approximate identity* on G, we mean a net (φ_n) in $\mathcal{A}G$ which converges to 1 uniformly on compacta. It is *completely bounded* if

$$\|(\varphi_n)\|_{\mathrm{cb}} := \sup_n \|\varphi_n\|_{\mathrm{cb}} < +\infty.$$

A group G is said to be *weakly amenable* if there is a completely bounded approximate identity on G. The Cowling–Haagerup constant $\Lambda_{\mathrm{cb}}(G)$ is defined to be

$$\Lambda_{\mathrm{cb}}(G) = \inf\bigl\{\|(\varphi_n)\|_{\mathrm{cb}} : (\varphi_n) \text{ a c.b.a.i. on } G\bigr\}.$$

Note that the above infimum is attained (see [CH], [BO] for more information).

It is easy to see that if $H \le G$ is a closed subgroup, then $\Lambda_{\mathrm{cb}}(H) \le \Lambda_{\mathrm{cb}}(G)$. On this occasion, we record that the same inequality holds also for a "random" or "measure equivalence" subgroup in the sense of [Mo] and [Sa] (cf. [CZ]). For this, we consider only countable discrete groups Λ and Γ. Recall that Λ is an ME subgroup of Γ if there is a standard measure space Ω on which $\Lambda \times \Gamma$ acts by measure-preserving transformations in such a way that each of the of Λ- and Γ-actions admits a fundamental domain and the measure of $\Omega_\Gamma := \Omega/\Gamma$ is finite. The action $\Lambda \curvearrowright \Omega$ gives rise to a measure-preserving action $\Lambda \curvearrowright \Omega_\Gamma$ and a measurable cocycle $\alpha\colon \Lambda \times \Omega_\Gamma \to \Gamma$ such that the action $\Lambda \curvearrowright \Omega$ is isomorphic (up to null sets) to the twisted action $\Lambda \curvearrowright \Omega_\Gamma \times \Gamma$, given by $a(t,g) = (at, \alpha(a,t)g)$ for $a \in \Lambda$, $t \in \Omega_\Gamma$, and $g \in \Gamma$. The map α satisfies the cocycle identity $\alpha(ab,t) = \alpha(a,bt)\alpha(b,t)$ for every $a, b \in \Lambda$ and almost every $t \in \Omega_\Gamma$. For $\varphi \in B_2(\Gamma)$, we denote the "induced" function on Λ by φ_α:

$$\varphi_\alpha(a) = \int_{\Omega_\Gamma} \varphi\bigl(\alpha(a,t)\bigr)\, dt.$$

Here, we normalized the measure so that $|\Omega_\Gamma| = 1$. Since

$$\varphi_\alpha(b^{-1}a) = \int_{\Omega_\Gamma} \varphi\bigl(\alpha(b, b^{-1}at)^{-1}\alpha(a,t)\bigr)\, dt = \int_{\Omega_\Gamma} \varphi\bigl(\alpha(b, b^{-1}t)^{-1}\alpha(a, a^{-1}t)\bigr)\, dt,$$

one has $\varphi_\alpha \in B_2(\Lambda)$ and $\|\varphi_\alpha\|_{\mathrm{cb}} \le \|\varphi\|_{\mathrm{cb}}$. Suppose now that $\varphi \in \mathcal{A}\Gamma$. Then, φ_α is a coefficient of the unitary Λ-representation σ on $L^2(\Omega)$ induced by the measure-preserving action $\Lambda \curvearrowright \Omega$; that is, there are $\xi, \eta \in L^2(\Omega)$ such that $\varphi_\alpha(a) = \langle \sigma(a)\xi, \eta\rangle$. Since Ω admits a Λ-fundamental domain, σ is a multiple of the regular representation and $\varphi_\alpha \in \mathcal{A}\Lambda$. By inducing an approximate identity on Γ, one sees that if Γ is weakly amenable, then so is Λ and $\Lambda_{\mathrm{cb}}(\Lambda) \le \Lambda_{\mathrm{cb}}(\Gamma)$.

3. Proof of Theorem A

LEMMA 1

Let N be an amenable closed normal subgroup of G, and let $\varphi \in B_2(G)$. Then, there are a Hilbert space $\mathcal{H}$, functions $\xi, \eta \in C(G, \mathcal{H})$, and a continuous unitary representation π of N on $\mathcal{H}$ such that

- $\|\xi\|_\infty = \|\eta\|_\infty = \|\varphi\|_{\mathrm{cb}}^{1/2}$;
- $\varphi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle$ *for every* $x, y \in G$;
- $\pi(a)\xi(x) = \xi(ax)$ *and* $\pi(a)\eta(y) = \eta(ay)$ *for every* $a \in N$ *and* $x, y \in G$.

Proof

We follow Jolissaint's [Jo] simple proof of the inequality $\|\varphi\|_{\mathrm{cb}} \le \|m_\varphi\|_{\mathrm{cb}}$. Since N is amenable, the quotient map $q\colon G \to G/N$ extends to a $*$-homomorphism $q\colon C^*_\lambda G \to C^*_\lambda(G/N)$ between the reduced group C*-algebras. Since $q \circ m_\varphi$ is completely bounded on $C^*_\lambda G$, a Stinespring-type factorization theorem (see [BO, Theorem B.7]) yields a $*$-representation $\pi\colon C^*_\lambda G \to \mathbb{B}(\mathcal{H})$ and operators $V, W \in \mathbb{B}(L^2(G/N), \mathcal{H})$ such that $\|V\| = \|W\| \le \|q \circ m_\varphi\|_{\mathrm{cb}}^{1/2}$ and $(q \circ m_\varphi)(X) = W^* \times \pi(X)V$ for $X \in C^*_\lambda G$. We view π as a continuous unitary representation of G. Then, for a fixed unit vector $\zeta \in L^2(G/N)$, the maps $\xi(x) = \pi(x)V\lambda_{G/N}(q(x^{-1}))\zeta$ and $\eta(y) = \pi(y)W\lambda_{G/N}(q(y^{-1}))\zeta$ are continuous, $\|\xi\|_\infty, \|\eta\|_\infty \le \|m_\varphi\|_{\mathrm{cb}}^{1/2}$, and $\varphi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle$ for every $x, y \in G$. Moreover, $\pi(a)\xi(x) = \xi(ax)$ for $a \in N$, because $\lambda_{G/N}(a) = 1$. □

We denote by φ^g the right translation of a function φ by $g \in G$; that is, $\varphi^g(x) = \varphi(xg^{-1})$.

LEMMA 2

Let N be an amenable group, let $\varphi \in B_2(N)$, and let $a \in N$. Then,

$$\Big\|\frac{1}{2}(\varphi + \varphi^a)\Big\|_{\mathrm{cb}}^2 + \Big\|\frac{1}{2}(\varphi - \varphi^a)\Big\|_{\mathrm{cb}}^2 \le \|\varphi\|_{\mathrm{cb}}^2.$$

Proof

There are a continuous unitary representation π of N on a Hilbert space $\mathcal{H}$ and vectors $\xi, \eta \in \mathcal{H}$ such that $\|\xi\| = \|\eta\| = \|\varphi\|_{\mathrm{cb}}^{1/2}$ and $\varphi(x) = \langle \pi(x)\xi, \eta \rangle$ for every $x \in N$. Since $(\varphi \pm \varphi^a)(x) = \langle \pi(x)(\xi \pm \pi(a^{-1})\xi), \eta \rangle$, one has

$$\|\varphi + \varphi^a\|_{\mathrm{cb}}^2 + \|\varphi - \varphi^a\|_{\mathrm{cb}}^2 \le \|\xi + \pi(a^{-1})\xi\|^2\|\eta\|^2 + \|\xi - \pi(a^{-1})\xi\|^2\|\eta\|^2 = 4\|\varphi\|_{\mathrm{cb}}^2.$$

□

For $\varphi \in B_2(G)$, we define $\varphi^*(x) := \overline{\varphi(x^{-1})}$ and say that φ is *self-adjoint* if $\varphi^* = \varphi$. For any $\varphi \in B_2(G)$, the function $(\varphi + \varphi^*)/2$ is self-adjoint and $\|(\varphi + \varphi^*)/2\|_{\mathrm{cb}} \le \|\varphi\|_{\mathrm{cb}}$. Thus every approximate identity can be made self-adjoint without increasing norm. We fix a closed subgroup N of G. A completely bounded approximate identity (φ_n) on G is said to be *N-optimal* if all φ_n are self-adjoint,

$\|(\varphi_n)\|_{\text{cb}} = \Lambda_{\text{cb}}(G)$ and

$$\|(\varphi_n|_N)\|_{\text{cb}} = \inf\big\{\|(\psi_n|_N)\|_{\text{cb}} : (\psi_n) \text{ a c.b.a.i. such that } \|(\psi_n)\|_{\text{cb}} = \Lambda_{\text{cb}}(G)\big\}.$$

Note that an N-optimal approximate identity exists (if G is weakly amenable).

PROPOSITION 3

Let G be a weakly amenable group, and let N be an amenable closed normal subgroup of G. Let (φ_n) be an N-optimal approximate identity on G. Then, for every $g \in G$ and $a \in N$,

$$\lim_n \|(\varphi_n - \varphi_n \circ \text{Ad}_g)|_N\|_{\text{cb}} = 0 \qquad \text{and} \qquad \lim_n \|(\varphi_n - \varphi_n^a)|_N\|_{\text{cb}} = 0.$$

Proof

We apply Lemma 1 for each φ_n and find $(\pi_n, \mathcal{H}_n, \xi_n, \eta_n)$ satisfying the conditions stated there. In particular, $\|\xi\|_\infty = \|\eta\|_\infty \le \Lambda_{\text{cb}}(G)^{1/2}$ and $\varphi_n(y^{-1}x) = \langle \xi_n(x), \eta_n(y)\rangle$ for every $x, y \in G$. Let $g \in G$ be given, and consider $\psi_n = (\varphi_n + \varphi_n^g)/2$. Since (ψ_n) is a completely bounded approximate identity, one must have $\liminf_n \|\psi_n\|_{\text{cb}} \ge \Lambda_{\text{cb}}(G)$. Meanwhile, since φ_n is self-adjoint,

$$\psi_n(y^{-1}x) = \frac{1}{4}\big(\langle \xi_n(x) + \xi_n(xg^{-1}), \eta_n(y)\rangle + \langle \eta_n(x) + \eta_n(xg^{-1}), \xi_n(y)\rangle\big),$$

and hence

$$\begin{aligned}\|\psi_n\|_{\text{cb}} &\le \Big\|\frac{1}{\sqrt{2}}\Big(\frac{\xi_n + \xi_n^g}{2}, \frac{\eta_n + \eta_n^g}{2}\Big)\Big\|_{L^\infty(G, \mathcal{H}\oplus\mathcal{H})} \Big\|\frac{1}{\sqrt{2}}(\eta_n, \xi_n)\Big\|_{L^\infty(G, \mathcal{H}\oplus\mathcal{H})} \\ &\le \Lambda_{\text{cb}}(G).\end{aligned}$$

It follows that

$$\lim_n \Big\|\frac{1}{\sqrt{2}}\Big(\frac{\xi_n + \xi_n^g}{2}, \frac{\eta_n + \eta_n^g}{2}\Big)\Big\|_{L^\infty(G, \mathcal{H}\oplus\mathcal{H})} = \Lambda_{\text{cb}}(G)^{1/2},$$

which means that there is a net $z_n \in G$ such that

$$\lim_n \Big\|\frac{\xi_n(z_n) + \xi_n(z_n g^{-1})}{2}\Big\| = \Lambda_{\text{cb}}(G)^{1/2}$$

and

$$\lim_n \Big\|\frac{\eta_n(z_n) + \eta_n(z_n g^{-1})}{2}\Big\| = \Lambda_{\text{cb}}(G)^{1/2}.$$

By the parallelogram identity, this implies that

$$\lim_n \|\xi_n(z_n) - \xi_n(z_n g^{-1})\| = 0 \qquad \text{and} \qquad \lim_n \|\eta_n(z_n) - \eta_n(z_n g^{-1})\| = 0.$$

The unitary N-representation $\pi_n' = \pi_n \circ \text{Ad}_{z_n}$ satisfies $\pi_n'(a)\xi_n(x) = \xi_n(z_n a z_n^{-1} x)$,

$$\varphi_n(a) = \langle \pi_n'(a)\xi_n(z_n), \eta_n(z_n)\rangle$$

and

$$(\varphi_n \circ \text{Ad}_g)(a) = \langle \pi_n'(a)\xi_n(z_n g^{-1}), \eta_n(z_n g^{-1})\rangle$$

for $a \in N$. It follows that $\|(\varphi_n - \varphi_n \circ \mathrm{Ad}_g)|_N\|_{\mathrm{cb}} \to 0$. That $\|(\varphi_n - \varphi_n^a)|_N\|_{\mathrm{cb}} \to 0$ follows from N-optimality of (φ_n) and Lemma 2. $\square$

Proof of Theorem A

Let (φ_n) be an N-optimal approximate identity on G, and consider linear functionals $\omega_n = \tau_0 \circ m_{\varphi_n}$ on $C^*_\lambda N$, where τ_0 is the unit character on N (see Section 2). Since $\varphi_n \in \mathcal{A}G$, the linear functionals ω_n extend to ultraweakly continuous linear functionals on the group von Neumann algebra $\mathcal{L}N$. Indeed, they are nothing but $\varphi_n|_N \in \mathcal{A}N = (\mathcal{L}N)_*$. One has $\|\omega_n\| \le \Lambda_{\mathrm{cb}}(G)$, $\omega_n(1_{\mathcal{L}N}) = \varphi_n(1_N)$, and, by Proposition 3, $\|\omega_n - \omega_n \circ \mathrm{Ad}_g\| \to 0$ and $\|\omega_n - \omega_n^a\| \to 0$ for every $g \in G$ and $a \in N$. We consider $\zeta_n := |\omega_n|^{1/2} \in L^2(N)$ and $\zeta_n' := \omega_n|\omega_n|^{-1/2} \in L^2(N)$ so that $\omega_n(X) = \langle X\zeta_n, \zeta_n'\rangle$ for $X \in \mathcal{L}N$. Here the absolute value and the square root are taken in the sense of the standard representation $\mathcal{L}N \subset \mathbb{B}(L^2(N))$. (In the case where N is abelian, the Fourier transform $L^2(N) \cong L^2(\widehat{N})$ implements $\mathcal{L}N \cong L^\infty(\widehat{N})$ and $(\mathcal{L}N)_* \cong L^1(\widehat{N})$, and the absolute value and square root are computed as ordinary functions on the Pontrjagin dual $\widehat{N}$.) We note that $\varphi_n(1) \le \|\zeta_n\|_2^2 \le \Lambda_{\mathrm{cb}}(G)$. By continuity of the absolute value (see Proposition [Ta, III.4.10]) and the Powers–Størmer inequality, one has $\|\zeta_n - \mathrm{Ad}_g\,\zeta_n\|_2 \to 0$ for every $g \in G$. Moreover, since

$$\|\zeta_n\|_2\|\zeta_n'\|_2 - \left\|\frac{\zeta_n + \lambda(a^{-1})\zeta_n}{2}\right\|_2 \|\zeta_n'\|_2 \le \|\omega_n\| - \left\|\frac{\omega_n + \omega_n^a}{2}\right\| \to 0,$$

one has $\|\zeta_n - \lambda(a^{-1})\zeta_n\|_2 \to 0$ for every $a \in N$. Thus, any limit point of (ζ_n^2) in $L^\infty(N)^*$ is a nonzero positive $(G \ltimes N)$-invariant linear functional on $L^\infty(N)$. $\square$

COROLLARY 4

Let Γ and Λ be discrete groups with Λ nontrivial and Γ nonamenable. Then the wreath product $\Lambda \wr \Gamma$ is not weakly amenable. Also, the group $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ is not weakly amenable.

Proof

The proof is the same as that of [OP, Corollary 2.12]. We note that the stabilizer of a nonneutral element in $\mathbb{Z}^2$ is an abelian (amenable) subgroup of $\mathrm{SL}(2,\mathbb{Z})$. $\square$

4. Proof of Theorem B

We first fix notation. Throughout this section, M is a finite von Neumann algebra with a distinguished faithful normal tracial state τ, and P is an amenable von Neumann subalgebra of M. The *normalizer* $\mathcal{N}(P)$ of P in M is

$$\mathcal{N}(P) = \{u \in \mathcal{U}(M) : \mathrm{Ad}_u(P) = P\},$$

where $\mathcal{U}(M)$ is the group of the unitary elements of M and $\mathrm{Ad}_u(x) = uxu^*$. The GNS Hilbert space with respect to the trace τ is denoted by $L^2(M)$, and the vector in $L^2(M)$ associated with $x \in M$ is denoted by $\hat{x}$, that is, $\langle \hat{x}, \hat{y}\rangle = \tau(y^*x)$, for $x, y \in M$. The complex conjugate $\bar{M} = \{\bar{a} : a \in M\}$ of M acts on $L^2(M)$ from the right. Thus there is a $*$-representation ς of the algebraic tensor product $M \otimes \bar{M}$ on $L^2(M)$ defined by $\varsigma(a \otimes \bar{b})\hat{x} = \widehat{axb^*}$ for $a, b, x \in M$. We also use the bimodule notation $a\hat{x}b^*$ for $\varsigma(a \otimes \bar{b})\hat{x}$. Since P is amenable, the $*$-homomorphism $\varsigma|_{M \otimes \bar{P}}$ is continuous with respect to the minimal tensor norm.

DEFINITION

A von Neumann algebra M is said to have the *weak* completely bounded approximation property*, or W*CBAP in short, if there is a net of ultraweakly continuous finite-rank maps (φ_n) on M such that $\varphi_n \to \mathrm{id}_M$ in the point-ultraweak topology and $\sup \|\varphi_n\|_{\mathrm{cb}} < +\infty$.

Recall that a finite von Neumann algebra P is amenable (i.e., hyperfinite, injective, AFD, etc.) if the trace τ on P extends to a P-central state ω on $\mathbb{B}(L^2(P))$. Here, a state ω is said to be *P-central* if $\omega \circ \mathrm{Ad}_u = \omega$ for every $u \in \mathcal{U}(P)$ or, equivalently, $\omega(ax) = \omega(xa)$ for every $a \in P$ and $x \in \mathbb{B}(L^2(P))$.

DEFINITION

Let P be a finite von Neumann algebra, and let $\mathcal{G}$ be a group acting on P by trace-preserving $*$-automorphisms. We denote by σ the corresponding unitary representation of $\mathcal{G}$ on $L^2(P)$. The action $\mathcal{G} \curvearrowright P$ is said to be *weakly compact* if there is a state ω on $\mathbb{B}(L^2(P))$ such that $\omega|_P = \tau$ and $\omega \circ \mathrm{Ad}_u = \omega$ for every $u \in \sigma(\mathcal{G}) \cup \mathcal{U}(P)$. (This forces P to be amenable.) A von Neumann subalgebra P of a finite von Neumann algebra M is said to be *weakly compact* in M if the conjugate action by the normalizer $\mathcal{N}(P)$ is weakly compact (see [OP] for more information).

If M admits a crossed product decomposition $M = P \rtimes \Lambda$ such that the "core" P is nonatomic and weakly compact in M, then M does not have property (T). Indeed, the hypothesis implies that $\mathcal{L}\Lambda$ is coamenable in M (see [OP, Proposition 3.2]); that is, the M-M module $L^2\langle M, e_{\mathcal{L}\Lambda}\rangle$ contains an approximately central vector (see [OP, Theorem 2.1]). But since $L^2\langle M, e_{\mathcal{L}\Lambda}\rangle \cong \bigoplus_{t\in\Lambda} L^2(P) \otimes L^2(P)$ as a P-P module, it does not contain a nonzero central vector. This proves that M does not have property (T).

LEMMA 5

Every P-central state ω on $\mathbb{B}(L^2(P))$ decomposes uniquely as a sum $\omega = \omega_{\mathrm{n}} + \omega_{\mathrm{s}}$ of P-central positive linear functionals such that $\omega_{\mathrm{n}}|_P$ is normal and $\omega_{\mathrm{s}}|_P$ is singular. A trace-preserving action $\mathcal{G} \curvearrowright P$ is weakly compact if there is a positive linear functional ω on $\mathbb{B}(L^2(P))$ such that

- *$\omega(p) > 0$ for every nonzero central projection p in P,*
- *$\omega \circ \mathrm{Ad}_u = \omega$ for every $u \in \sigma(\mathcal{G}) \cup \mathcal{U}(P)$.*

Proof

We denote by Z the center of P. Recall that every tracial state τ' on P satisfies $\tau' = \tau'|_Z \circ E_Z$, where $E_Z \colon P \to Z$ is the center-valued trace. In particular, τ' is normal on P if and only if it is normal on Z. Let ω be a P-central state, and consider the normal/singular decomposition of the state $\omega|_Z$ (see [Ta, Definition III.2.15]). There is an increasing sequence (p_n) of projections in Z such that $p_n \nearrow 1$ and $(\omega|_Z)_{\mathrm{s}}(p_n) = 0$ for all n (see [Ta, Theorem III.3.8]). We fix an ultralimit Lim on $\mathbb{N}$ and let $\omega_{\mathrm{n}}(x) = \mathrm{Lim}\,\omega(p_n x)$ and $\omega_{\mathrm{s}} = \omega - \omega_{\mathrm{n}}$. Since ω is P-central, these are P-central positive linear functionals on $\mathbb{B}(L^2(P))$, and $\omega|_Z = \omega_{\mathrm{n}}|_Z + \omega_{\mathrm{s}}|_Z$ is the normal/singular decomposition of $\omega|_Z$. Suppose that $\omega = \omega'_{\mathrm{n}} + \omega'_{\mathrm{s}}$ is another such decomposition. Then, since $\omega_{\mathrm{s}} + \omega'_{\mathrm{s}}$ is singular on Z, there is an increasing sequence (q_n) of projections in Z such that $q_n \nearrow 1$ and $(\omega_{\mathrm{s}} + \omega'_{\mathrm{s}})(q_n) = 0$ for all n. It follows that $\omega'_{\mathrm{n}}(x) = \lim \omega(q_n x) = \omega_{\mathrm{n}}(x)$ for every $x \in \mathbb{B}(L^2(P))$. This proves the first half of this lemma. For the second half, we first observe that we may assume that ω is normal on P by uniqueness of the normal/singular decomposition. Thus, there is $h \in L^1(Z)_+$ such that $\omega(z) = \tau(hz)$ for $z \in Z$. By assumption, h has full support and is $\mathcal{G}$-invariant. Thus, $\tilde{\omega}(x) := \mathrm{Lim}\,\omega((h + n^{-1})^{-1}x)$ defines a $\mathcal{G}$-invariant P-central state on $\mathbb{B}(L^2(P))$ such that $\tilde{\tau}|_Z = \tau|_Z$. □

LEMMA 6

Let φ be a completely bounded map on M. Then, there are a $*$-representation of the minimal tensor product $M \otimes_{\min} \bar{P}$ on a Hilbert space $\mathcal{H}$ and operators $V, W \in \mathbb{B}(L^2(M), \mathcal{H})$ such that $\|V\| = \|W\| \le \|\varphi\|_{\mathrm{cb}}^{1/2}$ and

$$\tau\big(y^*\varphi(a)xb^*\big) = \langle \varphi(a)\hat{x}b^*, \hat{y}\rangle = \langle \pi(a \otimes \bar{b})V\hat{x}, W\hat{y}\rangle$$

for every $a, x, y \in M$ and $b \in P$.

Proof

Since the $*$-representation $\varsigma \colon M \otimes_{\min} \bar{P} \to \mathbb{B}(L^2(M))$ is continuous, a Stinespring-type factorization theorem ([BO, Theorem B.7]), applied to the completely bounded map $\varsigma \circ (\varphi \otimes \mathrm{id}_{\bar{P}})$ yields a $*$-representation $\pi \colon M \otimes_{\min} \bar{P} \to \mathbb{B}(\mathcal{H})$ and operators $V, W \in \mathbb{B}(L^2(M), \mathcal{H})$ such that $\|V\|\|W\| \le \|\varphi\|_{\mathrm{cb}}$ and

$$\varphi(a)\hat{x}b^* = \varsigma\big((\varphi \otimes \mathrm{id}_{\bar{P}})(a \otimes \bar{b})\big)\hat{x} = W^*\pi(a \otimes \bar{b})V\hat{x}$$

for $a, x \in M$ and $b \in P$. □

Since W*CBAP passes to a subalgebra (which is the range of a conditional expectation), we assume from now on that P is *regular* in M; that is, $\mathcal{N}(P)$ generates M as a von Neumann algebra. We say that a linear map φ on M is *P-cb* if there are a $*$-representation π of $M \otimes_{\min} \bar{P}$ on a Hilbert space $\mathcal{H}$ and functions $V, W \in \ell_\infty(\mathcal{N}(P), \mathcal{H})$ such that

$$(*) \qquad \langle \varphi(a)\hat{x}b^*, \hat{y}\rangle = \langle \pi(a \otimes \bar{b})V(x), W(y)\rangle$$

for every $a \in M$, $x, y \in \mathcal{N}(P)$, and $b \in P$. The P-cb norm of φ is defined as

$$\|\varphi\|_P = \inf\{\|V\|_\infty \|W\|_\infty : (\pi, \mathcal{H}, V, W) \text{ satisfies } (*)\}.$$

It is indeed a norm, and the infimum is attained. (For the latter fact, use the ultraproduct.) By the above lemma, $\|\varphi\|_P \leq \|\varphi\|_{\mathrm{cb}}$. By an *approximate identity*, we mean a net (φ_n) of ultraweakly continuous finite-rank maps such that $\varphi_n \to \mathrm{id}_M$ in the point-ultraweak topology and $\sup \|\varphi_n\|_P < +\infty$. It exists if M has the W*CBAP. We define

$$\Lambda_P(M) = \inf\Big\{\sup_n \|\varphi_n\|_P : (\varphi_n) \text{ an approximate identity}\Big\}.$$

For a map φ on M, we define $\varphi^*(a) = \varphi(a^*)^*$ and say that φ is *self-adjoint* if $\varphi = \varphi^*$. We note that if $(\pi, \mathcal{H}, V, W)$ satisfies $(*)$ for φ, then $(\pi, \mathcal{H}, W, V)$ satisfies $(*)$ for φ^*. In particular, $(\varphi + \varphi^*)/2$ is self-adjoint and $\|(\varphi + \varphi^*)/2\|_P \leq \|\varphi\|_P$. Thus, any approximate identity can be made self-adjoint without increasing the norm. For a P-cb map φ, we define a bounded linear functional μ_φ on $M \otimes_{\min} \bar{P}$ by

$$\mu_\varphi(a \otimes \bar{b}) := \tau\big(\varphi(a)b^*\big) = \langle \varphi(a)\hat{1}b^*, \hat{1}\rangle = \langle \pi(a \otimes \bar{b})V(1), W(1)\rangle.$$

Note that $\|\mu_\varphi\| \leq \|\varphi\|_P$. If φ is ultraweakly continuous and finite-rank, then μ_φ extend to an ultraweakly continuous linear functional on the von Neumann algebra $M \bar{\otimes} \bar{P}$.

PROPOSITION 7

*Let M be a finite von Neumann algebra having the W*CBAP, and let (φ_n) be a self-adjoint approximate identity such that $\sup_n \|\varphi_n\|_P = \Lambda_P(M)$. Then, the net $\mu_n := \mu_{\varphi_n}|_{P\bar{\otimes}\bar{P}}$ satisfies the following properties:*

- *μ_n are self-adjoint and ultraweakly continuous for all n;*
- *$\sup \|\mu_n\| \leq \Lambda_P(M)$ and $\mu_n(a \otimes \bar{1}) \to \tau(a)$ for every $a \in P$;*
- *$\|\mu_n - \mu_n^{v\otimes\bar{v}}\| \to 0$ for every $v \in \mathcal{U}(P)$, where $\mu_n^{v\otimes\bar{v}}(a \otimes \bar{b}) = \mu_n((a \otimes \bar{b})(v \otimes \bar{v})^*)$;*
- *$\|\mu_n - \mu_n \circ \mathrm{Ad}_{u\otimes\bar{u}}\| \to 0$ for every $u \in \mathcal{N}(P)$.*

Proof

The first two conditions are easy to see. Let $u \in \mathcal{N}(P)$ be given, and define φ_n^u by $\varphi_n^u(a) = \varphi_n(au^*)u$ for $a \in M$. We note that $\mu_{\varphi_n^u}|_{P\bar{\otimes}\bar{P}} = \mu_n^{u\otimes\bar{u}}$ if $u \in \mathcal{U}(P)$. Thus, it suffices to show

$$\lim_n \|\mu_{\varphi_n} - \mu_{\varphi_n^u}\| = 0 \qquad \text{and} \qquad \lim_n \|\mu_{\varphi_n} - \mu_{\varphi_n} \circ \mathrm{Ad}_{u\otimes\bar{u}}\| = 0.$$

Take $(\pi_n, \mathcal{H}_n, V_n, W_n)$ satisfying $(*)$ and $\lim \|V_n\|_\infty = \lim \|W_n\|_\infty = \Lambda_P(M)^{1/2}$. It follows that

$$\langle \varphi_n^u(a)\hat{x}b^*, \hat{y}\rangle = \langle \varphi_n(au^*)\widehat{ux}b^*, \hat{y}\rangle = \langle \pi_n(a \otimes \bar{b})\pi_n(u^* \otimes \bar{1})V_n(ux), W_n(y)\rangle$$

for every $a \in M$, $b \in P$, and $x, y \in \mathcal{N}(P)$. Hence with $V_n^u(x) = \pi_n(u^* \otimes \bar{1})V_n(ux)$, the quadruplet $(\pi_n, \mathcal{H}_n, V_n^u, W_n)$ satisfies $(*)$ for φ_n^u. Note that $\|V_n^u\|_\infty = \|V_n\|_\infty$.

We define W_n^u similarly. Since φ_n is self-adjoint, $(\pi_n, \mathcal{H}_n, W_n, V_n)$ (resp., $(\pi_n, \mathcal{H}_n, W_n^u, V_n)$) satisfies $(*)$ for φ_n (resp., φ_n^u), too. Thus, for $\psi_n = (\varphi_n + \varphi_n^u)/2$, one has

$$\|\psi_n\|_P \le \Big\| \frac{1}{\sqrt{2}}\Big(\frac{V_n + V_n^u}{2}, \frac{W_n + W_n^u}{2}\Big)\Big\|_{\ell_\infty(\mathcal{N}(P), \mathcal{H}\oplus\mathcal{H})} \times \Big\| \frac{1}{\sqrt{2}}(W_n, V_n)\Big\|_{\ell_\infty(\mathcal{N}(P), \mathcal{H}\oplus\mathcal{H})}.$$

Meanwhile, since (ψ_n) is an approximate identity, one must have $\liminf \|\psi_n\|_P \ge \Lambda_P(M)$. It follows that

$$\lim_n \Big\| \frac{1}{\sqrt{2}}\Big(\frac{V_n + V_n^u}{2}, \frac{W_n + W_n^u}{2}\Big)\Big\|_{\ell_\infty(\mathcal{N}(P), \mathcal{H}\oplus\mathcal{H})} = \Lambda_P(M)^{1/2}$$

and hence there is a net (z_n) in $\mathcal{N}(P)$ such that

$$\lim_n \Big\| \frac{1}{\sqrt{2}}\Big(\frac{(V_n + V_n^u)(z_n)}{2}, \frac{(W_n + W_n^u)(z_n)}{2}\Big)\Big\|_{\mathcal{H}\oplus\mathcal{H}} = \Lambda_P(M)^{1/2}.$$

By the parallelogram identity, this implies that

$$\lim_n \|V_n(z_n) - V_n^u(z_n)\| = 0 \qquad \text{and} \qquad \lim_n \|W_n(z_n) - W_n^u(z_n)\| = 0.$$

Let $\pi_n' = \pi_n \circ (\mathrm{id}_M \otimes \mathrm{Ad}_{\bar{z}_n^{-1}})$. Since

$$\mu_{\varphi_n}(a \otimes \bar{b}) = \langle \varphi_n(a)\hat{z}_n \,\mathrm{Ad}_{z_n^{-1}}(b)^*, \hat{z}_n\rangle = \langle \pi_n'(a\otimes\bar{b})V_n(z_n), W_n(z_n)\rangle,$$

$$\mu_{\varphi_n^u}(a \otimes \bar{b}) = \langle \varphi_n(au^*)\widehat{uz_n}\,\mathrm{Ad}_{z_n^{-1}}(b)^*, \hat{z}_n\rangle = \langle \pi_n'(a\otimes\bar{b})V_n^u(z_n), W_n(z_n)\rangle,$$

and

$$(\mu_{\varphi_n} \circ \mathrm{Ad}_{u\otimes\bar{u}})(a \otimes \bar{b}) = \langle \varphi_n(uau^*)\widehat{uz_n}\,\mathrm{Ad}_{z_n^{-1}}(b)^*, \widehat{uz_n}\rangle = \langle \pi_n'(a\otimes\bar{b})V_n^u(z_n), W_n^u(z_n)\rangle,$$

we conclude that $\|\mu_{\varphi_n} - \mu_{\varphi_n^u}\| \to 0$ and $\|\mu_{\varphi_n} - \mu_{\varphi_n} \circ \mathrm{Ad}_{u\otimes\bar{u}}\| \to 0$. □

Proof of Theorem B

Since M has the W*CBAP, there is a net (μ_n) satisfying the conclusion of Proposition 7. We view μ_n as an element in $L^1(P \mathbin{\bar{\otimes}} \bar{P})$ (see Section 2 in [OP]) and let $\zeta_n = |\mu_n|^{1/2} \in L^2(P \mathbin{\bar{\otimes}} \bar{P})$ and $\zeta_n' = \mu_n|\mu_n|^{-1/2} \in L^2(P \mathbin{\bar{\otimes}} \bar{P})$ so that $\mu_n(X) = \langle X\zeta_n, \zeta_n'\rangle$ for $X \in P \mathbin{\bar{\otimes}} \bar{P}$. By continuity of the absolute value (see [Ta, Proposition III.4.10]) and the Powers–Størmer inequality, one has $\|\zeta_n - \mathrm{Ad}_{u\otimes\bar{u}}\,\zeta_n\|_2 \to 0$ for every $u \in \mathcal{N}(P)$. Since

$$2\|\mu_n\| \approx \|\mu_n + \mu_n^{v\otimes\bar{v}}\| \le \|\zeta_n + (v\otimes\bar{v})\zeta_n\|_2\|\zeta_n'\|_2 \le 2\|\zeta_n\|_2\|\zeta_n'\|_2 = 2\|\mu_n\|,$$

one also has $\|\zeta_n - (v\otimes\bar{v})\zeta_n\| \to 0$ for every $v \in \mathcal{U}(P)$. Now, fix an ultralimit Lim, and define ω on $\mathbb{B}(L^2(P))$ by $\omega(x) = \mathrm{Lim}\langle (x\otimes\bar{1})\zeta_n, \zeta_n\rangle$. Then ω is an $\mathcal{N}(P)$-

invariant P-central positive linear functional satisfying

$$\omega(p) = \mathrm{Lim}_n\,|\mu_n|(p \otimes \bar{1}) \geq \mathrm{Lim}_n\,|\mu_n(p \otimes \bar{1})| = \tau(p)$$

for every central projection p in P. By Lemma 5, we are done. □

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