Gabor families in $l^2(\mathbb{Z}^d)$

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Abstract This paper addresses Gabor families in $l^2(\mathbb{Z}^d)$. The discrete Gabor families have interested many researchers due to their good potential for digital signal processing. Gabor analysis in $l^2(\mathbb{Z}^d)$ is more complicated than that in $l^2(\mathbb{Z})$ since the geometry of the lattices generated by time-frequency translation matrices can be quite complex in this case. In this paper, we characterize window functions such that they correspond to complete Gabor families (Gabor frames) in $l^2(\mathbb{Z}^d)$; obtain a necessary and sufficient condition on time-frequency translation for the existence of complete Gabor families (Gabor frames, Gabor Riesz bases) in $l^2(\mathbb{Z}^d)$; characterize duals with Gabor structure for Gabor frames; derive an explicit expression of the canonical dual for a Gabor frame; and prove its norm minimality among all Gabor duals.

1. Introduction

To begin, we introduce some notions and notation. Let $\mathcal{H}$ be a separable Hilbert space. An at most countable sequence $\{g_n\}_{n\in\mathcal{I}}$ in $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist $0<C\le D<\infty$ such that

$$C\|f\|^2 \le \sum_{n\in\mathcal{I}} |\langle f, g_n\rangle|^2 \le D\|f\|^2 \tag{1.1}$$

for $f\in\mathcal{H}$, where C, D are called *frame bounds*. In particular, it is called a *tight frame* (*normalized tight frame*) for $\mathcal{H}$ if $C=D$ ($C=D=1$) in (1.1). The sequence $\{g_n\}_{n\in\mathcal{I}}$ is called a *Bessel sequence* in $\mathcal{H}$ if the right-hand side inequality in (1.1) holds, where D is called its *Bessel bound*. A frame for $\mathcal{H}$ is called a *Riesz basis* for $\mathcal{H}$ if it ceases to be a frame whenever an arbitrary element is removed. Let $\{g_n\}_{n\in\mathcal{I}}$ be a frame for $\mathcal{H}$. A sequence $\{h_n\}_{n\in\mathcal{I}}$ in $\mathcal{H}$ is called a *dual frame* of $\{g_n\}_{n\in\mathcal{I}}$ if it is a frame for $\mathcal{H}$, and $f=\sum_{n\in\mathcal{I}}\langle f,h_n\rangle g_n$ for $f\in\mathcal{H}$. It is easy to check that $\{g_n\}_{n\in\mathcal{I}}$ is also a dual frame of $\{h_n\}_{n\in\mathcal{I}}$ in this case, so we say that $\{h_n\}_{n\in\mathcal{I}}$ and $\{g_n\}_{n\in\mathcal{I}}$ are dual frames for $\mathcal{H}$.

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We always denote by I the identity operator. Let $\{g_n\}_{n\in\mathcal{I}}$ and $\{h_n\}_{n\in\mathcal{I}}$ be Bessel sequences in $\mathcal{H}$. Define the operator $\mathcal{S}_{h,g}:\mathcal{H}\to\mathcal{H}$ by

$$\mathcal{S}_{h,g}f := \sum_{n\in\mathcal{I}}\langle f, h_n\rangle g_n \tag{1.2}$$

for $f\in\mathcal{H}$. Then $\mathcal{S}_{h,g}$ is a bounded operator on $\mathcal{H}$. By definition $\{h_n\}_{n\in\mathcal{I}}$ and $\{g_n\}_{n\in\mathcal{I}}$ are dual frames for $\mathcal{H}$ if and only if $\mathcal{S}_{h,g}=I$. In particular, if $g_n=h_n$ and $\{g_n\}_{n\in\mathcal{I}}$ is a frame for $\mathcal{H}$ with frame bounds C and D, then $\mathcal{S}_{g,g}$ is also invertible and $\{\mathcal{S}_{g,g}^{-1}g_n\}_{n\in\mathcal{I}}$ is a frame for $\mathcal{H}$ with frame bounds D^{-1} and C^{-1}, which is a dual of $\{g_n\}_{n\in\mathcal{I}}$ (called the *canonical dual*). The fundamentals of frames can be found in [2], [7], [34].

Let $\mathbb{Z}$ denote the set of integers, and let $\mathbb{N}$ be the set of positive integers. For $L\in\mathbb{N}$, we write

$$\mathbb{N}_L := \{0,1,\dots,L-1\}.$$

Given $d\in\mathbb{N}$, we denote by e_i with $i\in\mathbb{N}_d$ the vectors in $\mathbb{C}^d$,

$$\mathrm{e}_i = (0,0,\dots,1,0,\dots,0)^t, \tag{1.3}$$

with the ith component being 1 and the others being zero, by $\mathbb{T}^d$ the set $[0,1)^d$, by $\mathrm{GL}(\mathbb{R}^d)$ and $\mathrm{GL}(\mathbb{Z}^d)$ the set of $d\times d$ invertible real matrices and the set of $d\times d$ invertible integer matrices, respectively, by $l^2(\mathbb{Z}^d)$ the Hilbert space of square-summable sequences on $\mathbb{Z}^d$, and by $l^2(\mathcal{I})$ the closed subspace of $l^2(\mathbb{Z}^d)$,

$$l^2(\mathcal{I}) := \{f : f\in l^2(\mathbb{Z}^d), f(j)=0 \text{ for } j\notin\mathcal{I}\},$$

for a nonempty set $\mathcal{I}$ in $\mathbb{Z}^d$, which is obviously a Hilbert space. For $Q\in\mathrm{GL}(\mathbb{Z}^d)$, a set $\mathcal{E}$ is called a *full set* of $\mathbb{Z}^d/Q\mathbb{Z}^d$ if it is a set of representatives of distinct cosets in $\mathbb{Z}^d/Q\mathbb{Z}^d$. Such an $\mathcal{E}$ has many different choices for a given $Q\in\mathrm{GL}(\mathbb{Z}^d)$. It is easy to see that $\mathcal{E}$ can be given by $\mathcal{E}=(Q\mathbb{T}^d)\cap\mathbb{Z}^d$ as an example. For a Lebesgue measurable set E in $\mathbb{R}^d$, we denote by $|E|$ its Lebesgue measure, by $L^\infty(E)$ the Banach space of all essentially bounded measurable functions with essential supremum norm, by $\chi_{_E}$ the characteristic function of E, and by $E+x_0$ the set $E+x_0=\{x+x_0 : x\in E\}$ for an arbitrary $x_0\in\mathbb{R}^d$. Similarly, if E is a subset in $\mathbb{Z}^d$, we also denote by $\chi_{_E}$ the characteristic function of E. Let A, $B\in\mathrm{GL}(\mathbb{R}^d)$, and let N, $M\in\mathrm{GL}(\mathbb{Z}^d)$. The continuous Gabor family, also known as the Weyl-Heisenberg family, generated by a function g in $L^2(\mathbb{R}^d)$ is the following family of functions in $L^2(\mathbb{R}^d)$:

$$\mathcal{G}(g,A,B) := \{e^{2\pi i\langle Bm,\cdot\rangle}g(\cdot - An) : m,n\in\mathbb{Z}^d\}. \tag{1.4}$$

Similarly, the discrete Gabor family (or Weyl-Heisenberg family) generated by a function g in $l^2(\mathbb{Z}^d)$ is the following family of functions in $l^2(\mathbb{Z}^d)$:

$$\mathcal{G}(g,N,M) := \{E_{M^{-1}m}T_{Nn}g : m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d, n\in\mathbb{Z}^d\}, \tag{1.5}$$

where $E_{M^{-1}m}$ and T_{Nn} denote the modulation operator and translation operator, respectively,

$$E_{M^{-1}m}f(\cdot) := e^{2\pi i\langle M^{-1}m,\cdot\rangle}f(\cdot), \qquad T_{Nn}f(\cdot) := f(\cdot - Nn),$$

for $f \in l^2(\mathbb{Z}^d)$. For $g,\ h \in l^2(\mathbb{Z}^d)$ with $\mathcal{G}(g,N,M)$ and $\mathcal{G}(h,N,M)$ both being Bessel sequences in $l^2(\mathbb{Z}^d)$, the operator $\mathcal{S}_{h,g}$ defined as in (1.2) can be written as

$$\mathcal{S}_{h,g} f := \sum_{n\in\mathbb{Z}^d} \sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} \langle f, E_{M^{-1}m} T_{Nn} h\rangle E_{M^{-1}m} T_{Nn} g \tag{1.6}$$

for $f \in l^2(\mathbb{Z}^d)$. When $\mathcal{S}_{h,g} = I$ on $l^2(\mathbb{Z}^d)$, h is called a *Gabor dual* of g. When $\mathcal{G}(g,N,M)$ is a frame for $l^2(\mathbb{Z}^d)$, we usually call $\mathcal{S}_{g,g}^{-1} g$ the *canonical Gabor dual* of g since

$$\mathcal{S}_{g,g}^{-1} E_{M^{-1}m} T_{Nn} g = E_{M^{-1}m} T_{Nn} \mathcal{S}_{g,g}^{-1} g$$

for $m \in (M\mathbb{T}^d) \cap \mathbb{Z}^d$ and $n \in \mathbb{Z}^d$.

The continuous Gabor families were introduced by Gabor in [12] and have been extensively studied (especially for the case $d = 1$). We refer readers to [8], [9], [13], [14], [17], [21], [27], [28] for details. Under certain conditions, there is a way to obtain discrete Gabor frames via Gabor frames for $L^2(\mathbb{R})$ through sampling (see [19], [25], [31]). One can also consider Gabor frames in $l^2(\mathbb{Z})$ without referring to frames in $L^2(\mathbb{R})$. The general theory of discrete Gabor analysis is somewhat similar to the continuous case. However, its transference to the discrete case is not all direct or trivial, and sometimes major differences occur. In 1989, Heil [18] showed that while Gabor frames in the continuous case are bases only if they are generated by functions that are not smooth or have poor decay, it is possible in the discrete case to construct Gabor frames that are bases and are generated by sequences with good decay. The sampled Gaussian provides an example of such a signal. In recent years, the discrete Gabor families have interested many researchers due to their good potential for digital signal processing (for details see [1], [4], [5], [18]–[20], [22]–[25], [31], [33] and the references therein). Interestingly, to our knowledge, all these results are concentrated on one-dimensional periodic sequences or $l^2(\mathbb{Z})$ instead of general $l^2(\mathbb{Z}^d)$. This paper addresses Gabor analysis in $l^2(\mathbb{Z}^d)$ with d being an arbitrarily fixed positive integer.

The density problem is an important one in Gabor analysis; that is, under what conditions on A and B in (1.4) (N and M in (1.5)) we can find a function $g \in L^2(\mathbb{R}^d)$ ($g \in l^2(\mathbb{Z}^d)$) such that the Gabor family $\mathcal{G}(g,A,B)$ in (1.4) ($\mathcal{G}(g,N,M)$ in (1.5)) is an orthonormal basis (a frame, complete) for $L^2(\mathbb{R}^d)$ ($l^2(\mathbb{Z}^d)$). For $\mathcal{G}(g,A,B)$ with $d = 1$, it is easy to check that $g = \sqrt{|B|}\chi_{[0,|A|)}$ is such that $\mathcal{G}(g,A,B)$ is a tight frame (an orthonormal basis) for $L^2(\mathbb{R})$ if $|AB| \le 1$ (if $|AB| = 1$); and conversely, Rieffel [29] proved that $|AB| \le 1$ if $\mathcal{G}(g,A,B)$ is complete in $L^2(\mathbb{R})$ for some $g \in L^2(\mathbb{R})$. For $\mathcal{G}(g,A,B)$ with $d > 1$, an analogous necessary condition was established by Ramanathan and Steger in [26], Ron and Shen in [30], and Christensen, Deng, and Heil in [3], where they proved that $|\det(AB)| = 1$ ($|\det(AB)| \le 1$) if there exists $g \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(g,A,B)$ is an orthonormal basis (a frame) for $L^2(\mathbb{R}^d)$. In particular, Gabardo and Han in [11] and [10] proved this result by a simple and general approach to the incompleteness property for arbitrary grouplike unitary systems. The converse had not been resolved until Han and Wang [16] proved the following proposition by studying a problem concerning lattice tiling in $\mathbb{R}^d$.

PROPOSITION 1.1 ([16, THEOREM 3.3])
For $A, B \in \mathrm{GL}(\mathbb{R}^d)$, the following are equivalent.

(i) *There exists $g \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(g,A,B)$ is a normalized tight frame for $L^2(\mathbb{R}^d)$.*
(ii) *There exists $g \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(g,A,B)$ is complete in $L^2(\mathbb{R}^d)$.*
(iii) *We have $|\det(AB)| \leq 1$.*

Let us turn to the discrete Gabor family $\mathcal{G}(g,N,M)$ in (1.5). When $d=1$, the density problem has been completely answered. As a consequence of [20, Section 1.6.5], we have $|N| \leq |M|$ if there exists $g \in l^2(\mathbb{Z})$ such that $\mathcal{G}(g,N,M)$ is a frame for $l^2(\mathbb{Z})$. As a special case of [22, Theorems 4.3, 4.4, 5.2, 5.3], we have the following propositions.

PROPOSITION 1.2
For $N, M \in \mathbb{Z}\backslash\{0\}$, the following are equivalent.

(i) *There exists $g \in l^2(\mathbb{Z})$ such that $\mathcal{G}(g,N,M)$ is a normalized tight frame for $l^2(\mathbb{Z})$.*
(ii) *There exists $g \in l^2(\mathbb{Z})$ such that $\mathcal{G}(g,N,M)$ is complete in $l^2(\mathbb{Z})$.*
(iii) *We have $|N| \leq |M|$.*

PROPOSITION 1.3
Given $N, M \in \mathbb{Z}\backslash\{0\}$, let $\mathcal{G}(g,N,M)$ be a frame for $l^2(\mathbb{Z})$. Then $\mathcal{G}(g,N,M)$ is a Riesz basis for $l^2(\mathbb{Z})$ if and only if $|N| = |M|$.

PROPOSITION 1.4
For $N, M \in \mathbb{Z}\backslash\{0\}$, there exists $g \in l^2(\mathbb{Z})$ such that $\mathcal{G}(g,N,M)$ is a Riesz basis for $l^2(\mathbb{Z})$ if and only if $|N| = |M|$.

When $d > 1$, the density problem of $\mathcal{G}(g,N,M)$ is much more complicated since the geometry of the lattices $N\mathbb{Z}^d$ and $M^{-1}\mathbb{Z}^d$ can be quite complex. The main goal of this paper is to study the density problem of $\mathcal{G}(g,N,M)$ and related problems.

In Section 2, for given $N, M \in \mathrm{GL}(\mathbb{Z}^d)$ we characterize g with $\mathcal{G}(g,N,M)$ complete in $l^2(\mathbb{Z}^d)$. In Section 3, we characterize $N, M \in \mathrm{GL}(\mathbb{Z}^d)$ for the existence of complete Gabor families (Gabor frames, Gabor Riesz bases) of the form $\mathcal{G}(g,N,M)$ in $l^2(\mathbb{Z}^d)$. In Section 4, we obtain a characterization of Gabor frames and their Gabor duals, give a formula about the frame bounds of tight Gabor frames, derive an explicit expression of the canonical dual for a Gabor frame, and prove its norm minimality among all Gabor duals.

2. Completeness of $\mathcal{G}(g, N, M)$ in $l^2(\mathbb{Z}^d)$

Let $N,\ M \in \mathrm{GL}(\mathbb{Z}^d)$. This section is devoted to the characterization of g such that $\mathcal{G}(g, N, M)$ in (1.5) is complete in $l^2(\mathbb{Z}^d)$. In order to state our result, we first introduce a matrix-valued function related to $\mathcal{G}(g, N, M)$.

For a complex matrix P, let P^t be its transpose, and let P^* be its conjugate transpose. For $f \in l^2(\mathbb{Z}^d)$, define its *Fourier transform* by

$$\hat{f}(\xi) := \sum_{j \in \mathbb{Z}^d} f(j) e^{-2\pi i \langle j, \xi \rangle} \tag{2.1}$$

for ae $\xi \in \mathbb{R}^d$. We associate $\mathcal{G}(g, N, M)$ with a $|\det N| \times |\det M|$ matrix-valued function $G(\cdot)$ defined by

(2.2)

$$\big(G(\xi)\big)_{k,m} := \hat{g}\big(\xi - M^{-1}m - (N^t)^{-1}k\big), \quad k \in (N^t\mathbb{T}^d) \cap \mathbb{Z}^d, m \in (M\mathbb{T}^d) \cap \mathbb{Z}^d$$

for ae $\xi \in \mathbb{R}^d$. Herein the order of rows and columns of $G(\cdot)$ can be arbitrarily chosen for our convenience since only the rank of $G(\cdot)$ and the spectrum of $G(\cdot)G^*(\cdot)$ are involved in the paper, which are not changed under row or column permutations to $G(\cdot)$. Similarly, we associate a Gabor family $\mathcal{G}(h, N, M)$ generated by an arbitrary $h \in l^2(\mathbb{Z}^d)$ with $H(\cdot)$. For a vector-valued function $\mathcal{F}(\cdot) = (\mathcal{F}_k(\cdot))_{k \in (N^t\mathbb{T}^d) \cap \mathbb{Z}^d}$, the order of columns of $G^*(\cdot)$ is required to adapt to the order of components of $\mathcal{F}(\cdot)$ when we compute the vector $G^*(\cdot)\mathcal{F}(\cdot)$; that is, its mth component is

$$\big(G^*(\cdot)\mathcal{F}(\cdot)\big)_m = \sum_{k \in (N^t\mathbb{T}^d) \cap \mathbb{Z}^d} \overline{\hat{g}\big(\cdot - M^{-1}m - (N^t)^{-1}k\big)} \mathcal{F}_k(\cdot)$$

for $m \in (M\mathbb{T}^d) \cap \mathbb{Z}^d$. The following theorem provides us with a characterization of the completeness of $\mathcal{G}(g, N, M)$ in $l^2(\mathbb{Z}^d)$.

THEOREM 2.1

For $N,\ M \in \mathrm{GL}(\mathbb{Z}^d)$ and $g \in l^2(\mathbb{Z}^d)$, $\mathcal{G}(g, N, M)$ is complete in $l^2(\mathbb{Z}^d)$ if and only if $\mathrm{rank}(G(\cdot)) = |\det N|$ *ae on $(N^t)^{-1}\mathbb{T}^d$, where $G(\cdot)$ is defined as in (2.2).*

To prove Theorem 2.1, we need some more notation and lemmas, which are also used in the following sections. Without specification, relations between two measurable sets in $\mathbb{R}^d$, such as equality, disjointness, or inclusion, are always understood up to a set of measure zero.

DEFINITION 2.1

Let $Q \in \mathrm{GL}(\mathbb{R}^d)$. For a measurable set Ω in $\mathbb{R}^d$, we say Ω *tiles* $\mathbb{R}^d$ by $Q\mathbb{Z}^d$ if

(i) $\bigcup_{\ell \in \mathbb{Z}^d} (\Omega + Q\ell) = \mathbb{R}^d$;
(ii) $(\Omega + Q\ell) \cap (\Omega + Q\ell') = \emptyset$ for any $\ell \neq \ell'$ in $\mathbb{Z}^d$.

One says that Ω *packs* $\mathbb{R}^d$ by $Q\mathbb{Z}^d$ if only the condition (ii) holds.

DEFINITION 2.2

For a measurable set S in $\mathbb{R}^d$, a collection $\{S_i : i \in \mathcal{I}\}$ of at most countably measurable sets is called a *partition* of S if $S = \bigcup_{i\in\mathcal{I}} S_i$ and $S_i \cap S_{i'} = \emptyset$ for any $i \neq i'$ in $\mathcal{I}$.

DEFINITION 2.3

Let $Q \in \mathrm{GL}(\mathbb{R}^d)$. For two measurable sets S and $\tilde{S}$ in $\mathbb{R}^d$, we say that S and $\tilde{S}$ are *$Q\mathbb{Z}^d$-congruent* if there exists a partition $\{S_k : k \in \mathbb{Z}^d\}$ of S such that $\{S_k + Qk : k \in \mathbb{Z}^d\}$ is a partition of $\tilde{S}$. Similarly, for Q_1, $Q_2 \in \mathrm{GL}(\mathbb{R}^d)$, S and $\tilde{S}$ are said to be *$Q_1\mathbb{Z}^d + Q_2\mathbb{Z}^d$-congruent* if there exists a partition $\{S_{k,\ell} : (k,\ell) \in \mathbb{Z}^d \times \mathbb{Z}^d\}$ of S such that $\{S_{k,\ell} + Q_1 k + Q_2 \ell : (k,\ell) \in \mathbb{Z}^d \times \mathbb{Z}^d\}$ is a partition of $\tilde{S}$.

To better understand the congruence between two sets, we see a simple example. Consider two sets $[0,1]$ and $[1,3/2)\cup[7/2,4]$. Take $S_1 = [0,1/2)$, $S_3 = [1/2,1]$, and $S_k = \emptyset$ for $k \in \mathbb{Z}\setminus\{1,3\}$. Then $S_1 + 1 = [1,3/2)$, $S_3 + 3 = [7/2,4]$, and $S_k + k = \emptyset$ for $k \in \mathbb{Z}\setminus\{1,3\}$. So $\{S_k : k \in \mathbb{Z}\}$ is a partition of $[0,1]$, and $\{S_k + k : k \in \mathbb{Z}\}$ is a partition of $[1,3/2)\cup[7/2,4]$ by Definition 2.2. Therefore, $[0,1]$ and $[1,3/2)\cup[7/2,4]$ are $\mathbb{Z}$-congruent by Definition 2.3.

It is obvious that Ω tiles $\mathbb{R}^d$ by $Q\mathbb{Z}^d$ if and only if $\{\Omega + Q\ell : \ell \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. Observing that $Q\mathbb{T}^d$ tiles $\mathbb{R}^d$ by $Q\mathbb{Z}^d$, we also have that Ω tiles $\mathbb{R}^d$ by $Q\mathbb{Z}^d$ if and only if Ω and $Q\mathbb{T}^d$ are $Q\mathbb{Z}^d$-congruent.

LEMMA 2.1

Given N, $M \in \mathrm{GL}(\mathbb{Z}^d)$, let $\mathbb{T}^d \cap (N^t)^{-1}\mathbb{Z}^d \subset \mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$. Then there exist finitely many $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{L-1}$ in $\mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$ with $\varepsilon_0 = 0$ and mutually disjoint subsets $\Omega_0, \Omega_1, \ldots, \Omega_{L-1}$ of $\mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$ such that

$$\mathbb{T}^d \cap M^{-1}\mathbb{Z}^d = \bigcup_{\ell\in\mathbb{N}_L} \Omega_\ell,$$

and each Ω_ℓ is $\mathbb{Z}^d$-congruent to $(\mathbb{T}^d \cap (N^t)^{-1}\mathbb{Z}^d) + \varepsilon_\ell$.

Proof

Write $\tilde{\Omega} = \mathbb{T}^d \cap (N^t)^{-1}\mathbb{Z}^d$ and $\Omega = \mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$. Let $\Omega_0 = \tilde{\Omega}$ and $\varepsilon_0 = 0$. If $\Omega = \Omega_0$, the lemma holds. If $\Omega_0 \subsetneq \Omega$, we choose an arbitrary $\varepsilon_1 \in \Omega\backslash\Omega_0$. Then, to each $\eta \in \tilde{\Omega}$ there corresponds a unique $k_{\eta,\varepsilon_1} \in \mathbb{Z}^d$ such that $\eta + \varepsilon_1 + k_{\eta,\varepsilon_1} \in \mathbb{T}^d$. Since $\eta \in \tilde{\Omega} \subset \Omega$ and $\varepsilon_1 \in \Omega$, both $M\eta$ and $M\varepsilon_1$ are in $M\mathbb{T}^d \cap \mathbb{Z}^d$, which implies that $M(\eta + \varepsilon_1) \in \mathbb{Z}^d$. Observe that $Mk_{\eta,\varepsilon_1} \in \mathbb{Z}^d$ due to the fact that $M \in \mathrm{GL}(\mathbb{Z}^d)$ and $k_{\eta,\varepsilon_1} \in \mathbb{Z}^d$. It follows that $M(\eta + \varepsilon_1 + k_{\eta,\varepsilon_1}) \in \mathbb{Z}^d$; equivalently, $\eta + \varepsilon_1 + k_{\eta,\varepsilon_1} \in M^{-1}\mathbb{Z}^d$. Also observing that $\varepsilon_1 \notin (N^t)^{-1}\mathbb{Z}^d$, we have $\eta + \varepsilon_1 + k_{\eta,\varepsilon_1} \in \Omega \setminus \Omega_0$. For $\eta \neq \tilde{\eta}$ in $\tilde{\Omega}$, $(\eta + \varepsilon_1 + k_{\eta,\varepsilon_1}) - (\tilde{\eta} + \varepsilon_1 + k_{\tilde{\eta},\varepsilon_1}) \notin \mathbb{Z}^d$ since η, $\tilde{\eta} \in \mathbb{T}^d$. Define $\Omega_1 := \{\eta + \varepsilon_1 + k_{\eta,\varepsilon_1} : \eta \in \tilde{\Omega}\}$. Then $\Omega_1 \subset \Omega \setminus \Omega_0$, and Ω_1 is $\mathbb{Z}^d$-congruent to $\tilde{\Omega} + \varepsilon_1$. If $\Omega = \Omega_0 \cup \Omega_1$, the lemma follows. If $\Omega_0 \cup \Omega_1 \subsetneq \Omega$, we choose an arbitrary $\varepsilon_2 \in \Omega\backslash(\Omega_0 \cup \Omega_1)$. Then, to each $\eta \in \tilde{\Omega}$ there corresponds a unique $k_{\eta,\varepsilon_2} \in \mathbb{Z}^d$ such that $\eta + \varepsilon_2 + k_{\eta,\varepsilon_2} \in \mathbb{T}^d$. Define $\Omega_2 := \{\eta + \varepsilon_2 + k_{\eta,\varepsilon_2} : \eta \in \tilde{\Omega}\}$. Similarly, we have

$\Omega_2 \subset \Omega \setminus (\Omega_0 \cup \Omega_1)$, and Ω_2 is $\mathbb{Z}^d$-congruent to $\tilde{\Omega} + \varepsilon_2$. If $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$, the lemma follows. If $\Omega_0 \cup \Omega_1 \cup \Omega_2 \subsetneqq \Omega$, we can obtain Ω_3 similarly. Since Ω is a finite set, there exists $L \in \mathbb{N}$ such that this procedure stops when we obtain Ω_{L-1}. This finishes the proof. $\square$

LEMMA 2.2
Let $Q \in \mathrm{GL}(\mathbb{Z}^d)$. Then $\mathbb{T}^d$ and

$$\Delta := \bigcup_{k \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d} (Q^{-1}\mathbb{T}^d - Q^{-1}k)$$

are $\mathbb{Z}^d$-congruent, and the union here is a disjoint union.

Proof
For $k \neq k'$ in $(Q\mathbb{T}^d) \cap \mathbb{Z}^d$, since $(\mathbb{T}^d - k) \cap (\mathbb{T}^d - k') = \emptyset$, we have

$$(Q^{-1}\mathbb{T}^d - Q^{-1}k) \cap (Q^{-1}\mathbb{T}^d - Q^{-1}k') = Q^{-1}[(\mathbb{T}^d - k) \cap (\mathbb{T}^d - k')] = \emptyset.$$

This implies that $\bigcup_{k \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d} (Q^{-1}\mathbb{T}^d - Q^{-1}k)$ is a disjoint union, and thus,

$$|\Delta| = \sum_{k \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d} |Q^{-1}\mathbb{T}^d - Q^{-1}k| = \sum_{k \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d} |Q^{-1}\mathbb{T}^d| = 1.$$

Let $\mathbb{K} = \{k \in \mathbb{Z}^d : (\Delta - k) \cap \mathbb{T}^d \neq \emptyset\}$. Then $\mathbb{K}$ is a finite set since both Δ and $\mathbb{T}^d$ are bounded sets. Define

$$S_k := \begin{cases} (\Delta - k) \cap \mathbb{T}^d & \text{if } k \in \mathbb{K}, \\ \emptyset & \text{if } k \notin \mathbb{K}, \end{cases}$$

for $k \in \mathbb{Z}^d$. Observe that $\{\mathbb{T}^d + k : k \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$ and that $S_k + k = \Delta \cap (\mathbb{T}^d + k)$ for $k \in \mathbb{Z}^d$. It follows that $\{S_k + k : k \in \mathbb{Z}^d\}$ forms a partition of Δ, and thus

$$\sum_{k \in \mathbb{Z}^d} |S_k| = \sum_{k \in \mathbb{Z}^d} |S_k + k| = |\Delta| = 1. \tag{2.3}$$

Next we prove that $\{S_k : k \in \mathbb{Z}^d\}$ is a partition of $\mathbb{T}^d$. By (2.3) and the definition of S_k, we only need to prove that $S_k \cap S_{k'} = \emptyset$ for $k \neq k'$ in $\mathbb{K}$. For such k and k', since $(Q\mathbb{T}^d) \cap \mathbb{Z}^d$ is a full set of $\mathbb{Z}^d/Q\mathbb{Z}^d$, we have $\tilde{k} + Qk \neq \tilde{k'} + Qk'$, and thus

$$(\mathbb{T}^d - \tilde{k} - Qk) \cap (\mathbb{T}^d - \tilde{k'} - Qk') = \emptyset$$

for $\tilde{k}$, $\tilde{k'} \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d$. It follows that

$$(Q^{-1}\mathbb{T}^d - Q^{-1}\tilde{k} - k) \cap (Q^{-1}\mathbb{T}^d - Q^{-1}\tilde{k'} - k') = \emptyset$$

for $\tilde{k}$, $\tilde{k'} \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d$. So $(\Delta - k) \cap (\Delta - k') = \emptyset$ by the definition of Δ, which implies that $S_k \cap S_{k'} = \emptyset$. The proof is completed. $\square$

EXAMPLE 2.1
Let $Q = \left(\begin{smallmatrix} 4 & 3 \\ 3 & 2 \end{smallmatrix}\right)$ in Lemma 2.2. Then $(Q\mathbb{T}^2) \cap \mathbb{Z}^2 = \{(0,0)^t\}$, $Q^{-1} = \left(\begin{smallmatrix} -2 & 3 \\ 3 & -4 \end{smallmatrix}\right)$, and

$$\Delta = \{(x,y)^t \in \mathbb{R}^2 : 0 \le 3x + 2y < 1, 0 \le 4x + 3y < 1\}.$$

Under the notation of Lemma 2.2, we have

$$\mathbb{K}=\big\{(-2,2)^t,(-2,1)^t,(-1,1)^t,(-1,0)^t,(0,0)^t,(0,-1)^t,\\ (0,-2)^t,(1,-2)^t,(1,-3)^t,(2,-3)^t,(2,-4)^t\big\},$$

and $\{S_k : k\in\mathbb{Z}^2\}$ has the following form:

$$\begin{aligned}
S_{(-2,2)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : \frac{2}{3}(1-y)\le x<\frac{3}{4}(1-y), 0\le y\le 1\Big\},\\
S_{(-2,1)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : \frac{2}{3}(2-y)\le x<\frac{3}{4}(2-y), \frac{2}{3}\le y\le 1\Big\}\\
&\quad\cup\Big\{(x,y)^t\in\mathbb{R}^2 : \frac{2}{3}(2-y)\le x<1, \frac{1}{2}\le y\le \frac{2}{3}\Big\},\\
S_{(-1,1)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : 0\le x<\frac{1}{2}-\frac{3}{4}y, \frac{1}{2}\le y\le \frac{2}{3}\Big\}\\
&\quad\cup\Big\{(x,y)^t\in\mathbb{R}^2 : \frac{1}{3}-\frac{2}{3}y\le x<\frac{1}{2}-\frac{3}{4}y, 0\le y\le \frac{1}{2}\Big\},\\
S_{(-1,0)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : 1-\frac{2}{3}y\le x<\frac{5}{4}-\frac{3}{4}y, \frac{1}{3}\le y\le 1\Big\}\\
&\quad\cup\Big\{(x,y)^t\in\mathbb{R}^2 : 1-\frac{2}{3}y\le x<1, 0\le y\le \frac{1}{3}\Big\},\\
S_{(0,0)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : 0\le x<\frac{1}{4}-\frac{3}{4}y, 0\le y\le \frac{1}{3}\Big\},\\
S_{(0,-1)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : \frac{3}{4}(1-y)\le x<1-\frac{3}{4}y, 0\le y\le 1\Big\},\\
S_{(0,-2)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : \frac{3}{4}(2-y)\le x<1, \frac{2}{3}\le y\le 1\Big\},\\
S_{(1,-2)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : 0\le x<\frac{2}{3}(1-y), \frac{2}{3}\le y\le 1\Big\}\\
&\quad\cup\Big\{(x,y)^t\in\mathbb{R}^2 : \frac{1}{2}-\frac{3}{4}y\le x<\frac{2}{3}(1-y), 0\le y\le \frac{2}{3}\Big\},\\
S_{(1,-3)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : \frac{5}{4}-\frac{3}{4}y\le x<\frac{4}{3}-\frac{2}{3}y, \frac{1}{2}\le y\le 1\Big\}\\
&\quad\cup\Big\{(x,y)^t\in\mathbb{R}^2 : \frac{5}{4}-\frac{3}{4}y\le x<1, \frac{1}{3}\le y\le \frac{1}{2}\Big\},\\
S_{(2,-3)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : 0\le x<\frac{1}{3}-\frac{2}{3}y, \frac{1}{3}\le y\le \frac{1}{2}\Big\}\\
&\quad\cup\Big\{(x,y)^t\in\mathbb{R}^2 : \frac{1}{4}-\frac{3}{4}y\le x<\frac{1}{3}-\frac{2}{3}y, 0\le y\le \frac{1}{3}\Big\},\\
S_{(2,-4)^t} &= \Big\{(x,y)^t\in\mathbb{R}^2 : 1-\frac{3}{4}y\le x<1-\frac{2}{3}y, 0\le y\le 1\Big\},\\
S_k &= \emptyset \quad \text{for } k\notin\mathbb{K}.
\end{aligned}$$

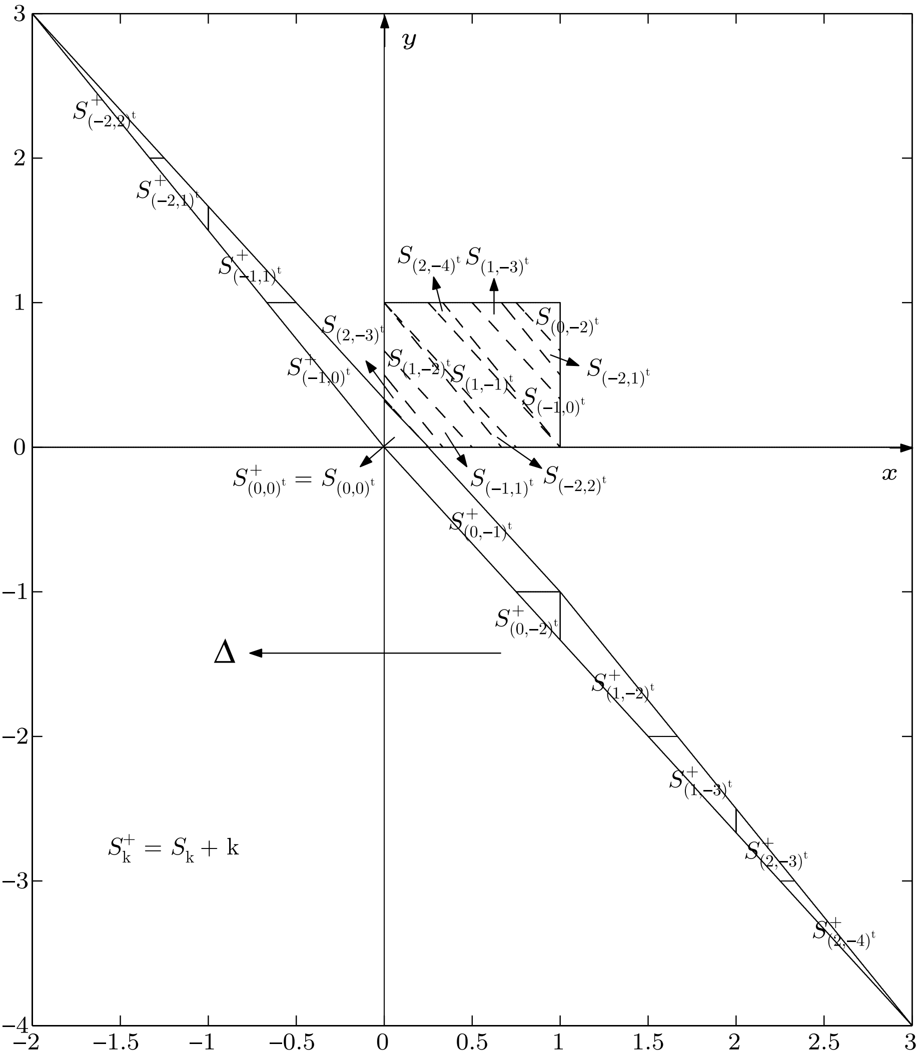

Figure 1. $\{S_k : k \in \mathbb{Z}^2\}$ is a partition of $\mathbb{T}^2$, and $\{S_k^+ = S_k + k : k \in \mathbb{Z}^2\}$ is a partition of Δ

By the proof of Lemma 2.2, $\{S_k : k \in \mathbb{Z}^2\}$ is a partition of $\mathbb{T}^2$, and $\{S_k^+ = S_k + k : k \in \mathbb{Z}^2\}$ is a partition of Δ (see also Figure 1). Therefore, $\mathbb{T}^2$ and Δ are $\mathbb{Z}^2$-congruent.

LEMMA 2.3
Let N, $M \in \mathrm{GL}(\mathbb{Z}^d)$, and let $g \in l^2(\mathbb{Z}^d)$. Define $G(\cdot)$ as in (2.2). Then

$$\langle f, E_{M^{-1}m} T_{Nn} g\rangle = e^{-2\pi i \langle M^{-1}m, Nn\rangle} \int_{(N^t)^{-1}\mathbb{T}^d} \big(G^*(\xi)\mathcal{F}(\xi)\big)_m e^{2\pi i \langle Nn, \xi\rangle}\, d\xi$$

for $f \in l^2(\mathbb{Z}^d)$, $m \in (M\mathbb{T}^d) \cap \mathbb{Z}^d$, and $n \in \mathbb{Z}^d$, where $\mathcal{F}(\cdot) := \big(\hat{f}(\cdot - (N^t)^{-1} \times k)\big)_{k \in (N^t\mathbb{T}^d) \cap \mathbb{Z}^d}$.

Proof
A simple computation shows that

$$(E_{M^{-1}m}T_{Nn}g)\hat{}(\xi)=e^{2\pi i\langle M^{-1}m,Nn\rangle}e^{-2\pi i\langle Nn,\xi\rangle}\hat{g}(\xi-M^{-1}m) \tag{2.4}$$

for $m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d$, $n\in\mathbb{Z}^d$, and ae $\xi\in\mathbb{T}^d$. It follows that

$$\langle f,E_{M^{-1}m}T_{Nn}g\rangle=e^{-2\pi i\langle M^{-1}m,Nn\rangle}\int_{\mathbb{T}^d}\hat{f}(\xi)\overline{\hat{g}(\xi-M^{-1}m)}e^{2\pi i\langle Nn,\xi\rangle}\,d\xi \tag{2.5}$$

for $m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d$ and $n\in\mathbb{Z}^d$. Letting $Q=N^t$ in Lemma 2.2, we have $\mathbb{T}^d$ and

$$\Delta:=\bigcup_{k\in(N^t\mathbb{T}^d)\cap\mathbb{Z}^d}\big((N^t)^{-1}\mathbb{T}^d-(N^t)^{-1}k\big) \tag{2.6}$$

are $\mathbb{Z}^d$-congruent, and the union here is a disjoint union. So, by Definition 2.3, there exists a partition $\{\Delta_\ell:\ell\in\mathbb{Z}^d\}$ of Δ such that $\{\Delta_\ell+\ell:\ell\in\mathbb{Z}^d\}$ is a partition of $\mathbb{T}^d$. It follows that $\mathbb{T}^d=\bigcup_{\ell\in\mathbb{Z}^d}(\Delta_\ell+\ell)$, where the union is a disjoint union. Also observing that the integrand in (2.5) is $\mathbb{Z}^d$-periodic, we have

$$\begin{aligned}\int_{\mathbb{T}^d}\hat{f}(\xi)\overline{\hat{g}(\xi-M^{-1}m)}e^{2\pi i\langle Nn,\xi\rangle}\,d\xi&=\sum_{\ell\in\mathbb{Z}^d}\int_{\Delta_\ell+\ell}\hat{f}(\xi)\overline{\hat{g}(\xi-M^{-1}m)}e^{2\pi i\langle Nn,\xi\rangle}\,d\xi\\&=\sum_{\ell\in\mathbb{Z}^d}\int_{\Delta_\ell}\hat{f}(\xi)\overline{\hat{g}(\xi-M^{-1}m)}e^{2\pi i\langle Nn,\xi\rangle}d\xi \qquad (2.7)\\&=\int_{\Delta}\hat{f}(\xi)\overline{\hat{g}(\xi-M^{-1}m)}e^{2\pi i\langle Nn,\xi\rangle}d\xi,\end{aligned}$$

where we use the fact that Δ is a disjoint union of Δ_ℓ, $\ell\in\mathbb{Z}^d$, in the last equality. Since the union in (2.6) is a disjoint one, we arrive at

$$\begin{aligned}&\int_{\Delta}\hat{f}(\xi)\overline{\hat{g}(\xi-M^{-1}m)}e^{2\pi i\langle Nn,\xi\rangle}d\xi\\&\quad=\sum_{k\in(N^t\mathbb{T}^d)\cap\mathbb{Z}^d}\int_{(N^t)^{-1}\mathbb{T}^d-(N^t)^{-1}k}\hat{f}(\xi)\overline{\hat{g}(\xi-M^{-1}m)}e^{2\pi i\langle Nn,\xi\rangle}d\xi\\&\quad=\sum_{k\in(N^t\mathbb{T}^d)\cap\mathbb{Z}^d}\int_{(N^t)^{-1}\mathbb{T}^d}\hat{f}\big(\xi-(N^t)^{-1}k\big)\overline{\hat{g}\big(\xi-M^{-1}m-(N^t)^{-1}k\big)}e^{2\pi i\langle Nn,\xi\rangle}d\xi.\end{aligned}$$

This, together with (2.7) and (2.5), leads to

$$\begin{aligned}&\langle f,E_{M^{-1}m}T_{Nn}g\rangle\\&\quad=e^{-2\pi i\langle M^{-1}m,Nn\rangle}\\&\qquad\times\int_{(N^t)^{-1}\mathbb{T}^d}\Big(\sum_{k\in(N^t\mathbb{T}^d)\cap\mathbb{Z}^d}\hat{f}\big(\xi-(N^t)^{-1}k\big)\overline{\hat{g}\big(\xi-M^{-1}m-(N^t)^{-1}k\big)}\Big)\\&\qquad\times e^{2\pi i\langle Nn,\xi\rangle}d\xi\\&\quad=e^{-2\pi i\langle M^{-1}m,Nn\rangle}\int_{(N^t)^{-1}\mathbb{T}^d}\big(G^*(\xi)\mathcal{F}(\xi)\big)_m e^{2\pi i\langle Nn,\xi\rangle}d\xi\end{aligned}$$

for $m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d$ and $n\in\mathbb{Z}^d$. The proof is completed. □

Proof of Theorem 2.1
By Lemma 2.3, $\mathcal{G}(g,N,M)$ is complete in $l^2(\mathbb{Z}^d)$ if and only if $f=0$ is the unique solution to the equation about $f\in l^2(\mathbb{Z}^d)$:

$$G^*(\cdot)\mathcal{F}(\cdot)=0 \text{ ae} \quad \text{on } (N^t)^{-1}\mathbb{T}^d, \tag{2.8}$$

where $\mathcal{F}(\cdot):=\big(\hat{f}(\cdot-(N^t)^{-1}k)\big)_{k\in(N^t\mathbb{T}^d)\cap\mathbb{Z}^d}$.

For sufficiency, suppose that $\operatorname{rank}(G(\cdot))=|\det N|$ ae on $(N^t)^{-1}\mathbb{T}^d$, and suppose that $f\in l^2(\mathbb{Z}^d)$ satisfies the equation (2.8). Then $\mathcal{F}(\cdot)=0$ ae on $(N^t)^{-1}\mathbb{T}^d$, which together with Lemma 2.2 implies that $\hat{f}(\cdot)=0$ ae on $\mathbb{T}^d$. It follows that $f=0$, and thus $\mathcal{G}(g,N,M)$ is complete in $l^2(\mathbb{Z}^d)$.

For necessity, we argue by contradiction. Suppose that $\mathcal{G}(g,N,M)$ is complete in $l^2(\mathbb{Z}^d)$, and suppose that $\operatorname{rank}(G(\cdot))<|\det N|$ on E for some measurable set $E\subset(N^t)^{-1}\mathbb{T}^d$ with $|E|>0$. Then $\ker(G^*(\xi))\neq\{0\}$ for $\xi\in E$. Define

$$\mathcal{P}(\cdot):=\lim_{n\to\infty}e^{-nG(\cdot)G^*(\cdot)}.$$

Then $\mathcal{P}(\cdot)$ is measurable and the orthogonal projection onto the kernel of $G^*(\cdot)$ by an easy application of the spectral theorem for self-adjoint matrices (see also [6, p. 978]). Therefore there exist an $i\in\mathbb{N}_{|\det N|}$ and a measurable set $\tilde{E}\subset E$ with $|\tilde{E}|>0$ such that $\mathcal{P}(\xi)\mathrm{e}_i\neq0$ for $\xi\in\tilde{E}$, where e_i is defined as in (1.3) with d replaced by $|\det N|$. Define $f\in l^2(\mathbb{Z}^d)$ via its Fourier transform by

$$\big(\hat{f}(\cdot-(N^t)^{-1}k)\big)_{k\in(N^t\mathbb{T}^d)\cap\mathbb{Z}^d}:=\begin{cases}\mathcal{P}(\cdot)\mathrm{e}_i & \text{on } \tilde{E},\\ 0 & \text{otherwise,}\end{cases}$$

ae on $(N^t)^{-1}\mathbb{T}^d$. Then f is well defined by Lemma 2.2, and f is a nonzero solution to (2.8). This is in contradiction to the first paragraph of the proof. The proof is completed. □

3. Density results

In this section, we characterize N, $M\in\mathrm{GL}(\mathbb{Z}^d)$ for the existence of complete Gabor families (Gabor frames, Gabor Riesz bases) of the form $\mathcal{G}(g,N,M)$ in $l^2(\mathbb{Z}^d)$. We first introduce some definitions and lemmas.

DEFINITION 3.1
Let $Q\in\mathrm{GL}(\mathbb{R}^d)$, and let e_i, $i\in\mathbb{N}_d$, be as in (1.3). Given $x\in\mathbb{R}^d$, the vector $(x_0,\ldots,x_{d-1})^t\in\mathbb{R}^d$ is called the *Q-coordinate* of x if $x=\sum_{i\in\mathbb{N}_d}x_iQ\mathrm{e}_i$. The set

$$\Big\{x\in\mathbb{R}^d: x=\sum_{i\in\mathbb{N}_d}x_iQ\mathrm{e}_i, x_i>0, i\in\mathbb{N}_d\Big\}$$

is called the *first Q-quadrant.*

Definition 3.1 is well defined. In fact, $Q\mathrm{e}_i$, $i\in\mathbb{N}_d$, are linearly independent, and consequently, for each $x\in\mathbb{R}^d$, there corresponds a unique $(x_0,\ldots,x_{d-1})^t\in\mathbb{R}^d$ such that $x=\sum_{i\in\mathbb{N}_d}x_iQ\mathrm{e}_i$. In particular, the Q-coordinate is identical with the usual rectangular coordinate when Q is the identity matrix. Without specification, a point $(x_0,\dots,x_{d-1})^t$ or a set S in $\mathbb{R}^d$ is always referred to as the one defined according to the rectangular coordinate system. The following lemma is another equivalent statement of [16, Theorem 1.2].

LEMMA 3.1

Let Q_1 and Q_2 be two matrices in $\mathrm{GL}(\mathbb{R}^d)$ such that $|\det Q_1|\le|\det Q_2|$. Then there exists a measurable set Ω in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by $Q_1\mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $Q_2\mathbb{Z}^d$.

LEMMA 3.2

Given $Q_1,\ Q_2\in\mathrm{GL}(\mathbb{R}^d)$, let S and $\tilde{S}$ be two bounded measurable sets in $\mathbb{R}^d$ such that they are $(Q_1\mathbb{Z}^d+Q_2\mathbb{Z}^d)$-congruent. Then there exist a finite subset $\{(k_i,\ell_i):1\le i\le n\}$ of $\mathbb{Z}^d\times\mathbb{Z}^d$ and a partition $\{S_i:1\le i\le n\}$ of S such that $\{S_i+Q_1k_i+Q_2\ell_i:1\le i\le n\}$ is a partition of $\tilde{S}$.

Proof

Since S and $\tilde{S}$ are $Q_1\mathbb{Z}^d+Q_2\mathbb{Z}^d$-congruent, there exists a partition $\{S_{k,\ell}:(k,\ell)\in\mathbb{Z}^d\times\mathbb{Z}^d\}$ of S such that $\{S_{k,\ell}+Q_1k+Q_2\ell:(k,\ell)\in\mathbb{Z}^d\times\mathbb{Z}^d\}$ is a partition of $\tilde{S}$. Also recall that by assumption S and $\tilde{S}$ are both bounded. It follows that

$$\Lambda=\big\{\lambda:\lambda=Q_1k+Q_2\ell \text{ for some } (k,\ell)\in\mathbb{Z}^d\times\mathbb{Z}^d \text{ with } |S_{k,\ell}|>0\big\}$$

is a finite set. Suppose that $\{(k_i,\ell_i):1\le i\le n\}$ is a finite subset of $\mathbb{Z}^d\times\mathbb{Z}^d$ such that $\Lambda=\{\lambda_i=Q_1k_i+Q_2\ell_i:1\le i\le n\}$. Define

$$S_i:=\bigcup_{(k,\ell)\in\mathbb{Z}^d:Q_1k+Q_2\ell=\lambda_i}S_{k,\ell}$$

for $1\le i\le n$. Then $\{S_i:1\le i\le n\}$ is as desired. The proof is completed. $\square$

LEMMA 3.3

For arbitrary $Q_1,\,Q_2\in\mathrm{GL}(\mathbb{Z}^d)$ with $|\det Q_1|\le|\det Q_2|$, there exists a set $\mathcal{E}$ in $\mathbb{Z}^d$ such that $\mathcal{E}$ is a full set of $\mathbb{Z}^d/Q_1\mathbb{Z}^d$ and is a subset of some full set of $\mathbb{Z}^d/Q_2\mathbb{Z}^d$. In particular, when $|\det Q_1|=|\det Q_2|$, $\mathcal{E}$ is also a full set of $\mathbb{Z}^d/Q_2\mathbb{Z}^d$.

Proof

By Lemma 3.1, there exists a measurable set Ω in $\mathbb{R}^d$ such that Ω tiles $\mathbb{R}^d$ by $Q_1\mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $Q_2\mathbb{Z}^d$. Note that $Q_1\mathbb{T}^d$ and $Q_2\mathbb{T}^d$ tile $\mathbb{R}^d$ by $Q_1\mathbb{Z}^d$ and $Q_2\mathbb{Z}^d$, respectively. It follows that Ω is $Q_1\mathbb{Z}^d$-congruent to $Q_1\mathbb{T}^d$ and $Q_2\mathbb{Z}^d$-congruent to a subset of $Q_2\mathbb{T}^d$, and thus $Q_1\mathbb{T}^d$ is $(Q_1\mathbb{Z}^d+Q_2\mathbb{Z}^d)$-congruent to a subset of $Q_2\mathbb{T}^d$. Then, by Lemma 3.2, there exist a finite subset $\{(k_i,\ell_i):1\le i\le n\}$ of $\mathbb{Z}^d\times\mathbb{Z}^d$ and a partition $\{S_i:1\le i\le n\}$ of $Q_1\mathbb{T}^d$ such that $\{S_i+Q_1k_i+Q_2\ell_i:1\le i\le n\}$ is a mutually disjoint measurable collection of subsets of $Q_2\mathbb{T}^d$. Fix an arbitrary $\gamma\in(Q_1\mathbb{T}^d)\cap\mathbb{Z}^d$. We denote by $U(\gamma,\delta)$ the δ neighborhood of γ and define

$$U_{Q_1}(\gamma,\delta)=U(\gamma,\delta)\cap\{x+\gamma:x \text{ is in the first } Q_1\text{-quadrant}\}$$

for $\delta > 0$. We claim that there exists i_γ with $1 \le i_\gamma \le n$ such that $S_{i_\gamma} \cap U_{Q_1}(\gamma, \delta)$ has positive measure for any $\delta > 0$. If, for each $1 \le i \le n$, there exists $\delta_i > 0$ such that $|S_i \cap U_{Q_1}(\gamma, \delta_i)| = 0$, then by taking $\delta = \min_{1\le i\le n} \delta_i$, we have $|S_i \cap U_{Q_1}(\gamma, \delta)| = 0$ for each $1 \le i \le n$, which contradicts the fact that $\{S_i : 1 \le i \le n\}$ is a partition of $Q_1\mathbb{T}^d$. So we can choose some $1 \le i_\gamma \le n$ such that $|S_{i_\gamma} \cap U_{Q_1}(\gamma, \delta)| > 0$ for each $\delta > 0$, and consequently,

$$|(S_{i_\gamma} + Q_1 k_{i_\gamma} + Q_2 \ell_{i_\gamma}) \cap U_{Q_1}(\gamma + Q_1 k_{i_\gamma} + Q_2 \ell_{i_\gamma}, \delta)| > 0 \quad \text{for each } \delta > 0. \tag{3.1}$$

Also observe that $S_{i_\gamma} + Q_1 k_{i_\gamma} + Q_2 \ell_\gamma \subset Q_2\mathbb{T}^d$ leads to

$$\gamma + Q_1 k_{i_\gamma} + Q_2 \ell_{i_\gamma} \in Q_2\mathbb{T}^d. \tag{3.2}$$

We define $\mathcal{E} = \{\gamma + Q_1 k_{i_\gamma} : \gamma \in (Q_1\mathbb{T}^d) \cap \mathbb{Z}^d\}$. Since $(Q_1\mathbb{T}^d) \cap \mathbb{Z}^d$ is a full set of $\mathbb{Z}^d/Q_1\mathbb{Z}^d$, so does $\mathcal{E}$. For $\gamma \ne \tilde{\gamma}$ in $(Q_1\mathbb{T}^d) \cap \mathbb{Z}^d$, we have $\gamma + Q_1 k_{i_\gamma} + Q_2 \ell_{i_\gamma} \ne \tilde{\gamma} + Q_1 k_{i_{\tilde{\gamma}}} + Q_2 \ell_{i_{\tilde{\gamma}}}$ by (3.1) and the fact that $\{S_i + Q_1 k_i + Q_2 \ell_i : 1 \le i \le n\}$ is a mutually disjoint measurable collection of subsets of $Q_2\mathbb{T}^d$. Combined with (3.2) and the fact that $(Q_2\mathbb{T}^d) \cap \mathbb{Z}^d$ is a full set of $\mathbb{Z}^d/Q_2\mathbb{Z}^d$, it follows that $\mathcal{E}$ is a subset of some full set of $\mathbb{Z}^d/Q_2\mathbb{Z}^d$. In particular, when $|\det Q_1| = |\det Q_2|$, the cardinality of $\mathcal{E}$ is $|\det Q_2|$, and so $\mathcal{E}$ is also a full set of $\mathbb{Z}^d/Q_2\mathbb{Z}^d$. The proof is completed. $\square$

LEMMA 3.4

Given $Q \in \mathrm{GL}(\mathbb{Z}^d)$, let $\mathcal{E}$ be a full set of $\mathbb{Z}^d/Q^t\mathbb{Z}^d$, and let $\mathcal{E}'$ be a nonempty subset of $\mathcal{E}$. Then

(i) *$\{(1/\sqrt{|\det Q|})E_{Q^{-1}m}\chi_{\mathcal{E}} : m \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d\}$ is an orthonormal basis for $l^2(\mathcal{E})$.*

(ii) *$\{(1/\sqrt{|\det Q|})E_{Q^{-1}m}\chi_{\mathcal{E}'} : m \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d\}$ is a normalized tight frame for $l^2(\mathcal{E}')$.*

Proof

Clearly, the condition (i) is an equivalent statement of [32, Lemma 2.1], a special case of which was obtained in [15, Lemma 5.1]. Since $l^2(\mathcal{E}')$ is a closed subspace of $l^2(\mathcal{E})$, and the family in (ii) is the orthogonal projection onto $l^2(\mathcal{E}')$ of the one in (i), it is a normalized tight frame for $l^2(\mathcal{E}')$ by [2, Proposition 5.3.5]. $\square$

Note that $G(\cdot)$ in Theorem 2.1 is a $(|\det N| \times |\det M|)$ matrix. By Theorem 2.1, the condition $|\det N| \le |\det M|$ is necessary for the existence of a complete Gabor family $\mathcal{G}(g, N, M)$ in $l^2(\mathbb{Z}^d)$. It turns out that such a condition is also sufficient, and we can say much more about it.

THEOREM 3.1

For N, $M \in \mathrm{GL}(\mathbb{Z}^d)$, the following are equivalent.

(i) *There exists $g \in l^2(\mathbb{Z}^d)$ such that $\mathcal{G}(g, N, M)$ is a frame for $l^2(\mathbb{Z}^d)$.*
(ii) *There exists $g \in l^2(\mathbb{Z}^d)$ such that $\mathcal{G}(g, N, M)$ is complete in $l^2(\mathbb{Z}^d)$.*
(iii) *We have $|\det N| \le |\det M|$.*

Proof
It is obvious that (i) implies (ii). Also observing that the matrix $G(\cdot)$ in (2.2) is a $|\det N| \times |\det M|$ matrix-valued function, we have $\mathrm{rank}(G(\cdot)) \le |\det M|$. So (ii) implies (iii) by Theorem 2.1. Next we prove that (iii) implies (i). Suppose that (iii) holds. Then, by Lemma 3.3, there exists a set $\mathcal{E}$ in $\mathbb{Z}^d$ such that $\mathcal{E}$ is a full set of $\mathbb{Z}^d/N\mathbb{Z}^d$ and is a subset of some full set of $\mathbb{Z}^d/M^t\mathbb{Z}^d$. It follows that $l^2(\mathbb{Z}^d) = \bigoplus_{n\in\mathbb{Z}^d} l^2(\mathcal{E}+Nn)$ and that $\{E_{M^{-1}m}g : m \in (M\mathbb{T}^d)\cap\mathbb{Z}^d\}$ with $g = (1/\sqrt{|\det M|})\chi_{\mathcal{E}}$ is a normalized tight frame for $l^2(\mathcal{E})$ by Lemma 3.4. So $\{T_{Nn}E_{M^{-1}m}g : m \in (M\mathbb{T}^d)\cap\mathbb{Z}^d, n\in\mathbb{Z}^d\}$ is a normalized tight frame for $l^2(\mathbb{Z}^d)$; equivalently, $\mathcal{G}(g,N,M)$ is a normalized tight frame for $l^2(\mathbb{Z}^d)$. The proof is completed. □

LEMMA 3.5
Let N, $M \in \mathrm{GL}(\mathbb{Z}^d)$, and let $g \in l^2(\mathbb{Z}^d)$. Suppose that $\mathcal{G}(g,N,M)$ is a frame for $l^2(\mathbb{Z}^d)$. Then $\|\mathcal{S}_{g,g}^{-1/2}g\|^2 = |\det N|/|\det M|$.

Proof
Arbitrarily fix $\gamma \in (N\mathbb{T}^d)\cap\mathbb{Z}^d$. It is easy to check that

$$\sum_{n\in\mathbb{Z}^d}\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d}\Big|\Big\langle f, E_{M^{-1}m}T_{Nn}\frac{1}{\sqrt{|\det M|}}\chi_{\{\gamma\}}\Big\rangle\Big|^2 = \sum_{n\in\mathbb{Z}^d}|f(\gamma+Nn)|^2 = \|f\|^2$$

for $f \in l^2(\{\gamma\}+N\mathbb{Z}^d)$. So $\mathcal{G}((1/\sqrt{|\det M|})\chi_{\{\gamma\}}, N, M)$ is a normalized tight frame for $l^2(\{\gamma\}+N\mathbb{Z}^d)$. Also note that $l^2(\mathbb{Z}^d) = \bigoplus_{\gamma\in(N\mathbb{T}^d)\cap\mathbb{Z}^d} l^2(\{\gamma\}+N\mathbb{Z}^d)$. The family

$$\Big\{E_{M^{-1}m}T_{Nn}\frac{1}{\sqrt{|\det M|}}\chi_{\{\gamma\}} : m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d, n\in\mathbb{Z}^d, \gamma\in(N\mathbb{T}^d)\cap\mathbb{Z}^d\Big\}$$

is a normalized tight frame for $l^2(\mathbb{Z}^d)$. It follows that

$$\begin{aligned}
&\|\mathcal{S}_{g,g}^{-1/2}g\|^2\\
&\quad=\frac{1}{|\det M|}\sum_{\gamma\in(N\mathbb{T}^d)\cap\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d}|\langle \mathcal{S}_{g,g}^{-1/2}g, E_{M^{-1}m}T_{Nn}\chi_{\{\gamma\}}\rangle|^2\\
&\quad=\frac{1}{|\det M|}\sum_{\gamma\in(N\mathbb{T}^d)\cap\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d}|\langle E_{M^{-1}(-m)}T_{-Nn}\mathcal{S}_{g,g}^{-1/2}g, \chi_{\{\gamma\}}\rangle|^2\\
&\quad=\frac{1}{|\det M|}\sum_{\gamma\in(N\mathbb{T}^d)\cap\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d}|\langle E_{M^{-1}(M\tilde{m}-m)}T_{-Nn}\mathcal{S}_{g,g}^{-1/2}g, \chi_{\{\gamma\}}\rangle|^2,
\end{aligned}$$

where $\tilde{m} = (\mathrm{sgn}(m_0),\dots,\mathrm{sgn}(m_{d-1}))^t$, $(m_0,\dots,m_{d-1})^t$ denotes the M-coordinate of m and $\mathrm{sgn}(\cdot)$ denotes the sign function. Note that $x\in M\mathbb{T}^d$ if and only if its M-coordinate $(x_0,\dots,x_{d-1})^t$ satisfies $0\le x_i<1$ for $i\in\mathbb{N}_d$. It follows that

$\{M\tilde{m} - m : m \in (M\mathbb{T}^d) \cap \mathbb{Z}^d\} = (M\mathbb{T}^d) \cap \mathbb{Z}^d$, and consequently,

$$\begin{aligned} \|\mathcal{S}_{g,g}^{-1/2} g\|^2 = & \frac{1}{|\det M|} \\ & \times \sum_{\gamma \in (N\mathbb{T}^d)\cap\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} \sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} |\langle E_{M^{-1}m} T_{Nn} \mathcal{S}_{g,g}^{-1/2} g, \chi_{\{\gamma\}}\rangle|^2. \end{aligned} \tag{3.3}$$

However, $\mathcal{G}(\mathcal{S}_{g,g}^{-1/2} g, N, M)$ is a normalized tight frame for $l^2(\mathbb{Z}^d)$ since $\mathcal{G}(g, N, M)$ is a frame for $l^2(\mathbb{Z}^d)$. So, by (3.3), we have

$$\|\mathcal{S}_{g,g}^{-1/2} g\|^2 = \frac{1}{|\det M|} \sum_{\gamma \in (N\mathbb{T}^d)\cap\mathbb{Z}^d} \|\chi_{\{\gamma\}}\|^2 = \frac{|\det N|}{|\det M|}.$$

The proof is completed. □

THEOREM 3.2

Let N, $M \in \mathrm{GL}(\mathbb{Z}^d)$, and let $g \in l^2(\mathbb{Z}^d)$. Suppose that $\mathcal{G}(g, N, M)$ forms a frame for $l^2(\mathbb{Z}^d)$. Then $\mathcal{G}(g, N, M)$ is a Riesz basis for $l^2(\mathbb{Z}^d)$ if and only if $|\det N| = |\det M|$.

Proof

Note that $\mathcal{G}(g, N, M)$ is a Riesz basis for $l^2(\mathbb{Z}^d)$ if and only if $\mathcal{G}(\mathcal{S}_{g,g}^{-1/2} g, N, M)$ is an orthonormal basis for $l^2(\mathbb{Z}^d)$, which is also equivalent to $\|\mathcal{S}_{g,g}^{-1/2} g\| = 1$ since $\mathcal{G}(\mathcal{S}_{g,g}^{-1/2} g, N, M)$ is a normalized tight frame for $l^2(\mathbb{Z}^d)$. The theorem therefore follows by Lemma 3.5. □

THEOREM 3.3

For N, $M \in \mathrm{GL}(\mathbb{Z}^d)$, there exists $g \in l^2(\mathbb{Z}^d)$ such that $\mathcal{G}(g, N, M)$ is a Riesz basis for $l^2(\mathbb{Z}^d)$ if and only if $|\det N| = |\det M|$.

Proof

The necessity is an immediate consequence of Theorem 3.2. Next we turn to the sufficiency. Suppose $|\det N| = |\det M|$. Then, by Lemma 3.3, there exists a set $\mathcal{E}$ in $\mathbb{Z}^d$ such that $\mathcal{E}$ is a full set of both $\mathbb{Z}^d/N\mathbb{Z}^d$ and $\mathbb{Z}^d/M^t\mathbb{Z}^d$. It follows that $l^2(\mathbb{Z}^d) = \bigoplus_{n\in\mathbb{Z}^d} l^2(\mathcal{E} + Nn)$ and that $\{E_{M^{-1}m} g : m \in (M\mathbb{T}^d) \cap \mathbb{Z}^d\}$ with $g = (1/\sqrt{|\det M|})\chi_{\mathcal{E}}$ is an orthonormal basis for $l^2(\mathcal{E})$ by Lemma 3.4. So $\{T_{Nn} E_{M^{-1}m} g : m \in (M\mathbb{T}^d) \cap \mathbb{Z}^d, n \in \mathbb{Z}^d\}$ is an orthonormal basis for $l^2(\mathbb{Z}^d)$; equivalently, $\mathcal{G}(g, N, M)$ is an orthonormal basis for $l^2(\mathbb{Z}^d)$. The proof is completed. □

4. Frames and duals

In this section, we characterize Gabor frames and their Gabor duals, obtain a formula about the frame bounds of tight Gabor frames, derive an explicit expression of the canonical dual for a Gabor frame, and prove its norm minimality among all Gabor duals.

LEMMA 4.1

Let $Q \in \mathrm{GL}(\mathbb{Z}^d)$, and let $k' \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d$. Define

$$\mathcal{N}_1 := \big\{k \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d : k_i < 1 - k'_i \text{ for each } i \in \mathbb{N}_d\big\},$$
$$\mathcal{N}_2 := \big\{k \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d : 1 - k'_i \le k_i \text{ for some } i \in \mathbb{N}_d\big\},$$

where $(k_0, \dots, k_{d-1})^t$ and $(k'_0, \dots, k'_{d-1})^t$ denote the Q-coordinates of k and k', respectively. Then

$$(Q\mathbb{T}^d) \cap \mathbb{Z}^d = \{k + k' : k \in \mathcal{N}_1\} \cup \Big\{k + k' - \sum_{i \in \mathcal{I}_{k,k'}} Q\mathrm{e}_i : k \in \mathcal{N}_2\Big\}, \tag{4.1}$$

where e_i is defined by (1.3) for each $i \in \mathbb{N}_d$, and $\mathcal{I}_{k,k'} = \{i \in \mathbb{N}_d : 1 - k'_i \le k_i\}$ for k, $k' \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d$.

Proof

Define a mapping $\tau : (Q\mathbb{T}^d) \cap \mathbb{Z}^d \to \mathbb{Z}^d$ by

$$\tau(k) := \begin{cases} k + k' & \text{if } k \in \mathcal{N}_1, \\ k + k' - \sum_{i \in \mathcal{I}_{k,k'}} Q\mathrm{e}_i & \text{if } k \in \mathcal{N}_2, \end{cases}$$

for $k \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d$. Then $\mathrm{range}(\tau) \subset (Q\mathbb{T}^d) \cap \mathbb{Z}^d$ since $x \in Q\mathbb{T}^d$ if and only if its Q-coordinate $(x_0, \dots, x_{d-1})^t$ satisfies $0 \le x_i < 1$ for $i \in \mathbb{N}_d$. Also observing that $k - \tilde{k} \notin Q\mathbb{Z}^d$ for $k \ne \tilde{k}$ in $(Q\mathbb{T}^d) \cap \mathbb{Z}^d$, we have that τ is injective. The equation (4.1) therefore follows. $\square$

LEMMA 4.2

For $Q \in \mathrm{GL}(\mathbb{Z}^d)$ and $J \in \mathbb{N}$, let $(Q\mathbb{T}^d) \cap \mathbb{Z}^d = \{n_\ell : \ell \in \mathbb{N}_{|\det Q|}\}$, let $E = \{\epsilon_j : j \in \mathbb{N}_J\}$ be a finite set in $\mathbb{R}^d$, and let $\lambda(\cdot)$ be a $\mathbb{Z}^d$-periodic Lebesgue measurable function on $\mathbb{R}^d$. Define a $|\det Q| \times J$ matrix-valued function $\Lambda(\cdot)$ by

$$\Lambda(\cdot) := \big(\lambda(\cdot - Q^{-1}n_\ell - \epsilon_j)\big)_{0 \le \ell \le |\det Q|-1, 0 \le j \le J-1}$$

ae on $\mathbb{R}^d$. Then:

(i) *To each $k' \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d$ there corresponds a row permutation matrix $U_{k'}$ such that $\Lambda(\cdot - Q^{-1}k') = U_{k'}\Lambda(\cdot)$ ae on $\mathbb{R}^d$.*

(ii) *The rank of $\Lambda(\cdot)$ and the spectrum of $\Lambda(\cdot)\Lambda^*(\cdot)$ on $Q^{-1}\mathbb{T}^d$ completely determine the ones on $\mathbb{R}^d$.*

Proof

We use the notation in Lemma 4.1. Fix arbitrarily $k' \in (Q\mathbb{T}^d) \cap \mathbb{Z}^d$. Since $\lambda(\cdot)$ is $\mathbb{Z}^d$-periodic, we can rewrite (ℓ, j)-entries of $\Lambda(\cdot - Q^{-1}k')$ as

$$\big(\Lambda(\cdot - Q^{-1}k')\big)_{\ell,j} = \begin{cases} \lambda(\cdot - Q^{-1}(n_\ell + k') - \epsilon_j) & n_\ell \in \mathcal{N}_1, \\ \lambda\big(\cdot - Q^{-1}(n_\ell + k' - \sum_{i \in \mathcal{I}_{n_\ell,k'}} Q\mathrm{e}_i) - \epsilon_j\big) & n_\ell \in \mathcal{N}_2. \end{cases}$$

So $\Lambda(\cdot - Q^{-1}k')$ can be obtained by a permutation of the rows of $\Lambda(\cdot)$ by Lemma 4.1, which gives (i). It follows from (i) that

$$\langle \Lambda(\cdot - Q^{-1}k')\Lambda^*(\cdot - Q^{-1}k')x, x\rangle = \langle \Lambda(\cdot)\Lambda^*(\cdot)U_{k'}^* x, U_{k'}^* x\rangle$$

ae on $\mathbb{R}^d$ for $x \in \mathbb{C}^d$. Therefore, the rank of $\Lambda(\cdot)$ and the spectrum of $\Lambda(\cdot)\Lambda^*(\cdot)$ on $Q^{-1}\mathbb{T}^d$ completely determine the ones on $\mathbb{T}^d$ and thus on $\mathbb{R}^d$ by Lemma 2.2 and $\mathbb{Z}^d$-periodicity of $\lambda(\cdot)$. □

Let N, $M \in \mathrm{GL}(\mathbb{Z}^d)$, and let $g \in l^2(\mathbb{Z}^d)$. Define $G(\cdot)$ as in (2.2). Note that only the spectrum of $G(\cdot)G^*(\cdot)$ is involved in what follows. All conditions in our theorems are stated on $(N^t)^{-1}\mathbb{T}^d$ instead of $\mathbb{T}^d$ or $\mathbb{R}^d$ since they are equivalent to each other by Lemma 4.2.

LEMMA 4.3

For N, $M \in \mathrm{GL}(\mathbb{Z}^d)$ and $g \in l^2(\mathbb{Z}^d)$, $\mathcal{G}(g, N, M)$ is a Bessel sequence in $l^2(\mathbb{Z}^d)$ with Bessel bound D if and only if

$$G(\cdot)G^*(\cdot) \le D|\det N| I$$

almost everywhere on $(N^t)^{-1}\mathbb{T}^d$, where $G(\cdot)$ is defined as in (2.2).

Proof

Let $\Gamma = \{f \in l^2(\mathbb{Z}^d) : \hat{f} \in L^\infty(\mathbb{T}^d)\}$. Then Γ is dense in $l^2(\mathbb{Z}^d)$. For $f \in \Gamma$, by Lemma 2.3, we have

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^d}\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} |\langle f, E_{M^{-1}m}T_{Nn}g\rangle|^2 \\ &\quad= \sum_{n\in\mathbb{Z}^d}\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} \left|\int_{(N^t)^{-1}\mathbb{T}^d} \big(G^*(\xi)\mathcal{F}(\xi)\big)_m e^{2\pi i\langle Nn,\xi\rangle}\, d\xi\right|^2,\end{aligned}$$

where $\mathcal{F}(\xi) := \big(\hat{f}(\xi - (N^t)^{-1}k)\big)_{k\in(N^t\mathbb{T}^d)\cap\mathbb{Z}^d}$. Also observe that each component of $G^*(\cdot)\mathcal{F}(\cdot)$ is in $L^2((N^t)^{-1}\mathbb{T}^d)$, and $\{\sqrt{|\det N|}e^{2\pi i\langle Nn,\cdot\rangle} : n \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2((N^t)^{-1}\mathbb{T}^d)$. It follows that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^d}\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} |\langle f, E_{M^{-1}m}T_{Nn}g\rangle|^2 \\ &\quad= |\det N|^{-1} \sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} \int_{(N^t)^{-1}\mathbb{T}^d} |(G^*(\xi)\mathcal{F}(\xi))_m|^2\, d\xi \qquad (4.2)\\ &\quad= |\det N|^{-1} \int_{(N^t)^{-1}\mathbb{T}^d} \langle G(\xi)G^*(\xi)\mathcal{F}(\xi), \mathcal{F}(\xi)\rangle\, d\xi\end{aligned}$$

for $f \in \Gamma$. However,

$$\begin{aligned}\|f\|^2 &= \int_{\mathbb{T}^d} |\hat{f}(\xi)|^2\, d\xi = \sum_{k\in(N^t\mathbb{T}^d)\cap\mathbb{Z}^d} \int_{(N^t)^{-1}\mathbb{T}^d} \left|\hat{f}(\xi - (N^t)^{-1}k)\right|^2 d\xi \qquad (4.3)\\ &= \int_{(N^t)^{-1}\mathbb{T}^d} \|\mathcal{F}(\xi)\|^2\, d\xi\end{aligned}$$

by Lemma 2.2, where the norm of $\mathcal{F}(\xi)$ is taken in $\mathbb{C}^{|\det N|}$. Therefore, by [2, Lemma 3.2.6], $\mathcal{G}(g,N,M)$ is a Bessel sequence in $l^2(\mathbb{Z}^d)$ with Bessel bound D if and only if

$$\int_{(N^t)^{-1}\mathbb{T}^d} \langle G(\xi)G^*(\xi)\mathcal{F}(\xi), \mathcal{F}(\xi)\rangle \, d\xi \leq D|\det N| \int_{(N^t)^{-1}\mathbb{T}^d} \|\mathcal{F}(\xi)\|^2 \, d\xi \tag{4.4}$$

for $f \in \Gamma$. So the sufficiency obviously holds. Now we turn to the necessity. Suppose that $\mathcal{G}(g,N,M)$ is a Bessel sequence in $l^2(\mathbb{Z}^d)$ with Bessel bound D. We only need to prove that

$$\langle G(\xi)G^*(\xi)x, x\rangle \leq D|\det N|\|x\|^2 \tag{4.5}$$

for $x \in \mathbb{C}^{|\det N|}$ and ae $\xi \in (N^t)^{-1}\mathbb{T}^d$. Note that all entries of $G(\cdot)G^*(\cdot)$ are in $L^1(\mathbb{T}^d)$, and thus they are locally integrable by their $\mathbb{Z}^d$-periodicity. Then almost every interior point of $(N^t)^{-1}\mathbb{T}^d$ is a Lebesgue point of all entries of $G(\xi)G^*(\xi)$. Let ξ_0 be such an arbitrary point, and let x be an arbitrary vector in $\mathbb{C}^{|\det N|}$. To finish the proof, next we only need to prove that (4.5) holds for ξ_0 and x. Let $\epsilon > 0$ be such that the ϵ-neighborhood of ξ_0, denoted by $U(\xi_0,\epsilon)$, is contained in $(N^t)^{-1}\mathbb{T}^d$. Define $f \in l^2(\mathbb{Z}^d)$ by

$$\mathcal{F}(\xi) := |U(\xi_0,\epsilon)|^{-1/2}\chi_{U(\xi_0,\epsilon)}(\xi)x$$

for $\xi \in (N^t)^{-1}\mathbb{T}^d$. Then $f \in \Gamma$, and by (4.4) we have

$$\frac{1}{|U(\xi_0,\epsilon)|}\int_{U(\xi_0,\epsilon)} \langle G(\xi)G^*(\xi)x, x\rangle \leq D|\det N|\|x\|^2.$$

Letting $\epsilon \to 0$ leads to (4.5) for ξ_0 and x. The proof is completed. □

REMARK 4.1

By Lemma 4.3, for N, $M \in \mathrm{GL}(\mathbb{Z}^d)$ and $g \in l^2(\mathbb{Z}^d)$, $\mathcal{G}(g,N,M)$ is a Bessel sequence in $l^2(\mathbb{Z}^d)$ if and only if each entry of $G(\cdot)$ is in $L^\infty((N^t)^{-1}\mathbb{T}^d)$. Indeed, $\mathcal{G}(g,N,M)$ being a Bessel sequence in $l^2(\mathbb{Z}^d)$ is equivalent to the existence of a constant $0 < D < \infty$ such that $G(\cdot)G^*(\cdot) \leq D|\det N|I$; equivalently, $\|G^*(\cdot)x\| \leq \sqrt{D|\det N|}\|x\|$ for $x \in \mathbb{C}^{|\det N|}$ ae on $(N^t)^{-1}\mathbb{T}^d$. This is also equivalent to each entry of $G(\cdot)$ being in $L^\infty((N^t)^{-1}\mathbb{T}^d)$.

THEOREM 4.1

For N, $M \in \mathrm{GL}(\mathbb{Z}^d)$ and $g \in l^2(\mathbb{Z}^d)$, $\mathcal{G}(g,N,M)$ is a frame for $l^2(\mathbb{Z}^d)$ with frame bounds $0 < C \leq D < \infty$ if and only if

$$C|\det N|I \leq G(\cdot)G^*(\cdot) \leq D|\det N|I$$

ae on $(N^t)^{-1}\mathbb{T}^d$, where $G(\cdot)$ is defined as in (2.2).

Proof

By Lemma 4.3, we may as well assume that $\mathcal{G}(g,N,M)$ is a Bessel sequence in $l^2(\mathbb{Z}^d)$ with Bessel bound D. We use the notation in Lemma 4.3. Next we only need to prove that

$$C\|f\|^2 \le \sum_{n\in\mathbb{Z}^d} \sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} |\langle f, E_{M^{-1}m}T_{Nn}g\rangle|^2 \quad \text{for } f\in\Gamma \tag{4.6}$$

if and only if

$$C|\det N|\|x\|^2 \le \langle G(\xi)G^*(\xi)x, x\rangle \quad \text{for } x\in\mathbb{C}^{|\det N|} \text{ and ae } \xi\in(N^t)^{-1}\mathbb{T}^d \tag{4.7}$$

by [2, Lemmas 3.2.6, 5.1.7]. By (4.2) and (4.3), (4.6) can be rewritten as

$$\begin{aligned} &C|\det N|\int_{(N^t)^{-1}\mathbb{T}^d} \|\mathcal{F}(\xi)\|^2\,d\xi \\ &\le \int_{(N^t)^{-1}\mathbb{T}^d} \langle G(\xi)G^*(\xi)\mathcal{F}(\xi), \mathcal{F}(\xi)\rangle\,d\xi \quad \text{for } f\in\Gamma. \end{aligned} \tag{4.8}$$

So we only need to prove the equivalence between (4.8) and (4.7). This can be done by the same procedure as in Lemma 4.3. The proof is completed. □

COROLLARY 4.1

Let N, $M\in\mathrm{GL}(\mathbb{Z}^d)$, and let $g\in l^2(\mathbb{Z}^d)$ with $\hat{g}(\cdot)\in L^\infty(\mathbb{T}^d)$. For $L\in\mathbb{N}$, let $\{S_\ell : \ell\in\mathbb{N}_L\}$ be a partition of $(N^t)^{-1}\mathbb{T}^d$ such that, to each $\ell\in\mathbb{N}_L$, there corresponds a $(|\det N|\times|\det N|)$-invertible submatrix $G_\ell(\cdot)$ of $G(\cdot)$ satisfying that all entries of $(G_\ell(\cdot))^{-1}$ are in $L^\infty(S_\ell)$. Then $\mathcal{G}(g,N,M)$ is a frame for $l^2(\mathbb{Z}^d)$. In particular, when $|\det N| = |\det M|$, $\mathcal{G}(g,N,M)$ is a Riesz basis for $l^2(\mathbb{Z}^d)$.

Proof

By Theorems 4.1 and 3.2, we only need to prove that, to each $\ell\in\mathbb{N}_L$, there corresponds $0 < C_\ell \le D_\ell < \infty$ such that

$$C_\ell\|x\|^2 \le \langle G(\cdot)G^*(\cdot)x, x\rangle \le D_\ell\|x\|^2 \quad \text{for } x\in\mathbb{C}^{|\det N|} \tag{4.9}$$

ae on S_ℓ. Since $\hat{g}(\cdot)\in L^\infty(\mathbb{T}^d)$, the right-hand-side inequality in (4.9) holds. Now we prove the left-hand-side inequality in (4.9). Fix $\ell\in\mathbb{N}_L$. By Theorem 3.1 and the argument ahead of Theorem 2.1, we may as well assume that $G(\cdot) = (G_\ell(\cdot), \tilde{G}_\ell(\cdot))$ ae on S_ℓ when $|\det M| > |\det N|$, where the size of $\tilde{G}_\ell(\cdot)$ is $|\det N|\times(|\det M| - |\det N|)$. In particular, $G(\cdot) = G_\ell(\cdot)$ when $|\det M| = |\det N|$. Then

$$\langle G(\cdot)G^*(\cdot)x, x\rangle \ge \langle G_\ell(\cdot)G_\ell^*(\cdot)x, x\rangle = \|G_\ell^*(\cdot)x\|^2 \quad \text{for } x\in\mathbb{C}^{|\det N|} \tag{4.10}$$

ae on S_ℓ. Since each entry of $(G_\ell(\cdot))^{-1}$ is in $L^\infty(S_\ell)$, there exists a constant $0 < C_\ell < \infty$ such that $\|(G_\ell^*(\cdot))^{-1}x\|^2 \le C_\ell^{-1}\|x\|^2$ for $x\in\mathbb{C}^{|\det N|}$ ae on S_ℓ. So $\|G_\ell^*(\cdot)x\|^2 \ge C_\ell\|x\|^2$ for $x\in\mathbb{C}^{|\det N|}$ ae on S_ℓ, which together with (4.10) leads to the left-hand-side inequality in (4.9). □

THEOREM 4.2

Let N, $M\in\mathrm{GL}(\mathbb{Z}^d)$, and let $g\in l^2(\mathbb{Z}^d)$. Suppose that $\mathcal{G}(g,N,M)$ is a tight frame for $l^2(\mathbb{Z}^d)$ with frame bound C. Then $C = (|\det M|/|\det N|)\|g\|^2$.

Proof
By Theorem 4.1, we have $G(\xi)G^*(\xi) = C|\det N|I$, and consequently,

$$\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} |\hat{g}(\xi - M^{-1}m)|^2 = C|\det N| \tag{4.11}$$

for ae $\xi \in (N^t)^{-1}\mathbb{T}^d$. By Lemma 2.2, $\mathbb{T}^d$ is $\mathbb{Z}^d$-congruent to the set $\bigcup_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d}[M^{-1}\mathbb{T}^d - M^{-1}m]$, where the union is a disjoint one. It follows that

$$\|g\|^2 = \|\hat{g}\|^2 = \int_{\mathbb{T}^d} |\hat{g}(\xi)|^2\, d\xi = \sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} \int_{M^{-1}\mathbb{T}^d} |\hat{g}(\xi - M^{-1}m)|^2\, d\xi,$$

which together with (4.11) yields

$$\|g\|^2 = C\frac{|\det N|}{|\det M|}.$$

The proof is completed. □

THEOREM 4.3
Let N, $M \in \mathrm{GL}(\mathbb{Z}^d)$, and let g, $h \in l^2(\mathbb{Z}^d)$ be such that $\mathcal{G}(g, N, M)$ and $\mathcal{G}(h, N, M)$ are both Bessel sequences in $l^2(\mathbb{Z}^d)$. Let $G(\cdot)$ be defined as in (2.2), and let $H(\cdot)$ be defined analogously. Then $\mathcal{S}_{h,g} = I$ on $l^2(\mathbb{Z}^d)$ if and only if

$$G(\cdot)H^*(\cdot) = |\det N|I \tag{4.12}$$

ae on $(N^t)^{-1}\mathbb{T}^d$.

Proof
By Lemma 2.3 and (2.4), we have

$$\begin{aligned}(\mathcal{S}_{h,g}f)\hat{}\big(\xi - (N^t)^{-1}k\big) &= \sum_{n\in\mathbb{Z}^d}\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d}\Big(\int_{(N^t)^{-1}\mathbb{T}^d}\big(H^*(\xi')\mathcal{F}(\xi')\big)_m \\ &\quad \times e^{2\pi i\langle Nn,\xi'\rangle}\, d\xi'\Big)e^{-2\pi i\langle Nn,\xi\rangle}\big(G(\xi)\big)_{k,m}\end{aligned} \tag{4.13}$$

for $f \in l^2(\mathbb{Z}^d)$, $k \in (N^t\mathbb{T}^d) \cap \mathbb{Z}^d$, and ae $\xi \in (N^t)^{-1}\mathbb{T}^d$, where $\mathcal{F}(\xi) = (\hat{f}(\xi - (N^t)^{-1}k))_{k\in(N^t\mathbb{T}^d)\cap\mathbb{Z}^d}$. Since all entries of $H^*(\cdot)$ are in $L^\infty((N^t)^{-1}\mathbb{T}^d)$ by Remark 4.1, all entries of $H^*(\cdot)\mathcal{F}(\cdot)$ are in $L^2((N^t)^{-1}\mathbb{T}^d)$. However, $\{\sqrt{|\det N|} \times e^{-2\pi i\langle Nn,\cdot\rangle} : n \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2((N^t)^{-1}\mathbb{T}^d)$. So (4.13) can be rewritten as

$$\begin{aligned}(\mathcal{S}_{h,g}f)\hat{}\big(\xi - (N^t)^{-1}k\big) &= |\det N|^{-1}\sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d}\big(H^*(\xi)\mathcal{F}(\xi)\big)_m\big(G(\xi)\big)_{k,m} \\ &= |\det N|^{-1}\big(G(\xi)H^*(\xi)\mathcal{F}(\xi)\big)_k\end{aligned} \tag{4.14}$$

for $f \in l^2(\mathbb{Z}^d)$, $k \in (N^t\mathbb{T}^d) \cap \mathbb{Z}^d$, and ae $\xi \in (N^t)^{-1}\mathbb{T}^d$. It follows that $\mathcal{S}_{h,g} = I$ on $l^2(\mathbb{Z}^d)$ if and only if

$$G(\xi)H^*(\xi)\mathcal{F}(\xi) = |\det N|\mathcal{F}(\xi) \tag{4.15}$$

for $f \in l^2(\mathbb{Z}^d)$ and ae $\xi \in (N^t)^{-1}\mathbb{T}^d$. Obviously, (4.12) implies (4.15). Conversely, for an arbitrary $x \in \mathbb{C}^{|\det N|}$, we define $f \in l^2(\mathbb{Z}^d)$ by $\mathcal{F}(\xi) = x$ for ae $\xi \in (N^t)^{-1}\mathbb{T}^d$. Then applying (4.15) to such f gives (4.12). The proof is completed. $\square$

THEOREM 4.4

Let N, $M \in \mathrm{GL}(\mathbb{Z}^d)$, and let $g \in l^2(\mathbb{Z}^d)$ be such that $\mathcal{G}(g, N, M)$ is a frame for $l^2(\mathbb{Z}^d)$. Let $G(\cdot)$ be defined as in (2.2). Define $\gamma_0 \in l^2(\mathbb{Z}^d)$ via its Fourier transform by

$$\begin{aligned} &\big(\hat{\gamma}_0(\cdot - (N^t)^{-1}k)\big)_{k \in (N^t\mathbb{T}^d)\cap\mathbb{Z}^d} \\ &\quad = |\det N| \big(G(\cdot)G^*(\cdot)\big)^{-1} \big(\hat{g}(\cdot - (N^t)^{-1}k)\big)_{k \in (N^t\mathbb{T}^d)\cap\mathbb{Z}^d} \end{aligned} \tag{4.16}$$

ae on $(N^t)^{-1}\mathbb{T}^d$. Then

(i) *γ_0 is the canonical Gabor dual of g, i.e., $\gamma_0 = \mathcal{S}_{g,g}^{-1} g$;*

(ii) *$\|\gamma_0\| \le \|\gamma\|$ for an arbitrary Gabor dual γ of g, and the equality holds if and only if $\gamma = \gamma_0$.*

Proof

(i) By Theorem 4.1, there exists $0 < C < \infty$ such that

$$\big(G(\cdot)G^*(\cdot)\big)^{-1} \le C^{-1} |\det N|^{-1} I$$

ae on $(N^t)^{-1}\mathbb{T}^d$, which together with Lemma 2.2 implies that $\hat{\gamma}_0(\cdot)$ is an essentially bounded measurable function. It follows that γ_0 is well defined, and $\mathcal{G}(\gamma_0, N, M)$ is a Bessel sequence in $l^2(\mathbb{Z}^d)$ by Remark 4.1. From (4.14), we have

$$\begin{aligned} &\big((\mathcal{S}_{g,g} f)\hat{}(\cdot - (N^t)^{-1}k)\big)_{k \in (N^t\mathbb{T}^d)\cap\mathbb{Z}^d} \\ &\quad = |\det N|^{-1} G(\cdot)G^*(\cdot) \big(\hat{f}(\cdot - (N^t)^{-1}k)\big)_{k \in (N^t\mathbb{T}^d)\cap\mathbb{Z}^d} \end{aligned}$$

for $f \in l^2(\mathbb{Z}^d)$ ae on $(N^t)^{-1}\mathbb{T}^d$. Then taking $f = \mathcal{S}_{g,g}^{-1} g$ leads to

$$\begin{aligned} &\big(\hat{g}(\cdot - (N^t)^{-1}k)\big)_{k \in (N^t\mathbb{T}^d)\cap\mathbb{Z}^d} \\ &\quad = |\det N|^{-1} G(\cdot)G^*(\cdot) \big((\mathcal{S}_{g,g}^{-1} g)\hat{}(\cdot - (N^t)^{-1}k)\big)_{k \in (N^t\mathbb{T}^d)\cap\mathbb{Z}^d} \end{aligned}$$

ae on $(N^t)^{-1}\mathbb{T}^d$. So $\hat{\gamma}_0(\cdot - (N^t)^{-1}k) = (\mathcal{S}_{g,g}^{-1} g)\hat{}(\cdot - (N^t)^{-1}k)$ for $k \in (N^t\mathbb{T}^d) \cap \mathbb{Z}^d$ ae on $(N^t)^{-1}\mathbb{T}^d$, and consequently, $\gamma_0 = \mathcal{S}_{g,g}^{-1} g$.

(ii) Since $\mathcal{S}_{\gamma,g} = \mathcal{S}_{\gamma_0,g} = I$ on $l^2(\mathbb{Z}^d)$, we have

$$\begin{aligned} 0 &= \langle \mathcal{S}_{\gamma-\gamma_0,g}\gamma_0, \gamma_0 \rangle \\ &= \sum_{n \in \mathbb{Z}^d} \sum_{m \in (M\mathbb{T}^d)\cap\mathbb{Z}^d} \langle \gamma_0, E_{M^{-1}m} T_{Nn}(\gamma - \gamma_0) \rangle \langle E_{M^{-1}m} T_{Nn} g, \gamma_0 \rangle \\ &= \sum_{n \in \mathbb{Z}^d} \sum_{m \in (M\mathbb{T}^d)\cap\mathbb{Z}^d} \langle E_{M^{-1}(-m)} T_{-Nn}\gamma_0, \gamma - \gamma_0 \rangle \langle g, E_{M^{-1}(-m)} T_{-Nn}\gamma_0 \rangle \end{aligned} \tag{4.17}$$

$$= \sum_{n\in\mathbb{Z}^d} \sum_{m\in(M\mathbb{T}^d)\cap\mathbb{Z}^d} \langle E_{M^{-1}(M\tilde{m}-m)} T_{Nn}\gamma_0, \gamma-\gamma_0\rangle$$
$$\times \langle g, E_{M^{-1}(M\tilde{m}-m)} T_{Nn}\gamma_0\rangle,$$

where $\tilde{m} = (\mathrm{sgn}(m_0), \ldots, \mathrm{sgn}(m_{d-1}))^t$, $(m_0, \ldots, m_{d-1})^t$ denotes the M-coordinate of m, and $\mathrm{sgn}(\cdot)$ denotes the sign function. Note that $x \in M\mathbb{T}^d$ if and only if its M-coordinate $(x_0, \ldots, x_{d-1})^t$ satisfies $0 \le x_i < 1$ for $i \in \mathbb{N}_d$. It follows that $\{M\tilde{m} - m : m \in (M\mathbb{T}^d) \cap \mathbb{Z}^d\} = (M\mathbb{T}^d) \cap \mathbb{Z}^d$, and consequently, (4.17) can be rewritten as

$$0 = \langle g, \mathcal{S}_{\gamma_0,\gamma_0}(\gamma-\gamma_0)\rangle = \langle \mathcal{S}_{\gamma_0,\gamma_0} g, \gamma-\gamma_0\rangle.$$

However, $\mathcal{S}_{\gamma_0,\gamma_0} = \mathcal{S}_{g,g}^{-1}$ by [2, Lemma 5.1.5]. So $\mathcal{S}_{\gamma_0,\gamma_0} g = \mathcal{S}_{g,g}^{-1} g = \gamma_0$, and thus $\langle \gamma_0, \gamma-\gamma_0\rangle = 0$, which implies that

$$\|\gamma\|^2 = \|\gamma_0\|^2 + \|\gamma-\gamma_0\|^2 \ge \|\gamma_0\|^2,$$

and the equality holds if and only if $\gamma = \gamma_0$. The proof is completed. □

We conclude this section with two classes of examples.

EXAMPLE 4.1

Let N, $M \in \mathrm{GL}(\mathbb{Z}^d)$ with $\mathbb{T}^d \cap (N^t)^{-1}\mathbb{Z}^d \subset \mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$, and let $g \in l^2(\mathbb{Z}^d)$ be such that $\mathrm{supp}(\hat{g}(\cdot)) = (N^t)^{-1}\mathbb{T}^d + \mathbb{Z}^d$. Then we have the following.

(i) $\mathcal{G}(g, N, M)$ is complete in $l^2(\mathbb{Z}^d)$.

(ii) When

$$C\sqrt{|\det N|}\chi_{(N^t)^{-1}\mathbb{T}^d+\mathbb{Z}^d}(\cdot) \le |\hat{g}(\cdot)| \le D\sqrt{|\det N|}\chi_{(N^t)^{-1}\mathbb{T}^d+\mathbb{Z}^d}(\cdot) \tag{4.18}$$

ae on $\mathbb{R}^d$ for some constants $0 < C \le D < \infty$, $\mathcal{G}(g, N, M)$ is a frame for $l^2(\mathbb{Z}^d)$. In particular, it is a Riesz basis for $l^2(\mathbb{Z}^d)$ if $\mathbb{T}^d \cap (N^t)^{-1}\mathbb{Z}^d = \mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$ in addition.

(iii) When $\hat{g}(\cdot) = \sqrt{|\det N|}\chi_{(N^t)^{-1}\mathbb{T}^d+\mathbb{Z}^d}(\cdot)$ ae on $\mathbb{R}^d$, then $\mathcal{G}(g, N, M)$ is a tight frame for $l^2(\mathbb{Z}^d)$. In particular, it is an orthonormal basis for $l^2(\mathbb{Z}^d)$ if $\mathbb{T}^d \cap (N^t)^{-1}\mathbb{Z}^d = \mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$ in addition.

Proof

Let $G(\cdot)$ be defined as in (2.2). We use the notation in Lemma 2.1. By $\mathbb{Z}^d$-periodicity of $\hat{g}$ and the argument ahead of Theorem 2.1, we may as well assume that

$$G(\cdot) = \big(G_0(\cdot), G_0(\cdot - \varepsilon_1), \ldots, G_0(\cdot - \varepsilon_{L-1})\big),$$

where $G_0(\cdot) = (\hat{g}(\cdot - m - k))_{k\in\Omega_0, m\in\Omega_0}$ is a $|\det N| \times |\det N|$ matrix-valued function. Since $-N^t\Omega_0$ and $N^t\Omega_0$ are both full sets of $\mathbb{Z}^d/(N^t\mathbb{Z}^d)$, we can define a bijection τ: $\Omega_0 \to \Omega_0$ by $N^t\tau(k) + N^t k \in N^t\mathbb{Z}^d$, that is, $\tau(k) + k \in \mathbb{Z}^d$. Then we claim that

$$\hat{g}(\cdot - m - k)\begin{cases}\neq 0 & \text{if } m = \tau(k),\\ = 0 & \text{if } m \neq \tau(k),\end{cases} \tag{4.19}$$

ae on $(N^t)^{-1}\mathbb{T}^d$ for k, $m \in \Omega_0$. Indeed, given arbitrarily $k, m \in \Omega_0$. If $m = \tau(k)$, then $\hat{g}(\cdot - m - k) = \hat{g}(\cdot) \neq 0$ ae on $(N^t)^{-1}\mathbb{T}^d$. If $m \neq \tau(k)$, then $m + k \in \Omega_0 \backslash \{0\} + \mathbb{Z}^d$. Also observing that $(N^t)^{-1}\mathbb{T}^d \cap ((N^t)^{-1}\mathbb{T}^d + \Omega_0\backslash\{0\} + \mathbb{Z}^d) = \emptyset$ by Lemma 2.2, we have $\hat{g}(\cdot - m - k) = 0$ ae on $(N^t)^{-1}\mathbb{T}^d$. So (4.19) holds.

(i) From (4.19), it follows that $\operatorname{rank}(G_0(\cdot)) = |\det N|$ ae on $(N^t)^{-1}\mathbb{T}^d$, and thus $\operatorname{rank}(G(\cdot)) = |\det N|$ ae on $(N^t)^{-1}\mathbb{T}^d$. By Theorem 2.1, $\mathcal{G}(g, N, M)$ is complete in $l^2(\mathbb{Z}^d)$.

(ii) Suppose that (4.18) holds. Note that $|\det N| = |\det M|$ when $\mathbb{T}^d \cap (N^t)^{-1}\mathbb{Z}^d = \mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$. We only need to prove that $\mathcal{G}(g, N, M)$ is a frame for $l^2(\mathbb{Z}^d)$ by Theorem 3.2. From (4.19), we have $C^2|\det N| I \leq G_0(\cdot)G_0^*(\cdot) \leq D^2|\det N| I$ ae on $(N^t)^{-1}\mathbb{T}^d$ and thus on $\mathbb{R}^d$ by Lemma 4.2. Therefore, we have

$$LC^2|\det N| I \leq G(\cdot)G^*(\cdot) = \sum_{\ell \in \mathbb{N}_L} G_0(\cdot - \varepsilon_\ell)G_0^*(\cdot - \varepsilon_\ell) \leq LD^2|\det N| I$$

ae on $\mathbb{R}^d$, which implies that $\mathcal{G}(g, N, M)$ is a frame for $l^2(\mathbb{Z}^d)$ with frame bounds LC^2 and LD^2 by Theorem 4.1.

(iii) Suppose $\hat{g}(\cdot) = \sqrt{|\det N|}\chi_{(N^t)^{-1}\mathbb{T}^d + \mathbb{Z}^d}(\cdot)$ ae on $\mathbb{R}^d$. Take $C = D = 1$ in (ii). Then $\mathcal{G}(g, N, M)$ is a tight frame for $l^2(\mathbb{Z}^d)$ by (ii). In particular, when $\mathbb{T}^d \cap (N^t)^{-1}\mathbb{Z}^d = \mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$, $L = 1$. Also observing that $\|g\| = 1$ leads to the fact that $\mathcal{G}(g, N, M)$ is an orthonormal basis for $l^2(\mathbb{Z}^d)$. □

REMARK 4.2

Let $N \in \mathrm{GL}(\mathbb{Z}^d)$ be a triangular matrix with the ith diagonal element being r, other diagonal elements being 1, and all elements of the ith row being in $r\mathbb{Z}$, and let $M \in \mathrm{GL}(\mathbb{Z}^d)$ be a triangular matrix with the ith diagonal element being r and all elements of the ith column being in $r\mathbb{Z}$. Then $\mathbb{T}^d \cap (N^t)^{-1}\mathbb{Z}^d \subset \mathbb{T}^d \cap M^{-1}\mathbb{Z}^d$ as required in Example 4.1.

Next we turn to another class of examples, where the Gabor generators g are finitely supported.

EXAMPLE 4.2

Let $r, l_1, l_2, l_3 \in \mathbb{Z}\backslash\{0\}$ and $a, b, c \in \mathbb{R}$. Let $N = \left(\begin{smallmatrix} 1 & 1\\ 1 & -1\end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} 0 & 1\\ 2r & 0\end{smallmatrix}\right)$. Define $g \in l^2(\mathbb{Z}^2)$ by

$$g\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = 1, \qquad g\left(\begin{pmatrix}-1\\0\end{pmatrix}\right) = g\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = a, \qquad g\left(\begin{pmatrix}2l_3+1\\0\end{pmatrix}\right) = -c,$$

$$g\left(\begin{pmatrix}-2l_1\\-2l_2\end{pmatrix}\right) = g\left(\begin{pmatrix}2l_1\\2l_2\end{pmatrix}\right) = b, \qquad g\left(-\begin{pmatrix}2l_3+1\\0\end{pmatrix}\right) = c,$$

$$g(j)=0, \quad j\in\mathbb{Z}^2\setminus\left\{\begin{pmatrix}0\\0\end{pmatrix}, \pm\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-2l_1\\-2l_2\end{pmatrix}, \begin{pmatrix}2l_1\\2l_2\end{pmatrix}, \pm\begin{pmatrix}2l_3+1\\0\end{pmatrix}\right\}.$$

Then

(i) when $a\neq 0$ or $c\neq 0$, $\mathcal{G}(g,N,M)$ is complete in $l^2(\mathbb{Z}^2)$;

(ii) when $|b|<1/2$ and $ac\neq 0$, $\mathcal{G}(g,N,M)$ is a frame for $l^2(\mathbb{Z}^2)$. In particular, when $|r|=1$, $\mathcal{G}(g,N,M)$ is a Riesz basis for $l^2(\mathbb{Z}^2)$.

Proof

Let $G(\cdot)$ be defined as in (2.2). It is easy to check that

$$\mathbb{T}^2\cap(N^t)^{-1}\mathbb{Z}^2=\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1/2\\1/2\end{pmatrix}\right\}, \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1/2\\0\end{pmatrix}\right\}\subset\mathbb{T}^2\cap M^{-1}\mathbb{Z}^2.$$

Then, by the argument ahead of Theorem 2.1, we may as well assume that $G(\cdot)$ have the following form:

$$G(\cdot)=\big(G_0(\cdot),G_1(\cdot)\big),$$

where

$$G_0(\cdot)=\begin{pmatrix}\hat{g}(\cdot) & \hat{g}\left(\cdot-\begin{pmatrix}1/2\\0\end{pmatrix}\right)\\ \hat{g}\left(\cdot-\begin{pmatrix}1/2\\1/2\end{pmatrix}\right) & \hat{g}\left(\cdot-\begin{pmatrix}0\\1/2\end{pmatrix}\right)\end{pmatrix}.$$

A simple computation shows

$$\begin{aligned}\hat{g}(\xi)=\hat{g}\left(\xi-\begin{pmatrix}0\\1/2\end{pmatrix}\right)&=1+2a\cos 2\pi\xi_0+2b\cos(4l_1\pi\xi_0+4l_2\pi\xi_1)\\&\quad+i2c\sin(4l_3\pi\xi_0+2\pi\xi_0),\\ \hat{g}\left(\xi-\begin{pmatrix}1/2\\0\end{pmatrix}\right)&=\hat{g}\left(\xi-\begin{pmatrix}1/2\\1/2\end{pmatrix}\right)\\&=1-2a\cos 2\pi\xi_0+2b\cos(4l_1\pi\xi_0+4l_2\pi\xi_1)\\&\quad-i2c\sin(4l_3\pi\xi_0+2\pi\xi_0)\end{aligned}$$

for ae $\xi=\begin{pmatrix}\xi_0\\\xi_1\end{pmatrix}\in\mathbb{R}^2$. It follows that

$$\det\big(G_0(\xi)\big)=8\big(1+2b\cos(4l_1\pi\xi_0+4l_2\pi\xi_1)\big)\big(a\cos 2\pi\xi_0+ic\sin(4l_3\pi\xi_0+2\pi\xi_0)\big).$$

(i) When $a\neq 0$ or $c\neq 0$, the set of zeros of $\det(G_0(\xi))$ has measure zero. So $\operatorname{rank}(G_0(\cdot))=2$, and thus $\operatorname{rank}(G(\cdot))=2$ ae on $\mathbb{R}^2$. Therefore $\mathcal{G}(g,N,M)$ is complete in $l^2(\mathbb{Z}^2)$ by Theorem 2.1.

(ii) When $|b|<1/2$ and $ac\neq 0$, $\det(G_0(\cdot))\neq 0$ on $\mathbb{R}^2$. Note that $\det(G_0(\cdot))$ is a $\mathbb{Z}^d$-periodic continuous function. It follows that $|\det(G_0(\cdot))|$ has a positive bound from below on $\mathbb{R}^2$, which implies that all entries of $(G_0(\cdot))^{-1}$ are essentially bounded measurable functions. Therefore, by Corollary 4.1, $\mathcal{G}(g,N,M)$ is a frame for $l^2(\mathbb{Z}^2)$. In particular, when $|r|=1$, $\mathcal{G}(g,N,M)$ is a Riesz basis for $l^2(\mathbb{Z}^2)$ by Corollary 4.1. $\square$

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