Vanishing mean oscillation spaces associated with operators satisfying Davies–Gaffney estimates

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Abstract Let $(\mathcal{X}, d, \mu)$ be a metric measure space, let L be a linear operator that has a bounded H_∞-functional calculus and satisfies the Davies–Gaffney estimate, let Φ be a concave function on $(0,\infty)$ of critical lower type $p_\Phi^- \in (0,1]$, and let $\rho(t) \equiv t^{-1}/\Phi^{-1}(t^{-1})$ for all $t \in (0,\infty)$. In this paper, the authors introduce the generalized VMO space $\mathrm{VMO}_{\rho,L}(\mathcal{X})$ associated with L and establish its characterization via the tent space. As applications, the authors show that $(\mathrm{VMO}_{\rho,L}(\mathcal{X}))^* = B_{\Phi,L^*}(\mathcal{X})$, where L^* denotes the adjoint operator of L in $L^2(\mathcal{X})$ and $B_{\Phi,L^*}(\mathcal{X})$ the Banach completion of the Orlicz–Hardy space $H_{\Phi,L^*}(\mathcal{X})$.

Contents

1. Introduction

John and Nirenberg [24] introduced the *space* $\mathrm{BMO}(\mathbb{R}^n)$, which is defined to be the space of all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \equiv \sup_{\text{ball } B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x) - f_B|\,dx < \infty,$$

where in what follows, $f_B \equiv \frac{1}{|B|}\int_B f(x)\,dx$. The space $\mathrm{BMO}(\mathbb{R}^n)$ was proved to be the dual of the Hardy space $H^1(\mathbb{R}^n)$ by Fefferman and Stein [14].

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Sarason [28] introduced the *space* $\mathrm{VMO}(\mathbb{R}^n)$, which is defined to be the space of all $f \in \mathrm{BMO}(\mathbb{R}^n)$ such that

$$\lim_{c\to 0} \sup_{\substack{\text{ball } B\subset\mathbb{R}^n \\ r_B\le c}} \frac{1}{|B|}\int_B |f(x)-f_B|\,dx = 0,$$

where r_B denotes the radius of the ball B. In order to represent $H^1(\mathbb{R}^n)$ as a dual space, Coifman and Weiss [8] introduced the *space* $\mathrm{CMO}(\mathbb{R}^n)$, which is defined to be the closure of all infinitely differentiable functions with compact support in the $\mathrm{BMO}(\mathbb{R}^n)$-norm and was originally denoted by the *symbol* $\mathrm{VMO}(\mathbb{R}^n)$ in [8], and they proved that $(\mathrm{CMO}(\mathbb{R}^n))^* = H^1(\mathbb{R}^n)$. For more properties of $\mathrm{BMO}(\mathbb{R}^n)$, $\mathrm{VMO}(\mathbb{R}^n)$, and $\mathrm{CMO}(\mathbb{R}^n)$, we refer the reader to Janson [18] and Bourdaud [5].

Let L be a *linear operator* in $L^2(\mathbb{R}^n)$ that generates an analytic semigroup $\{e^{-tL}\}_{t\ge 0}$ with kernels satisfying an *upper bound of Poisson type*. The Hardy space $H^1_L(\mathbb{R}^n)$, the BMO space $\mathrm{BMO}_L(\mathbb{R}^n)$, and Morrey spaces associated with L were introduced and studied in [4], [11], [13]. Duong and Yan [12] further proved that $(H^1_L(\mathbb{R}^n))^* = \mathrm{BMO}_{L^*}(\mathbb{R}^n)$, where L^* denotes the *adjoint operator* of L in $L^2(\mathbb{R}^n)$. Moreover, recently, Deng et al. [9] introduced the *space* $\mathrm{VMO}_L(\mathbb{R}^n)$, the space of vanishing mean oscillation associated with the operator L, and proved that $(\mathrm{VMO}_L(\mathbb{R}^n))^* = H^1_{L^*}(\mathbb{R}^n)$ and also

$$\mathrm{VMO}_\Delta(\mathbb{R}^n) = \mathrm{CMO}(\mathbb{R}^n) = \mathrm{VMO}_{\sqrt{\Delta}}(\mathbb{R}^n)$$

with equivalent norms, where Δ is the Laplace operator $-\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Let Φ on $(0,\infty)$ be a continuous, strictly increasing, subadditive function of upper type 1 and of critical lower type $p_\Phi^- \le 1$ but near to 1 (see Section 2.4 below for the definition). Let $\rho(t) \equiv t^{-1}/\Phi^{-1}(t^{-1})$ for all $t \in (0,\infty)$. A typical example of such Orlicz functions is $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$ and $p \le 1$ but near to 1. Jiang and Yang [22] introduced the VMO-type space $\mathrm{VMO}_{\rho,L}(\mathbb{R}^n)$ and proved that the dual space of $\mathrm{VMO}_{\rho,L^*}(\mathbb{R}^n)$ is the space $B_{\Phi,L}(\mathbb{R}^n)$, where $B_{\Phi,L}(\mathbb{R}^n)$ denotes the *Banach completion* of the Orlicz–Hardy space $H_{\Phi,L}(\mathbb{R}^n)$ in [23].

Let L be a *second-order divergence form elliptic operator* with complex bounded measurable coefficients, and let Φ be a *continuous, strictly increasing, concave function* of critical lower-type $p_\Phi^- \in (0,1]$. Jiang and Yang [19] studied the VMO-type spaces $\mathrm{VMO}_{\rho,L}(\mathbb{R}^n)$ and proved that the dual space of $\mathrm{VMO}_{\rho,L^*}(\mathbb{R}^n)$ is the space $B_{\Phi,L}(\mathbb{R}^n)$, where $B_{\Phi,L}(\mathbb{R}^n)$ denotes the *Banach completion* of the Orlicz–Hardy space $H_{\Phi,L}(\mathbb{R}^n)$ in [20]. (We remark that the *assumptions on* p_Φ *in [19], [20] can be relaxed into the same assumptions on* p_Φ^-; see Remark 2.2(ii) below.) In particular, when $\Phi(t) \equiv t$ for all $t \in (0,\infty)$, then $\rho(t) \equiv 1$ and $(\mathrm{VMO}_{1,L}(\mathbb{R}^n))^* = H^1_{L^*}(\mathbb{R}^n)$, which was also independently obtained by Song and Xu [29], where $H^1_{L^*}(\mathbb{R}^n)$ denotes the Hardy space first introduced by Hofmann and Mayboroda [16] (see also [17]).

Let $(\mathcal{X}, d)$ be a *metric space endowed with a doubling measure* μ, and let L be a *nonnegative self-adjoint operator satisfying Davies–Gaffney estimates*. Hofmann et al. [15] introduced the Hardy space $H^1_L(\mathcal{X})$ associated to L. Jiang and Yang [21] further introduced the Orlicz–Hardy space $H_{\Phi,L}(\mathcal{X})$. Anh [1] studied the VMO space $\mathrm{VMO}_L(\mathcal{X})$ associated to L and proved that the dual space of $\mathrm{VMO}_L(\mathcal{X})$ is the Hardy space $H^1_L(\mathcal{X})$. Recently, Duong and Li [10] observed that the assumption "L is a nonnegative self-adjoint operator" in [15] can be replaced by a weaker assumption that "*L has a bounded H_∞-functional calculus on $L^2(\mathcal{X})$*" and introduced the Hardy space $H^p_L(\mathcal{X})$ with $p \in (0,1]$, which was further generalized by Anh and Li [2] to the Orlicz–Hardy spaces $H_{\Phi,L}(\mathcal{X})$.

From now on, we always assume that L *is a linear operator which has a bounded H_∞-functional calculus and satisfies Davies–Gaffney estimates* and that Φ *is a continuous, strictly increasing, concave function of critical lower-type $p_\Phi^- \in (0,1]$*. In this paper, we introduce the generalized VMO space $\mathrm{VMO}_{\rho,L}(\mathcal{X})$ associated with L and establish its characterization via the tent space in [21]. Then, we further prove that $(\mathrm{VMO}_{\rho,L}(\mathcal{X}))^* = B_{\Phi,L^*}(\mathcal{X})$, where $B_{\Phi,L^*}(\mathcal{X})$ denotes the *Banach completion* of the Orlicz–Hardy space $H_{\Phi,L^*}(\mathcal{X})$ in [2]. When $\Phi(t) \equiv t$ for all $t \in (0,\infty)$, we denote $\mathrm{VMO}_{\rho,L}(\mathcal{X})$ simply by $\mathrm{VMO}_L(\mathcal{X})$. As a special case of the main results in this paper, we show that $(\mathrm{VMO}_L(\mathcal{X}))^* = H^1_{L^*}(\mathcal{X})$, which, when L is nonnegative self-adjoint, was already obtained by Anh [1].

Precisely, the paper is organized as follows. In Section 2, we recall some known notions and notation concerning metric measure spaces $\mathcal{X}$, then describe some basic assumptions on the considered operator L and the Orlicz function Φ and present some properties of the operator L and the Orlicz function Φ considered in this paper.

In Section 3, we first obtain the ρ-Carleson measure characterization (see Theorem 3.1 below) of the space $\mathrm{BMO}_{\rho,L}(\mathcal{X})$ in [2] via first establishing a Calderón reproducing formula (see Proposition 3.3 below). Differently from the Calderón reproducing formula in [21, Proposition 4.6], the Calderón reproducing formula in Proposition 3.3 below holds for all molecules instead of atoms in [21], which brings us some extra difficulty due to the lack of the support of molecules. Then we introduce the generalized VMO space $\mathrm{VMO}_{\rho,L}(\mathcal{X})$ associated with L, and the tent space $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$, and establish some basic properties of these spaces. In particular, we characterize the space $\mathrm{VMO}_{\rho,L}(\mathcal{X})$ via $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$ (see Theorem 3.4 below). To this end, we first need to make clear the dual relation between $H_{\Phi,L^*}(\mathcal{X})$ and $\mathrm{BMO}_{\rho,L}(\mathcal{X})$ (see Theorem 3.2 below), which is deduced from a technical result on the optimal representation of finite linear combinations of molecules (see Theorem 3.1 below). We remark that variants of Theorems 3.1 and 3.2 below have already been given, respectively, in [2, Theorems 3.15, 3.13, 3.16] without a detailed proof of [2, Theorem 3.15]. We give a detailed proof of Theorem 3.1 below which induces *more accurate indices* appearing in Theorems 3.1 and 3.2 below, comparing with [2, Theorems 3.13, 3.15] (see Remark 3.2 below). Moreover, the proof of Theorem 3.1 below simplifies the proof of [15, Theorem 5.4] in a subtle way, and the proof of [15, Theorem 5.4] strongly depends on the support of atoms (see Remark 3.1 below).

In Section 4, we first obtain, in Theorem 4.1 below, the dual space of the tent space $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$ in Definition 3.4 below, from which we further deduce that $(\mathrm{VMO}_{\rho,L}(\mathcal{X}))^* = B_{\Phi,L^*}(\mathcal{X})$ in Theorem 4.2 below, where $B_{\Phi,L^*}(\mathcal{X})$ denotes the

Banach completion of $H_{\Phi,L^*}(\mathcal{X})$. In particular, we obtain $(\mathrm{VMO}_L(\mathcal{X}))^* = H^1_{L^*}(\mathcal{X})$.

Finally we make some conventions on notation. Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *constant with subscripts*, such as C_1, does not change in different occurrences. We also use $C(\gamma,\ldots)$ to denote *a positive constant depending on the indicated parameters* $\gamma,\ldots$. The *symbol* $A\lesssim B$ means that $A\le CB$. If $A\lesssim B$ and $B\lesssim A$, then we write $A\sim B$. We also set $\mathbb{N}\equiv\{1,2,\ldots\}$ and $\mathbb{Z}_+\equiv\mathbb{N}\cup\{0\}$. The *symbol* $B(x,r)$ denotes the ball $\{y\in\mathcal{X}: d(x,y)<r\}$; moreover, let $CB(x,r)\equiv B(x,Cr)$. For a measurable set E, denote by χ_E the *characteristic function* of E and by $E^{\complement}$ the *complement* of E in $\mathcal{X}$.

2. Preliminaries

In this section, we first recall some notions and notation on metric measure spaces and then describe some basic assumptions on the operator L considered in this paper and its functional calculus; finally, we also present some basic assumptions and properties on Orlicz functions.

2.1. Metric measure spaces

Throughout the whole paper, let $\mathcal{X}$ be a *set*, let d be a *metric* on $\mathcal{X}$, and let μ be a *nonnegative Borel regular measure* on $\mathcal{X}$. Moreover, assume that there exists a constant $C_1\ge 1$ such that for all $x\in\mathcal{X}$ and $r>0$,

$$V(x,2r)\le C_1V(x,r)<\infty, \tag{2.1}$$

where $B(x,r)\equiv\{y\in\mathcal{X}: d(x,y)<r\}$ and

$$V(x,r)\equiv\mu\big(B(x,r)\big). \tag{2.2}$$

Observe that if d is further assumed to be a quasi-metric, then $(\mathcal{X},d,\mu)$ is called a *space of homogeneous type* in the sense of Coifman and Weiss [7] (see also [8]).

Notice that the doubling property (2.1) implies the following *strong homogeneity property*: there exist some positive constants C and n, depending on C_1, such that

$$V(x,\lambda r)\le C\lambda^nV(x,r) \tag{2.3}$$

uniformly for all $\lambda\ge 1$, $x\in\mathcal{X}$, and $r>0$. The parameter n measures the *dimension* of the space $\mathcal{X}$ in some sense. Also, there exist constants $C\in(0,\infty)$ and $N\in[0,n]$, depending on C_1, such that

$$V(x,r)\le C\Big(1+\frac{d(x,y)}{r}\Big)^NV(y,r) \tag{2.4}$$

uniformly for all $x,y\in\mathcal{X}$ and $r>0$. Indeed, the property (2.4) with $N=n$ is a simple corollary of the strong homogeneity property (2.3). In the case of Euclidean spaces, Lie groups of polynomial growth and, more generally, Ahlfors regular spaces, N can be chosen to be zero.

In what follows, for any ball $B \subset \mathcal{X}$, we set

$$U_0(B) \equiv B \qquad \text{and} \qquad U_j(B) \equiv 2^j B \backslash 2^{j-1} B \quad \text{for } j \in \mathbb{N}. \tag{2.5}$$

The following covering lemma established in [1, Lemma 2.1] plays a key role in the sequel.

LEMMA 2.1

For any $\ell > 0$, there exists $N_\ell \in \mathbb{N}$, depending on ℓ, such that for all balls $B(x_B, \ell r)$, with $x_B \in \mathcal{X}$ and $r > 0$, there exists a family $\{B(x_{B,i}, r)\}_{i=1}^{N_\ell}$ of balls such that

(i) $B(x_B, \ell r) \subset \bigcup_{i=1}^{N_\ell} B(x_{B,i}, r)$;
(ii) $N_\ell \le C\ell^n$;
(iii) $\sum_{i=1}^{N_\ell} \chi_{B(x_{B,i}, r)} \le C$.

Here C is a positive constant independent of x_B, r, and ℓ.

2.2. Holomorphic functional calculi

We now recall some basic notions of holomorphic functional calculi introduced by McIntosh [25].

Let $0 < \nu < \gamma < \pi$. Define the *closed sector* S_ν in the complex plane $\mathbb{C}$ by setting $S_\nu \equiv \{z \in \mathbb{C} : |\arg z| \le \nu\} \cup \{0\}$, and denote by S_ν^0 its *interior*. We employ the following *subspaces*, $H_\infty(S_\nu^0)$ and $\Psi(S_\nu^0)$, of the *space* $H(S_\nu^0)$ of all holomorphic functions on S_ν^0:

$$H_\infty(S_\nu^0) \equiv \Big\{ b \in H(S_\nu^0) : \|b\|_{L^\infty(S_\nu^0)} \equiv \sup_{z \in S_\nu^0} |b(z)| < \infty \Big\}$$

and

$$\Psi(S_\nu^0) \equiv \Big\{ \psi \in H(S_\nu^0) : \text{there exist } s \in (0,\infty) \text{ and } C \in (0,\infty) \text{ such that}$$
$$\text{for all } z \in S_\nu^0, |\psi(z)| \le C|z|^s (1 + |z|^{2s})^{-1} \Big\}.$$

Given $\nu \in (0, \pi)$, a closed operator L in $L^2(\mathbb{R}^n)$ is said to be of *type* ν if $\sigma(L) \subset S_\nu$, where $\sigma(L)$ denotes its *spectra* and if, for all $\gamma > \nu$, there exists a positive constant C_γ such that for all $\lambda \notin S_\gamma$, $\|(L - \lambda I)^{-1}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \le C_\gamma |\lambda|^{-1}$. Let $\mathscr{X}$ and $\mathscr{Y}$ be two *linear normed spaces*, and let T be a *continuous linear operator* from $\mathscr{X}$ to $\mathscr{Y}$. Here and in what follows, $\|T\|_{\mathscr{X} \to \mathscr{Y}}$ denotes the *operator norm of T from $\mathscr{X}$ to $\mathscr{Y}$*. Let $\theta \in (\nu, \gamma)$, and let Γ be the *contour* $\{\xi = re^{\pm i\theta} : r \ge 0\}$ parameterized clockwise around S_ν. Then if L is of type ν and $\psi \in \Psi(S_\nu^0)$, the *operator* $\psi(L)$ is defined by

$$\psi(L) \equiv \frac{1}{2\pi i} \int_\Gamma (L - \lambda I)^{-1} \psi(\lambda)\, d\lambda,$$

where the integral is absolutely convergent in $\mathfrak{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ (the *class of all bounded linear operators in* $L^2(\mathbb{R}^n)$). By the Cauchy theorem, we know that $\psi(L)$ is independent of the choices of ν and γ such that $\theta \in (\nu, \gamma)$. Moreover, if L is one-to-one and has dense range, and $b \in H_\infty(S_\gamma^0)$, then $b(L)$ is defined by setting $b(L) \equiv [\psi(L)]^{-1}(b\psi)(L)$, where $\psi(z) \equiv z(1+z)^{-2}$ for all $z \in S_\gamma^0$. It was proved by McIntosh [25] that $b(L)$ is a well-defined linear operator in $L^2(\mathbb{R}^n)$. Moreover, the operator L is said to have a *bounded H_∞-calculus* in $L^2(\mathbb{R}^n)$ if, for all $\gamma \in (\nu, \pi)$, there exists a positive constant $\widetilde{C}_\gamma$ such that for all $b \in H_\infty(S_\gamma^0)$, $b(L) \in \mathfrak{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ and

$$\|b\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le \widetilde{C}_\gamma \|b\|_{L^\infty(S_\gamma^0)}. \tag{2.6}$$

2.3. Assumptions on the operator L

Throughout the whole paper, we always suppose that the considered operators L satisfy the following *assumptions*.

ASSUMPTION $(L)_1$

The operator L has a bounded H_∞-calculus in $L^2(\mathcal{X})$.

ASSUMPTION $(L)_2$

The semigroup $\{e^{-tL}\}_{t>0}$ generated by L is analytic on $L^2(\mathcal{X})$ and satisfies the *Davies–Gaffney estimate*; namely, there exist positive constants C_2 and C_3 such that for all closed sets E and F in $\mathcal{X}$, $t \in (0,\infty)$ and $f \in L^2(E)$,

$$\|e^{-tL}f\|_{L^2(F)} \le C_2 \exp\Big\{-\frac{[\mathrm{dist}(E,F)]^2}{C_3 t}\Big\}\|f\|_{L^2(E)}, \tag{2.7}$$

where $\mathrm{dist}(E,F) \equiv \inf_{x\in E, y\in F} d(x,y)$ and the *space* $L^2(E)$ denotes the set of all μ-measurable functions on E such that $\|f\|_{L^2(E)} \equiv \{\int_E |f(x)|^2\,d\mu(x)\}^{1/2} < \infty$.

REMARK 2.1

By the functional calculus of L on $L^2(\mathcal{X})$, it is easy to see that if an operator L satisfies Assumptions $(L)_1$ and $(L)_2$, the adjoint operator L^* also satisfies Assumptions $(L)_1$ and $(L)_2$, and, therefore, the following Lemmas 2.2 and 2.3 also hold for L^*.

By Assumptions $(L)_1$ and $(L)_2$, we have the following technical result which was obtained by Anh and Li [2, Proposition 2.2].

LEMMA 2.2

Let L satisfy Assumptions $(L)_1$ and $(L)_2$. Then for any fixed $k \in \mathbb{Z}_+$ (resp., $j,k \in \mathbb{Z}_+$ with $j \le k$), the family $\{(t^2L)^k e^{-t^2L}\}_{t>0}$ (resp., $\{(t^2L)^j(I+t^2L)^{-k}\}_{t>0}$) of operators also satisfies the Davies–Gaffney estimate (2.7) with positive constants C_2, C_3 depending only on n and k (resp., n, j, and k).

By (2.6), we have the following useful lemma.

LEMMA 2.3

Let L satisfy Assumptions $(L)_1$ and $(L)_2$. Then for any fixed $k \in \mathbb{N}$, the operator *given by setting, for all $f \in L^2(\mathcal{X})$ and $x \in \mathcal{X}$,*

$$S_L^k f(x) \equiv \Big(\iint_{\Gamma(x)} |(t^2L)^k e^{-t^2L} f(y)|^2 \,\frac{d\mu(y)}{V(x,t)}\,\frac{dt}{t} \Big)^{1/2},$$

is bounded on $L^2(\mathcal{X})$.

2.4. Orlicz functions

Let Φ be a positive function on $\mathbb{R}_+ \equiv (0,\infty)$. The function Φ is said to be of *upper* (resp., *lower*) *type* p for some $p \in [0,\infty)$, if there exists a positive constant C such that for all $t \in [1,\infty)$ (resp., $t \in (0,1]$) and $s \in (0,\infty)$,

$$\Phi(st) \le Ct^p \Phi(s). \tag{2.8}$$

Obviously, if Φ is of lower type p for some $p \in (0,\infty)$, then $\lim_{t\to 0_+} \Phi(t) = 0$. So for the sake of convenience, if it is necessary, we may *assume* that $\Phi(0) = 0$. If Φ is of both upper-type p_1 and lower-type p_0, then Φ is said to be of *type* (p_0, p_1). Let

$$\begin{aligned} p_\Phi^+ \equiv \inf\big\{&p \in (0,\infty) : \text{there exists a positive constant } C \\ &\text{such that (2.8) holds for all } t \in [1,\infty) \text{ and } s \in (0,\infty)\big\} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} p_\Phi^- \equiv \sup\big\{&p \in (0,\infty) : \text{there exists a positive constant } C \\ &\text{such that (2.8) holds for all } t \in (0,1) \text{ and } s \in (0,\infty)\big\}. \end{aligned} \tag{2.10}$$

It is easy to see that $p_\Phi^- \le p_\Phi^+$ for all Φ. In what follows, p_Φ^- and p_Φ^+ are respectively called the *critical lower-type index* and the *critical upper-type index* of Φ.

Throughout the whole paper, we always assume that Φ satisfies the following *assumption.*

ASSUMPTION (Φ)

Let Φ be a positive, continuous, strictly increasing function on $(0,\infty)$ which is of critical lower type $p_\Phi^- \in (0,1]$. Also assume that Φ is concave.

REMARK 2.2

(i) Recall that the function Φ is called of *strictly lower-type* p if (2.8) holds with $C \equiv 1$ for all $t \in (0,1)$ and $s \in (0,\infty)$. Then the *strictly critical lower-type index* p_Φ of Φ is defined by

$$p_\Phi \equiv \sup\big\{p \in (0,\infty) : \Phi(st) \le t^p \Phi(s) \text{ holds for all } t \in (0,1) \text{ and } s \in (0,\infty)\big\}.$$

Obviously, $p_\Phi \le p_\Phi^- \le p_\Phi^+$. Moreover, it was proved in [20, Remark 2.1] that Φ is also of strictly lower-type p_Φ. In other words, p_Φ is *attainable.*

However, p_Φ^- and p_Φ^+ may not be attainable. For example, for $p \in (0,1]$, if $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$, then Φ satisfies Assumption (Φ) and $p_\Phi = p_\Phi^- = p_\Phi^+ = p$; for $p \in [1/2,1]$, if $\Phi(t) \equiv t^p/\ln(e+t)$ for all $t \in (0,\infty)$, then Φ satisfies

Assumption (Φ) and $p_\Phi^- = p = p_\Phi^+$, p_Φ^- is not attainable but p_Φ^+ is attainable; for $p \in (0, 1/2]$, if $\Phi(t) \equiv t^p \ln(e+t)$ for all $t \in (0,\infty)$, then Φ satisfies Assumption (Φ) and $p_\Phi^- = p = p_\Phi^+$, p_Φ^- is attainable but p_Φ^+ is not attainable.

(ii) We observe that, via the Aoki–Rolewicz theorem in [3] and [26], all results in [2], [19], [20], and [21] are still true if the *assumptions on* p_Φ *are replaced by the same assumptions on* p_Φ^-.

Notice that if Φ satisfies Assumption (Φ), then $\Phi(0) = 0$. For any positive function $\widetilde{\Phi}$ of critical lower-type $p_{\widetilde{\Phi}}^-$, if we set $\Phi(t) \equiv \int_0^t (\widetilde{\Phi}(s)/s)\, ds$ for $t \in [0,\infty)$, then by [30, Proposition 3.1], Φ is equivalent to $\widetilde{\Phi}$; namely, there exists a positive constant C such that $C^{-1}\widetilde{\Phi}(t) \le \Phi(t) \le C\widetilde{\Phi}(t)$ for all $t \in [0,\infty)$; moreover, Φ is a positive, strictly increasing, concave, and continuous function of critical lower-type $p_{\widetilde{\Phi}}^-$. Notice that all our results of this paper are invariant on equivalent Orlicz functions. From this, we deduce that *all results with* Φ *as in Assumption* (Φ) *also hold for all positive functions* $\widetilde{\Phi}$ *of the same critical lower-type* p_Φ^- *as* Φ.

Let Φ satisfy Assumption (Φ). A measurable function f on $\mathcal{X}$ is said to be in the *space* $L^\Phi(\mathcal{X})$ if $\int_{\mathcal{X}} \Phi(|f(x)|)\, d\mu(x) < \infty$. Moreover, for any $f \in L^\Phi(\mathcal{X})$, define

$$\|f\|_{L^\Phi(\mathcal{X})} \equiv \inf\Big\{\lambda \in (0,\infty) : \int_{\mathcal{X}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu(x) \le 1\Big\}.$$

Since Φ is strictly increasing, we define the function $\rho(t)$ on $(0,\infty)$ by

$$\rho(t) \equiv \frac{t^{-1}}{\Phi^{-1}(t^{-1})} \tag{2.11}$$

for all $t \in (0,\infty)$, where Φ^{-1} is the *inverse function* of Φ. Then the types of Φ and ρ have the following relation. *If* $0 < p_0 \le p_1 \le 1$ *and* Φ *is an increasing function, then* Φ *is of type* (p_0, p_1) *if and only if* ρ *is of type* $(p_1^{-1} - 1, p_0^{-1} - 1)$ (see [30] for its proof).

3. The space $\mathrm{VMO}_{\rho,L}(\mathcal{X})$

In this section, we introduce the generalized vanishing mean oscillation spaces associated with L. Throughout this section, we *always assume* that L satisfies Assumptions $(L)_1$ and $(L)_2$.

We first recall the notion of tent spaces in [27], which, when $\mathcal{X} \equiv \mathbb{R}^n$, were first introduced by Coifman, Meyer, and Stein [6].

For any $\nu > 0$ and $x \in \mathcal{X}$, let $\Gamma_\nu(x) \equiv \{(y,t) \in \mathcal{X} \times (0,\infty) : d(x,y) < \nu t\}$ denote the *cone of aperture* ν *with vertex* $x \in \mathcal{X}$. For any closed set F of $\mathcal{X}$, denote by $\mathcal{R}_\nu F$ the *union of all cones with vertices in* F, namely, $\mathcal{R}_\nu F \equiv \bigcup_{x\in F} \Gamma_\nu(x)$; and for any open set O in $\mathcal{X}$, denote the *tent over* O by $T_\nu(O)$, which is defined as $T_\nu(O) \equiv [\mathcal{R}_\nu(O^\complement)]^\complement$. It is easy to see that $T_\nu(O) = \{(x,t) \in \mathcal{X} \times (0,\infty) : d(x, O^\complement) \ge \nu t\}$. In what follows, we denote $\mathcal{R}_1(F)$, $\Gamma_1(x)$, and $T_1(O)$ simply by $\mathcal{R}(F)$, $\Gamma(x)$, and $\widehat{O}$, respectively.

For all measurable functions g on $\mathcal{X}\times(0,\infty)$ and $x\in\mathcal{X}$, define

$$\mathcal{A}_\nu(g)(x)\equiv\Big(\iint_{\Gamma_\nu(x)}|g(y,t)|^2\,\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\Big)^{1/2}$$

and

$$\mathcal{C}_\rho(g)(x)\equiv\sup_{B\ni x}\frac{1}{\rho(\mu(B))}\Big(\frac{1}{\mu(B)}\iint_{\widehat{B}}|g(y,t)|^2\,\frac{d\mu(y)\,dt}{t}\Big)^{1/2},$$

where the supremum is taken over all balls B containing x. We denote $\mathcal{A}_1(g)$ simply by $\mathcal{A}(g)$.

Recall that for $p\in(0,\infty)$, the *tent space* $T_2^p(\mathcal{X})$ is defined to be the space of all measurable functions g on $\mathcal{X}\times(0,\infty)$ such that $\|g\|_{T_2^p(\mathcal{X})}\equiv\|\mathcal{A}(g)\|_{L^p(\mathcal{X})}<\infty$, which was introduced by Coifman, Meyer, and Stein [6] for $\mathcal{X}\equiv\mathbb{R}^n$ and by Russ [27] for a space $\mathcal{X}$ of homogeneous type. Let Φ satisfy Assumption (Φ). In what follows, we denote by $T_\Phi(\mathcal{X})$ the *space of all measurable functions g on $\mathcal{X}\times(0,\infty)$ such that* $\mathcal{A}(g)\in L^\Phi(\mathcal{X})$, and for any $g\in T_\Phi(\mathcal{X})$, we define its *norm* by

$$\|g\|_{T_\Phi(\mathcal{X})}\equiv\|\mathcal{A}(g)\|_{L^\Phi(\mathcal{X})}=\inf\Big\{\lambda>0:\int_{\mathcal{X}}\Phi\Big(\frac{\mathcal{A}(g)(x)}{\lambda}\Big)\,d\mu(x)\le 1\Big\};$$

the *space* $T_\Phi^\infty(\mathcal{X})$ is defined to be the space of all measurable functions g on $\mathcal{X}\times(0,\infty)$ satisfying $\|g\|_{T_\Phi^\infty(\mathcal{X})}\equiv\|\mathcal{C}_\rho(g)\|_{L^\infty(\mathcal{X})}<\infty$.

Recall that a function a on $\mathcal{X}\times(0,\infty)$ is called a $T_\Phi(\mathcal{X})$-*atom* if

(i) there exists a ball $B\subset\mathcal{X}$ such that $\operatorname{supp} a\subset\widehat{B}$;
(ii) $\iint_{\widehat{B}}|a(x,t)|^2\,\frac{d\mu(x)\,dt}{t}\le[\mu(B)]^{-1}[\rho(\mu(B))]^{-2}$.

Since Φ is concave, from Jensen's inequality and Hölder's inequality we deduce that for all $T_\Phi(\mathcal{X})$-atoms a, $\|a\|_{T_\Phi(\mathcal{X})}\le 1$ (see [21] for the details). Moreover, the following atomic decomposition for elements in $T_\Phi(\mathcal{X})$ is just [21, Theorem 3.1].

LEMMA 3.1
Let Φ satisfy Assumption (Φ). Then for any $f\in T_\Phi(\mathcal{X})$, there exist $T_\Phi(\mathcal{X})$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty\subset\mathbb{C}$ such that for almost every $(x,t)\in\mathcal{X}\times(0,\infty)$,

$$f(x,t)=\sum_{j=1}^\infty\lambda_j a_j(x,t),\tag{3.1}$$

and the series converges in $T_\Phi(\mathcal{X})$. Moreover, there exists a positive constant C such that for all $f\in T_\Phi(\mathcal{X})$,

$$\begin{aligned}\Lambda(\{\lambda_j a_j\}_{j=1}^\infty)&\equiv\inf\Big\{\lambda>0:\sum_{j=1}^\infty\mu(B_j)\Phi\Big(\frac{|\lambda_j|}{\lambda\mu(B_j)\rho(\mu(B_j))}\Big)\le 1\Big\}\\&\le C\|f\|_{T_\Phi(\mathcal{X})},\end{aligned}\tag{3.2}$$

where $\widehat{B}_j$ appears as the support of a_j.

DEFINITION 3.1

Let L satisfy Assumptions $(L)_1$ and $(L)_2$, let Φ satisfy Assumption (Φ), let ρ be as in (2.11), let $M \in \mathbb{N}$, $\epsilon \in (0,\infty)$, and let B be a ball. A function $\beta \in L^2(\mathcal{X})$ is called a $(\Phi, M, \epsilon)_L$*-molecule adapted to the ball* B if there exists a function $b \in \mathcal{D}(L^M)$ such that

(i) $\beta = L^M b$;

(ii) For every $k \in \{0, 1, \dots, M\}$ and $j \in \mathbb{Z}_+$, there holds

$$\|(r_B^2 L)^k b\|_{L^2(U_j(B))} \le r_B^{2M} 2^{-j\epsilon} [\mu(2^j B)]^{-1/2} \big[\rho\big(\mu(2^j B)\big)\big]^{-1},$$

where $U_j(B)$ for $j \in \mathbb{Z}_+$ is as in (2.5).

Let $\phi = L^M \nu$ be a function in $L^2(\mathcal{X})$, where $\nu \in \mathcal{D}(L^M)$. Following [15] and [16], for $\epsilon > 0$, $M \in \mathbb{N}$, and a fixed $x_0 \in \mathcal{X}$, we introduce the *space*

$$\mathcal{M}_\Phi^{M,\epsilon}(L) \equiv \big\{\phi = L^M \nu \in L^2(\mathcal{X}) : \|\phi\|_{\mathcal{M}_\Phi^{M,\epsilon}(L)} < \infty\big\}, \tag{3.3}$$

where

$$\|\phi\|_{\mathcal{M}_\Phi^{M,\epsilon}(L)} \equiv \sup_{j \in \mathbb{Z}_+} \Big\{2^{j\epsilon} [V(x_0, 2^j)]^{1/2} \rho\big(V(x_0, 2^j)\big) \sum_{k=0}^{M} \|L^k \nu\|_{L^2(U_j(B(x_0,1)))}\Big\}$$

(see also [2]).

Notice that if $\phi \in \mathcal{M}_\Phi^{M,\epsilon}(L)$ for some $\epsilon > 0$ with norm 1, then ϕ is a $(\Phi, M, \epsilon)_L$-molecule adapted to the ball $B(x_0, 1)$. Conversely, if β is a $(\Phi, M, \epsilon)_L$-molecule adapted to any ball, then $\beta \in \mathcal{M}_\Phi^{M,\epsilon}(L)$.

Let A_t denote either $(I + t^2 L)^{-1}$ or $e^{-t^2 L}$, and let A_t^* denote either $(I + t^2 L^*)^{-1}$ or $e^{-t^2 L^*}$. For any $f \in (\mathcal{M}_\Phi^{M,\epsilon}(L^*))^*$, the *dual space* of $\mathcal{M}_\Phi^{M,\epsilon}(L^*)$, we claim that $(I - A_t)^M f \in L^2_{\mathrm{loc}}(\mathcal{X})$ *in the sense of distributions*. Indeed, for any ball B, if $\psi \in L^2(B)$, then it follows from the Davies–Gaffney estimate (2.7) and Remark 2.1 that $(I - A_t^*)^M \psi \in \mathcal{M}_\Phi^{M,\epsilon}(L^*)$ for every $\epsilon > 0$. Thus, there exists a nonnegative constant $C(t, r_B, \mathrm{dist}(B, x_0))$, depending on t, r_B, and $\mathrm{dist}(B, x_0)$, such that for all $\psi \in L^2(B)$,

$$\begin{aligned} \big|\langle (I - A_t)^M f, \psi\rangle\big| &\equiv \big|\langle f, (I - A_t^*)^M \psi\rangle\big| \\ &\le C\big(t, r_B, \mathrm{dist}(B, x_0)\big) \|f\|_{(\mathcal{M}_\Phi^{M,\epsilon}(L^*))^*} \|\psi\|_{L^2(B)}, \end{aligned}$$

which implies that $(I - A_t)^M f \in L^2_{\mathrm{loc}}(\mathcal{X})$ in the sense of distributions.

Finally, for any $M \in \mathbb{N}$, define

$$\mathcal{M}_{\Phi,L}^M(\mathcal{X}) \equiv \bigcap_{\epsilon > n(1/p_\Phi^- - 1/p_\Phi^+)} \big(\mathcal{M}_\Phi^{M,\epsilon}(L^*)\big)^*, \tag{3.4}$$

where p_Φ^+ and p_Φ^- are, respectively, as in (2.9) and (2.10).

DEFINITION 3.2

Let L, Φ, and ρ be as in Definition 3.1, and let $M > (1/p_\Phi^- - 1/2)n/2$. A function

$f \in \mathcal{M}^M_{\Phi,L}(\mathcal{X})$ is said to be in the *space* $\mathrm{BMO}^M_{\rho,L}(\mathcal{X})$ if

$$\|f\|_{\mathrm{BMO}^M_{\rho,L}(\mathcal{X})} \equiv \sup_{B\subset\mathcal{X}} \frac{1}{\rho(\mu(B))}\Big[\frac{1}{\mu(B)}\int_B |(I-e^{-r_B^2 L})^M f(x)|^2\,d\mu(x)\Big]^{1/2} < \infty,$$

where the supremum is taken over all balls B of $\mathcal{X}$.

Now, let us recall some notions on the Orlicz–Hardy spaces associated with L. For all $f \in L^2(\mathcal{X})$ and $x \in \mathcal{X}$, define

$$\mathcal{S}_L f(x) \equiv \Big(\iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \,\frac{d\mu(y)\,dt}{t}\Big)^{1/2}.$$

The *Orlicz–Hardy space* $H_{\Phi,L}(\mathcal{X})$ is defined to be the completion of the set $\{f \in L^2(\mathcal{X}) : \mathcal{S}_L f \in L^\Phi(\mathcal{X})\}$ with respect to the quasi-norm $\|f\|_{H_{\Phi,L}(\mathcal{X})} \equiv \|\mathcal{S}_L f\|_{L^\Phi(\mathcal{X})}$.

The Orlicz–Hardy space $H_{\Phi,L}(\mathcal{X})$ was introduced and studied in [2] (see also [21]). If $\Phi(t) \equiv t^p$ for $p \in (0,1]$ and all $t \in (0,\infty)$, then the space $H_{\Phi,L}(\mathcal{X})$ coincides with the Hardy space $H^p_L(\mathcal{X})$, which was introduced and studied by Duong and Li [10].

Let the *space* $H^{\mathrm{mol},\epsilon,M}_{\Phi,\mathrm{fin},L}(\mathcal{X})$ denote the space of finite linear combinations of $(\Phi,M,\epsilon)_L$-molecules. By [2, Corollary 3.8], we obtain that $H^{\mathrm{mol},\epsilon,M}_{\Phi,\mathrm{fin},L}(\mathcal{X})$ is dense in $H_{\Phi,L}(\mathcal{X})$ (see also [21, Corollary 4.2]).

In what follows, for $M \in \mathbb{N}$, let $C(M)$ be the *positive constant* such that

$$C(M)\int_0^\infty t^{2(M+1)} e^{-2t^2}\,\frac{dt}{t} = 1. \tag{3.5}$$

Recall that a variant of the following representation of finite linear combinations of molecules was given by [2, Theorem 3.15] without a detailed proof. The following Theorem 3.1 gives more *accurate ranges* of ϵ and M, comparing with [2, Theorem 3.15].

THEOREM 3.1

Let L, Φ, and M be as in Definition 3.2, and let $\epsilon \in (0, M-(1/p_\Phi^- - 1/2)n/2)$. Assume that $f = \sum_{i=0}^N \lambda_i a_i$, where $N \in \mathbb{N}$, $\{a_i\}_{i=0}^N$ is a family of $(\Phi,2M,\epsilon)_L$-molecules, $\{\lambda_i\}_{i=0}^N \subset \mathbb{C}$, and $\sum_{i=0}^N |\lambda_i| < \infty$. Then there exists a representation of $f = \sum_{i=0}^{2N} \mu_i m_i$, where $\{m_i\}_{i=1}^{2N}$ are $(\Phi,M,\epsilon)_L$-molecules, $\{\mu_i\}_{i=0}^{2N} \subset \mathbb{C}$, and $\sum_{i=0}^{2N} |\mu_i| \le C\|f\|_{H_{\Phi,L}(\mathcal{X})}$, where C is a positive constant, depending only on $\mathcal{X}, L, M, \epsilon$, and n.

Proof

Throughout this proof, we choose $\widetilde{p}_\Phi \in (0, p_\Phi^-)$ such that $M > (1/\widetilde{p}_\Phi - 1/2)n/2$ and $\epsilon \in (0, M-(1/\widetilde{p}_\Phi - 1/2)n/2)$. Therefore, Φ is of *lower-type* $\widetilde{p}_\Phi$, and hence ρ is of *upper-type* $1/\widetilde{p}_\Phi - 1$.

Since $\{a_i\}_{i=0}^N$ is a family of $(\Phi,2M,\epsilon)_L$-molecules, by definition there exist a family $\{b_i\}_{i=0}^N$ of functions and a family $\{B_i\}_{i=0}^N$ of balls such that for every $i \in \{0,1,\ldots,N\}$, $a_i = L^{2M} b_i$ satisfies Definition 3.1(ii). Fix a point $x_0 \in \mathcal{X}$. Let

$\widetilde{C}(M) \equiv 2C(M)/(M+1)$, where $C(M)$ is as in (3.5). Then

$$\widetilde{C}(M) \int_0^\infty t^{2(M+2)} e^{-2t^2} \frac{dt}{t} = 1.$$

By this and the L^2-functional calculus, for $f = \sum_{i=0}^N \lambda_i a_i \in L^2(\mathcal{X})$, we have

$$\begin{aligned} f &= \widetilde{C}(M) \int_0^\infty (t^2 L)^{M+2} e^{-2t^2 L} f \frac{dt}{t} \\ &= \widetilde{C}(M) \int_{K_1}^\infty (t^2 L)^{M+2} e^{-2t^2 L} f \frac{dt}{t} + \widetilde{C}(M) \int_0^{K_1} \cdots \equiv f_1 + f_2, \end{aligned}$$

where K_1 is a *positive constant* which is determined later.

Let us start with the term f_1. Set $\mu \equiv N^{-1} \|f\|_{H_{\Phi,L}(\mathcal{X})}$. Substituting $f = \sum_{i=0}^N \lambda_i a_i$ into f_1, we have

$$f_1 = \widetilde{C}(M) \sum_{i=0}^N \lambda_i \int_{K_1}^\infty (t^2 L)^{M+2} e^{-2t^2 L} a_i \frac{dt}{t} = \sum_{i=0}^N \mu_i m_{i,K_1},$$

where $\mu_i \equiv \widetilde{C}(M)\mu$, $m_{i,K_1} \equiv L^M f_{i,K_1}$, and

$$f_{i,K_1} \equiv \mu^{-1} \lambda_i \int_{K_1}^\infty t^{2(M+2)} L^2 e^{-2t^2 L} a_i \frac{dt}{t}.$$

Then, obviously, $\sum_{i=0}^N |\mu_i| = \sum_{i=0}^N \mu_i = C(M) \|f\|_{H_{\Phi,L}(\mathcal{X})}$. We now claim that for an appropriate choice of K_1 and $i \in \{0, 1, \ldots, N\}$, m_{i,K_1} is a $(\Phi, M, \epsilon)_L$-molecule adapted to the ball B_i. Observe that $a_i = L^{2M} b_i$, for $i \in \{0, 1, \ldots, N\}$. By Minkowski's inequality, for $k \in \{0, 1, \ldots, M\}$, $i \in \{0, 1, \ldots, N\}$, and $j \in \mathbb{Z}_+$,

$$\begin{aligned} &\|(r_{B_i}^2 L)^k f_{i,K_1}\|_{L^2(U_j(B_i))} \\ &\quad \le \mu^{-1} |\lambda_i| \int_{K_1}^\infty t^{-2M} \|(t^2 L)^{2(M+1)} e^{-2t^2 L} (r_{B_i}^2 L)^k b_i\|_{L^2(U_j(B_i))} \frac{dt}{t} \\ &\quad \le \mu^{-1} |\lambda_i| \sum_{l=0}^\infty \int_{K_1}^\infty t^{-2M} \\ &\qquad \times \left\| (t^2 L)^{2(M+1)} e^{-2t^2 L} \big(\chi_{U_l(B_i)} [(r_{B_i}^2 L)^k b_i] \big) \right\|_{L^2(U_j(B_i))} \frac{dt}{t} \\ &\quad \equiv \mu^{-1} |\lambda_i| \sum_{l=0}^\infty \mathrm{H}_l, \end{aligned}$$

where $U_l(B_i)$ for $l \in \mathbb{Z}_+$ is as in (2.5). When $l < j-1$, by Lemma 2.2, $\mu(2^j B_i) \lesssim 2^{n(j-l)} \mu(2^l B_i)$, $\rho(\mu(2^j B_i)) \lesssim 2^{n(j-l)(1/\widetilde{p}_\Phi - 1)} \rho(\mu(2^l B_i))$, and Definition 3.1(ii), we conclude that

$$\begin{aligned} \mathrm{H}_l &\lesssim \int_{K_1}^\infty t^{-2M} \|(r_{B_i}^2 L)^k b_i\|_{L^2(U_l(B_i))} \Big(\frac{t}{2^j r_{B_i}} \Big)^{\epsilon + n(1/\widetilde{p}_\Phi - 1/2)} \frac{dt}{t} \\ &\lesssim \int_{K_1}^\infty t^{-2M} r_{B_i}^{4M} 2^{-l\epsilon} [\mu(2^l B_i)]^{-1/2} \big[\rho\big(\mu(2^l B_i)\big)\big]^{-1} \Big(\frac{t}{2^j r_{B_i}} \Big)^{\epsilon + n(1/\widetilde{p}_\Phi - 1/2)} \frac{dt}{t} \end{aligned}$$

$$\lesssim r_{B_i}^{2M} 2^{-j\epsilon}[\mu(2^j B_i)]^{-1/2}\big[\rho\big(\mu(2^j B_i)\big)\big]^{-1}$$
$$\times 2^{-l(\epsilon+(1/\widetilde{p}_\Phi-1/2)n/2)}\Big(\frac{r_{B_i}}{K_1}\Big)^{2[M-\epsilon/2-(1/\widetilde{p}_\Phi-1/2)n/2]}.$$

When $l\in\{j-1,j,j+1\}$, from Lemma 2.2 and Definition 3.1(ii), it follows that

$$\mathrm{H}_l \lesssim \int_{K_1}^{\infty} t^{-2M}\|(r_{B_i}^2 L)^k b_i\|_{L^2(U_j(B_i))}\,\frac{dt}{t}$$
$$\lesssim r_{B_i}^{2M} 2^{-j\epsilon}[\mu(2^j B_i)]^{-1/2}\big[\rho\big(\mu(2^j B_i)\big)\big]^{-1}\Big(\frac{r_{B_i}}{K_1}\Big)^{2M}.$$

When $l>j+1$, by Lemma 2.2, $\mu(2^j B_i)\lesssim\mu(2^l B_i)$, $\rho(\mu(2^j B_i))\lesssim\rho(\mu(2^l B_i))$, and Definition 3.1(ii), we obtain

$$\mathrm{H}_l \lesssim \int_{K_1}^{\infty} t^{-2M}\|(r_{B_i}^2 L)^k b_i\|_{L^2(U_l(B_i))}\Big(\frac{t}{2^l r_{B_i}}\Big)^{\epsilon}\frac{dt}{t}$$
$$\lesssim r_{B_i}^{2M} 2^{-j\epsilon}[\mu(2^j B_i)]^{-1/2}\big[\rho\big(\mu(2^j B_i)\big)\big]^{-1} 2^{-l\epsilon}\Big(\frac{r_{B_i}}{K_1}\Big)^{2M-\epsilon}.$$

Combining these estimates, by choosing $K_1>\max\{r_{B_1},\dots,r_{B_N}\}$, we further conclude that there exists a positive constant $\widetilde{C}$, independent of i, such that

$$\|(r_{B_i}^2 L)^k f_{i,K_1}\|_{L^2(U_j(B_i))}\le \widetilde{C} r_{B_i}^{2M} 2^{-j\epsilon}[\mu(2^j B_i)]^{-1/2}\big[\rho\big(\mu(2^j B_i)\big)\big]^{-1}$$
$$\times \mu^{-1}|\lambda_i|\Big(\frac{r_{B_i}}{K_1}\Big)^{2[M-\epsilon/2-(1/\widetilde{p}_\Phi-1/2)n/2]}.$$

Then, by choosing

$$K_1\equiv \max_{0\le i\le N}\Big\{r_{B_i}\Big[\widetilde{C}\mu^{-1}\max_{0\le i\le N}|\lambda_i|\Big]^{1/(2[M-\epsilon/2-(1/\widetilde{p}_\Phi-1/2)n/2])}\Big\},$$

we see that for $i\in\{0,1,\dots,N\}$, m_{i,K_1} is a $(\Phi,M,\epsilon)_L$-molecule adapted to the ball B_i, which shows the claim.

We now consider the term f_2. Set $\mu\equiv N^{-1}\|f\|_{H_{\Phi,L}(\mathcal{X})}$. Substituting $f=\sum_{i=0}^N\lambda_i a_i$ into f_2, we have

$$f_2=\widetilde{C}(M)\sum_{i=0}^N\lambda_i\int_0^{K_1}(t^2L)^{M+1}e^{-t^2L}(t^2Le^{-t^2L}a_i)\,\frac{dt}{t}=\sum_{i=0}^N\mu_i m_{i,K_1},$$

where $\mu_i\equiv C(M)\mu$, $m_{i,K_1}\equiv L^M f_{i,K_1}$, and

$$f_{i,K_1}\equiv\mu^{-1}\lambda_i\int_0^{K_1}t^{2(M+1)}Le^{-t^2L}(t^2Le^{-t^2L}a_i)\,\frac{dt}{t}.$$

Then, obviously, $\sum_{i=0}^N|\mu_i|=\sum_{i=0}^N\mu_i=C(M)\|f\|_{H_{\Phi,L}(\mathcal{X})}$. We now claim that for K_1 as above and $i\in\{0,1,\dots,N\}$, m_{i,K_1} is a $(\Phi,M,\epsilon)_L$-molecule adapted to the ball $2^{K_0}B_i$, where $K_0\in(0,\infty)$ is determined later. To show the claim, for $i\in\{0,1,\dots,N\}$ and $j\in\mathbb{Z}_+$, set $\Omega_{j,K_0}\equiv 2^{j+K_0+2}B_i\backslash 2^{j+K_0-2}B_i$, and write

$$f_{i,K_1}=\mu^{-1}\lambda_i\int_0^{K_1}t^{2(M+1)}Le^{-t^2L}([t^2Le^{-t^2L}a_i]\chi_{\Omega_{j,K_0}})\,\frac{dt}{t}$$

$$
\begin{aligned}
&+\mu^{-1}\lambda_i\int_0^{K_1} t^{2(M+1)}Le^{-t^2L}([t^2Le^{-t^2L}a_i]\chi_{\Omega_{j,K_0}^\complement})\,\frac{dt}{t}\\
&\equiv g_{i,K_1,K_0}+h_{i,K_1,K_0}.
\end{aligned}
$$

Then, by Minkowski's inequality, for $k\in\{0,1,\ldots,M\}$, $i\in\{0,1,\ldots,N\}$, and $j\in\mathbb{Z}_+$,

$$
\begin{aligned}
&\|(2^{2K_0}r_{B_i}^2L)^k g_{i,K_1,K_0}\|_{L^2(U_j(2^{K_0}B_i))}\\
&\quad\le \mu^{-1}|\lambda_i|r_{B_i}^{2M}\Big\|\int_0^{K_1}\Big(\frac{t}{r_{B_i}}\Big)^{2M-2k}2^{2kK_0}\\
&\qquad\times (t^2L)^{k+1}e^{-t^2L}([t^2Le^{-t^2L}a_i]\chi_{\Omega_{j,K_0}})\,\frac{dt}{t}\Big\|_{L^2(U_j(2^{K_0}B_i))}\\
&\quad\le \mu^{-1}|\lambda_i|\sum_{l=0}^{\infty}\int_0^{K_1}\Big(\frac{t}{r_{B_i}}\Big)^{2M-2k}2^{2kK_0}\|\chi_{U_l(2^{K_0}B_i)}t^2Le^{-t^2L}a_i\|_{L^2(\Omega_{j,K_0})}\,\frac{dt}{t}\\
&\quad\equiv \mu^{-1}|\lambda_i|\sum_{l=0}^{\infty}\mathrm{H}_l.
\end{aligned}
$$

When $l<j-2$, from Lemma 2.2, $\mu(2^{j+K_0}B_i)\lesssim 2^{n(j-l)}\mu(2^{l+K_0}B_i)$, $\rho(\mu(2^{j+K_0}B_i))\lesssim 2^{n(j-l)(1/\widetilde{p}_\Phi-1)}\rho(\mu(2^{l+K_0}B_i))$, and Definition 3.1(ii), it follows that

$$
\begin{aligned}
\mathrm{H}_l&\lesssim\int_0^{K_1}\Big(\frac{t}{r_{B_i}}\Big)^{2M-2k}2^{2kK_0}\|a_i\|_{L^2(U_l(2^{K_0}B_i))}\Big(\frac{t}{2^{j+K_0}r_{B_i}}\Big)^{\epsilon+n(1/\widetilde{p}_\Phi-1/2)}\,\frac{dt}{t}\\
&\lesssim\int_0^{K_1}\Big(\frac{t}{r_{B_i}}\Big)^{2M-2k}2^{2kK_0}r_{B_i}^{4M}2^{-(l+K_0)\epsilon}[\mu(2^{l+K_0}B_i)]^{-1/2}\big[\rho\big(\mu(2^{l+K_0}B_i)\big)\big]^{-1}\\
&\qquad\times\Big(\frac{t}{2^{j+K_0}r_{B_i}}\Big)^{\epsilon+n(1/\widetilde{p}_\Phi-1/2)}\,\frac{dt}{t}\\
&\lesssim(2^{K_0}r_{B_i})^{2M}2^{-j\epsilon}[\mu(2^{j+K_0}B_i)]^{-1/2}\big[\rho\big(\mu(2^{j+K_0}B_i)\big)\big]^{-1}2^{-l[\epsilon+(1/\widetilde{p}_\Phi-1/2)n/2]}\\
&\qquad\times 2^{-2K_0[M-k+\epsilon+(1/\widetilde{p}_\Phi-1/2)n/2]}K_1^{2M-2k+\epsilon+n(1/\widetilde{p}_\Phi-1/2)}\\
&\qquad\times r_{B_i}^{2M+2k-\epsilon-n(1/\widetilde{p}_\Phi-1/2)}.
\end{aligned}
$$

When $l\in\{j-2,\ldots,j+2\}$, by Lemma 2.2 and Definition 3.1(ii), we see that

$$
\begin{aligned}
\mathrm{H}_l&\lesssim\int_0^{K_1}\Big(\frac{t}{r_{B_i}}\Big)^{2M-2k}2^{2kK_0}\|a_i\|_{L^2(U_j(2^{K_0}B_i))}\,\frac{dt}{t}\\
&\lesssim(2^{K_0}r_{B_i})^{2M}2^{-j\epsilon}[\mu(2^{j+K_0}B_i)]^{-1/2}\\
&\qquad\times\big[\rho\big(\mu(2^{j+K_0}B_i)\big)\big]^{-1}2^{-2K_0(M-k+\epsilon/2)}K_1^{2M-2k}r_{B_i}^{2M+2k}.
\end{aligned}
$$

When $l>j+2$, from Lemma 2.2, $\mu(2^jB_i)\lesssim\mu(2^lB_i)$, $\rho(\mu(2^{j+K_0}B_i))\lesssim\rho(\mu(2^{l+K_0}B_i))$, and Definition 3.1(ii), we infer that

$$
\mathrm{H}_l\lesssim\int_0^{K_1}\Big(\frac{t}{r_{B_i}}\Big)^{2M-2k}2^{2kK_0}\|a_i\|_{L^2(U_l(2^{K_0}B_i))}\Big(\frac{t}{2^{l+K_0}r_{B_i}}\Big)^{\epsilon}\,\frac{dt}{t}
$$

$$\lesssim \int_0^{K_1} \Big(\frac{t}{r_{B_i}}\Big)^{2M-2k} 2^{2kK_0} r_{B_i}^{4M} 2^{-(l+K_0)\epsilon}[\mu(2^{l+K_0}B_i)]^{-1/2}\big[\rho\big(\mu(2^{l+K_0}B_i)\big)\big]^{-1}$$
$$\times \Big(\frac{t}{2^{l+K_0} r_{B_i}}\Big)^{\epsilon} \frac{dt}{t}$$
$$\lesssim (2^{K_0} r_{B_i})^{2M} 2^{-j\epsilon}[\mu(2^{j+K_0}B_i)]^{-1/2}\big[\rho\big(\mu(2^{j+K_0}B_i)\big)\big]^{-1} 2^{-l\epsilon}$$
$$\times 2^{-2K_0(M-k+\epsilon)} K_1^{2M-2k+\epsilon} r_{B_i}^{2M+2k-\epsilon}.$$

Then we estimate h_{i,K_1,K_0}. By Minkowski's inequality and Definition 3.1(ii), for $k \in \{0,1,\dots,M\}$, $i \in \{0,1,\dots,N\}$, and $j \in \mathbb{Z}_+$, we conclude that

$$\|(2^{2K_0} r_{B_i}^2 L)^k h_{i,K_1,K_0}\|_{L^2(U_j(2^{K_0}B_i))}$$
$$\le \mu^{-1}|\lambda_i| r_{B_i}^{2M} \Big\| \int_0^{K_1} \Big(\frac{t}{r_{B_i}}\Big)^{2M-2k} 2^{2kK_0}$$
$$\times (t^2 L)^{k+1} e^{-t^2 L}([t^2 L e^{-t^2 L} a_i]\chi_{\Omega^{\complement}_{j,K_0}}) \frac{dt}{t}\Big\|_{L^2(U_j(2^{K_0}B_i))}$$
$$\le \mu^{-1}|\lambda_i| \int_0^{K_1} \Big(\frac{t}{r_{B_i}}\Big)^{2M-2k} 2^{2kK_0} \Big(\frac{t}{2^{j+K_0} r_{B_i}}\Big)^{\epsilon+n(1/\widetilde{p}_\Phi-1/2)}$$
$$\times \|t^2 L e^{-t^2 L} a_i\|_{L^2(\mathcal{X})} \frac{dt}{t}$$
$$\lesssim (2^{K_0} r_{B_i})^{2M} 2^{-j\epsilon}[\mu(2^{j+K_0}B_i)]^{-1/2}\big[\rho\big(\mu(2^{j+K_0}B_i)\big)\big]^{-1}$$
$$\times 2^{-2K_0[M-k+\epsilon+(1/\widetilde{p}_\Phi-1/2)n/2]}$$
$$\times K_1^{2M-2k+\epsilon+n(1/\widetilde{p}_\Phi-1/2)} r_{B_i}^{2M+2k-\epsilon-n(1/\widetilde{p}_\Phi-1/2)}.$$

Combining these estimates, by choosing $K_1 > \max\{r_{B_1},\dots,r_{B_N}\}$, we further see that

$$\|(2^{2K_0} r_{B_i}^2 L)^k f_{i,K_1}\|_{L^2(U_j(2^{K_0}B_i))}$$
$$\lesssim (2^{K_0} r_{B_i})^{2M} 2^{-j\epsilon}[\mu(2^{j+K_0}B_i)]^{-1/2}\big[\rho\big(\mu(2^{j+K_0}B_i)\big)\big]^{-1}$$
$$\times 2^{-2K_0(M-k+\epsilon/2)} K_1^{2M-2k+\epsilon+(1/\widetilde{p}_\Phi-1/2)n/2} r_{B_i}^{2M+2k}.$$

Then, by choosing

$$K_0 \equiv \max_{0\le k\le M}\Big(\frac{\ln(K_1^{2M-2k+\epsilon+(1/\widetilde{p}_\Phi-1/2)n/2}\max_{0\le i\le N}\{r_{B_i}^{2M+2k}\})}{2\ln 2(M-k+\epsilon/2)}\Big),$$

we conclude that for $i \in \{0,1,\dots,N\}$, m_{i,K_1} is a $(\Phi,M,\epsilon)_L$-molecule adapted to the ball $2^{K_0}B_i$, which shows the claim and hence completes the proof of Theorem 3.1. $\square$

REMARK 3.1

We point out that the proof of Theorem 3.1 also works for [15, Theorem 5.4]. Moreover, due to the lack of the support of molecules, we show that m_{i,K_1} for

$i \in \{1, \ldots, N\}$ is a $(\Phi, M, \epsilon)_L$-molecule adapted to the ball $2^{K_0} B_i$, instead of B_i as in the proof of [15, Theorem 5.4], which also simplifies the proof of [15, Theorem 5.4].

By Theorem 3.1, with the argument the same as for the proofs of [2, Theorems 3.13, 3.16], we obtain the following dual theorem. We omit the details.

THEOREM 3.2

Let L, Φ, ρ, and M be as in Definition 3.2. Then for any function $f \in \mathrm{BMO}^M_{\rho,L}(\mathcal{X})$, the linear functional ℓ, defined by $\ell(g) \equiv \langle f, g\rangle$ initially on $H^{\mathrm{mol},\epsilon,2\widetilde{M}}_{\Phi,\mathrm{fin},L^*}(\mathcal{X})$ with $\widetilde{M} > M$ and $\epsilon \in (0, \widetilde{M} - (1/p_\Phi^- - 1/2)n/2)$, has a unique extension to $H_{\Phi,L^*}(\mathcal{X})$ and, moreover, $\|\ell\|_{(H_{\Phi,L^*}(\mathcal{X}))^*} \le C\|f\|_{\mathrm{BMO}^M_{\rho,L}(\mathcal{X})}$ for some nonnegative constant C independent of f.

Conversely, for any $\ell \in (H_{\Phi,L^*}(\mathcal{X}))^*$, there exists $f \in \mathrm{BMO}^M_{\rho,L}(\mathcal{X})$ such that $\ell(g) \equiv \langle f, g\rangle$ for all $g \in H^{\mathrm{mol},\epsilon,M}_{\Phi,\mathrm{fin},L^*}(\mathcal{X})$ and $\|f\|_{\mathrm{BMO}^M_{\rho,L}(\mathcal{X})} \le C\|\ell\|_{(H_{\Phi,L^*}(\mathcal{X}))^*}$, where C is a nonnegative constant independent of ℓ.

REMARK 3.2

(i) Theorem 3.1 is just [2, Theorems 3.15] but with the ranges of indices M and ϵ replaced, respectively, by $M > (1/p_\Phi^- - 1/2)n/2$ and $\epsilon \in (0, M - (1/p_\Phi^- - 1/2)n/2)$.

(ii) By Theorem 3.2, we see that for all $M > (1/p_\Phi^- - 1/2)n/2$, the spaces $\mathrm{BMO}^M_{\rho,L}(\mathcal{X})$ for different M coincide with equivalent norms; thus, in what follows, we denote $\mathrm{BMO}^M_{\rho,L}(\mathcal{X})$ simply by $\mathrm{BMO}_{\rho,L}(\mathcal{X})$.

The following two propositions are just [2, Propositions 3.11, 3.12] (see also [21, Propositions 4.4, 4.5]).

PROPOSITION 3.1

Let L, Φ, ρ, and M be as in Definition 3.2. Then $f \in \mathrm{BMO}_{\rho,L}(\mathcal{X})$ if and only if $f \in \mathcal{M}^M_{\Phi,L}(\mathcal{X})$ and

$$\sup_{B\subset\mathcal{X}} \frac{1}{\rho(\mu(B))}\Big[\frac{1}{\mu(B)}\int_B \big|[I - (I + r_B^2 L)^{-1}]^M f(x)\big|^2\, d\mu(x)\Big]^{1/2} < \infty.$$

Moreover, the quantity appearing in the left-hand side of the above formula is equivalent to $\|f\|_{\mathrm{BMO}^M_{\rho,L}(\mathcal{X})}$.

PROPOSITION 3.2

Let L, Φ, ρ, and M be as in Definition 3.2. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}_{\rho,L}(\mathcal{X})$,

$$\sup_{B\subset\mathcal{X}} \frac{1}{\rho(\mu(B))}\Big[\frac{1}{\mu(B)}\iint_{\widehat{B}} |(t^2L)^M e^{-t^2L} f(x)|^2\, \frac{d\mu(x)\,dt}{t}\Big]^{1/2} \le C\|f\|_{\mathrm{BMO}^M_{\rho,L}(\mathcal{X})}.$$

The following Proposition 3.3 and Lemma 3.2 are kinds of Calderón reproducing formulae.

PROPOSITION 3.3

Let L, Φ, ρ, and M be as in Definition 3.2, let $\epsilon, \epsilon_1 \in (0,\infty)$, and let $\widetilde{M} > M + \epsilon_1 + n/4 + (1/p_\Phi^- - 1)N/2$, where N is as in (2.4). Fix $x_0 \in \mathcal{X}$. Assume that $f \in \mathcal{M}_{\Phi,L}^M(\mathcal{X})$ satisfies

$$\int_{\mathcal{X}} \frac{|(I-(I+L)^{-1})^M f(x)|^2}{1+[d(x,x_0)]^{n+\epsilon_1+2N(1/p_\Phi^- - 1)}}\, d\mu(x) < \infty. \tag{3.6}$$

Then for all $(\Phi, \widetilde{M}, \epsilon)_{L^*}$-molecules α,

$$\langle f, \alpha\rangle = C(M) \iint_{\mathcal{X}\times(0,\infty)} (t^2L)^M e^{-t^2L} f(x) \overline{t^2L^* e^{-t^2L^*}\alpha(x)}\, \frac{d\mu(x)\, dt}{t},$$

where $C(M)$ is as in (3.5).

Proof

For $R > \delta > 0$, write

$$\begin{aligned}
C(M) &\int_\delta^R \int_{\mathcal{X}} (t^2L)^M e^{-t^2L} f(x) \overline{t^2L^* e^{-t^2L^*}\alpha(x)}\, \frac{d\mu(x)\, dt}{t}\\
&= \left\langle f, C(M) \int_\delta^R (t^2L^*)^{M+1} e^{-2t^2L^*} \alpha \frac{dt}{t}\right\rangle\\
&= \langle f, \alpha\rangle - \left\langle f, \alpha - C(M) \int_\delta^R (t^2L^*)^{M+1} e^{-2t^2L^*} \alpha \frac{dt}{t}\right\rangle.
\end{aligned}$$

Since α is a $(\Phi, \widetilde{M}, \epsilon)_{L^*}$-molecule, by Definition 3.1, there exists $b \in L^2(\mathcal{X})$ such that $\alpha = (L^*)^{\widetilde{M}} b$. Notice that

$$\begin{aligned}
f &= [I-(I+L)^{-1} + (I+L)^{-1}]^M f\\
&= \sum_{k=0}^M \binom{M}{k} [I-(I+L)^{-1}]^{M-k} (I+L)^{-k} f = \sum_{k=0}^M \binom{M}{k} [I-(I+L)^{-1}]^M L^{-k} f,
\end{aligned}$$

where $\binom{M}{k}$ denotes the *binomial coefficient*, which, together with the H_∞-functional calculus, further implies that

$$\begin{aligned}
\Big\langle f, \alpha - C(M) &\int_\delta^R (t^2L^*)^{M+1} e^{-2t^2L^*} \alpha \frac{dt}{t}\Big\rangle\\
&= \sum_{k=0}^M \binom{M}{k} \Big\langle [I-(I+L)^{-1}]^M f, L^{\widetilde{M}-k} b - C(M)\\
&\qquad \times \int_\delta^R (t^2L^*)^{M+1} e^{-2t^2L^*} (L^*)^{\widetilde{M}-k} b \frac{dt}{t}\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{M} \binom{M}{k} \Big\langle [I-(I+L)^{-1}]^M f, C(M) \\
&\quad \times \int_0^{\delta} (t^2L^*)^{M+1} e^{-2t^2L^*} (L^*)^{\widetilde{M}-k} b \, \frac{dt}{t} \Big\rangle \\
&\quad + \sum_{k=0}^{M} \binom{M}{k} \Big\langle [I-(I+L)^{-1}]^M f, C(M) \\
&\quad \times \int_R^{\infty} (t^2L^*)^{M+1} e^{-2t^2L^*} (L^*)^{\widetilde{M}-k} b \, \frac{dt}{t} \Big\rangle \\
&\equiv \sum_{k=0}^{M} \binom{M}{k} (\mathrm{H}+\mathrm{J}).
\end{aligned}
$$

For J, by (3.6) and Hölder's inequality, we conclude that

$$
\begin{aligned}
|\mathrm{J}| &\lesssim \Big\{ \int_{\mathcal{X}} \frac{|(I-(I+L)^{-1})^M f(x)|^2}{1+[d(x,x_0)]^{n+\epsilon_1+2N(1/p_\Phi^- -1)}} \, d\mu(x) \Big\}^{1/2} \\
&\quad \times \Big\{ \int_{\mathcal{X}} \Big| \int_R^{\infty} (t^2L^*)^{M+\widetilde{M}-k+1} e^{-2t^2L^*} b(x) \frac{1}{t^{2(\widetilde{M}-k)+1}} \, dt \Big|^2 \\
&\quad \times \big(1+[d(x,x_0)]^{n+\epsilon_1+2N(1/p_\Phi^- -1)}\big) \, d\mu(x) \Big\}^{1/2} \\
&\lesssim \int_R^{\infty} \big\| (t^2L^*)^{M+\widetilde{M}-k+1} e^{-2t^2L^*} b \big(1+[d(\cdot,x_0)]^{n+\epsilon_1+2N(1/p_\Phi^- -1)}\big)^{1/2} \big\|_{L^2(\mathcal{X})} \\
&\quad \times \frac{1}{t^{2(\widetilde{M}-k)+1}} \, dt.
\end{aligned}
$$

Let $B_0 \equiv B(x_0,1)$. Notice that there exist $\widetilde{N}$, $d \in \mathbb{N}$ such that for all $j \in \mathbb{N}$, $j \geq \widetilde{N}$,

$$
U_j(B_0) \subset \bigcup_{i=-d}^{d} U_{j+i}(B),
$$

where B is the ball adapted to α and $U_j(B)$ for $j \in \mathbb{Z}_+$ is as in (2.5). By choosing $j_0 \geq \widetilde{N}$, we conclude that

$$
\begin{aligned}
|\mathrm{J}| &\lesssim \int_R^{\infty} \big\| (t^2L^*)^{M+\widetilde{M}-k+1} e^{-2t^2L^*} b \\
&\quad \times \big(1+[d(\cdot,x_0)]^{n+\epsilon_1+2N(1/p_\Phi^- -1)}\big)^{1/2} \big\|_{L^2(2^{j_0}B_0)} \frac{1}{t^{2(\widetilde{M}-k)+1}} \, dt \\
&\quad + \sum_{j=j_0+1}^{\infty} \int_R^{\infty} \big\| (t^2L^*)^{M+\widetilde{M}-k+1} e^{-2t^2L^*} b \\
&\quad \times \big(1+[d(\cdot,x_0)]^{n+\epsilon_1+2N(1/p_\Phi^- -1)}\big)^{1/2} \big\|_{L^2(U_j(B_0))} \frac{1}{t^{2(\widetilde{M}-k)+1}} \, dt \equiv \mathrm{J}_1+\mathrm{J}_2.
\end{aligned}
$$

For all $\widetilde{\epsilon}>0$, let $C_1\equiv 2^{(n+\epsilon_1+2N(1/p_\Phi^- -1))j_0/2}\|b\|_{L^2(\mathcal{X})}$ and $R_1\equiv (C_1/\widetilde{\epsilon})^{1/(2(\widetilde{M}-k))}$; then for all $R>R_1$, we obtain

$$\mathrm{J}_1\lesssim 2^{j_0/2(n+\epsilon_1+2N(1/p_\Phi^- -1))}\int_R^\infty \frac{dt}{t^{2(\widetilde{M}-k)+1}}\|b\|_{L^2(\mathcal{X})}\lesssim\widetilde{\epsilon}.$$

Letting $C_2\equiv r_B^{(1/p_\Phi^- -1/2)n/2+2\widetilde{M}}$ and $R_1\equiv (C_2/\widetilde{\epsilon})^{1/(2(\widetilde{M}-k))}$, we then know that for all $R>R_1$,

$$\begin{aligned}\mathrm{J}_2\lesssim &\sum_{j=j_0+1}^\infty 2^{(n+\epsilon_1+2N(1/p_\Phi^- -1))j/2}\\ &\times\sum_{i=-d}^d\Big\{\int_R^\infty \|(t^2L^*)^{M+\widetilde{M}-k+1}e^{-2t^2L^*}(\chi_{\widetilde{U}_{j+i}(B)}b)\|_{L^2(U_{j+i}(B))}\frac{1}{t^{2(\widetilde{M}-k)+1}}\,dt\\ &+\int_R^\infty \|(t^2L^*)^{M+\widetilde{M}-k+1}e^{-2t^2L^*}(\chi_{(\widetilde{U}_{j+i}(B))^\complement}b)\|_{L^2(U_{j+i}(B))}\frac{1}{t^{2(\widetilde{M}-k)+1}}\,dt\Big\},\end{aligned}$$

where $\widetilde{U}_{j+i}(B)\equiv 2^{j+i+1}B\backslash 2^{j+i-1}B$. Then, since

$$\begin{aligned}&\int_R^\infty \|(t^2L^*)^{M+\widetilde{M}-k+1}e^{-2t^2L^*}(\chi_{\widetilde{U}_{j+i}(B)}b)\|_{L^2(U_{j+i}(B))}\frac{1}{t^{2(\widetilde{M}-k)+1}}\,dt\\ &\quad\lesssim\frac{1}{R^{2(\widetilde{M}-k)}}\|b\|_{L^2(\widetilde{U}_{j+i}(B))}\lesssim 2^{-(n+\epsilon_1+2N(1/p_\Phi^- -1))j/2}\widetilde{\epsilon},\end{aligned}$$

and $\int_R^\infty \|(t^2L^*)^{M+\widetilde{M}-k+1}e^{-2t^2L^*}(\chi_{(\widetilde{U}_{j+i}(B))^\complement}b)\|_{L^2(U_{j+i}(B))}1/(t^{2(\widetilde{M}-k)+1})\,dt$ satisfies the same estimate, we see that $\mathrm{J}_2\lesssim\widetilde{\epsilon}$. Thus, $\lim_{R\to\infty}\mathrm{J}=0$.

To consider H, let $\widetilde{f}\equiv[I-(I+L)^{-1}]^M f$. Then

$$\begin{aligned}S_{M+1}&\equiv\Big\langle\widetilde{f},\int_0^\delta (t^2L^*)^{M+1}e^{-2t^2L^*}(L^*)^{\widetilde{M}-k}b\,\frac{dt}{t}\Big\rangle\\ &=-\frac14\Big\langle\widetilde{f},\int_0^\delta (t^2L^*)^M\frac{\partial}{\partial t}(e^{-2t^2L^*})(L^*)^{\widetilde{M}-k}b\,\frac{dt}{t}\Big\rangle\\ &=-\frac14\langle\widetilde{f},(\delta^2L^*)^Me^{-2\delta^2L^*}(L^*)^{\widetilde{M}-k}b\rangle\\ &\quad+\frac{M}{2}\Big\langle\widetilde{f},\int_0^\delta (t^2L^*)^Me^{-2t^2L^*}(L^*)^{\widetilde{M}-k}b\,\frac{dt}{t}\Big\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}S_{M+1}&=-\frac14\langle\widetilde{f},(\delta^2L^*)^Me^{-2\delta^2L^*}(L^*)^{\widetilde{M}-k}b\rangle+\frac{M}{2}S_M\\ &=\sum_{\ell=1}^M\frac{-M!}{2^{\ell+1}(M-\ell+1)!}\langle\widetilde{f},(\delta^2L^*)^{M-\ell+1}e^{-2\delta^2L^*}(L^*)^{\widetilde{M}-k}b\rangle+\frac{M!}{2^M}S_1.\end{aligned}$$

For all $\ell\in\{1,\dots,M\}$, from Hölder's inequality, we infer that

$$|\langle\widetilde{f},(\delta^2L^*)^{M-\ell+1}e^{-2\delta^2L^*}(L^*)^{\widetilde{M}-k}b\rangle|$$

$$
\begin{aligned}
&\lesssim \Big\{\int_{\mathcal{X}} \frac{|(I-(I+L)^{-1})^M f(x)|^2}{1+[d(x,x_0)]^{n+\epsilon_1+2N(1/p_\Phi^- -1)}}\,d\mu(x)\Big\}^{1/2}\\
&\quad\times\Big\{\int_{\mathcal{X}} |(\delta^2L^*)^{M-\ell+1}e^{-2\delta^2L^*}(L^*)^{\widetilde{M}-k}b(x)|^2\\
&\quad\times\big(1+[d(x,x_0)]^{n+\epsilon_1+2N(1/p_\Phi^- -1)}\big)\,d\mu(x)\Big\}^{1/2}\\
&\lesssim 2^{[n+\epsilon_1+2N(1/p_\Phi^- -1)]j_0/2}\|(\delta^2L^*)^{M-\ell+1}e^{-2\delta^2L^*}(L^*)^{\widetilde{M}-k}b\|_{L^2(2^{j_0}B_0)}\\
&\quad+\sum_{j=j_0+1}^{\infty}2^{[n+\epsilon_1+2N(1/p_\Phi^- -1)]j/2}\\
&\quad\times\Big\{\big\|(\delta^2L^*)^{M-\ell+1}e^{-2\delta^2L^*}\big(\chi_{\bigcup_{i=j-d-1}^{j+d+1}U_i(B)}(L^*)^{\widetilde{M}-k}b\big)\big\|_{L^2(U_j(B_0))}\\
&\quad+\big\|(\delta^2L^*)^{M-\ell+1}e^{-2\delta^2L^*}\big(\chi_{(\bigcup_{i=j-d-1}^{j+d+1}U_i(B))^{\complement}}(L^*)^{\widetilde{M}-k}b\big)\big\|_{L^2(U_j(B_0))}\Big\}.
\end{aligned}
$$

By the L^2-functional calculus, we see that $\lim_{\delta\to 0}(\delta^2L^*)^{M-\ell+1}e^{-2\delta^2L^*}(L^*)^{\widetilde{M}-k}b=0$ in $L^2(\mathcal{X})$, and by Lemma 2.2, we know that

$$
\begin{aligned}
&\sum_{j=j_0+1}^{\infty}2^{[n+\epsilon_1+2N(1/p_\Phi^- -1)]j/2}\\
&\qquad\times\Big\{\big\|(\delta^2L^*)^{M-\ell+1}e^{-2\delta^2L^*}\big(\chi_{\bigcup_{i=j-d-1}^{j+d+1}U_i(B)}(L^*)^{\widetilde{M}-k}b\big)\big\|_{L^2(U_j(B_0))}\\
&\qquad+\big\|(\delta^2L^*)^{M-\ell+1}e^{-2\delta^2L^*}\big(\chi_{(\bigcup_{i=j-d-1}^{j+d+1}U_i(B))^{\complement}}(L^*)^{\widetilde{M}-k}b\big)\big\|_{L^2(U_j(B_0))}\Big\}\\
&\quad\lesssim\sum_{j=j_0+1}^{\infty}2^{[n+\epsilon_1+2N(1/p_\Phi^- -1)]j/2}\big[\|(L^*)^{\widetilde{M}-k}b\|_{L^2(\bigcup_{i=j-d-1}^{j+d+1}U_i(B))}\\
&\qquad+e^{-(2^jr_B)/\delta}\|(L^*)^{\widetilde{M}-k}b\|_{L^2(\mathcal{X})}\big]\\
&\quad\lesssim\widetilde{\epsilon}.
\end{aligned}
$$

From

$$
S_1=\Big\langle\widetilde{f},\int_0^{\delta}(t^2L^*)e^{-2t^2L^*}(L^*)^{\widetilde{M}-k}b\,\frac{dt}{t}\Big\rangle=\langle\widetilde{f},(e^{-2\delta^2L^*}-I)(L^*)^{\widetilde{M}-k}b\rangle
$$

and

$$
\lim_{\delta\to 0}\|(e^{-2\delta^2L^*}-I)(L^*)^{\widetilde{M}-k}b\|_{L^2(\mathcal{X})}=0,
$$

it follows that $\lim_{\delta\to 0}H=0$, which completes the proof of Proposition 3.3. □

Instead of [21, Proposition 4.6] by Proposition 3.3 here, repeating the proof of [21, Corollary 4.3], we obtain the following Lemma 3.2. The details are omitted.

LEMMA 3.2

Let L, Φ, ρ, and M be as in Definition 3.2, and let $\epsilon\in(0,\infty)$. If $f\in \mathrm{BMO}_{\rho,L}(\mathcal{X})$, *then for any $(\Phi, M, \epsilon)_{L^*}$-molecule α, there holds*

$$\langle f, \alpha\rangle = C(M)\iint_{\mathcal{X}\times(0,\infty)} (t^2L)^M e^{-t^2L} f(x)\overline{t^2L^* e^{-t^2L^*}\alpha(x)}\,\frac{d\mu(x)\,dt}{t}.$$

Recall that a measure $d\mu$ on $\mathcal{X}\times(0,\infty)$ is called a *ρ-Carleson measure* if

$$\|d\mu\|_\rho \equiv \sup_{B\subset\mathcal{X}}\Big\{\frac{1}{\mu(B)[\rho(\mu(B))]^2}\iint_{\widehat{B}} |d\mu|\Big\}^{1/2} < \infty,$$

where the supremum is taken over all balls B of $\mathcal{X}$.

Using Theorem 3.2 and Proposition 3.2, similarly to the proof of [21, Theorem 4.2], we obtain the following ρ-Carleson measure characterization of $\mathrm{BMO}_{\rho,L}(\mathcal{X})$.

THEOREM 3.3

Let L, Φ, ρ, and M be as in Definition 3.2. Fix $x_0\in\mathcal{X}$. Then the following are equivalent:

(i) $f\in\mathrm{BMO}_{\rho,L}(\mathcal{X})$*;*

(ii) $f\in\mathcal{M}^M_{\Phi,L}(\mathcal{X})$ *satisfies*

$$\int_{\mathcal{X}} \frac{|(I-(I+L)^{-1})^M f(x)|^2}{1+[d(x,x_0)]^{n+\epsilon_1+2N(1/p_\Phi^- -1)}}\,d\mu(x) < \infty$$

for some $\epsilon_1\in(0,\infty)$, and $d\mu_f$ is a ρ-Carleson measure, where $d\mu_f$ is defined by $d\mu_f(x,t)\equiv|(t^2L)^M e^{-t^2L}f(x)|^2\,\frac{d\mu(x)\,dt}{t}$ for all $(x,t)\in\mathcal{X}\times(0,\infty)$.

Moreover, $\|d\mu_f\|_\rho$ is equivalent to $\|f\|_{\mathrm{BMO}_{\rho,L}(\mathcal{X})}$.

Proof

It follows from Proposition 3.1 and the proof of Lemma 3.2 that (i) implies (ii).

To show that (ii) implies (i), let $\widetilde{M} > M+\epsilon_1+n/4+(1/p_\Phi^- -1)N/2$. From Proposition 3.3, we deduce that

$$\langle f, g\rangle = C(M)\iint_{\mathcal{X}\times(0,\infty)} (t^2L)^M e^{-t^2L} f(x)\overline{t^2L^* e^{-t^2L^*}g(x)}\,\frac{d\mu(x)\,dt}{t},$$

where g is any finite combination of $(\Phi, M, \epsilon)_{L^*}$-molecules. Then $t^2L^*e^{-t^2L^*}g\in T_\Phi(\mathcal{X})$. By Lemma 3.1, there exist $\{\lambda_j\}_{j=1}^\infty\subset\mathbb{C}$ and $T_\Phi(\mathcal{X})$-atoms $\{a_j\}_{j=1}^\infty$ supported in $\{\widehat{B}_j\}_{j=1}^\infty$ such that (3.1) and (3.2) hold. This, together with Fatou's lemma and Hölder's inequality, implies that

$$\begin{aligned}|\langle f,g\rangle| &= \Big|C(M)\iint_{\mathcal{X}\times(0,\infty)} (t^2L)^M e^{-t^2L} f(x)\overline{t^2L^* e^{-t^2L^*}g(x)}\,\frac{d\mu(x)\,dt}{t}\Big|\\ &\lesssim \sum_j |\lambda_j|\int_0^\infty\int_{\mathcal{X}} |(t^2L)^M e^{-t^2L}f(x)\overline{a_j(x,t)}|\,\frac{d\mu(x)\,dt}{t}\\ &\lesssim \sum_j |\lambda_j|\|a_j\|_{T_2^2(\mathcal{X})}\Big(\iint_{\widehat{B}_j} |(t^2L)^M e^{-t^2L}f(x)|^2\,\frac{d\mu(x)\,dt}{t}\Big)^{1/2}\end{aligned}$$

$$\lesssim \sum_j |\lambda_j| \|d\mu_f\|_\rho \lesssim \|(t^2L^*)^M e^{-t^2L^*} g\|_{T_\Phi(\mathcal{X})} \|d\mu_f\|_\rho \sim \|g\|_{H_{\Phi,L^*}(\mathcal{X})} \|d\mu_f\|_\rho.$$

By this and Theorem 3.2, we conclude that $f \in (H_{\Phi,L^*}(\mathcal{X}))^* = \mathrm{BMO}_{\rho,L}(\mathcal{X})$, which completes the proof of Theorem 3.3. □

Now we introduce the space $\mathrm{VMO}_{\rho,L}(\mathcal{X})$.

DEFINITION 3.3

Let L, Φ, ρ, and M be as in Definition 3.2. An element $f \in \mathrm{BMO}_{\rho,L}(\mathcal{X})$ is said to be in the *space* $\mathrm{VMO}^M_{\rho,L}(\mathcal{X})$ if it satisfies the limiting conditions $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where $x_0 \in \mathcal{X}$ is a fixed point, $c \in (0,\infty)$,

$$\gamma_1(f) \equiv \lim_{c\to 0} \sup_{B: r_B \le c} \frac{1}{\rho(\mu(B))} \Big[\frac{1}{\mu(B)} \int_B |(I - e^{-r_B^2 L})^M f(x)|^2 \, d\mu(x)\Big]^{1/2},$$

$$\gamma_2(f) \equiv \lim_{c\to \infty} \sup_{B: r_B \ge c} \frac{1}{\rho(\mu(B))} \Big[\frac{1}{\mu(B)} \int_B |(I - e^{-r_B^2 L})^M f(x)|^2 \, d\mu(x)\Big]^{1/2},$$

and

$$\gamma_3(f) \equiv \lim_{c\to \infty} \sup_{B: B \subset [B(x_0,c)]^\complement} \frac{1}{\rho(\mu(B))} \Big[\frac{1}{\mu(B)} \int_B |(I - e^{-r_B^2 L})^M f(x)|^2 \, d\mu(x)\Big]^{1/2}.$$

For any $f \in \mathrm{VMO}^M_{\rho,L}(\mathcal{X})$, define $\|f\|_{\mathrm{VMO}^M_{\rho,L}(\mathcal{X})} \equiv \|f\|_{\mathrm{BMO}_{\rho,L}(\mathcal{X})}$.

DEFINITION 3.4

Let Φ satisfy Assumption (Φ), and let ρ be as in (2.11). The *space* $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$ is defined to be the space of all $f \in T^\infty_\Phi(\mathcal{X})$ satisfying $\eta_1(f) = \eta_2(f) = \eta_3(f) = 0$ with the same norm as the space $T^\infty_\Phi(\mathcal{X})$, where $x_0 \in \mathcal{X}$ is a fixed point, $c \in (0,\infty)$,

$$\eta_1(f) \equiv \lim_{c\to 0} \sup_{B: r_B \le c} \frac{1}{\rho(\mu(B))} \Big[\frac{1}{\mu(B)} \iint_{\widehat{B}} |f(y,t)|^2 \frac{d\mu(y)\, dt}{t}\Big]^{1/2},$$

$$\eta_2(f) \equiv \lim_{c\to \infty} \sup_{B: r_B \ge c} \frac{1}{\rho(\mu(B))} \Big[\frac{1}{\mu(B)} \iint_{\widehat{B}} |f(y,t)|^2 \frac{d\mu(y)\, dt}{t}\Big]^{1/2},$$

and

$$\eta_3(f) \equiv \lim_{c\to \infty} \sup_{B: B \subset [B(x_0,c)]^\complement} \frac{1}{\rho(\mu(B))} \Big[\frac{1}{\mu(B)} \iint_{\widehat{B}} |f(y,t)|^2 \frac{d\mu(y)\, dt}{t}\Big]^{1/2}.$$

It is easy to see that $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$ is a *closed linear subspace* of $T^\infty_\Phi(\mathcal{X})$.

Further, denote by $T^\infty_{\Phi,1}(\mathcal{X})$ the *space of all* $f \in T^\infty_\Phi(\mathcal{X})$ *with* $\eta_1(f) = 0$, and denote by $T^2_{2,b}(\mathcal{X})$ the *space of all* $f \in T^2_2(\mathcal{X})$ *with bounded support*. Obviously, we have $T^2_{2,b}(\mathcal{X}) \subset T^\infty_{\Phi,\mathrm{v}}(\mathcal{X}) \subset T^\infty_{\Phi,1}(\mathcal{X})$. Finally, denote by $T^\infty_{\Phi,0}(\mathcal{X})$ the *closure of* $T^2_{2,b}(\mathcal{X})$ *in the space* $T^\infty_{\Phi,1}(\mathcal{X})$.

LEMMA 3.3
Let L and Φ be as in Definition 3.1, and let $T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$ and $T^{\infty}_{\Phi,0}(\mathcal{X})$ be defined as above. Then $T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$ and $T^{\infty}_{\Phi,0}(\mathcal{X})$ coincide with equivalent norms.

Proof
Since $T^2_{2,b}(\mathcal{X}) \subset T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$ and $T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$ is a closed linear subspace of $T^{\infty}_{\Phi}(\mathcal{X})$, we conclude that $T^{\infty}_{\Phi,0}(\mathcal{X}) = \overline{T^2_{2,b}(\mathcal{X})} \subset T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$.

Conversely, for any $f \in T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$, by the definition of $T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$, for any $\epsilon > 0$, there exist positive constants a_0, b_0, and c_0 such that

$$\sup_{B: r_B \le a_0} \frac{1}{\mu(B)[\rho(\mu(B))]^2} \iint_{\widehat{B}} |f(y,t)|^2 \frac{d\mu(y)\,dt}{t} < \epsilon, \tag{3.7}$$

$$\sup_{B: r_B \ge b_0} \frac{1}{\mu(B)[\rho(\mu(B))]^2} \iint_{\widehat{B}} |f(y,t)|^2 \frac{d\mu(y)\,dt}{t} < \epsilon, \tag{3.8}$$

and

$$\sup_{B: B \subset [B(x_0,c_0)]^{\complement}} \frac{1}{\mu(B)[\rho(\mu(B))]^2} \iint_{\widehat{B}} |f(y,t)|^2 \frac{d\mu(y)\,dt}{t} < \epsilon. \tag{3.9}$$

Let $K_0 \equiv \max\{a_0^{-1}, b_0, c_0\}$, and for all $(y,t) \in \mathcal{X} \times (0,\infty)$, let

$$g(y,t) \equiv f(y,t)\chi_{B(x_0,2K_0)\times((2K_0)^{-1},2K_0)}(y,t).$$

Obviously, $g \in T^2_{2,b}(\mathcal{X})$. To complete the proof of Lemma 3.3, we need to show that

$$\|f-g\|^2_{T^{\infty}_{\Phi}(\mathcal{X})} \lesssim \epsilon.$$

We consider the following three cases for all balls B in (3.7), (3.8), and (3.9).

Case (i): $r_B < a_0$ or $r_B > b_0$. In this case, from (3.7) and (3.8), we deduce that

$$\|f-g\|^2_{T^{\infty}_{\Phi}(\mathcal{X})} \le \frac{2}{\mu(B)[\rho(\mu(B))]^2} \iint_{\widehat{B}} |f(y,t)|^2 \frac{d\mu(y)\,dt}{t} \le 2\epsilon.$$

Case (ii): $a_0 \le r_B \le b_0$ and $B \subset [B(x_0,c_0)]^{\complement}$. In this case, by (3.9), we conclude that

$$\|f-g\|^2_{T^{\infty}_{\Phi}(\mathcal{X})} \le \frac{2}{\mu(B)[\rho(\mu(B))]^2} \iint_{\widehat{B}} |f(y,t)|^2 \frac{d\mu(y)\,dt}{t} \le 2\epsilon.$$

Case (iii): $a_0 \le r_B \le b_0$ and $B \cap B(x_0,c_0) \ne \emptyset$. In this case, we have

$$\begin{aligned}\iint_{\widehat{B}} |f(y,t)-g(y,t)|^2 \frac{d\mu(y)\,dt}{t} &\le \int_0^{(2K_0)^{-1}} \int_B |f(y,t)|^2 \frac{d\mu(y)\,dt}{t} \\ &\le \int_0^{(2K_0)^{-1}} \int_{B(x_B,2^k a_0)} |f(y,t)|^2 \frac{d\mu(y)\,dt}{t},\end{aligned}$$

where x_B is the *center* of B and k is the *smallest integer* such that $2^k a_0 > r_B$. Then, by Lemma 2.1, we pick a family of balls with the same radius a_0,

$\{B(x_{B,i},a_0)\}_{i=1}^{N_k}$, such that $B(x_B,2^k a_0)\subset\bigcup_{i=1}^{N_k}B(x_{B,i},a_0)$, $N_k\lesssim 2^{kn}$, and $\sum_{i=1}^{N_k}\chi_{B(x_{B,i},a_0)}\lesssim 1$. Therefore, combining the fact that ρ is an increasing function, we obtain

$$\begin{aligned}\iint_{\widehat{B}}|f(y,t)-g(y,t)|^2\,\frac{d\mu(y)\,dt}{t} &\le \int_0^{(2K_0)^{-1}}\int_{\bigcup_{i=1}^{N_k}B(x_{B,i},a_0)}|f(y,t)|^2\,\frac{d\mu(y)\,dt}{t}\\ &\le \sum_{i=1}^{N_k}\iint_{\widehat{B}(x_{B,i},a_0)}|f(y,t)|^2\,\frac{d\mu(y)\,dt}{t}\\ &\lesssim \epsilon\sum_{i=1}^{N_k}\mu\big(B(x_{B,i},a_0)\big)\big[\rho\big(\mu(B(x_{B,i},a_0))\big)\big]^2\\ &\lesssim \epsilon[\rho(\mu(B))]^2\sum_{i=1}^{N_k}\mu\big(B(x_{B,i},a_0)\big)\\ &\lesssim \epsilon\mu(B)\big[\rho\big(\mu(B)\big)\big]^2,\end{aligned}$$

which completes the proof of Lemma 3.3. □

DEFINITION 3.5

Let L, Φ, ρ, and M be as in Definition 3.2. The *space* $\widetilde{\mathrm{VMO}}_{\rho,L}^M(\mathcal{X})$ is defined to be the space of all elements $f\in\mathrm{BMO}_{\rho,L}^M(\mathcal{X})$ that satisfy the limiting conditions $\widetilde{\gamma}_1(f)=\widetilde{\gamma}_2(f)=\widetilde{\gamma}_3(f)=0$, where $c\in(0,\infty)$,

$$\widetilde{\gamma}_1(f)\equiv\lim_{c\to 0}\sup_{B:r_B\le c}\frac{1}{\rho(\mu(B))}\Big[\frac{1}{\mu(B)}\int_B\big|(I-[I+r_B^2L]^{-1})^M f(x)\big|^2\,d\mu(x)\Big]^{1/2},$$

$$\widetilde{\gamma}_1(f)\equiv\lim_{c\to \infty}\sup_{B:r_B\ge c}\frac{1}{\rho(\mu(B))}\Big[\frac{1}{\mu(B)}\int_B\big|(I-[I+r_B^2L]^{-1})^M f(x)\big|^2\,d\mu(x)\Big]^{1/2},$$

and

$$\begin{aligned}\widetilde{\gamma}_1(f)\equiv\lim_{c\to \infty}\sup_{B:B\subset[B(0,c)]^{\complement}}&\frac{1}{\rho(\mu(B))}\\ &\times\Big[\frac{1}{\mu(B)}\int_B\big|(I-[I+r_B^2L]^{-1})^M f(x)\big|^2\,d\mu(x)\Big]^{1/2}.\end{aligned}$$

PROPOSITION 3.4

Let L, Φ, ρ, and M be as in Definition 3.2. Then $f\in\mathrm{VMO}_{\rho,L}^M(\mathcal{X})$ if and only if $f\in\widetilde{\mathrm{VMO}}_{\rho,L}^M(\mathcal{X})$.

Proof

Suppose that $f\in\widetilde{\mathrm{VMO}}_{\rho,L}^M(\mathcal{X})$. To see $f\in\mathrm{VMO}_{\rho,L}^M(\mathcal{X})$, it suffices to show that

$$\frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}}\Big[\int_B|(I-e^{-r_B^2L})^M f(x)|^2\,d\mu(x)\Big]^{1/2}\lesssim\sum_{k=0}^{\infty}2^{-k}\delta_k(f,B), \tag{3.10}$$

where

$$\delta_k(f,B) \equiv \sup_{\{B'\subset 2^{k+1}B:\, r_{B'}\in[2^{-1}r_B,\, r_B]\}} \frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \times \left[\int_B \left|(I-[I+r_B^2L]^{-1})^M f(x)\right|^2 d\mu(x)\right]^{1/2}. \tag{3.11}$$

Indeed, since $f\in \widetilde{\mathrm{VMO}}^M_{\rho,L}(\mathcal{X})$, by Definition 3.5 and Proposition 3.1, we conclude that $\delta_k(f,B)\lesssim \|f\|_{\mathrm{BMO}_{\rho,L}(\mathcal{X})}$ and for all $k\in\mathbb{Z}_+$,

$$\lim_{c\to0}\sup_{B:\, r_B\le c}\delta_k(f,B)=\lim_{c\to\infty}\sup_{B:\, r_B\ge c}\delta_k(f,B)=\lim_{c\to\infty}\sup_{B:\, B\subset[B(x_0,c)]^{\complement}}\delta_k(f,B)=0.$$

Then by the dominated convergence theorem for series, we have

$$\begin{aligned}\gamma_1(f)&=\lim_{c\to0}\sup_{B:\, r_B\le c}\frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}}\left[\int_B |(I-e^{-r_B^2L})^M f(x)|^2\,d\mu(x)\right]^{1/2}\\&\lesssim\sum_{k=1}^{\infty}2^{-k}\lim_{c\to0}\sup_{B:\, r_B\le c}\delta_k(f,B)=0.\end{aligned}$$

Similarly we see that $\gamma_2(f)=\gamma_3(f)=0$, and hence $f\in \mathrm{VMO}^M_{\rho,L}(\mathcal{X})$.

Let us now prove (3.10). Write

$$f=(I-[I+r_B^2L]^{-1})^M f+\left\{I-(I-[I+r_B^2L]^{-1})^M\right\}f\equiv f_1+f_2. \tag{3.12}$$

By Lemma 2.2, we have

$$\begin{aligned}&\|(I-e^{-r_B^2L})^M f_1\|_{L^2(B)}\\&\quad\le\sum_{k=0}^{\infty}\|(I-e^{-r_B^2L})^M(f_1\chi_{U_k(B)})\|_{L^2(B)}\\&\quad\lesssim\sum_{k=0}^{\infty}e^{-c2^{2k}}\|f_1\chi_{U_k(B)}\|_{L^2(\mathcal{X})}\\&\quad\lesssim\rho\big(\mu(B)\big)[\mu(B)]^{1/2}\sum_{k=0}^{\infty}e^{-c2^{2k}}2^{kn}\delta_k(f,B)\\&\quad\lesssim\rho\big(\mu(B)\big)[\mu(B)]^{1/2}\sum_{k=0}^{\infty}2^{-k}\delta_k(f,B),\end{aligned} \tag{3.13}$$

where $U_k(B)$ for all $k\in\mathbb{Z}_+$ is as in (2.5), c is a positive constant, and the third inequality follows from Lemma 2.1 that there exists a collection, $\{B_{k,1}, B_{k,2},\dots,B_{k,N_k}\}$, of balls such that each ball $B_{k,i}$ is of radius r_B, $B(x_B,2^{k+1}r_B)\subset \bigcup_{i=1}^{N_k}B_{k,i}$, and $N_k\lesssim 2^{nk}$.

To estimate the remaining term, by the formula

$$I-(I-[I+r_B^2L]^{-1})^M=\sum_{j=1}^{M}\frac{M!}{j!(M-j)!}(r_B^2L)^{-j}(I-[I+r_B^2L]^{-1})^M \tag{3.14}$$

(which relies on the fact that $(I-(I+r^2L)^{-1})(r^2L)^{-1}=(I+r^2L)^{-1}$ for all $r\in(0,\infty)$) and Minkowski's inequality, we obtain

$$
\begin{aligned}
&\|(I-e^{-r_B^2L})^M f_2\|_{L^2(B)}\\
&\quad\lesssim \sum_{j=1}^M \Big\{\int_B \Big|(I-e^{-r_B^2L})^{M-j}\Big[-\int_0^{r_B}\frac{s}{r_B^2}e^{-s^2L}\,ds\Big]^j f_1(x)\Big|^2\,d\mu(x)\Big\}^{1/2}\\
&\quad\lesssim \sum_{j=1}^M\sum_{i=0}^{M-j}\int_0^{r_B}\cdots\int_0^{r_B}\frac{s_1}{r_B^2}\cdots\frac{s_j}{r_B^2}\|e^{-(ir_B^2+s_1^2+\cdots+s_j^2)L}f_1\|_{L^2(B)}\,ds_1\cdots ds_j\\
&\quad\lesssim \sum_{j=1}^M\sum_{i=0}^{M-j}\int_0^{r_B}\cdots\int_0^{r_B}\frac{s_1}{r_B^2}\cdots\frac{s_j}{r_B^2}\\
&\qquad\times\sum_{k=0}^\infty e^{-c(2^kr_B)^2/(ir_B^2+s_1^2+\cdots+s_j^2)}\|f_1\chi_{U_k(B)}\|_{L^2(\mathcal{X})}\,ds_1\cdots ds_j \qquad (3.15)\\
&\quad\lesssim \rho\big(\mu(B)\big)[\mu(B)]^{1/2}\sum_{k=0}^\infty e^{-(c2^{2k})/M}2^{kn}\delta_k(f,B)\\
&\quad\lesssim \rho\big(\mu(B)\big)[\mu(B)]^{1/2}\sum_{k=0}^\infty 2^{-k}\delta_k(f,B),
\end{aligned}
$$

where c is a positive constant and in the penultimate inequality, we used the fact that $\int_0^{r_B}\cdots\int_0^{r_B}(s_1/r_B^2)\cdots(s_j/r_B^2)\,ds_1\cdots ds_j\sim 1$. Combining the estimates (3.13) and (3.15), we obtain (3.10), which further implies that $\widetilde{\mathrm{VMO}}{}^M_{\rho,L}(\mathcal{X})\subset \mathrm{VMO}^M_{\rho,L}(\mathcal{X})$.

By borrowing some ideas from the proof of [16, Lemma 8.1], similarly to the proof above, we conclude that $\mathrm{VMO}^M_{\rho,L}(\mathcal{X})\subset\widetilde{\mathrm{VMO}}{}^M_{\rho,L}(\mathcal{X})$ and the details are omitted. This finishes the proof of Proposition 3.4. □

We now characterize the space $\mathrm{VMO}^M_{\rho,L}(\mathcal{X})$ via the tent space.

THEOREM 3.4

Let L, Φ, and ρ be as in Definition 3.1, let M, $M_1\in\mathbb{N}$, and let $M_1\ge M>(1/p_\Phi^- - 1/2)n/2$. Then the following are equivalent:

(i) $f\in\mathrm{VMO}^M_{\rho,L}(\mathcal{X})$;

(ii) $f\in\mathcal{M}^{M_1}_{\Phi,L}(\mathcal{X})$ *and* $(t^2L)^{M_1}e^{-t^2L}f\in T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$.

Moreover, $\|(t^2L)^{M_1}e^{-t^2L}f\|_{T^\infty_\Phi(\mathcal{X})}$ *is equivalent to* $\|f\|_{\mathrm{BMO}_{\rho,L}(\mathcal{X})}$.

Proof

We first show that (i) implies (ii). Let $f\in\mathrm{VMO}^M_{\rho,L}(\mathcal{X})$. By Proposition 3.2, we know that $(t^2L)^{M_1}e^{-t^2L}f\in T^\infty_\Phi(\mathcal{X})$. To see that $(t^2L)^{M_1}e^{-t^2L}f\in T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$, we claim that it suffices to show that for all balls B,

$$\begin{aligned}\frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}}\left[\iint_{\widehat{B}}|(t^2L)^{M_1}e^{-t^2L}f(x)|^2\,\frac{d\mu(x)\,dt}{t}\right]^{1/2}\\ \lesssim\sum_{k=0}^{\infty}2^{-k}\delta_k(f,B),\end{aligned}\tag{3.16}$$

where $\delta_k(f,B)$ is as in (3.11). Indeed, since $f\in \mathrm{VMO}^M_{\rho,L}(\mathcal{X})=\widetilde{\mathrm{VMO}}^M_{\rho,L}(\mathcal{X})$, we conclude that for each $k\in\mathbb{N}$, $\delta_k(f,B)\lesssim\|f\|_{\mathrm{BMO}_{\rho,L}(\mathcal{X})}$ and

$$\begin{aligned}\lim_{c\to0}\sup_{B:r_B\le c}\delta_k(f,B)&=\lim_{c\to\infty}\sup_{B:r_B\ge c}\delta_k(f,B)\\ &=\lim_{c\to\infty}\sup_{B:B\subset[B(x_0,c)]^\complement}\delta_k(f,B)=0.\end{aligned}$$

Then from the dominated convergence theorem for series, we infer that

$$\begin{aligned}\eta_1(f)&=\lim_{c\to0}\sup_{B:r_B\le c}\frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}}\left[\iint_{\widehat{B}}|(t^2L)^{M_1}e^{-t^2L}f(x)|^2\,\frac{d\mu(x)\,dt}{t}\right]^{1/2}\\ &\lesssim\sum_{k=1}^{\infty}2^{-k}\lim_{c\to0}\sup_{B:r_B\le c}\delta_k(f,B)=0.\end{aligned}$$

Similarly we see that $\eta_2(f)=\eta_3(f)=0$, and hence $(t^2L)^{M_1}e^{-t^2L}f\in T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$.

Let us now prove (3.16). Write $f\equiv f_1+f_2$ as in (3.12). Then by Lemmas 2.2 and 2.3, similarly to the estimate of (3.13), we have

$$\begin{aligned}&\left\{\iint_{\widehat{B}}|(t^2L)^{M_1}e^{-t^2L}f_1(x)|^2\,\frac{d\mu(x)\,dt}{t}\right\}^{1/2}\\ &\quad\le\sum_{k=0}^{\infty}\left\{\iint_{\widehat{B}}|(t^2L)^{M_1}e^{-t^2L}(f_1\chi_{U_k(B)})(x)|^2\,\frac{d\mu(x)\,dt}{t}\right\}^{1/2}\\ &\quad\lesssim\|f_1\|_{L^2(4B)}+\sum_{k=3}^{\infty}\left[\int_0^{r_B}\exp\left\{-\frac{(2^kr_B)^2}{ct^2}\right\}\frac{dt}{t}\right]^{1/2}\|f_1\chi_{U_k(B)}\|_{L^2(\mathcal{X})}\\ &\quad\lesssim\|f_1\|_{L^2(4B)}+\sum_{k=3}^{\infty}\left\{\int_0^{r_B}\left[\frac{t^2}{(2^kr_B)^2}\right]^{n+1}\frac{dt}{t}\right\}^{1/2}\|f_1\chi_{U_k(B)}\|_{L^2(\mathcal{X})}\\ &\quad\lesssim\rho\big(\mu(B)\big)[\mu(B)]^{1/2}\sum_{k=0}^{\infty}2^{-k}\delta_k(f,B),\end{aligned}\tag{3.17}$$

where $U_k(B)$ for all $k\in\mathbb{Z}_+$ is as in (2.5) and c is a positive constant. Applying (3.14), Lemma 2.2, and $M_1>M$ to f_2, we see that

$$\begin{aligned}&\left\{\iint_{\widehat{B}}|(t^2L)^{M_1}e^{-t^2L}f_2(x)|^2\,\frac{d\mu(x)\,dt}{t}\right\}^{1/2}\\ &\quad\lesssim\sum_{j=1}^{M}\left\{\iint_{\widehat{B}}|(t^2L)^{M_1}e^{-t^2L}(r_B^2L)^{-j}f_1(x)|^2\,\frac{d\mu(x)\,dt}{t}\right\}^{1/2}\end{aligned}$$

$$\lesssim \sum_{j=1}^{M}\sum_{k=0}^{\infty}\Big\{\int\!\!\int_{\widehat{B}}\Big[\frac{t^2}{r_B^2}\Big]^{2j}|(t^2L)^{M_1-j}e^{-t^2L}(f_1\chi_{U_k(B)})(x)|^2\,\frac{d\mu(x)\,dt}{t}\Big\}^{1/2}$$

$$\lesssim \sum_{j=1}^{M}\Big\{\sum_{k=0}^{2}\Big[\int_0^{r_B}\Big(\frac{t^2}{r_B^2}\Big)^{2j}\frac{dt}{t}\Big]^{1/2}\|f_1\|_{L^2(4B)} \tag{3.18}$$

$$+\sum_{k=3}^{\infty}\Big[\int_0^{r_B}\exp\Big\{-\frac{(2^kr_B)^2}{ct^2}\Big\}\frac{dt}{t}\Big]^{1/2}\|f_1\chi_{U_k(B)}\|_{L^2(\mathcal{X})}\Big\}$$

$$\lesssim \|f_1\|_{L^2(4B)}+\sum_{k=3}^{\infty}\Big\{\int_0^{r_B}\Big[\frac{t^2}{(2^kr_B)^2}\Big]^{n+1}\frac{dt}{t}\Big\}^{1/2}\|f_1\chi_{U_k(B)}\|_{L^2(\mathcal{X})}$$

$$\lesssim \rho\big(\mu(B)\big)[\mu(B)]^{1/2}\sum_{k=0}^{\infty}2^{-k}\delta_k(f,B).$$

The estimates (3.17) and (3.18) imply (3.16), which completes the proof that (i) implies (ii).

Conversely, let $f\in\mathcal{M}^{M_1}_{\Phi,L}(\mathcal{X})$ and $(t^2L)^{M_1}e^{-t^2L}f\in T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$. By Proposition 3.2, we conclude that $f\in \mathrm{BMO}_{\rho,L}(\mathcal{X})$. For any ball B, write

$$\Big(\int_B|(I-e^{-r_B^2L})^Mf(x)|^2\,d\mu(x)\Big)^{1/2}$$
$$=\sup_{\|g\|_{L^2(B)}\le1}\Big|\int_B(I-e^{-r_B^2L})^Mf(x)\overline{g(x)}\,d\mu(x)\Big|$$
$$=\sup_{\|g\|_{L^2(B)}\le1}\Big|\int_B f(x)\overline{(I-e^{-r_B^2L^*})^Mg(x)}\,d\mu(x)\Big|.$$

Notice that for any $g\in L^2(B)$, $(I-e^{-r_B^2L^*})^Mg$ is a multiple of a $(\Phi,M,\epsilon)_{L^*}$-molecule (see [16, p. 43]). Then by Lemma 3.2 and Hölder's inequality, we obtain

$$\Big[\int_B|(I-e^{-r_B^2L})^Mf(x)|^2\,d\mu(x)\Big]^{1/2}$$
$$\sim\sup_{\|g\|_{L^2(B)}\le1}\Big|\int\!\!\int_{\mathcal{X}\times(0,\infty)}(t^2L)^{M_1}e^{-t^2L}$$
$$\times f(x)t^2L^*e^{-t^2L^*}\overline{(I-e^{-r_B^2L^*})^Mg(x)}\,\frac{d\mu(x)\,dt}{t}\Big|$$
$$\sim\sum_{k=0}^{\infty}\Big\{\int\!\!\int_{V_k(B)}|(t^2L)^{M_1}e^{-t^2L}f(x)|^2\,\frac{d\mu(x)\,dt}{t}\Big\}^{1/2}$$
$$\times\sup_{\|g\|_{L^2(B)}\le1}\Big\{\int\!\!\int_{V_k(B)}|t^2L^*e^{-t^2L^*}(I-e^{-r_B^2L^*})^Mg(x)|^2\,\frac{d\mu(x)\,dt}{t}\Big\}^{1/2}$$
$$\equiv\sum_{k=0}^{\infty}\sigma_k(f,B)\mathrm{I}_k,$$

where $V_0(B) \equiv \widehat{B}$ and $V_k(B) \equiv (\widehat{2^k B}) \backslash (\widehat{2^{k-1} B})$ for $k \in \mathbb{N}$. In what follows, for $k \geq 2$, let $V_{k,1} \equiv (\widehat{2^k B}) \backslash (2^{k-2} B \times (0, \infty))$ and $V_{k,2} \equiv V_k(B) \backslash V_{k,1}(B)$.

For $k \in \{0,1,2\}$, by Lemmas 2.2 and 2.3, we conclude that

$$\begin{aligned}
\mathrm{I}_k &= \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \iint_{V_k(B)} |t^2 L^* e^{-t^2 L^*} (I - e^{-r_B^2 L^*})^M g(x)|^2 \, \frac{d\mu(x)\,dt}{t} \right\}^{1/2} \\
&\lesssim \sup_{\|g\|_{L^2(B)} \leq 1} \|(I - e^{-r_B^2 L^*})^M g\|_{L^2(\mathcal{X})} \lesssim 1.
\end{aligned}$$

Now for $k \geq 3$, write

$$\begin{aligned}
\mathrm{I}_k &\lesssim \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \iint_{V_{k,1}(B)} |t^2 L^* e^{-t^2 L^*} (I - e^{-r_B^2 L^*})^M g(x)|^2 \, \frac{d\mu(x)\,dt}{t} \right\}^{1/2} \\
&\quad + \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \iint_{V_{k,2}(B)} \cdots \right\}^{1/2} \equiv \mathrm{I}_{k,1} + \mathrm{I}_{k,2}.
\end{aligned}$$

Since for any $(y,t) \in V_{k,2}(B)$, $t \geq 2^{k-2} r_B$, from Minkowski's inequality and Lemmas 2.2 and 2.3, it follows that

$$\begin{aligned}
\mathrm{I}_{k,2} &= \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \iint_{V_{k,2}(B)} |t^2 L^* e^{-t^2 L^*} (I - e^{-r_B^2 L^*})^M g(x)|^2 \, \frac{d\mu(x)\,dt}{t} \right\}^{1/2} \\
&= \sup_{\|g\|_{L^2(B)} \leq 1} \left\{ \iint_{V_{k,2}(B)} \left| t^2 L^* e^{-t^2 L^*} \left[- \int_0^{r_B^2} L^* e^{-s L^*} \, ds \right]^M g(x) \right|^2 \frac{d\mu(x)\,dt}{t} \right\}^{1/2} \\
&\lesssim \sup_{\|g\|_{L^2(B)} \leq 1} \int_0^{r_B^2} \cdots \int_0^{r_B^2} \left\{ \iint_{V_{k,2}(B)} |t^2 (L^*)^{M+1} \right. \\
&\quad \left. \times e^{-(t^2 + s_1 + \cdots + s_M) L^*} g(x)|^2 \, \frac{d\mu(x)\,dt}{t} \right\}^{1/2} ds_1 \cdots ds_M \\
&\lesssim \sup_{\|g\|_{L^2(B)} \leq 1} \int_0^{r_B^2} \cdots \int_0^{r_B^2} \left\{ \int_{2^{k-2} r_B}^{2^k r_B} \frac{t^4 \|g\|_{L^2(B)}^2}{(t^2 + s_1 + \cdots + s_M)^{2(M+1)}} \, \frac{dt}{t} \right\}^{1/2} ds_1 \cdots ds_M \\
&\lesssim 2^{-2kM}.
\end{aligned}$$

Similarly, we see that $\mathrm{I}_{k,1} \lesssim 2^{-2kM}$. Let $\widetilde{p}_\Phi \in (0, p_\Phi^-)$ be such that $M > (1/\widetilde{p}_\Phi - 1/2)n/2$. Combining the above estimates and the fact that ρ is of upper-type $1/\widetilde{p}_\Phi - 1$, we finally conclude that

$$\begin{aligned}
&\frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \left[\int_B |(I - e^{-r_B^2 L})^M f(x)|^2 \, d\mu(x) \right]^{1/2} \\
&\quad \lesssim \sum_{k=0}^{\infty} 2^{-2kM} \frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \sigma_k(f, B) \\
&\quad \lesssim \sum_{k=0}^{\infty} 2^{-k[2M - n(1/\widetilde{p}_\Phi - 1/2)]} \frac{\sigma_k(f, B)}{\rho(\mu(2^k B))[\mu(2^k B)]^{1/2}}.
\end{aligned}$$

Since $(t^2L)^{M_1}e^{-t^2L}f \in T^\infty_{\Phi,\mathrm{v}}(\mathcal{X}) \subset T_\Phi(\mathcal{X})$, from $M > (1/\widetilde{p}_\Phi - 1/2)n/2$ and the dominated convergence theorem for series, we infer that

$$\begin{aligned}\gamma_1(f) &= \lim_{c\to 0}\sup_{B:r_B\le c}\frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}}\Big[\int_B |(I-e^{-r_B^2L})^M f(x)|^2\,d\mu(x)\Big]^{1/2}\\ &\lesssim \sum_{k=1}^\infty 2^{-k[2M-n(1/\widetilde{p}_\Phi-1/2)]}\lim_{c\to 0}\sup_{B:r_B\le c}\frac{\sigma_k(f,B)}{\rho(\mu(2^kB))[\mu(2^kB)]^{1/2}} = 0.\end{aligned}$$

Similarly, $\gamma_2(f)=\gamma_3(f)=0$, which implies that $f\in \mathrm{VMO}^M_{\rho,L}(\mathcal{X})$ and hence completes the proof of Theorem 3.4. □

REMARK 3.3

It follows from Theorem 3.4 that for all $M\in\mathbb{N}$ and $M>(1/p_\Phi^- - 1/2)n/2$, the spaces $\mathrm{VMO}^M_{\rho,L}(\mathcal{X})$ coincide with equivalent norms. Thus, in what follows, we denote the $\mathrm{VMO}^M_{\rho,L}(\mathcal{X})$ simply by $\mathrm{VMO}_{\rho,L}(\mathcal{X})$.

4. The dual space of $\mathrm{VMO}_{\rho,L}(\mathcal{X})$

In this section, we show that the dual space of $\mathrm{VMO}_{\rho,L}(\mathcal{X})$ is $B_{\Phi,L^*}(\mathcal{X})$, where the *space* $B_{\Phi,L^*}(\mathcal{X})$ denotes the Banach completion of the space $H_{\Phi,L^*}(\mathcal{X})$ (see Definition 4.3 and Theorem 4.2 below).

The proof of the following proposition is similar to that of [23, Proposition 4.1]; we omit the details here.

PROPOSITION 4.1

Let Φ satisfy Assumption (Φ). Then the dual space of $T_\Phi(\mathcal{X})$ is $T^\infty_\Phi(\mathcal{X})$. Moreover, the pairing

$$\langle f,g\rangle \to \int_{\mathcal{X}\times(0,\infty)} f(y,t)g(y,t)\,\frac{d\mu(y)\,dt}{t}$$

for all $f\in\widetilde{T}_\Phi(\mathcal{X})$ and $g\in T^\infty_\Phi(\mathcal{X})$ realizes $T^\infty_\Phi(\mathcal{X})$ as being equivalent to the dual of $T_\Phi(\mathcal{X})$.

We now introduce a new tent space $\widetilde{T}_\Phi(\mathcal{X})$ and present some properties.

DEFINITION 4.1

Let $p\in(0,1)$. The *space* $\widetilde{T}_\Phi(\mathcal{X})$ is defined to be the space of all $f=\sum_{j=1}^\infty \lambda_j a_j$ in $(T^\infty_\Phi(\mathcal{X}))^*$, where $\{a_j\}_{j=1}^\infty$ are $T_\Phi(\mathcal{X})$-atoms and $\{\lambda_j\}_{j=1}^\infty\subset\mathbb{C}$ such that $\sum_{j=1}^\infty|\lambda_j|<\infty$. If $f\in\widetilde{T}_\Phi(\mathcal{X})$, then define $\|f\|_{\widetilde{T}_\Phi(\mathcal{X})}\equiv\inf\{\sum_{j=1}^\infty|\lambda_j|\}$, where the infimum is taken over all the possible decompositions of f as above.

By [16, Lemma 3.1], $\widetilde{T}_\Phi(\mathcal{X})$ is a Banach space. Moreover, from Definition 4.1, it is easy to deduce that $T_\Phi(\mathcal{X})$ is dense in $\widetilde{T}_\Phi(\mathcal{X})$; in other words, $\widetilde{T}_\Phi(\mathcal{X})$ is a *Banach completion* of $T_\Phi(\mathcal{X})$.

LEMMA 4.1

Let Φ satisfy Assumption (Φ). Then $T_\Phi(\mathcal{X})$ is a dense subspace of $\widetilde{T}_\Phi(\mathcal{X})$, and there exists a positive constant C such that for all $f \in T_\Phi(\mathcal{X})$, $\|f\|_{\widetilde{T}_\Phi(\mathcal{X})} \le C\|f\|_{T_\Phi(\mathcal{X})}$.

Proof

Let $f \in T_\Phi(\mathcal{X})$. By Theorem 3.1, there exist $T_\Phi(\mathcal{X})$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that (3.1) and (3.2) hold.

For any $L \in \mathbb{N}$, set $\sigma_L \equiv \sum_{j=1}^L |\lambda_j|$. Since Φ is of upper-type 1, by this together with $\rho(t) = t^{-1}/\Phi^{-1}(t^{-1})$ for all $t \in (0,\infty)$, we obtain

$$\sum_{j=1}^\infty \mu(B_j)\Phi\Big(\frac{|\lambda_j|}{\sigma_L\mu(B_j)\rho(\mu(B_j))}\Big) \ge \sum_{j=1}^L \mu(B_j)\Phi\Big(\frac{1}{\sigma_L\mu(B_j)\rho(\mu(B_j))}\Big)\frac{|\lambda_j|}{\sigma_L} \gtrsim 1,$$

which implies that

$$\sum_{j=1}^L |\lambda_j| \lesssim \Lambda(\{\lambda_j a_j\}_{j=1}^\infty) \lesssim \|f\|_{T_\Phi(\mathcal{X})}.$$

Letting $L \to \infty$, we further conclude that $\sum_{j=1}^\infty |\lambda_j| \lesssim \|f\|_{T_\Phi(\mathcal{X})}$.

Since $f \in T_\Phi(\mathcal{X})$ and $(T_\Phi(\mathcal{X}))^* = T_\Phi^\infty(\mathcal{X})$, we see that

$$f \in T_\Phi(\mathcal{X}) \subset \big((T_\Phi(\mathcal{X}))^*\big)^* = \big(T_\Phi^\infty(\mathcal{X})\big)^*.$$

Thus, $f \in (T_\Phi^\infty(\mathcal{X}))^*$ and $\|f\|_{(T_\Phi^\infty(\mathcal{X}))^*} \lesssim \|f\|_{T_\Phi(\mathcal{X})}$. Recall that for any $\ell \in (T_\Phi^\infty(\mathcal{X}))^*$, its $(T_\Phi^\infty(\mathcal{X}))^*$-norm is defined by

$$\|\ell\|_{(T_\Phi^\infty(\mathcal{X}))^*} = \sup_{\|g\|_{T_\Phi^\infty(\mathcal{X})}\le 1} |\ell(g)|.$$

Observe also that $a_j \in (T_\Phi^\infty(\mathcal{X}))^*$ for all $j \in \mathbb{N}$. Now, from these observations, the monotone convergence theorem, and Hölder's inequality, it follows that

$$\begin{aligned}
&\Big\|f - \sum_{j=1}^L \lambda_j a_j\Big\|_{(T_\Phi^\infty(\mathcal{X}))^*}\\
&= \sup_{\|g\|_{T_\Phi^\infty(\mathcal{X})}\le 1}\Big|\int_{\mathcal{X}\times(0,\infty)}\Big[f(x,t) - \sum_{j=1}^L \lambda_j a_j(x,t)\Big]g(x,t)\,\frac{d\mu(x)\,dt}{t}\Big|\\
&\le \sup_{\|g\|_{T_\Phi^\infty(\mathcal{X})}\le 1}\int_{\mathcal{X}\times(0,\infty)}\sum_{j=L+1}^\infty |\lambda_j||a_j(x,t)g(x,t)|\,\frac{d\mu(x)\,dt}{t}\\
&= \sup_{\|g\|_{T_\Phi^\infty(\mathcal{X})}\le 1}\sum_{j=L+1}^\infty |\lambda_j|\int_{\widehat{B_j}}|a_j(x,t)g(x,t)|\,\frac{d\mu(x)\,dt}{t}\\
&\le \sup_{\|g\|_{T_\Phi^\infty(\mathcal{X})}\le 1}\sum_{j=L+1}^\infty |\lambda_j|\|a_j\|_{T_2^2(\mathcal{X})}\|g\chi_{\widehat{B_j}}\|_{T_2^2(\mathcal{X})} \le \sum_{j=L+1}^\infty |\lambda_j| \to 0,
\end{aligned}$$

as $L \to \infty$. Thus, the series in (3.1) converges in $(T_\Phi^\infty(\mathcal{X}))^*$, which further implies that $f \in \widetilde{T}_\Phi(\mathcal{X})$ and $\|f\|_{\widetilde{T}_\Phi(\mathcal{X})} \le \sum_{j=1}^\infty |\lambda_j| \lesssim \|f\|_{T_\Phi(\mathcal{X})}$. This finishes the proof of Lemma 4.1. □

LEMMA 4.2
Let Φ satisfy Assumption (Φ). Then $T_{2,b}^2(\mathcal{X})$ is dense in $\widetilde{T}_\Phi(\mathcal{X})$.

Proof
Since $T_\Phi(\mathcal{X})$ is dense in $\widetilde{T}_\Phi(\mathcal{X})$, to prove this lemma, it suffices to prove that $T_{2,b}^2(\mathcal{X})$ is dense in $T_\Phi(\mathcal{X})$ in the norm $\|\cdot\|_{\widetilde{T}_\Phi(\mathcal{X})}$.

Fix $x_0 \in \mathcal{X}$. For any $g \in T_\Phi(\mathcal{X})$ and $k \in \mathbb{N}$, let $g_k \equiv g\chi_{O_k}$, where

$$O_k \equiv \{(x,t) \in \mathcal{X} \times (0,\infty) : \mathrm{dist}(x,x_0) < k, t \in (1/k,k)\}.$$

By the dominated convergence theorem and the continuity of Φ, we conclude that for any $\lambda > 0$,

$$\lim_{k\to\infty} \int_{\mathcal{X}} \Phi\Big(\frac{\mathcal{A}(g-g_k)(x)}{\lambda}\Big)\, d\mu(x) = \int_{\mathcal{X}} \lim_{k\to\infty} \Phi\Big(\frac{\mathcal{A}(g-g_k)(x)}{\lambda}\Big)\, d\mu(x) = 0,$$

which implies that $\lim_{k\to\infty} \|g-g_k\|_{\widetilde{T}_\Phi(\mathcal{X})} = 0$. Then, by Lemma 4.1, we see that

$$\|g-g_k\|_{\widetilde{T}_\Phi(\mathcal{X})} \lesssim \|g-g_k\|_{T_\Phi(\mathcal{X})} \to 0,$$

as $k \to \infty$, which completes the proof of Lemma 4.2. □

LEMMA 4.3
Let Φ satisfy Assumption (Φ). Then $(\widetilde{T}_\Phi(\mathcal{X}))^* = T_\Phi^\infty(\mathcal{X})$ via the pairing

$$\langle f, g\rangle \to \int_{\mathcal{X}\times(0,\infty)} f(y,t)g(y,t)\, \frac{d\mu(y)\,dt}{t}$$

for all $f \in \widetilde{T}_\Phi(\mathcal{X})$ and $g \in T_\Phi^\infty(\mathcal{X})$.

Proof
By Proposition 4.1 and the definition of $\widetilde{T}_\Phi(\mathcal{X})$, we see that $(T_\Phi(\mathcal{X}))^* = T_\Phi^\infty(\mathcal{X})$ and $T_\Phi(\mathcal{X}) \subset \widetilde{T}_\Phi(\mathcal{X})$, which further implies that $(\widetilde{T}_\Phi(\mathcal{X}))^* \subset T_\Phi^\infty(\mathcal{X})$.

Conversely, let $g \in T_\Phi^\infty(\mathcal{X})$. Then for any $f \in \widetilde{T}_\Phi(\mathcal{X})$, choose a sequence of $T_\Phi(\mathcal{X})$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_j \lambda_j a_j$ in $(T_\Phi^\infty(\mathcal{X}))^*$ and $\sum_j |\lambda_j| \lesssim \|f\|_{\widetilde{T}_\Phi(\mathcal{X})}$. Thus, by Hölder's inequality, we obtain

$$\begin{aligned} |\langle f, g\rangle| &\le \sum_j \int_{\mathcal{X}\times(0,\infty)} |a_j(x,t)g(x,t)|\, \frac{d\mu(x)\,dt}{t} \\ &\le \|g\|_{T_\Phi^\infty(\mathcal{X})} \sum_j |\lambda_j| \lesssim \|g\|_{T_\Phi^\infty(\mathcal{X})} \|f\|_{\widetilde{T}_\Phi(\mathcal{X})}, \end{aligned}$$

which implies that $g \in (\widetilde{T}_\Phi(\mathcal{X}))^*$ and hence completes the proof of Lemma 4.3. □

LEMMA 4.4

Let Φ satisfy Assumption (Φ). If $f \in \widetilde{T}_\Phi(\mathcal{X})$, then

$$\|f\|_{\widetilde{T}_\Phi(\mathcal{X})} = \sup_{g \in T^2_{2,b}(\mathcal{X}), \|g\|_{T^\infty_\Phi(\mathcal{X})} \le 1} \left| \int_{\mathcal{X} \times (0,\infty)} f(x,t) g(x,t) \,\frac{d\mu(x)\,dt}{t} \right|. \tag{4.1}$$

Proof

Let $f \in \widetilde{T}_\Phi(\mathcal{X})$. From Lemma 4.2, we deduce that

$$\|f\|_{\widetilde{T}_\Phi(\mathcal{X})} = \sup_{\|g\|_{T^\infty_\Phi(\mathcal{X})} \le 1} \left| \int_{\mathcal{X} \times (0,\infty)} f(x,t) g(x,t) \,\frac{d\mu(x)\,dt}{t} \right|.$$

Thus, for any $\beta > 0$, there exists $g \in T^\infty_\Phi(\mathcal{X})$ such that $\|g\|_{T^2_{2,b}(\mathcal{X})} \le 1$ and

$$\left| \int_{\mathcal{X} \times (0,\infty)} f(x,t) g(x,t) \,\frac{d\mu(x)\,dt}{t} \right| \ge \|f\|_{\widetilde{T}_\Phi(\mathcal{X})} - \frac{\beta}{2}.$$

Observe here that $fg \in L^1(\mathcal{X} \times (0,\infty))$. Fix $x_0 \in \mathcal{X}$. Let

$$O_k \equiv \big\{(x,t) \in \mathcal{X} \times (0,\infty) : \mathrm{dist}(x,x_0) < k, 1/k < t < k\big\}.$$

Then there exists $k \in \mathbb{N}$ such that

$$\left| \int_{\mathcal{X} \times (0,\infty)} f(x,t) g(x,t) \chi_{O_k} \,\frac{d\mu(x)\,dt}{t} \right| \ge \|f\|_{\widetilde{T}_\Phi(\mathcal{X})} - \beta.$$

Obviously, $g\chi_{O_k} \in T^2_{2,b}(\mathcal{X})$. Thus, (4.1) holds, which completes the proof of Lemma 4.4. □

The following lemma is a slight modification of [8, Lemma 4.2]; see also [22]. We omit the details here.

LEMMA 4.5

Let Φ satisfy Assumption (Φ). Suppose that $\{f_k\}_{k=1}^\infty$ is a bounded family of functions in $\widetilde{T}_\Phi(\mathcal{X})$. Then there exist $f \in \widetilde{T}_\Phi(\mathcal{X})$ and a subsequence $\{f_{k_j}\}_{j=1}^\infty$ of $\{f_k\}_{k=1}^\infty$ such that for all $g \in T^2_{2,b}(\mathcal{X})$,

$$\lim_{j\to\infty} \int_{\mathcal{X} \times (0,\infty)} f_{k_j}(x,t) g(x,t) \,\frac{d\mu(x)\,dt}{t} = \int_{\mathcal{X} \times (0,\infty)} f(x,t) g(x,t) \,\frac{d\mu(x)\,dt}{t}.$$

THEOREM 4.1

Let Φ satisfy Assumption (Φ). Then $(T^\infty_{\Phi,\mathrm{v}}(\mathcal{X}))^*$, the dual space of the space $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$, coincides with $\widetilde{T}_\Phi(\mathcal{X})$ in the following sense.

For any $g \in \widetilde{T}_\Phi(\mathcal{X})$, define the linear function ℓ by setting, for all $f \in T^\infty_\Phi(\mathcal{X})$,

$$\ell(f) \equiv \int_{\mathcal{X} \times (0,\infty)} f(x,t) g(x,t) \,\frac{d\mu(x)\,dt}{t}. \tag{4.2}$$

Then there exists a positive constant C, independent of g, such that

$$\|\ell\|_{(T^\infty_\Phi(\mathcal{X}))^*} \le C\|g\|_{\widetilde{T}_\Phi(\mathcal{X})}.$$

Conversely, for any $\ell \in (T_\Phi^\infty(\mathcal{X}))^*$, there exists $g \in \widetilde{T}_\Phi(\mathcal{X})$ such that (4.2) holds for all $f \in T_\Phi^\infty(\mathcal{X})$ and $\|g\|_{\widetilde{T}_\Phi(\mathcal{X})} \le C\|\ell\|_{(T_\Phi^\infty(\mathcal{X}))^*}$, where C is a positive constant independent of ℓ.

Proof
From Lemma 4.2, we infer that $T_{\Phi,\mathrm{v}}^\infty(\mathcal{X}) \subset T_\Phi^\infty(\mathcal{X}) = (\widetilde{T}_\Phi(\mathcal{X}))^*$, which further implies that $\widetilde{T}_\Phi(\mathcal{X}) \subset (\widetilde{T}_\Phi(\mathcal{X}))^* \subset (T_{\Phi,\mathrm{v}}^\infty(\mathcal{X}))^*$.

Conversely, let $\ell \in (T_{\Phi,\mathrm{v}}^\infty(\mathcal{X}))^*$. Notice that for any $f \in T_{2,b}^2(\mathcal{X})$, without loss of generality, we may assume that $\operatorname{supp} f \subset K$, where K is a bounded set in $\mathcal{X} \times (0,\infty)$. Then we have $\|f\|_{T_{\Phi,\mathrm{v}}^\infty(\mathcal{X})} = \|f\|_{T_\Phi^\infty(\mathcal{X})} \le C(K)\|f\|_{T_{2,b}^2(\mathcal{X})}$. Thus, ℓ induces a bounded linear functional on $T_{2,b}^2(\mathcal{X})$. Let O_k be as in the proof of Lemma 4.4. By the Riesz representation theorem, there exists a unique $g_k \in L^2(O_k)$ such that for all $f \in L^2(O_k)$,

$$\ell(f) = \int_{\mathcal{X}\times(0,\infty)} f(x,t) g_k(x,t)\,\frac{d\mu(x)\,dt}{t}.$$

Obviously, $g_{k+1}O_k = g_k$ for all $k \in \mathbb{N}$. Let $g \equiv g_1\chi_{O_1} + \sum_{k=2}^\infty g_k\chi_{O_k\setminus O_{k-1}}$. Then $g \in L_{\mathrm{loc}}^2(\mathcal{X} \times (0,\infty))$, and for any $f \in T_{2,b}^2(\mathcal{X})$, we have

$$\ell(f) = \int_{\mathcal{X}\times(0,\infty)} f(y,t) g(y,t)\,\frac{d\mu(y)\,dt}{t}.$$

Set $\widetilde{g}_k \equiv g\chi_{O_k}$. Then for each $k \in \mathbb{N}$, obviously, we see that $\widetilde{g}_k \in T_{2,b}^2(\mathcal{X}) \subset T_\Phi(\mathcal{X}) \subset \widetilde{T}_\Phi(\mathcal{X})$. Then from Lemma 4.4, it follows that

$$\begin{aligned}
\|\widetilde{g}_k\|_{\widetilde{T}_\Phi(\mathcal{X})} &= \sup_{f\in T_{2,b}^2(\mathcal{X}),\|f\|_{T_\Phi^\infty(\mathcal{X})}\le 1} \left|\int_{\mathcal{X}\times(0,\infty)} f(x,t)g(x,t)\chi_{O_k}(x,t)\,\frac{d\mu(x)\,dt}{t}\right| \\
&= \sup_{f\in T_{2,b}^2(\mathcal{X}),\|f\|_{T_\Phi^\infty(\mathcal{X})}\le 1} |\ell(f\chi_{O_k})| \\
&\le \sup_{f\in T_{2,b}^2(\mathcal{X}),\|f\|_{T_\Phi^\infty(\mathcal{X})}\le 1} \|\ell\|_{(T_{\Phi,\mathrm{v}}^\infty(\mathcal{X}))^*}\|f\|_{T_\Phi^\infty(\mathcal{X})} \le \|\ell\|_{(T_{\Phi,\mathrm{v}}^\infty(\mathcal{X}))^*}.
\end{aligned}$$

Thus, by Lemma 4.5, there exist $\widetilde{g} \in \widetilde{T}_\Phi(\mathcal{X})$ and $\{\widetilde{g}_{k_j}\}_{j=1}^\infty \subset \{\widetilde{g}_k\}_{k=1}^\infty$ such that for all $f \in T_{2,b}^2(\mathcal{X})$,

$$\lim_{j\to\infty}\int_{\mathcal{X}\times(0,\infty)} f(x,t)\widetilde{g}_{k_j}(x,t)\,\frac{d\mu(x)\,dt}{t} = \int_{\mathcal{X}\times(0,\infty)} f(x,t)\widetilde{g}(x,t)\,\frac{d\mu(x)\,dt}{t}.$$

On the other hand, notice that for sufficient large k_j, we have

$$\begin{aligned}
\ell(f) &= \int_{\mathcal{X}\times(0,\infty)} f(x,t)g(x,t)\,\frac{d\mu(x)\,dt}{t} \\
&= \int_{\mathcal{X}\times(0,\infty)} f(x,t)\widetilde{g}_{k_j}(x,t)\,\frac{d\mu(x)\,dt}{t} = \int_{\mathcal{X}\times(0,\infty)} f(x,t)\widetilde{g}(x,t)\,\frac{d\mu(x)\,dt}{t},
\end{aligned}$$

which implies that $g = \widetilde{g}$ almost everywhere, and hence $g \in \widetilde{T}_\Phi(\mathcal{X})$. By a density argument, we conclude that (4.2) also holds for g and all $f \in T_\Phi^\infty(\mathcal{X})$, which completes the proof of Theorem 4.1. □

DEFINITION 4.2

Let L satisfy Assumptions $(L)_1$ and $(L)_2$, let Φ satisfy Assumption (Φ), let $M \in \mathbb{N}$, $M > (1/p_\Phi^- - 1/2)n/2$, and let $\epsilon \in (n(1/p_\Phi^- - 1/p_\Phi^+), \infty)$. An element $f \in (\mathrm{BMO}_{\rho,L^*}(\mathcal{X}))^*$ is said to be in the *space* $H_{\Phi,L}^{M,\epsilon}(\mathcal{X})$ if there exist $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and $(\Phi, M, \epsilon)_L$-molecules $\{\alpha_j\}_{j=1}^\infty$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $(\mathrm{BMO}_{\rho,L^*}(\mathcal{X}))^*$ and

$$\Lambda(\{\lambda_j \alpha_j\}_{j=1}^\infty) \equiv \inf\Big\{\lambda > 0 : \sum_{j=1}^\infty \mu(B_j)\Phi\Big(\frac{|\lambda_j|}{\lambda\mu(B_j)\rho(\mu(B_j))}\Big) \le 1\Big\} < \infty,$$

where for each j, α_j is adapted to the ball B_j.

If $f \in H_{\Phi,L}^{M,\epsilon}(\mathcal{X})$, then its *norm* is defined by $\|f\|_{H_{\Phi,L}^{M,\epsilon}(\mathcal{X})} \equiv \inf\{\Lambda(\{\lambda_j\alpha_j\}_{j=1}^\infty)\}$, where the infimum is taken over all the possible decompositions of f as above.

By [21, Theorem 5.1], we see that for all $M > (1/p_\Phi^- - 1/2)n/2$ and $\epsilon \in (n(1/p_\Phi^- - 1/p_\Phi^+), \infty)$, the *spaces* $H_{\Phi,L}(\mathcal{X})$ *and* $H_{\Phi,L}^{M,\epsilon}(\mathcal{X})$ *coincide with equivalent norms.*

Let us introduce the Banach completion of the space $H_{\Phi,L}(\mathcal{X})$.

DEFINITION 4.3

Let L satisfy Assumptions $(L)_1$ and $(L)_2$, let Φ satisfy Assumption (Φ), let $\epsilon \in (n(1/p_\Phi^- - 1/p_\Phi^+), \infty)$, and let $M > (1/p_\Phi^- - 1/2)n/2$. The *space* $B_{\Phi,L}^{M,\epsilon}(\mathcal{X})$ is defined to be the space of all $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $(\mathrm{BMO}_{\rho,L^*}(\mathcal{X}))^*$, where $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ with $\sum_{j=1}^\infty |\lambda_j| < \infty$ and $\{\alpha_j\}_{j=1}^\infty$ are $(\Phi, M, \epsilon)_L$-molecules. If $f \in B_{\Phi,L}^{M,\epsilon}(\mathcal{X})$, define $\|f\|_{B_{\Phi,L}^{M,\epsilon}(\mathcal{X})} \equiv \inf\{\sum_{j=1}^\infty |\lambda_j|\}$, where the infimum is taken over all the possible decompositions of f as above.

By [16, Lemma 3.1], we know that $B_{\Phi,L}^{M,\epsilon}(\mathcal{X})$ is a Banach space. Moreover, from Definition 4.2, it is easy to deduce that $H_{\Phi,L}(\mathcal{X})$ is dense in $B_{\Phi,L}^{M,\epsilon}(\mathcal{X})$. More precisely, we have the following lemma.

LEMMA 4.6

Let L satisfy Assumptions $(L)_1$ and $(L)_2$, let Φ satisfy Assumption (Φ), let $\epsilon \in (n(1/p_\Phi^- - 1/p_\Phi^+), \infty)$, and let $M > (1/p_\Phi^- - 1/2)n/2$. Then

(i) $H_{\Phi,L}(\mathcal{X}) \subset B_{\Phi,L}^{M,\epsilon}(\mathcal{X})$ *and the inclusion is continuous;*

(ii) *for any $\epsilon_1 \in (n(1/p_\Phi^- - 1/p_\Phi^+), \infty)$ and $M_1 > (1/p_\Phi^- - 1/2)n/2$, the spaces $B_{\Phi,L}^{M,\epsilon}(\mathcal{X})$ and $B_{\Phi,L}^{M_1,\epsilon_1}(\mathcal{X})$ coincide with equivalent norms.*

Proof

From Definition 4.3 and the molecular characterization of $H_{\Phi,L}(\mathcal{X})$, it is easy to deduce (i).

Let us prove (ii). By symmetry, it suffices to show that $B_{\Phi,L}^{M,\epsilon}(\mathcal{X}) \subset B_{\Phi,L}^{M_1,\epsilon_1}(\mathcal{X})$. Let $f \in B_{\Phi,L}^{M,\epsilon}(\mathcal{X})$. By Definition 4.3, there exist $(\Phi, M, \epsilon)_L$-molecules $\{\alpha_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in $(\mathrm{BMO}_{\rho,L^*}(\mathcal{X}))^*$ and $\sum_{j=1}^\infty |\lambda_j| \lesssim$

$\|f\|_{B^{M,\epsilon}_{\Phi,L}(\mathcal{X})}$. By (i), for each $j \in \mathbb{N}$, we see that $\alpha_j \in H_{\Phi,L}(\mathcal{X}) \subset B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$ and $\|\alpha_j\|_{B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})} \lesssim \|\alpha_j\|_{H_{\Phi,L}(\mathcal{X})} \lesssim 1$. Since $B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$ is a Banach space, we see that $f \in B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$ and $\|f\|_{B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})} \le \sum_{j=1}^{\infty} |\lambda_j| \|\alpha_j\|_{B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})} \lesssim \|f\|_{B^{M,\epsilon}_{\Phi,L}(\mathcal{X})}$. Thus, $B^{M,\epsilon}_{\Phi,L}(\mathcal{X}) \subset B^{M_1,\epsilon_1}_{\Phi,L}(\mathcal{X})$, which completes the proof of Lemma 4.6. □

Since the spaces $B^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ coincide for all $\epsilon \in (n(1/p_\Phi^- - 1/p_\Phi^+), \infty)$ and $M > (1/p_\Phi^- - 1/2)n/2$, in what follows, we denote $B^{M,\epsilon}_{\Phi,L}(\mathcal{X})$ simply by $B_{\Phi,L}(\mathcal{X})$.

LEMMA 4.7

Let L satisfy Assumptions $(L)_1$ and $(L)_2$, and let Φ satisfy Assumption (Φ). Then $(B_{\Phi,L}(\mathcal{X}))^* = \mathrm{BMO}_{\rho,L^*}(\mathcal{X})$.

Proof

Since $(H_{\Phi,L}(\mathcal{X}))^* = \mathrm{BMO}_{\rho,L^*}(\mathcal{X})$ and $H_{\Phi,L}(\mathcal{X}) \subset B_{\Phi,L}(\mathcal{X})$, by duality, we conclude that $(B_{\Phi,L}(\mathcal{X}))^* \subset \mathrm{BMO}_{\rho,L^*}(\mathcal{X})$.

Conversely, let $\epsilon \in (n(1/p_\Phi^- - 1/p_\Phi^+), \infty)$, $M > (1/p_\Phi^- - 1/2)n/2$, and $f \in \mathrm{BMO}_{\rho,L^*}(\mathcal{X})$. For any $g \in B_{\Phi,L}(\mathcal{X})$, by Definition 4.3, there exist $(\Phi, M, \epsilon)_L$-molecules $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ such that $g = \sum_{j=1}^{\infty} \lambda_j \alpha_j$ in $(\mathrm{BMO}_{\rho,L^*}(\mathcal{X}))^*$ and $\sum_{j=1}^{\infty} |\lambda_j| \lesssim \|g\|_{B_{\Phi,L}(\mathcal{X})}$. Thus,

$$\begin{aligned}|\langle f, g\rangle| &\le \sum_{j=1}^{\infty} |\lambda_j| |\langle f, \alpha_j\rangle| \lesssim \sum_{j=1}^{\infty} |\lambda_j| \|f\|_{\mathrm{BMO}_{\rho,L^*}(\mathcal{X})} \|\alpha_j\|_{H_{\Phi,L}(\mathcal{X})} \\ &\lesssim \|f\|_{\mathrm{BMO}_{\rho,L^*}(\mathcal{X})} \|g\|_{B_{\Phi,L}(\mathcal{X})},\end{aligned}$$

which implies that $f \in (B_{\Phi,L}(\mathcal{X}))^*$ and hence completes the proof of Lemma 4.7. □

Let $M \in \mathbb{N}$. For all $F \in L^2(\mathcal{X} \times (0,\infty))$ with bounded support, define

$$\pi_{L,M} F \equiv C(M) \int_0^{\infty} (t^2 L)^M e^{-t^2 L} F(\cdot, t) \, \frac{dt}{t}, \tag{4.3}$$

where $C(M)$ is as in (3.5).

PROPOSITION 4.2

Let L satisfy Assumptions $(L)_1$ and $(L)_2$, let Φ satisfy Assumption (Φ), and let $M \in \mathbb{N}$. Then the operator $\pi_{L,M}$, initially defined on $T^2_{2,b}(\mathcal{X})$, extends to a bounded linear operator

- (i) *from $T^2_2(\mathcal{X})$ to $L^2(\mathcal{X})$;*
- (ii) *from $T_\Phi(\mathcal{X})$ to $H_{\Phi,L}(\mathcal{X})$, if $M > (1/p_\Phi^- - 1/2)n/2$;*
- (iii) *from $\widetilde{T}_\Phi(\mathcal{X})$ to $B_{\Phi,L}(\mathcal{X})$, if $M > (1/p_\Phi^- - 1/2)n/2$;*
- (iv) *from $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$ to $\mathrm{VMO}_{\rho,L}(\mathcal{X})$.*

Proof
Conclusions (i) and (ii) were established in [2, Proposition 3.6] (see also [21, Lemma 3.1]).

By Lemma 4.2, we know that $T^2_{2,b}(\mathcal{X})$ is dense in $\widetilde{T}_\Phi(\mathcal{X})$. Let $f\in T^2_{2,b}(\mathcal{X})$. From (ii) and Lemma 4.6, we deduce that $\pi_{L,M}f\in H_{\Phi,L}(\mathcal{X})\subset B_{\Phi,L}(\mathcal{X})$. Moreover, by Definition 4.1, there exist $T_\Phi(\mathcal{X})$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty\subset\mathbb{C}$ such that $f=\sum_{j=1}^\infty\lambda_j a_j$ in $(T^\infty_\Phi(\mathcal{X}))^*$ and $\sum_j|\lambda_j|\lesssim\|f\|_{\widetilde{T}_\Phi(\mathcal{X})}$. In addition, for any $g\in \mathrm{BMO}_{\rho,L^*}(\mathcal{X})$, we have $(t^2L^*)^Me^{-t^2L^*}g\in T^\infty_\Phi(\mathcal{X})$. Thus, by $(T_\Phi(\mathcal{X}))^*=T^\infty_\Phi(\mathcal{X})$, we conclude that

$$\begin{aligned}\langle\pi_{L,M}(f),g\rangle&=C(M)\int_{\mathcal{X}\times(0,\infty)}f(x,t)\overline{(t^2L^*)^Me^{-t^2L^*}g(x)}\,\frac{d\mu(x)\,dt}{t}\\&=\sum_{j=1}^\infty\lambda_jC(M)\int_{\mathcal{X}\times(0,\infty)}a_j(x,t)\overline{(t^2L^*)^Me^{-t^2L^*}g(x)}\,\frac{d\mu(x)\,dt}{t}\\&=\sum_{j=1}^\infty\lambda_j\langle\pi_{L,M}(a_j),g\rangle,\end{aligned}$$

which implies that $\pi_{L,M}(f)=\sum_{j=1}^\infty\lambda_j\pi_{L,M}(a_j)$ in $(\mathrm{BMO}_{\rho,L^*}(\mathcal{X}))^*$. By (ii), we further conclude that

$$\begin{aligned}\|\pi_{L,M}(f)\|_{B_{\Phi,L}(\mathcal{X})}&\le\sum_{j=1}^\infty|\lambda_j|\|\pi_{L,M}(a_j)\|_{B_{\Phi,L}(\mathcal{X})}\\&\lesssim\sum_{j=1}^\infty|\lambda_j|\|\pi_{L,M}(a_j)\|_{H_{\Phi,L}(\mathcal{X})}\lesssim\|f\|_{\widetilde{T}_\Phi(\mathcal{X})}.\end{aligned}$$

Since $T^2_{2,b}(\mathcal{X})$ is dense in $\widetilde{T}_\Phi(\mathcal{X})$, we see that $\pi_{L,M}$ extends to a bounded linear operator from $\widetilde{T}_\Phi(\mathcal{X})$ to $B_{\Phi,L}(\mathcal{X})$, which completes the proof of (iii).

Let us now prove (iv). From Lemma 3.3, we infer that $T^2_{2,b}(\mathcal{X})$ is dense in $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$. Thus, to prove (iv), it suffices to show that $\pi_{L,M}$ maps $T^2_{2,b}(\mathcal{X})$ continuously into $\mathrm{VMO}_{\rho,L}(\mathcal{X})$.

Let $f\in T^2_{2,b}(\mathcal{X})$. By (i), we see that $\pi_{L,M}f\in L^2(\mathcal{X})$. Notice that (3.3) and (3.4) with L and L^* exchanged imply that $L^2(\mathcal{X})\subset\mathcal{M}^{M_1}_{\Phi,L}(\mathcal{X})$, when $M_1\in\mathbb{N}$ and $M_1>(1/p^-_\Phi-1/2)n/2$. Thus, $\pi_{L,M}f\in\mathcal{M}^{M_1}_{\Phi,L}(\mathcal{X})$. To show $\pi_{L,M}f\in\mathrm{VMO}_{\rho,L}(\mathcal{X})$, by Theorem 3.4, we still need to show that $(t^2L)^{M_1}e^{-t^2L}\pi_{L,M}f\in T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$.

For any ball $B\equiv B(x_B,r_B)$, let $V_0(B)\equiv\widehat{B}$ and $V_k(B)\equiv\widehat{(2^kB)}\backslash\widehat{(2^{k-1}B)}$ for any $k\in\mathbb{N}$. For all $k\in\mathbb{Z}_+$, let $f_k\equiv f\chi_{V_k(B)}$. Thus, for $k\in\{0,1,2\}$, by Lemma 2.2 and (i), we see that

$$\left[\iint_{\widehat{B}}|(t^2L)^{M_1}e^{-t^2L}\pi_{L,M}f_k(x)|^2\,\frac{d\mu(x)\,dt}{t}\right]^{1/2}\lesssim\|\pi_{L,M}f_k\|_{L^2(\mathcal{X})}\lesssim\|f_k\|_{T^2_2(\mathcal{X})}.$$

For $k\ge3$, let $V_{k,1}(B)\equiv\widehat{(2^kB)}\backslash(2^{k-2}B\times(0,\infty))$ and $V_{k,2}(B)\equiv V_k(B)\backslash V_{k,1}(B)$. We further write $f_k=f_k\chi_{V_{k,1}(B)}+f_k\chi_{V_{k,2}(B)}\equiv f_{k,1}+f_{k,2}$. From Minkowski's inequality, Lemma 2.3, and Hölder's inequality, we deduce that

$$
\begin{aligned}
&\Big[\iint_{\widehat{B}} |(t^2L)^{M_1} e^{-t^2L}\pi_{L,M} f_{k,2}(x)|^2 \,\frac{d\mu(x)\,dt}{t}\Big]^{1/2} \\
&\quad\sim \Big[\iint_{\widehat{B}} \Big|\int_{2^{k-2}r_B}^{2^k r_B} (t^2L)^{M_1} e^{-t^2L}(s^2L)^M e^{-s^2L}\big(f_{k,2}(\cdot,s)\big)(x)\,\frac{ds}{s}\Big|^2 \,\frac{d\mu(x)\,dt}{t}\Big]^{1/2} \\
&\quad\lesssim \int_{2^{k-2}r_B}^{2^k r_B} \Big[\iint_{\widehat{B}} |t^{2M_1}s^{2M}L^{M+M_1}e^{-(s^2+t^2)L}\big(f_{k,2}(\cdot,s)\big)(x)|^2 \,\frac{d\mu(x)\,dt}{t}\Big]^{1/2} \frac{ds}{s} \\
&\quad\lesssim \int_{2^{k-2}r_B}^{2^k r_B} \Big[\int_0^{r_B} \Big|\frac{t^{2M_1}s^{2M}}{(s^2+t^2)^{M+M_1}}\Big|^2 \|f_{k,2}(\cdot,s)\|_{L^2(\mathcal{X})}^2 \,\frac{dt}{t}\Big]^{1/2} \frac{ds}{s} \\
&\quad\lesssim 2^{-2kM_1}\int_{2^{k-2}r_B}^{2^k r_B} \|f_{k,2}(\cdot,s)\|_{L^2(\mathcal{X})}\,\frac{ds}{s} \lesssim 2^{-2kM_1}\|f_{k,2}\|_{T_2^2(\mathcal{X})}.
\end{aligned}
$$

Similarly, we have

$$
\Big[\iint_{\widehat{B}} |(t^2L)^{M_1} e^{-t^2L}\pi_{L,M} f_{k,1}(x)|^2 \,\frac{d\mu(x)\,dt}{t}\Big]^{1/2} \lesssim 2^{-2kM_1}\|f_{k,1}\|_{T_2^2(\mathcal{X})}.
$$

Let $\widetilde{p}_\Phi \in (0, p_\Phi^-)$ be such that $M > (1/\widetilde{p}_\Phi - 1/2)n/2$ and $M_1 > (1/\widetilde{p}_\Phi - 1/2)n/2$. Combining the above estimates, since Φ is of lower-type $\widetilde{p}_\Phi$, we finally conclude that

$$
\begin{aligned}
&\frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \Big[\iint_{\widehat{B}} |(t^2L)^{M_1} e^{-t^2L}\pi_{L,M} f(x)|^2 \,\frac{d\mu(x)\,dt}{t}\Big]^{1/2} \\
&\quad\lesssim \sum_{k=0}^{2} \frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \Big[\iint_{\widehat{B}} |(t^2L)^{M_1} e^{-t^2L}\pi_{L,M} f_k(x)|^2 \,\frac{d\mu(x)\,dt}{t}\Big]^{1/2} \\
&\qquad + \sum_{k=3}^{\infty}\sum_{i=1}^{2} \frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \\
&\qquad \times \Big[\iint_{\widehat{B}} |(t^2L)^{M_1} e^{-t^2L}\pi_{L,M} f_{k,i}(x)|^2 \,\frac{d\mu(x)\,dt}{t}\Big]^{1/2} \\
&\quad\lesssim \sum_{k=0}^{2} \frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}} \|f_k\|_{T_2^2(\mathcal{X})} \\
&\qquad + \sum_{k=3}^{\infty}\sum_{i=1}^{2} \frac{2^{-2kM_1}}{\rho(\mu(B))[\mu(B)]^{1/2}} \|f_{k,i}\|_{T_2^2(\mathcal{X})} \\
&\quad\lesssim \sum_{k=0}^{\infty} 2^{-2k[M_1-(1/\widetilde{p}_\Phi-1/2)n/2]} \frac{1}{\rho(\mu(2^kB))[\mu(2^kB)]^{1/2}} \|f_k\|_{T_2^2(\mathcal{X})}.
\end{aligned}
$$

Since $f \in T_{\Phi,\mathrm{v}}^{\infty}(\mathcal{X}) \subset T_{\Phi}^{\infty}(\mathcal{X})$, we have

$$
\frac{1}{\rho(\mu(2^kB))[\mu(2^kB)]^{1/2}} \|f_k\|_{T_2^2(\mathcal{X})} \lesssim \|f\|_{T_\Phi^\infty(\mathcal{X})}
$$

and, for all fixed $k\in\mathbb{N}$,

$$\lim_{c\to 0}\sup_{B:r_B\le c}\frac{\|f_k\|_{T_2^2(\mathcal{X})}}{\rho(\mu(2^kB))[\mu(2^kB)]^{1/2}}=\lim_{c\to\infty}\sup_{B:r_B\ge c}\frac{\|f_k\|_{T_2^2(\mathcal{X})}}{\rho(\mu(2^kB))[\mu(2^kB)]^{1/2}}$$
$$=\lim_{c\to\infty}\sup_{B:B\subset[B(0,c)]^{\complement}}\frac{\|f_k\|_{T_2^2(\mathcal{X})}}{\rho(\mu(2^kB))[\mu(2^kB)]^{1/2}}=0.$$

Thus, by the dominated convergence theorem for series, we further conclude that

$$\begin{aligned}&\eta_1\big((t^2L)^{M_1}e^{-t^2L}\pi_{L,M}f\big)\\&\quad=\lim_{c\to 0}\sup_{B:r_B\le c}\frac{1}{\rho(\mu(B))[\mu(B)]^{1/2}}\bigg[\iint_{\widehat{B}}|(t^2L)^{M_1}e^{-t^2L}\pi_{L,M}f(x)|^2\,\frac{d\mu(x)\,dt}{t}\bigg]^{1/2}\\&\quad\lesssim\sum_{k=0}^{\infty}2^{-2k[M_1-(1/\widetilde{p}_\Phi-1/2)n/2]}\lim_{c\to 0}\sup_{B:r_B\le c}\frac{\|f_k\|_{T_2^2(\mathcal{X})}}{\rho(\mu(2^kB))[\mu(2^kB)]^{1/2}}=0.\end{aligned}$$

Similarly, we have $\eta_2((t^2L)^{M_1}e^{-t^2L}\pi_{L,M}f)=\eta_3((t^2L)^{M_1}e^{-t^2L}\pi_{L,M}f)=0$, and hence $(t^2L)^{M_1}e^{-t^2L}\pi_{L,M}f\in T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$, which completes the proof of Proposition 4.2. □

LEMMA 4.8

Let L satisfy Assumptions $(L)_1$ and $(L)_2$, and let Φ satisfy Assumption (Φ). Then $\mathrm{VMO}_{\rho,L}(\mathcal{X})\cap L^2(\mathcal{X})$ is dense in $\mathrm{VMO}_{\rho,L}(\mathcal{X})$.

Proof

Let $f\in\mathrm{VMO}_{\rho,L}(\mathcal{X})$ and $M>(1/p_\Phi^- -1/2)n/2$. Then by Theorem 3.4, we have $h\equiv(t^2L)^Me^{-t^2L}f\in T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$. Similarly to the proof of Proposition 4.2, by Lemma 3.3, there exist $\{h_k\}_{k\in\mathbb{N}}\subset T^2_{2,b}(\mathcal{X})\subset T^{\infty}_{\Phi,\mathrm{v}}(\mathcal{X})$ such that $\|h-h_k\|_{T^\infty_\Phi(\mathcal{X})}\to 0$, as $k\to\infty$. Thus, by (i) and (iv) of Proposition 4.2, we see that $\pi_{L,1}h_k\in L^2(\mathcal{X})\cap\mathrm{VMO}_{\rho,L}(\mathcal{X})$ and

$$\|\pi_{L,1}(h-h_k)\|_{\mathrm{BMO}_{\rho,L}(\mathcal{X})}\lesssim\|h-h_k\|_{T^\infty_\Phi(\mathcal{X})}\to 0,\tag{4.4}$$

as $k\to\infty$.

Let α be a $(\Phi,M,\epsilon)_L$-molecule. Then by the definition of $H_{\Phi,L}(\mathcal{X})$, we know that $e^{-t^2L}\alpha\in T_\Phi(\mathcal{X})$, which, together with Lemma 3.2, the fact that $(T_\Phi(\mathcal{X}))^*=T^\infty_\Phi(\mathcal{X})$, and $(H_{\Phi,L}(\mathcal{X}))^*=\mathrm{BMO}_{\rho,L}(\mathcal{X})$, further implies that

$$\begin{aligned}\langle f,\alpha\rangle&=C(M)\iint_{\mathcal{X}\times(0,\infty)}(t^2L)^Me^{-t^2L}f(x)\overline{t^2L^*e^{-t^2L^*}\alpha(x)}\,\frac{d\mu(x)\,dt}{t}\\&=\lim_{k\to\infty}C(M)\iint_{\mathcal{X}\times(0,\infty)}h_k(x)\overline{t^2L^*e^{-t^2L^*}\alpha(x)}\,\frac{d\mu(x)\,dt}{t}\\&=\frac{C(M)}{C_1}\lim_{k\to\infty}\int_{\mathcal{X}}\big(\pi_{L,1}h_k(x)\big)\overline{\alpha(x)}\,d\mu(x)=\frac{C(M)}{C_1}\langle\pi_{L,1}h,\alpha\rangle.\end{aligned}$$

Since the set of finite combinations of molecules is dense in $H_{\Phi,L}(\mathcal{X})$, we then see that $f=(C(M)/C_1)\pi_{L,1}h$ in $\mathrm{BMO}_{\rho,L}(\mathcal{X})$.

Now, for each $k \in \mathbb{N}$, let $f_k \equiv (C(M)/C_1)\pi_{L,1}h_k$. Then $f_k \in \mathrm{VMO}_{\rho,L}(\mathcal{X}) \cap L^2(\mathcal{X})$, and, moreover, by (4.4), we have $\|f - f_k\|_{\mathrm{BMO}_{\rho,L}(\mathcal{X})} \to 0$, as $k \to \infty$, which completes the proof of Lemma 4.8. $\square$

The *symbol* $\langle\cdot,\cdot\rangle$ in the following theorem means the duality between the space $\mathrm{BMO}_{\rho,L}(\mathcal{X})$ and the space $B_{\Phi,L^*}(\mathcal{X})$ in the sense of Lemma 4.7 with L and L^* exchanged.

THEOREM 4.2

Let L satisfy Assumptions $(L)_1$ and $(L)_2$, and let Φ satisfy Assumption (Φ). Then the dual space of $\mathrm{VMO}_{\rho,L}(\mathcal{X})$, $(\mathrm{VMO}_{\rho,L}(\mathcal{X}))^*$, coincides with the space $B_{\Phi,L^*}(\mathcal{X})$ in the following sense.

For any $g \in B_{\Phi,L^*}(\mathcal{X})$, define the linear functional ℓ by setting, for all $f \in \mathrm{VMO}_{\rho,L}(\mathcal{X})$,

$$\ell(f) \equiv \langle f, g\rangle. \tag{4.5}$$

Then there exists a positive constant C independent of g such that

$$\|\ell\|_{(\mathrm{VMO}_{\rho,L}(\mathcal{X}))^*} \le C\|g\|_{B_{\Phi,L^*}(\mathcal{X})}.$$

Conversely, for any $\ell \in (\mathrm{VMO}_{\rho,L}(\mathcal{X}))^*$, there exist $g \in B_{\Phi,L^*}(\mathcal{X})$ such that (4.5) holds and a positive constant C, independent of ℓ, such that

$$\|g\|_{B_{\Phi,L^*}(\mathcal{X})} \le C\|\ell\|_{(\mathrm{VMO}_{\rho,L}(\mathcal{X}))^*}.$$

Proof

By Lemma 4.7, we have $(B_{\Phi,L^*}(\mathcal{X}))^* = \mathrm{BMO}_{\rho,L}(\mathcal{X})$. Definition 3.3 implies that $\mathrm{VMO}_{\rho,L}(\mathcal{X}) \subset \mathrm{BMO}_{\rho,L}(\mathcal{X})$, which further implies that $B_{\Phi,L^*}(\mathcal{X}) \subset (\mathrm{VMO}_{\rho,L}(\mathcal{X}))^*$.

Conversely, let $M > (1/p_\Phi^- - 1/2)n/2$ and $\ell \in (\mathrm{VMO}_{\rho,L}(\mathcal{X}))^*$. By Proposition 4.2, $\pi_{L,1}$ is bounded from $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$ to $\mathrm{VMO}_{\rho,L}(\mathcal{X})$, which implies that $\ell \circ \pi_{L,1}$ is a bounded linear functional on $T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$. Thus, by Theorem 4.1, there exists $g \in \widetilde{T}_\Phi(\mathcal{X})$ such that for all $g \in T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$, $\ell \circ \pi_{L,1}(f) = \langle f, g\rangle$.

Now, suppose that $f \in \mathrm{VMO}_{\rho,L}(\mathcal{X}) \cap L^2(\mathcal{X})$. By Theorem 3.4, we conclude that $(t^2L)^Me^{-t^2L}f \in T^\infty_{\Phi,\mathrm{v}}(\mathcal{X})$. Moreover, from the proof of Lemma 4.8, we deduce that $f = (C(M)/C_1)\pi_{L,1}((t^2L)^Me^{-t^2L}f)$ in $\mathrm{BMO}_{\rho,L}(\mathcal{X})$. Thus

$$\begin{aligned}\ell(f) &= \frac{C(M)}{C_1}\ell \circ \pi_{L,1}\big((t^2L)^Me^{-t^2L}f\big)\\ &= \frac{C(M)}{C_1}\iint_{\mathcal{X}\times(0,\infty)}(t^2L)^Me^{-t^2L}f(x)g(x,t)\,\frac{d\mu(x)\,dt}{t}.\end{aligned} \tag{4.6}$$

By Lemma 4.2, $T^2_{2,b}(\mathcal{X})$ is dense in $\widetilde{T}_\Phi(\mathcal{X})$. Since $g \in \widetilde{T}_\Phi(\mathcal{X})$, we choose $\{g_k\}_{k\in\mathbb{N}} \subset T^2_{2,b}(\mathcal{X})$ such that $g_k \to g$ in $\widetilde{T}_\Phi(\mathcal{X})$. By Proposition 4.2(iii), we see that $\pi_{L^*,M}(g)$, $\pi_{L^*,M}(g_k) \in B_{\Phi,L^*}(\mathcal{X})$ and

$$\|\pi_{L^*,M}(g - g_k)\|_{B_{\Phi,L^*}(\mathcal{X})} \lesssim \|g - g_k\|_{\widetilde{T}_\Phi(\mathcal{X})} \to 0,$$

as $k \to \infty$. This, together with (4.6), Theorem 4.1, the dominated convergence theorem, and Lemma 4.7, implies that

$$\begin{aligned}\ell(f) &= \frac{C(M)}{C_1} \lim_{k\to\infty} \iint_{\mathcal{X}\times(0,\infty)} (t^2L)^M e^{-t^2L} f(x) g_k(x,t)\, \frac{d\mu(x)\,dt}{t} \\ &= \frac{C(M)}{C_1} \lim_{k\to\infty} \int_{\mathcal{X}} f(x) \int_0^\infty (t^2L^*)^M e^{-t^2L^*}\big(g_k(\cdot,t)\big)(x)\, \frac{dt}{t}\, d\mu(x) \qquad (4.7)\\ &= \frac{1}{C_1} \lim_{k\to\infty} \langle f, \pi_{L^*,M}(g_k)\rangle = \frac{1}{C_1} \langle f, \pi_{L^*,M}(g)\rangle.\end{aligned}$$

Since $\mathrm{VMO}_{\rho,L}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $\mathrm{VMO}_{\rho,L}(\mathcal{X})$, we finally conclude that (4.7) holds for all $f \in \mathrm{VMO}_{\rho,L}(\mathcal{X})$, and $\|\ell\|_{(\mathrm{VMO}_{\rho,L}(\mathcal{X}))^*} = (1/C_1)\|\pi_{L^*,M} g\|_{B_{\Phi,L^*}(\mathcal{X})}$. In this sense, we have $(\mathrm{VMO}_{\rho,L}(\mathcal{X}))^* \subset B_{\Phi,L^*}(\mathcal{X})$, which completes the proof of Theorem 4.2. $\square$

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