On the coefficients of Vilenkin-Fourier series with small gaps

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Abstract The Riemann-Lebesgue lemma shows that the Vilenkin-Fourier coefficient $\hat{f}(n)$ is of $o(1)$ as $n \to \infty$ for any integrable function f on Vilenkin groups. However, it is known that the Vilenkin-Fourier coefficients of integrable functions can tend to zero as slowly as we wish. The definitive result is due to B. L. Ghodadra for functions of certain classes of generalized bounded fluctuations. We prove that this is a matter only of local fluctuation for functions with the Vilenkin-Fourier series lacunary with small gaps. Our results, as in the case of trigonometric Fourier series, illustrate the interconnection between 'localness' of the hypothesis and *type of lacunarity* and allow us to interpolate the results.

1. Introduction

Let G be a Vilenkin group, that is, a compact metrizable zero-dimensional (infinite) abelian group. Then the dual group X of G is a discrete, countable, torsion, abelian group (see [4, Theorems 24.15, 24.26]). In 1947, N. Ja. Vilenkin [14] developed part of the Fourier theory on G, and later Onneweer and Waterman [5]–[7] introduced various classes of functions of bounded fluctuations. For functions of these classes, in [3], we have studied the order of magnitude of Vilenkin-Fourier coefficients and proved Vilenkin group analogues of the results of Schramm and Waterman [13]. Here we study the order of magnitude of Fourier coefficients of Vilenkin-Fourier series with small gaps for functions of various classes of bounded fluctuations and prove the Vilenkin group analogue (Corollary 2) of the results of Patadia and Vyas [8, Theorem 5]. As in the case of trigonometric Fourier series (see [8]), here also we give an interconnection between the 'type of lacunarity' in Vilenkin-Fourier series and the *localness* of the hypothesis to be satisfied by the generic functions, which allow us to interpolate results concerning order of magnitude of Fourier coefficients of lacunary and nonlacunary Vilenkin-Fourier series.

2. Notation and definitions

For G and X as above, Vilenkin [14, Sections 1.1, 1.2] proved the existence of a sequence $\{X_n\}$ of finite subgroups of X and of a sequence $\{\varphi_n\}$ in X such that the following hold:

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(i) $X_0 = \{\chi_0\}$, where χ_0 is the identity character on G;
(ii) $X_0 \subset X_1 \subset X_2 \subset \ldots$;
(iii) for each $n \geq 1$, the quotient group X_n / X_{n-1} is of prime order p_n;
(iv) $X = \bigcup_{n=0}^{\infty} X_n$;
(v) $\varphi_n \in X_{n+1} \setminus X_n$ for all $n \geq 0$;
(vi) $\varphi_n^{p_{n+1}} \in X_n$ for all $n \geq 0$.

The group G is bounded if

$$p_0 = \sup_{i=1,2,\ldots} p_i < \infty;$$

otherwise, G is said to be unbounded. Using the φ_n's, we can enumerate X as follows. Let $m_0 = 1$, and let $m_n = \prod_{i=1}^{n} p_i$ for $n = 1, 2, \ldots$. Then each $k \in \mathbb{N}$ can be uniquely represented as $k = \sum_{i=0}^{s} a_i m_i$ with $0 \leq a_i < p_{i+1}$ for $0 \leq i \leq s$; we define χ_k by the formula $\chi_k = \varphi_0^{a_0} \cdot \cdots \cdot \varphi_s^{a_s}$. Observe that $\chi_{m_n} = \varphi_n$ for each $n \geq 0$. For $\chi \in X$ the degree of χ is defined by $\deg \chi_0 = 0$ and $\deg \chi_k = s + 1$ if χ_k is written as the product of φ_n's as described in the preceding lines. Any complex linear combination of finitely many elements of X is called a Vilenkin polynomial on G, and the degree of such a polynomial is the maximum of the degree of elements of X appearing in the polynomial.

$G = \prod_{n=1}^{\infty} \mathbb{Z}_{p_n}$, $\{p_n\}$ – a sequence of prime numbers, is a standard example. If $p_n = 2$ for all n, X is the group of Walsh functions $\psi_n, n = 0, 1, 2, \ldots,$ and $X_n = \{\psi_0, \psi_1, \ldots, \psi_{2^n - 1}\}$ (using Payley enumeration; see [10]) described by Fine [2]. If $p_n = p$ for all n, X is the group of generalized Walsh functions [1].

Let dx or m denote the normalized Haar measure on G. For $f \in L^1(G)$, the Vilenkin-Fourier series of f is given by

$$S[f](x) = \sum_{n=0}^{\infty} \hat{f}(n) \chi_n(x), \quad \hat{f}(n) = \int_G f(x) \bar{\chi}_n(x)\, dx,$$

where $\hat{f}(n)$ $(n = 0, 1, 2, \ldots)$ is the nth Vilenkin-Fourier coefficient of f. It is said to be *lacunary with small gaps* if $\hat{f}(n) \neq 0$ for $n \neq n_k$, where $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence of positive integers satisfying the small gap condition

$$(n_{k+1} - n_k) \geq q \geq 1 \quad (k = 1, 2, \ldots) \tag{1}$$

or, in particular, a more stringent small gap condition

$$(n_{k+1} - n_k) \to \infty \quad \text{as} \quad k \to \infty. \tag{2}$$

Observe that for each n, $X_n = \{\chi_k : 0 \leq k < m_n\}$. Let G_n be the annihilator of X_n, that is,

$$G_n = \{x \in G : \chi(x) = 1, \chi \in X_n\} = \{x \in G : \chi_k(x) = 1, 0 \leq k < m_n\}.$$

Then obviously, $G = G_0 \supset G_1 \supset G_2 \supset \ldots, \bigcap_{n=0}^{\infty} G_n = \{0\}$, and the G_n's form a fundamental system of neighborhoods of zero in G which are compact open and closed subgroups of G. Further, the index of G_n in G is m_n, and since the Haar measure is translation invariant with $m(G) = 1$, one has $m(G_n) = 1/m_n$. In [14, Section 3.2] Vilenkin proved that for each $n \geq 0$ there exists $x_n \in G_n \setminus G_{n+1}$

such that $\chi_{m_n}(x_n) = \exp(2\pi i/p_{n+1})$ and observed that each $x \in G$ has a unique representation $x = \sum_{i=0}^{\infty} b_i x_i$ with $0 \le b_i < p_{i+1}$ for all $i \ge 0$. This representation of the elements of G enables one to order them by means of the lexicographic ordering of the corresponding sequence $\{b_n\}$ and one observes that for each $n = 1, 2, \ldots,$

$$G_n = \Big\{x \in G : x = \sum_{i=0}^{\infty} b_i x_i, b_0 = \cdots = b_{n-1} = 0\Big\} = \Big\{x \in G : x = \sum_{i=n}^{\infty} b_i x_i\Big\}.$$

Consequently, each coset of G_n in G has a representation of the form $z + G_n$, where $z = \sum_{i=0}^{n-1} b_i x_i$ for some choice of the b_i with $0 \le b_i < p_{i+1}$. These z, ordered lexicographically, are denoted by $\{z_\alpha^{(n)}\}$ $(0 \le \alpha < m_n)$.

It may be noted that the choice of $\varphi_n \in X_{n+1} \backslash X_n$ and of the $x_n \in G_n \backslash G_{n+1}$ is not uniquely determined by the groups X and G. In the following, it is assumed that a particular choice has been made.

Observe that for $l, N \in \mathbb{N}$ if $l > N$; then $G_l \subset G_N$, and therefore,

$$G_l = \Big\{x \in G : x = \sum_{i=l}^{\infty} b_i x_i\Big\} = \Big\{x \in G_N : x = \sum_{i=N}^{\infty} b_i x_i, b_N = \cdots = b_{l-1} = 0\Big\}.$$

Thus each coset of G_l in G_N has a representation of the form $z + G_l$, where $z = \sum_{i=N}^{l-1} b_i x_i$ for some choice of the b_i with $0 \le b_i < p_{i+1}$. These $(m_l/m_N) = p_{N+1} p_{N+2} \cdots p_l = L$ (say) cosets of G_l in G_N are precisely the cosets $z_\alpha^{(l)} + G_l$, $\alpha = 0, 1, \ldots, L-1$, of G_l in G in that order. Also observe that for a given $y_0 = \sum_{i=0}^{\infty} c_i x_i$ in G and $N \in \mathbb{N}$, the coset $y_0 + G_N$ given by

$$y_0 + G_N = \Big\{x = \sum_{i=0}^{\infty} b_i x_i \in G : b_i = c_i, i = 0, 1, \ldots, N-1\Big\}$$

contains y_0 and is of Haar measure $1/m_N$. Since G_N is the disjoint union of the cosets $z_\alpha^{(l)} + G_l$, $\alpha = 0, 1, \ldots, L-1$, for $l > N$, the coset $y_0 + G_N$ is the disjoint union of the cosets $y_0 + z_\alpha^{(l)} + G_l$, $\alpha = 0, 1, \ldots, L-1$.

Let f be a complex function on G, let $\Lambda = \{\lambda_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} (1/\lambda_n)$ diverges, and let $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing function. Customarily ϕ is considered to be a convex function such that

$$\phi(0) = 0, \qquad \frac{\phi(x)}{x} \to 0 \quad (x \to 0_+), \qquad \frac{\phi(x)}{x} \to \infty \quad (x \to \infty).$$

Such a function is called an N-function. It is necessarily continuous and strictly increasing on $[0, \infty)$. For $H \subset G$, the *oscillation* of f on H is defined as

$$\operatorname{osc}(f; H) = \sup\big\{|f(x_1) - f(x_2)| : x_1, x_2 \in H\big\}.$$

We define various classes of functions of bounded fluctuation on a coset of G as follows.

DEFINITION 1

We say that f is of *ϕ-bounded fluctuation* over $y_0 + G_N$ $(f \in \phi\mathrm{BF}(y_0 + G_N))$ if the *total ϕ-fluctuation* of f on y_0+G_N given by

$$\mathrm{F}_\phi(f;y_0+G_N)=\sup\Big\{\sum_{t=1}^{T}\phi\big(\mathrm{osc}(f;I_t)\big)\Big\}$$

is finite, where the supremum is taken over all finite disjoint collections $\{I_1,I_2,\ldots,I_T\}$ in which each I_t is a coset of some $G_{m(t)}$ and $\bigcup_{t=1}^{T}I_t=y_0+G_N$.

DEFINITION 2

We say that f is of ϕ-Λ-*bounded fluctuation* over y_0+G_N ($f\in\phi\Lambda\mathrm{BF}(y_0+G_N)$) if the *total ϕ-Λ-fluctuation* of f on y_0+G_N given by

$$\mathrm{F}_{\phi\Lambda}(f;y_0+G_N)=\sup_{\{I_n\}}\Big\{\sum_n\frac{\phi(\mathrm{osc}(f;I_n))}{\lambda_n}\Big\}$$

is finite, where the supremum is taken over all sequences $\{I_n\}$ of disjoint cosets in y_0+G_N.

DEFINITION 3

We say that f is of *ϕ-generalized bounded fluctuation* over y_0+G_N ($f\in\phi\mathrm{GBF}(y_0+G_N)$) if the *total generalized ϕ-fluctuation* of f on y_0+G_N given by

$$\mathrm{GF}_\phi(f;y_0+G_N)=\sup_{l\ge N}\sum_{\alpha=0}^{m_l/m_N-1}\phi\big(\mathrm{osc}(f;y_0+z_\alpha^{(l)}+G_l)\big)$$

is finite.

We observe that if $\lambda_n\equiv1$, $\phi\Lambda\mathrm{BF}=\phi\mathrm{BF}$. If $\phi(x)=x^p$ ($p\ge1$), then $\phi\mathrm{BF}$ (resp., $\phi\mathrm{GBF}$) is denoted as $\mathrm{BF}^{(p)}$ (resp., $\mathrm{GBF}^{(p)}$), and functions of this class are called functions of *p-bounded fluctuation* (resp., *p-generalized bounded fluctuation*). Also, when $p=1$, the class $\mathrm{BF}^{(p)}$ (resp., $\mathrm{GBF}^{(p)}$) is denoted as BF (resp., GBF), and functions of this class are called functions of *bounded fluctuation* (resp., *generalized bounded fluctuation*). Further, from Definitions 1 and 3, it is clear that $\phi\mathrm{BF}\subset\phi\mathrm{GBF}$.

When $y_0+G_N=G$, our Definitions 2 and 3 are the same as [7, Definition 3] and [6, Definition 6], respectively. For $y_0+G_N=G$ and $\phi(x)=x^p$, our Definition 3 is same as [5, Definition 4]. Further, when $y_0+G_N=G$ and $\phi(x)=x$, our Definitions 1 and 3 are the same as Definitions 4 and 5, respectively, in [6].

3. Results

We prove the following results.

THEOREM 1

Let $f\in L^1(G)$ possess a lacunary Vilenkin-Fourier series

$$\sum_{k=1}^{\infty}\hat f(n_k)\chi_{n_k}(x)\tag{3}$$

with small gaps (1), and let $I = y_0 + G_N$ *be the coset with Haar measure* $1/m_N \geq 1/q$*. Then* $f \in \phi\mathrm{GBF}(I)$ *implies* $\hat{f}(n_k) = O(\phi^{-1}(1/m_l))$*, where* $m_l \leq n_k < m_{l+1}$*. If, in addition, G is bounded, then* $\hat{f}(n_k) = O(\phi^{-1}(1/n_k))$*.*

Taking $\phi(x) = x^p$ $(p \geq 1)$ in Theorem 1, we get the following.

COROLLARY 1

Let f and I be as in Theorem 1. Then $f \in \mathrm{GBF}^{(p)}(I)$ $(p \geq 1)$ *implies* $\hat{f}(n_k) = O(1/(m_l)^{1/p})$*, where* $m_l \leq n_k < m_{l+1}$*. If, in addition, G is bounded, then* $\hat{f}(n_k) = O(1/(n_k)^{1/p})$*.*

REMARK 1

Since $\phi\mathrm{BF} \subset \phi\mathrm{GBF}$, Theorem 1 holds for functions in $\phi\mathrm{BF}$ also. Similarly, as $\mathrm{BF}^{(p)} \subset \mathrm{GBF}^{(p)}$, Corollary 1 holds for functions in $\mathrm{BF}^{(p)}$ also.

THEOREM 2

Let f and I be as in Theorem 1. Then $f \in \phi\Lambda\mathrm{BF}(I)$ *implies*

$$\hat{f}(n_k) = O\left(\phi^{-1}\left(1/\left(\sum_{j=1}^{m_l} \frac{1}{\lambda_j}\right)\right)\right),$$

where $m_l \leq n_k < m_{l+1}$*. If, in addition, G is bounded, then*

$$\hat{f}(n_k) = O\left(\phi^{-1}\left(1/\left(\sum_{j=1}^{n_k} \frac{1}{\lambda_j}\right)\right)\right).$$

Taking $\phi(x) = x^p$ $(p \geq 1)$ in Theorem 2, we get the following result, which is the Vilenkin group analogue of the result of Patadia and Vyas [8, Theorem 5].

COROLLARY 2

Let f and I be as in Theorem 1. Then $f \in \Lambda\mathrm{BF}^{(p)}(I)$ $(p \geq 1)$ *implies*

$$\hat{f}(n_k) = O\left(1/\left(\sum_{j=1}^{m_l} \frac{1}{\lambda_j}\right)^{1/p}\right),$$

where $m_l \leq n_k < m_{l+1}$*. If, in addition, G is bounded, then*

$$\hat{f}(n_k) = O\left(1/\left(\sum_{j=1}^{n_k} \frac{1}{\lambda_j}\right)^{1/p}\right).$$

REMARK 2

Observe that $n_k = k$ for all $k \implies q = 1$ in (1) $\implies I$ is of Haar measure 1 in the above theorems $\implies I = G$; and one gets corresponding results for nonlacunary Vilenkin-Fourier series (see [3]). On the other hand, if the Vilenkin-Fourier series (3) of $f \in L^1(G)$ has gaps (2), then the above results hold if the coset I is just of positive measure. Because if $|I| > 0$, by the form of I, $|I| = 1/m_N$, where

$N \in \mathbb{N}$ can be taken as large as required. In view of (2), one gets $(n_{k+1} - n_k) \geq m_N$ for all $k \geq k_0$ for a suitable $k_0 = k_0(N)$. Then adding to $f(x)$ the Vilenkin polynomial $\sum_{j=1}^{k_0}(-\hat{f}(n_j))\chi_{n_j}(x)$, one gets a function g whose Fourier series is lacunary of the form (3) having gaps (1) with $q = m_N$, and results are true for g. Since f and g differ by a polynomial, results are true for f as well. Our results thus interpolate lacunary and nonlacunary results concerning order of magnitude of Fourier coefficients—displaying beautiful interconnection between types of lacunarity (as determined by q in (1)) and localness of the hypothesis to be satisfied by the generic function (as determined by the q-dependent length of I).

4. Proofs of results

The following lemma due to Schramm and Waterman [12] is needed.

LEMMA 1

If $a_1 \geq a_2 \geq \cdots \geq a_n > 0$, $\sum_{i=1}^n a_i = 1$, *and* $b_1 \geq b_2 \geq \cdots \geq b_n$, *then*

$$\sum_{i=1}^n b_i \leq n \sum_{i=1}^n a_i b_i.$$

Proof of Theorem 1.

We may assume without loss of generality that $y_0 = 0$; otherwise, one works with $g = T_{y_0} f \in \phi\mathrm{GBF}(G_N)$, whose Fourier series also has gaps (1). Then $I = G_N$, and if we consider the polynomial $P_N(x)$ (see [9, Lemma 4]) defined by

$$\begin{aligned} P_N(x) &= \prod_{k=0}^{N-1} \left(1 + \varphi_k(x) + \varphi_k^2(x) + \cdots + \varphi_k^{p_k - 1}(x)\right) \\ &= 1 + \sum_{i=0}^{N-1} \varphi_i(x) + \sum_{i,j=0, i\neq j}^{N-1} \sum_{l=1}^{p_i - 1} \sum_{m=1}^{p_j - 1} \varphi_i^l(x) \cdot \varphi_j^m(x) + \cdots + \Big(\prod_{i=0}^{N-1} \varphi_i^{p_i - 1}(x)\Big) \end{aligned}$$

having constant term 1 and with degree less than or equal to N, then

$$P_N(x) = \begin{cases} m_N & \text{if } x \in I, \\ 0 & \text{if } x \in G \setminus I. \end{cases} \tag{4}$$

Note that if $k \in \mathbb{N}$ is such that $\hat{f}(n_k) \neq 0$, then $(f \cdot P_N)\hat{}\,(n_k) = \hat{f}(n_k)$. In fact,

$$\begin{aligned} (f \cdot P_N)\hat{}\,(n_k) &= \int_G f(x) P_N(x) \bar{\chi}_{n_k}(x)\, dx \\ &= \hat{f}(n_k) + \sum_{i=0}^{N-1} \hat{f}(\bar{\varphi}_i \chi_{n_k}) + \sum_{i,j=0, i\neq j}^{N-1} \sum_{l=1}^{p_i - 1} \sum_{m=1}^{p_j - 1} \hat{f}(\bar{\varphi}_i^l \bar{\varphi}_j^m \chi_{n_k}) \\ &\quad + \cdots + \hat{f}\Big(\prod_{i=0}^{N-1} \bar{\varphi}_i^{p_i - 1} \chi_{n_k}\Big). \end{aligned} \tag{5}$$

The characters appearing in the right-hand side of (5) are of the form $\chi_{n_k}\chi$ wherein χ is such that $\deg\chi$ is positive and less than or equal to N. Observe that for each $j \in \mathbb{N}$ there are totally $m_{j-1}(p_j - 1) = (m_j - m_{j-1})$ characters of degree j, namely, $\chi_i \varphi_{j-1}^{a_{j-1}}$, $0 \le i < m_{j-1}$, and $1 \le a_{j-1} \le (p_j - 1)$, and they constitute $(X_j - X_{j-1})$. Consequently, the total number of characters of positive degree less than or equal to N is given by

$$(m_1 - m_0) + (m_2 - m_1) + \cdots + (m_N - m_{N-1}) = m_N - 1;$$

they are from χ_1 to χ_{m_N-1}, and they constitute $\bigcup_{j=1}^{m_N}(X_j - X_{j-1})$. It follows that when χ_{n_k} is multiplied by any character of positive degree less than or equal to N, the resulting character χ_m is such that

$$n_k < m \le n_k + m_N - 1 < n_k + m_N \le n_k + q \le n_{k+1}$$

because the lacunary Vilenkin-Fourier series (3) of f has gaps (1) with $q \ge m_N$. Since $\hat{f}(n_k) \ne 0$, all the terms of the right-hand side of (5) vanish except the first.

Let k be large enough, and let $l \in \mathbb{N} \cup \{0\}$ be such that $\hat{f}(n_k) \ne 0, m_l \le n_k < m_{l+1}$, and $l > N$. Then, in view of (4),

$$\hat{f}(n_k) = (f \cdot P_N)\hat{\ }(n_k) = m_N \int_{G_N} f(x)\bar{\chi}_{n_k}(x)\,dx. \tag{6}$$

Since $n_k \ge m_l$ and the Haar measure is translation invariant, it follows (see, e.g., [11, p. 114, (15)]) that

$$\int_{z_\alpha^{(l)}+G_l} \chi_{n_k}(x)\,dx = 0$$

for all $\alpha = 0, 1, \ldots, m_l - 1$; hence

$$\int_{z_\alpha^{(l)}+G_l} \bar{\chi}_{n_k}(x)\,dx = 0 \quad (\alpha = 0, 1, \ldots, m_l - 1).$$

Now, put $L = m_l/m_N = (p_{N+1}p_{N+2}\cdots p_l)$, and define a step function g on G_N by $g(x) = f(z_\alpha^{(l)})$ for x in $z_\alpha^{(l)} + G_l$, $\alpha = 0, 1, \ldots, L-1$. Then

$$\int_{G_N} g(x)\bar{\chi}_{n_k}(x)\,dx = \sum_{\alpha=0}^{L-1} f(z_\alpha^{(l)}) \int_{z_\alpha^{(l)}+G_l} \bar{\chi}_{n_k}(x)\,dx = 0.$$

Therefore, in view of (6) we have

$$|\hat{f}(n_k)| = \Big| m_N \int_{G_N} [f(x) - g(x)]\bar{\chi}_{n_k}(x)\,dx \Big| \le m_N \int_{G_N} |f(x) - g(x)|\,dx. \tag{7}$$

Now, by Jensen's inequality, for $c > 0$,

$$\begin{aligned} \phi\Big(m_N \cdot c \cdot \int_{G_N} |f(x) - g(x)|\,dx\Big) &\le m_N \int_{G_N} \phi\big(c|f(x) - g(x)|\big)\,dx \\ &= m_N \sum_{\alpha=0}^{L-1} \int_{z_\alpha^{(l)}+G_l} \phi\big(c|f(x) - f(z_\alpha^{(l)})|\big)\,dx. \end{aligned} \tag{8}$$

Therefore,

$$\begin{aligned}\phi\Big(m_N\cdot c\cdot\int_{G_N}|f(x)-g(x)|\,dx\Big)&\le m_N\sum_{\alpha=0}^{L-1}\int_{z_\alpha^{(l)}+G_l}\phi\big(\operatorname{osc}(cf;z_\alpha^{(l)}+G_l)\big)\,dx\\&=m_N\sum_{\alpha=0}^{L-1}\phi\big(\operatorname{osc}(cf;z_\alpha^{(l)}+G_l)\big)\frac{1}{m_l},\end{aligned}$$

and hence

$$\phi\Big(m_N\cdot c\cdot\int_{G_N}|f(x)-g(x)|\,dx\Big)\le\Big(\frac{m_N}{m_l}\Big)\mathrm{GF}_\phi(cf;I). \tag{9}$$

Since ϕ is convex and $\phi(0)=0$, we have $\phi(ax)\le a\phi(x)$ for $0<a<1$ and for all $x\ge 0$. Therefore, choosing c in $(0,1)$ so small that $(m_N\cdot\mathrm{GF}_\phi(cf;I))\le 1$, one gets

$$|\hat f(n_k)|\le m_N\int_{G_N}|f(x)-g(x)|\,dx\le\Big(\frac{m_N}{m_N\cdot c}\Big)\phi^{-1}\Big(\frac{1}{m_l}\Big)$$

in view of (9) and (7). This shows that $\hat f(n_k)=O(\phi^{-1}(1/m_l))$.

Finally, if G is bounded, there is a positive integer p_0 such that $p_l\le p_0$ for all l. Thus $n_k<m_{l+1}=m_l\cdot p_{l+1}\le m_l\cdot p_0$, which shows that $1/m_l\le p_0/n_k$, and hence (9) gives

$$\phi\Big(m_N\cdot c\cdot\int_{G_N}|f(x)-g(x)|\,dx\Big)\le\Big(\frac{p_0\cdot m_N}{n_k}\Big)\mathrm{GF}_\phi(cf;I). \tag{10}$$

Choosing now c in $(0,1)$ so small that $(p_0\cdot m_N\cdot\mathrm{GF}_\phi(cf;I))\le 1$, one obtains

$$|\hat f(n_k)|\le m_N\int_{G_N}|f(x)-g(x)|\,dx\le\Big(\frac{m_N}{m_N\cdot c}\Big)\phi^{-1}\Big(\frac{1}{n_k}\Big)$$

in view of (10) and (7). This completes the proof of Theorem 1. □

Proof of Theorem 2.
Proceeding as in the proof of Theorem 1, for $c>0$ we get (7) and (8). Let α_i, $i=0,1,\dots,L-1$, denote a rearrangement of $0,1,\dots,L-1$ such that $\{b_i\}_{i=0}^{L-1}$ is nonincreasing, where

$$b_i=\int_{z_{\alpha_i}^{(l)}+G_l}\phi\big(c|f(x)-f(z_{\alpha_i}^{(l)})|\big)\,dx$$

for all i. For each $i=0,1,\dots,L-1$, put $a_i=1/(\lambda_{i+1}\theta_L)$, where $\theta_n=\sum_{j=1}^n 1/\lambda_j$, for all $n\in\mathbb{N}$. Then $\{a_i\}_{i=0}^{L-1}$ is nonincreasing, and $\sum_{i=0}^{L-1}a_i=1$. Therefore by the lemma,

$$\begin{aligned}\sum_{\alpha=0}^{L-1}\int_{z_\alpha^{(l)}+G_l}\phi\big(|f(x)-f(z_\alpha^{(l)})|\big)\,dx&=\sum_{i=0}^{L-1}b_i\le L\sum_{i=0}^{L-1}a_ib_i\\&=\frac{L}{\theta_L}\sum_{i=0}^{L-1}\int_{z_{\alpha_i}^{(l)}+G_l}\Big(\frac{\phi(c|f(x)-f(z_{\alpha_i}^{(l)})|)}{\lambda_{i+1}}\Big)\,dx\end{aligned}$$

$$\leq \frac{L}{\theta_L}\sum_{i=0}^{L-1}\int_{z_{\alpha_i}^{(l)}+G_l}\Big(\frac{\phi(\operatorname{osc}(cf;z_{\alpha_i}^{(l)}+G_l))}{\lambda_{i+1}}\Big)\,dx$$

$$=\frac{m_l}{m_N\theta_L}\sum_{i=0}^{L-1}\frac{\phi(\operatorname{osc}(cf;z_{\alpha_i}^{(l)}+G_l))}{\lambda_{i+1}}\cdot\frac{1}{m_l}$$

$$\leq \frac{\mathrm{F}_{\phi\Lambda}(cf;I)}{m_N\theta_L}.$$

Therefore,

$$\sum_{\alpha=0}^{L-1}\int_{z_{\alpha}^{(l)}+G_l}\phi\big(|f(x)-f(z_{\alpha}^{(l)})|\big)\,dx\leq\frac{\mathrm{F}_{\phi\Lambda}(cf;I)}{\theta_{m_l}},\tag{11}$$

since $\{\lambda_i\}$ is nondecreasing. In view of (11) and (8) we get

$$\phi\Big(m_N\cdot c\cdot\int_{G_N}|f(x)-g(x)|\,dx\Big)\leq\frac{m_N\cdot\mathrm{F}_{\phi\Lambda}(cf;I)}{\theta_{m_l}}.\tag{12}$$

Since ϕ is convex and $\phi(0)=0$, we can choose c in $(0,1)$ so small such that $(m_N\cdot\mathrm{F}_{\phi\Lambda}(cf;I))\leq 1$. This proves, in view of (12) and (7), that $\hat{f}(n_k)=O(\phi^{-1}(1/\theta_{m_l}))$.

Finally, if G is bounded, $1/\theta_{m_l}\leq p_0/\theta_{n_k}$, and hence by (8) and (11)

$$\phi\Big(m_N\cdot c\cdot\int_{G_N}|f(x)-g(x)|\,dx\Big)\leq\frac{m_N\cdot p_0\cdot\mathrm{F}_{\phi\Lambda}(cf;I)}{\theta_{n_k}}.$$

Choosing now $c\in(0,1)$ small enough such that $(m_N\cdot p_0\cdot\mathrm{F}_{\phi\Lambda}(cf;G))\leq 1$, we then get

$$\int_{G_N}|f(x)-g(x)|\,dx\leq\Big(\frac{1}{m_N\cdot c}\Big)\phi^{-1}\Big(\frac{1}{\theta_{n_k}}\Big),$$

and hence we have the theorem in view of (7). □

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