The CR almost Schur lemma and Lee conjecture

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Abstract In this paper, we first derive the CR analogue of the almost Schur lemma on a pseudo-Hermitian $(2n+1)$-manifold (M, J, θ) for $n \geq 2$. Second, we study a sufficient condition for the existence of a pseudo-Einstein contact form when the CR structure of M has vanishing first Chern class which is related to the J. M. Lee conjecture.

1. Introduction

Let (M^n, g) be a closed Riemannian manifold. The Schur lemma says that every Einstein manifold of dimension $n \geq 3$ has constant scalar curvature. Here g is defined to be Einstein if its Ricci tensor is proportional to the metric, that is, $Rc = (S/n)g$. Recently, C. De Lellis and P. Topping proved an interesting result that generalizes the Schur lemma.

PROPOSITION 1.1

(Almost Schur lemma [LT, Theorem 0.1]) For $n \geq 3$, if (M^n, g) is a closed Riemannian manifold with nonnegative Ricci tensor, then

$$\int_M (S - \overline{S})^2 \leq \frac{4n(n-1)}{(n-2)^2} \int_M \left| Rc - \frac{S}{n} g \right|^2,$$

where $\overline{S}$ is the average value of the scalar curvature S of g.

Obviously the classical Schur lemma follows directly from this theorem. Later, Y. Ge and G. Wang [GW] showed that Proposition 1.1 holds under the condition of nonnegativity of the scalar curvature for dimension $n = 4$ and equality holds if and only if (M^4, g) is an Einstein manifold.

Let (M, J, θ) be a closed (i.e., compact without boundary) pseudo-Hermitian $(2n+1)$-manifold (see [Le] and Section 2 for basic notions in pseudo-Hermitian geometry). In this paper, we first consider a CR analogue of the almost Schur lemma on a closed pseudo-Hermitian $(2n+1)$-manifold M for $n \geq 2$. Second, we study a sufficient condition for the existence of a global pseudo-Einstein contact

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form when the CR structure of M has vanishing first Chern class which is related to the J. M. Lee conjecture (see [Le]). The Lee conjecture says that any closed pseudo-Hermitian CR manifold M whose CR structure has vanishing first Chern class admits a global pseudo-Einstein contact form.

We recall that a contact form θ on M is said to be pseudo-Einstein if its Webster–Ricci tensor $R_{\alpha\overline{\beta}}$ is proportional to the Levi form $h_{\alpha\overline{\beta}}$, that is,

$$R_{\alpha\overline{\beta}} = \frac{R}{n} h_{\alpha\overline{\beta}}.$$

Here $R = h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$ is the Webster scalar curvature of θ: the pseudo-Einstein condition yet less rigid than the Einstein condition in Riemannian geometry. Indeed, the Bianchi identity (3.7) no longer implies that R is a constant due to the presence of pseudo-Hermitian torsion terms.

The natural problem is to find a global pseudo-Einstein contact form on a closed pseudo-Hermitian manifold. Note that any contact form on a closed pseudo-Hermitian 3-manifold is actually pseudo-Einstein (since the Webster–Ricci tensor has only one component $R_{1\overline{1}}$); hence we assume that M has CR dimension $n \geq 2$.

First we state the following CR analogue of the almost Schur lemma on a closed pseudo-Hermitian $(2n+1)$-manifold M for $n \geq 2$.

THEOREM 1.2

For $n \geq 2$, if (M, J, θ) is a closed pseudo-Hermitian $(2n+1)$-manifold with

$$\Big(\mathrm{Ric} - \frac{n+1}{2}\,\mathrm{Tor}\Big)(Z, Z) \geq 0 \quad \text{for all } Z \in T_{1,0}(M),$$

then

$$\begin{aligned} \int_M (R - \overline{R})^2 \leq{}& \frac{2n(n+1)}{(n-1)(n+2)} \int_M \sum_{\alpha,\beta} \Big| R_{\alpha\overline{\beta}} - \frac{R}{n} h_{\alpha\overline{\beta}} \Big|^2 \\ &+ 2in \int_M (A_{\overline{\alpha}\overline{\beta}} \varphi^{\overline{\alpha}\overline{\beta}} - A_{\alpha\beta}\varphi^{\alpha\beta}), \end{aligned} \tag{1.1}$$

where $\overline{R}$ is the average value of R over M and φ is the unique real solution of $\Delta_b \varphi = R - \overline{R}$ with $\int_M \varphi = 0$. Moreover, if the equality holds, then

$$\int_M (R - \overline{R})^2 = \frac{2n(n+1)}{(n-1)(n+2)} \int_M \sum_{\alpha,\beta} \Big| R_{\alpha\overline{\beta}} - \frac{R}{n} h_{\alpha\overline{\beta}} \Big|^2$$

and the contact form $e^{(1/(n+1))\varphi}\theta$ will be pseudo-Einstein.

This theorem gives a characterization of pseudo-Einstein contact forms. It is important to note that the Bianchi identity (3.7) implies that

$$\Big(R_{\alpha\overline{\beta}} - \frac{R}{n} h_{\alpha\overline{\beta}}\Big)^{,\alpha\overline{\beta}} + \Big(R_{\alpha\overline{\beta}} - \frac{R}{n} h_{\alpha\overline{\beta}}\Big)^{,\overline{\beta}\alpha} = \frac{n-1}{n} \Delta_b R + 2(n-1)\,\mathrm{Im}(A_{\alpha\beta}{}^{,\alpha\beta}).$$

Thus when a contact form θ is pseudo-Einstein, then R being a constant on M (or $\Delta_b R = 0$ on M) is equivalent to the condition $\operatorname{Im}(A_{\alpha\beta}{}^{,\alpha\beta}) = 0$. Therefore, after integration by parts of (1.1), we obtain the following.

COROLLARY 1.3

In addition to the same conditions as in Theorem 1.2, we assume that $\operatorname{Im}(A_{\alpha\beta}{}^{,\alpha\beta}) = 0$. *Then*

$$\int_M (R-\overline{R})^2 \le \frac{2n(n+1)}{(n-1)(n+2)} \int_M \sum_{\alpha,\beta} \Big| R_{\alpha\overline{\beta}} - \frac{R}{n} h_{\alpha\overline{\beta}} \Big|^2.$$

This corollary implies that in addition if the contact form θ is pseudo-Einstein, then R will be a constant on M. Since

$$\sum_{\alpha,\beta} \Big| R_{\alpha\overline{\beta}} - \frac{\overline{R}}{n} h_{\alpha\overline{\beta}} \Big|^2 = \sum_{\alpha,\beta} \Big| R_{\alpha\overline{\beta}} - \frac{R}{n} h_{\alpha\overline{\beta}} \Big|^2 + \frac{1}{n}(R-\overline{R})^2,$$

we immediately get the following.

COROLLARY 1.4

Under the same conditions as in Corollary 1.3, we have

$$\int_M \sum_{\alpha,\beta} \Big| R_{\alpha\overline{\beta}} - \frac{\overline{R}}{n} h_{\alpha\overline{\beta}} \Big|^2 \le \frac{n(n+3)}{(n-1)(n+2)} \int_M \sum_{\alpha,\beta} \Big| R_{\alpha\overline{\beta}} - \frac{R}{n} h_{\alpha\overline{\beta}} \Big|^2.$$

Now we study a sufficient condition for the existence of a pseudo-Einstein contact form. It was shown, by J. M. Lee in [Le], that if a pseudo-Hermitian manifold M admits a global pseudo-Einstein contact form, the first Chern class $c_1(T_{1,0}M)$ of the contact distribution vanishes. Conversely, we have the following, a sufficient condition for the existence of a pseudo-Einstein contact form.

THEOREM 1.5

Suppose that (M, J, θ) *is a closed pseudo-Hermitian* $(2n+1)$*-manifold whose CR structure has vanishing first Chern class and there exists a contact form* $\widehat{\theta}$ *on* M *which is conformal to* θ *such that*

$$A_{\alpha\beta}{}^{,\alpha} = 0 \qquad \text{and} \qquad [\overline{\partial}_b^*, \nabla_T] = 0. \tag{1.2}$$

Then M *admits a global pseudo-Einstein contact form.*

From Corollary 1.3, we believe that there are more general conditions than in Theorem 1.5 on M for the existence of a global pseudo-Einstein contact form.

In particular, it follows from (4.2) that conditions (1.2) are satisfied on a closed pseudo-Hermitian $(2n+1)$-manifold with vanishing pseudo-Hermitian torsion. Thus our Theorem 1.5 generalizes the following result of J. M. Lee.

COROLLARY 1.6

([Le, Theorem E, Part (ii)]) Suppose that (M, J, θ) is a closed pseudo-Hermitian $(2n+1)$-manifold whose CR structure has vanishing first Chern class and there exists a contact form $\widehat{\theta}$ on M which is conformal to θ with free pseudo-Hermitian torsion. Then M admits a global pseudo-Einstein contact form.

2. Preliminary

Let us give a brief introduction to pseudo-Hermitian geometry (see [Le] for more details). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ, $\dim_R \xi = 2n$. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. A CR structure J can extend to $\mathbb{C} \otimes \xi$ and decomposes $\mathbb{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to i and $-i$, respectively. A pseudo-Hermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ. Such a choice determines a unique real vector field T, which is called the characteristic vector field of θ, such that $\theta(T) = 1$ and $d\theta(T, \cdot) = 0$. Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}, Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$, and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies

$$d\theta = i h_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}, \tag{2.1}$$

for some positive-definite Hermitian matrix of functions $(h_{\alpha\bar{\beta}})$. Actually we can always choose Z_α such that $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$; hence, throughout this paper, we assume $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$.

The pseudo-Hermitian connection of (J, θ) is the connection ∇ on $TM \otimes \mathbb{C}$ given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta, \qquad \nabla Z_{\bar{\alpha}} = {\omega_{\bar{\alpha}}}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \qquad \nabla T = 0,$$

where ${\omega_\alpha}^\beta$ are the 1-forms uniquely determined by the following equations:

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta,$$

$$\tau_\alpha \wedge \theta^\alpha = 0, \qquad {\omega_\alpha}^\beta + \omega_{\overline{\beta}}{}^{\overline{\alpha}} = 0.$$

We can write $\tau_\alpha = A_{\alpha\beta}\theta^\beta$ with $A_{\alpha\beta} = A_{\beta\alpha}$. Here $A_{\alpha\beta}$ is called the pseudo-Hermitian torsion. The curvature of the Webster–Stanton connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar{\alpha}}\}$, is

$$\Pi_\beta{}^\alpha = \overline{\Pi_{\bar{\beta}}{}^{\bar{\alpha}}} = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha,$$

$$\Pi_0{}^\alpha = \Pi_\alpha{}^0 = \Pi_0{}^{\bar{\beta}} = \Pi_{\bar{\beta}}{}^0 = \Pi_0{}^0 = 0.$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written as

$$\Pi_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\bar{\sigma}}\theta^\rho \wedge \theta^{\bar{\sigma}} + W_\beta{}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\bar{\rho}}\theta^{\bar{\rho}} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha,$$

where the coefficients satisfy

$$R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}}, \qquad W_{\beta\bar{\alpha}\beta} = W_{\beta\bar{\alpha}\beta}.$$

We denote components of covariant derivatives with indices preceded by a comma; thus we write $A_{\alpha\beta},^{\beta}$. The indices $\{0, \alpha, \bar{\alpha}\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$. For derivatives of a scalar function, we often omit the comma; for instance, $\varphi_\alpha = Z_\alpha\varphi$, $\varphi_{\alpha\bar{\beta}} = Z_{\bar{\beta}}Z_\alpha\varphi - \omega_\alpha{}^\gamma(Z_{\bar{\beta}})Z_\gamma\varphi$, $\varphi_0 = T\varphi$ for a (smooth) function φ.

For a real function φ, the subgradient ∇_b and sub-Laplacian Δ_b are defined by

$$\nabla_b\varphi = \varphi^\alpha Z_\alpha + \varphi^{\overline{\alpha}} Z_{\bar{\alpha}}, \qquad \Delta_b\varphi = (\varphi_\alpha{}^\alpha + \varphi_{\overline{\alpha}}{}^{\overline{\alpha}}).$$

It follows from (2.1) that the following commutation identities hold (see [Le, Lemma 2.3]):

$$\varphi_\alpha{}^\alpha = \frac{1}{2}(\Delta_b\varphi + inT\varphi) \qquad \text{and} \qquad \varphi_{\overline{\alpha}}{}^{\overline{\alpha}} = \frac{1}{2}(\Delta_b\varphi - inT\varphi). \tag{2.2}$$

The Webster–Ricci tensor and the torsion tensor on $T_{1,0}$ are defined by

$$\begin{aligned}
&\operatorname{Ric}(X,Y) = R_{\alpha\bar{\beta}}X^\alpha Y^{\bar{\beta}},\\
&\operatorname{Tor}(X,Y) = i\sum_{\alpha,\beta}(A_{\bar{\alpha}\bar{\beta}}X^{\bar{\alpha}}Y^{\bar{\beta}} - A_{\alpha\beta}X^\alpha Y^\beta),
\end{aligned}$$

where $X = X^\alpha Z_\alpha$, $Y = Y^\beta Z_\beta$, $R_{\alpha\bar{\beta}} = R_\gamma{}^\gamma{}_{\alpha\bar{\beta}}$. The Webster scalar curvature is $R = R_\alpha{}^\alpha = h^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}$.

3. The proof of Theorem 1.2

In this section, we follow the same arguments as in [LT] to prove Theorem 1.2. Let us recall the following integral formula.

PROPOSITION 3.1

Let (M, J, θ) be a closed pseudo-Hermitian $(2n+1)$-manifold. Then for any constant $c \in \mathbb{R}$, we have

$$\begin{aligned}
&\Big(\frac{1}{2} + \frac{c}{n}\Big)\int_M (\Delta_b\varphi)^2\\
&\quad = \Big[1 + \frac{2(1-c)}{n}\Big]\int_M \sum_{\alpha,\beta}\varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta} + \Big[1 - \frac{2(1-c)}{n}\Big]\int_M \sum_{\alpha,\beta}\varphi_{\alpha\beta}\varphi_{\overline{\alpha}\overline{\beta}}\\
&\qquad + \Big[1 - \frac{2(1-c)}{n}\Big]\int_M \operatorname{Ric}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big) + \frac{c}{2n}\int (P_0\varphi)\varphi\\
&\qquad - \Big(\frac{n}{2} + c\Big)\int_M \operatorname{Tor}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big),
\end{aligned} \tag{3.1}$$

where $(\nabla_b\varphi)_{\mathbb{C}} = \varphi^\alpha Z_\alpha$ is the corresponding complex $(1,0)$-vector field of $\nabla_b\varphi$ and P_0 is the CR Paneitz operator defined by $P_0\varphi = 8(\varphi_{\overline{\alpha}}{}^{\overline{\alpha}}{}_\beta + inA_{\beta\alpha}\varphi^\alpha),^\beta$.

Proposition 3.1 follows from [CC, Theorem 3.1] and the following identity (see [CC, Corollary 2.4]):

$$\int_M \varphi_0^2 = \frac{1}{n^2}\int_M (\Delta_b\varphi)^2 + \frac{2}{n}\int_M \operatorname{Tor}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big) - \frac{1}{2n^2}\int_M (P_0\varphi)\varphi. \tag{3.2}$$

It is important to note that (see the proof of [CC, Theorem 3.2])

$$\begin{aligned} \frac{n-1}{8n}\int_M (P_0\varphi)\varphi &= \int_M \sum_{\alpha,\beta} \varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta} - \frac{1}{4n}(\Delta_b\varphi)^2 - \frac{n}{4}\varphi_0^2 \\ &= \int_M \sum_{\alpha,\beta} \varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta} - \frac{1}{n}\varphi_\gamma{}^\gamma\varphi_{\overline{\delta}}{}^{\overline{\delta}} \\ &= \int_M \sum_{\alpha,\beta} \Big|\varphi_{\overline{\alpha}\beta} - \frac{1}{n}\varphi_\gamma{}^\gamma h_{\overline{\alpha}\beta}\Big|^2, \end{aligned} \tag{3.3}$$

where we use the identities (2.2) in the second equation. This implies that the CR Paneitz operator P_0 is nonnegative for $n \geq 2$.

In order to prove Theorem 1.2, first we claim that, for $n \geq 2$,

$$\begin{aligned} &\frac{n+2}{n-1}\int_M \sum_{\alpha,\beta} \Big|\varphi_{\overline{\alpha}\beta} - \frac{1}{n}\varphi_\gamma{}^\gamma h_{\overline{\alpha}\beta}\Big|^2 \\ &\quad = \frac{n+1}{2n}\int_M (\Delta_b\varphi)^2 - \int_M \sum_{\alpha,\beta} \varphi_{\alpha\beta}\varphi_{\overline{\alpha}\overline{\beta}} \\ &\qquad - \int_M \Big(\operatorname{Ric} - \frac{n+1}{2}\operatorname{Tor}\Big)\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big). \end{aligned} \tag{3.4}$$

(i) For $n \geq 3$, let $c = 0$ in equation (3.1); we have

$$\begin{aligned} \frac{n+2}{n}\int_M \sum_{\alpha,\beta} \varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta} &= \frac{1}{2}\int_M (\Delta_b\varphi)^2 - \frac{n-2}{n}\int_M \sum_{\alpha,\beta}\varphi_{\alpha\beta}\varphi_{\overline{\alpha}\overline{\beta}} \\ &\quad - \int_M \Big(\frac{n-2}{n}\operatorname{Ric} - \frac{n}{2}\operatorname{Tor}\Big)\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big). \end{aligned} \tag{3.5}$$

Also let $c = 1 - n/2$ in equation (3.1); we get

$$\begin{aligned} &\frac{1}{2n}\int_M (\Delta_b\varphi)^2 \\ &\quad = \int_M \sum_{\alpha,\beta} \varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta} - \frac{1}{2}\int_M \operatorname{Tor}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big) - \frac{n-2}{8n}\int_M (P_0\varphi)\varphi. \end{aligned} \tag{3.6}$$

Thus, by (3.3) and substituting (3.5) into (3.6), we obtain

$$\begin{aligned} \frac{n-2}{n-1}\int_M \sum_{\alpha,\beta} \Big|\varphi_{\overline{\alpha}\beta} - \frac{1}{n}\varphi_\gamma{}^\gamma h_{\overline{\alpha}\beta}\Big|^2 &= \frac{n-2}{8n}\int_M (P_0\varphi)\varphi \\ = \int_M \sum_{\alpha,\beta} \varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta} - \frac{1}{2n}\int_M (\Delta_b\varphi)^2 &- \frac{1}{2}\int_M \operatorname{Tor}\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big) \end{aligned}$$

$$= \frac{n-2}{n+2}\left[\frac{n+1}{2n}\int_M (\Delta_b\varphi)^2 - \int_M \sum_{\alpha,\beta}\varphi_{\alpha\beta}\varphi_{\overline{\alpha}\overline{\beta}} - \int_M \Big(\mathrm{Ric} - \frac{n+1}{2}\ \mathrm{Tor}\Big)\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big)\right].$$

(ii) For $n=2$, for $c\in(0,2)$ in equation (3.1), and by (3.3), we have

$$\begin{aligned}\frac{1+c}{2}\int_M (\Delta_b\varphi)^2 &= (2-c)\int_M \sum_{\alpha,\beta}\varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta} + c\int_M \sum_{\alpha,\beta}\varphi_{\alpha\beta}\varphi_{\overline{\alpha}\overline{\beta}} \\ &\quad + \int_M [c\ \mathrm{Ric} - (1+c)\ \mathrm{Tor}]\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big) \\ &\quad + 4c\int \sum_{\alpha,\beta}\Big|\varphi_{\overline{\alpha}\beta} - \frac{1}{2}\varphi_\gamma{}^\gamma h_{\overline{\alpha}\beta}\Big|^2;\end{aligned}$$

thus

$$\begin{aligned}\int_M \sum_{\alpha,\beta}\Big|\varphi_{\overline{\alpha}\beta} - \frac{1}{2}\varphi_\gamma{}^\gamma h_{\overline{\alpha}\beta}\Big|^2 &= \int_M \sum_{\alpha,\beta}\varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta} - \frac{1}{8}\int_M (\Delta_b\varphi)^2 - \frac{1}{2}\int_M \varphi_0^2 \\ &= \frac{3c}{4(2-c)}\int_M (\Delta_b\varphi)^2 - \frac{c}{2-c}\int_M \Big(\mathrm{Ric} - \frac{3}{2}\ \mathrm{Tor}\Big)\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big) \\ &\quad - \frac{c}{2-c}\int_M \sum_{\alpha,\beta}\varphi_{\alpha\beta}\varphi_{\overline{\alpha}\overline{\beta}} + \Big(1 - \frac{4c}{2-c}\Big)\int_M \sum_{\alpha,\beta}\Big|\varphi_{\overline{\alpha}\beta} - \frac{1}{2}\varphi_\beta{}^\beta h_{\overline{\alpha}\beta}\Big|^2,\end{aligned}$$

where in the second equation we have used the identity (3.2) for $n=2$. It yields

$$\begin{aligned}4\int_M \sum_{\alpha,\beta}\Big|\varphi_{\overline{\alpha}\beta} - \frac{1}{2}\varphi_\gamma{}^\gamma h_{\overline{\alpha}\beta}\Big|^2 &= \frac{3}{4}\int_M (\Delta_b\varphi)^2 - \int_M \sum_{\alpha,\beta}\varphi_{\alpha\beta}\varphi_{\overline{\alpha}\overline{\beta}} - \int_M \Big(\mathrm{Ric} - \frac{3}{2}\ \mathrm{Tor}\Big)\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big).\end{aligned}$$

This completes the proof of the claim (3.4).

Next we denote the traceless Webster–Ricci tensor by $R^0_{\alpha\overline{\beta}} \triangleq R_{\alpha\overline{\beta}} - (R/n)h_{\alpha\overline{\beta}}$, and from the contracted Bianchi identity (see [Le, (2.11)]) $R_{\alpha\overline{\beta}}{}^{,\overline{\beta}} = R_\alpha - i(n-1)A_{\alpha\beta}{}^{,\beta}$, we get

$$R^0_{\alpha\overline{\beta}}{}^{,\overline{\beta}} = \Big(R_{\alpha\overline{\beta}} - \frac{R}{n}h_{\alpha\overline{\beta}}\Big)^{,\overline{\beta}} = \frac{n-1}{n}R_\alpha - i(n-1)A_{\alpha\beta}{}^{,\beta}. \tag{3.7}$$

Now we can prove our Theorem 1.2.

Proof of Theorem 1.2
Let φ be the unique solution of $\Delta_b\varphi = R - \overline{R}$ with $\int_M \varphi = 0$. By (3.7), we then compute

$$\begin{aligned}
&\int_M (R-\overline{R})^2 \\
&\quad= \int_M (R-\overline{R})\Delta_b\varphi = -\int_M \langle \nabla_b R, \nabla_b\varphi\rangle = -\int_M (R_\alpha \varphi^\alpha + R_{\overline{\alpha}}\varphi^{\overline{\alpha}}) \\
&\quad= \Big(-\frac{n}{n-1}\int_M R^0_{\alpha\overline{\beta}}{}^{,\overline{\beta}}\varphi^\alpha + in\int_M A_{\alpha\beta}{}^{,\beta}\varphi^\alpha\Big) \\
&\qquad + \text{complex conjugate} \\
&\quad= \Big(\frac{n}{n-1}\int_M R^0_{\alpha\overline{\beta}}\Big(\varphi^{\alpha\overline{\beta}} - \frac{1}{n}\varphi_\gamma{}^\gamma h^{\alpha\overline{\beta}}\Big) - in\int_M A_{\alpha\beta}\varphi^{\alpha\beta}\Big) \qquad (3.8)\\
&\qquad + \text{complex conjugate} \\
&\quad= \frac{2n}{n-1}\int_M R^0_{\alpha\overline{\beta}}\Big(\varphi^{\alpha\overline{\beta}} - \frac{1}{n}\varphi_\gamma{}^\gamma h^{\alpha\overline{\beta}}\Big) + in\int_M (A_{\overline{\alpha}\overline{\beta}}\varphi^{\overline{\alpha}\overline{\beta}} - A_{\alpha\beta}\varphi^{\alpha\beta}) \\
&\quad\le \frac{2n}{n-1}\|R^0_{\alpha\overline{\beta}}\|_{L^2}\Big\|\varphi_{\overline{\alpha}\beta} - \frac{1}{n}\varphi_\gamma{}^\gamma h_{\overline{\alpha}\beta}\Big\|_{L^2} + in\int_M (A_{\overline{\alpha}\overline{\beta}}\varphi^{\overline{\alpha}\overline{\beta}} - A_{\alpha\beta}\varphi^{\alpha\beta}).
\end{aligned}$$

Now from (3.4) and the condition $\big(\mathrm{Ric} - ((n+1)/2)\,\mathrm{Tor}\big)((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \ge 0$, we
obtain

$$\begin{aligned}
\int_M \sum_{\alpha,\beta}\Big|\varphi_{\overline{\alpha}\beta} - \frac{1}{n}\varphi_\gamma{}^\gamma h_{\overline{\alpha}\beta}\Big|^2 &\le \frac{(n-1)(n+1)}{2n(n+2)}\int_M (\Delta_b\varphi)^2 \\
&= \frac{(n-1)(n+1)}{2n(n+2)}\int_M (R-\overline{R})^2,
\end{aligned}$$

and thus

$$\Big\|\varphi_{\overline{\alpha}\beta} - \frac{1}{n}\varphi_\gamma{}^\gamma h_{\overline{\alpha}\beta}\Big\|_{L^2} \le \Big(\frac{(n-1)(n+1)}{2n(n+2)}\int_M (R-\overline{R})^2\Big)^{1/2},$$

which combined with (3.8) and applying Young's inequality $2ab \le \epsilon a^2 + \epsilon^{-1}b^2$ with $\epsilon = \sqrt{(2n(n+2))/((n-1)(n+1))}$, then gives the equation (1.1).

Moreover, if the equality holds, then φ will satisfy

$$\varphi_{\alpha\beta} = 0 \quad \text{for all } \alpha, \beta, \qquad \Big(\mathrm{Ric} - \frac{n+1}{2}\,\mathrm{Tor}\Big)\big((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}\big) = 0$$

and

$$R^0_{\alpha\overline{\beta}} = r\Big(\varphi_{\alpha\overline{\beta}} - \frac{1}{n}\varphi_\gamma{}^\gamma h_{\alpha\overline{\beta}}\Big) \quad \text{for some real constant } r.$$

Simple computation shows that r is the constant $(n+2)/(n+1)$. Therefore,

$$\int_M (A_{\overline{\alpha}\overline{\beta}}\varphi^{\overline{\alpha}\overline{\beta}} - A_{\alpha\beta}\varphi^{\alpha\beta}) = 0 \qquad \text{and} \qquad R^0_{\alpha\overline{\beta}} = \frac{n+2}{n+1}\Big(\varphi_{\alpha\overline{\beta}} - \frac{1}{n}\varphi_\gamma{}^\gamma h_{\alpha\overline{\beta}}\Big),$$

which implies that the contact form $e^{1/(n+1)\varphi}\theta$ will be pseudo-Einstein by [DT, Proposition 5.9]. This completes the proof of Theorem 1.2. □

4. The proof of Theorem 1.5

Let (M,J,θ) be a closed pseudo-Hermitian $(2n+1)$-manifold. We consider a conformal change $\widehat{\theta}=e^{2u}\theta$ of the contact form, following the method of [Le]. Under this deformation, the contact distribution $\xi=\ker\theta$ and the complex structure J are fixed.

In [Le], J. M. Lee proved that, under the conformal change of the contact form $\widehat{\theta}=e^{2u}\theta$, the Webster–Ricci tensor $R_{\alpha\overline{\beta}}$ changes as

$$\widehat{R}_{\alpha\overline{\beta}}=R_{\alpha\overline{\beta}}-\big(\Delta_b u+2(n+1)|\nabla_b u|^2\big)h_{\alpha\overline{\beta}}-(n+2)(u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha}).$$

On the other hand, if the first Chern class $c_1(T_{1,0}M)$ of M vanishes, there exists a real 1-form σ such that

$$R_{\alpha\overline{\beta}}\theta^{\alpha}\wedge\theta^{\overline{\beta}}=d\sigma\quad\text{on }\xi.$$

It can be easily shown that the $(0,1)$-part $\eta=\sigma^{(0,1)}$ is $\overline{\partial}_b$-closed, so that there exist a complex function $f=u+iv\in C^{\infty}_{\mathbb{C}}(M)$ and a $\Box_b$-harmonic form γ such that

$$\eta=\frac{n+2}{2\pi}\overline{\partial}_b f-\gamma.$$

Then Theorem 1.5 follows from the following theorem.

THEOREM 4.1 ([Le, LEMMA 6.2])
Let (M,J,θ) be a closed pseudo-Hermitian $(2n+1)$-manifold. Assume that there exists a 1-form σ such that

$$R_{\alpha\overline{\beta}}\theta^{\alpha}\wedge\theta^{\overline{\beta}}=d\sigma\quad\text{on }\xi,$$

and the $\Box_b$-harmonic part γ of $\sigma^{(0,1)}$ satisfies the condition

$$\gamma^{\alpha,\overline{\beta}}+\gamma^{\overline{\beta},\alpha}=0,$$

where $\sigma^{(0,1)}=((n+2)/2\pi)\overline{\partial}_b(u+iv)-\gamma$. Then $\widehat{\theta}=e^{2u}\theta$ is a pseudo-Einstein contact form.

Moreover, it was also shown in [Le] that

$$2\pi(\gamma_{\alpha,\overline{\beta}}+\gamma_{\overline{\beta},\alpha})(\gamma^{\alpha,\overline{\beta}}+\gamma^{\overline{\beta},\alpha})=2\,\mathrm{Re}\,\gamma^{\overline{\beta},\alpha}[2(n+2)u_{\alpha\overline{\beta}}-R_{\alpha\overline{\beta}}].$$

Therefore, using the divergence formula, we have

$$\begin{aligned}2\pi&\int_M(\gamma_{\alpha,\overline{\beta}}+\gamma_{\overline{\beta},\alpha})(\gamma^{\alpha,\overline{\beta}}+\gamma^{\overline{\beta},\alpha})\\&=2\,\mathrm{Re}\int_M\big\{2(n+2)[(u_{\alpha\overline{\beta}}\gamma^{\overline{\beta}})^{,\alpha}-(u_{\alpha}{}^{\alpha}\gamma^{\overline{\beta}})_{,\overline{\beta}}-i(n-1)A_{\overline{\alpha}\overline{\beta}}u^{\overline{\alpha}}\gamma^{\overline{\beta}}]\\&\qquad-[(R_{\alpha\overline{\beta}}\gamma^{\overline{\beta}})^{,\alpha}-(\rho\gamma^{\overline{\beta}})_{,\overline{\beta}}+i(n-1)A_{\overline{\alpha}\overline{\beta}}{}^{,\overline{\alpha}}\gamma^{\overline{\beta}}]\big\}\\&=-2\,\mathrm{Re}\int_M i(n-1)[2(n+2)A_{\overline{\alpha}\overline{\beta}}u^{\overline{\alpha}}\gamma^{\overline{\beta}}+A_{\overline{\alpha}\overline{\beta}}{}^{,\overline{\alpha}}\gamma^{\overline{\beta}}].\end{aligned}\tag{4.1}$$

Here, we are now in the position to prove our Theorem 1.5.

Proof of Theorem 1.5
By the assumption $A_{\overline{\alpha}\overline{\beta}}{}^{,\overline{\alpha}} = 0$ and (4.1), we only have to show

$$\int_M A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} \gamma^{\overline{\beta}} = 0.$$

Using the commutation formula in [Le, Lemma 2.3], for any $\overline{\partial}_b$-closed $(0,1)$-form η, we have

$$[\overline{\partial}_b^*, \nabla_T]\eta = \eta_{0\overline{\alpha}}{}^{\overline{\alpha}} - \eta_{\overline{\alpha}0}{}^{\overline{\alpha}} = (A_{\overline{\beta}\overline{\alpha}} \eta^{\overline{\alpha}})^{,\overline{\beta}}. \tag{4.2}$$

Thus,

$$\int_M A_{\overline{\alpha}\overline{\beta}} u^{\overline{\alpha}} \gamma^{\overline{\beta}} = \int_M (A_{\overline{\alpha}\overline{\beta}} \gamma^{\overline{\beta}} u)^{,\overline{\alpha}} - (A_{\overline{\alpha}\overline{\beta}} \gamma^{\overline{\beta}})^{,\overline{\alpha}} u = 0.$$

This completes the proof of Theorem 1.5. □

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