A sharp weak-type bound for Itō processes and subharmonic functions

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Abstract Let $\alpha \geq 0$, and let X, Y be Itō processes

$$dX_t = \phi_t\, dB_t + \psi_t\, dt, \qquad dY_t = \zeta_t\, dB_t + \xi_t\, dt$$

such that $|X_0| \geq |Y_0|$, $|\phi| \geq |\zeta|$, and $\alpha\psi \geq |\xi|$. The purpose of the paper is to determine the optimal universal constant C_α in the weak-type estimate

$$\sup_\lambda \lambda \mathbb{P}\Big(\sup_t |Y_t| \geq \lambda\Big) \leq C_\alpha \sup_t \mathbb{E}|X_t|.$$

Then the inequality is extended, with unchanged constant, to the more general setting when X is a submartingale and Y is α-strongly differentially subordinate to X. As an application, a related estimate for subharmonic functions is established. The inequalities generalize and unify the earlier results of Burkholder, Choi, and Hammack for Itō processes, stochastic integrals, and smooth functions on Euclidean domains.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. Assume, in addition, that $\mathcal{F}_0$ contains all the sets of probability zero. Let $B = (B_t)_{t\geq 0}$ be an adapted Brownian motion starting from zero such that $(B_t - B_s)_{t\geq s}$ is independent of $\mathcal{F}_s$ for all $s \geq 0$. Let $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ be Itō processes with respect to B (see Ikeda and Watanabe [12]):

$$\begin{aligned} X_t &= X_0 + \int_{0+}^t \phi_s\, dB_s + \int_{0+}^t \psi_s\, ds, \\ Y_t &= Y_0 + \int_{0+}^t \zeta_s\, dB_s + \int_{0+}^t \xi_s\, ds, \end{aligned} \tag{1.1}$$

where $(\phi_s)_{s\geq 0}$, $(\psi_s)_{s\geq 0}$, $(\zeta_s)_{s\geq 0}$, $(\xi_s)_{s\geq 0}$ are predictable and satisfy

$$\mathbb{P}\Big(\int_{0+}^t |\phi_s|^2\, ds < \infty \text{ and } \int_{0+}^t |\psi_s|\, ds < \infty \text{ for all } t > 0\Big) = 1,$$

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$$\mathbb{P}\Big(\int_{0+}^{t} |\zeta_s|^2\, ds < \infty \text{ and } \int_{0+}^{t} |\xi_s|\, ds < \infty \text{ for all } t > 0\Big) = 1.$$

Assuming control of X_0 over Y_0, ϕ over ζ, and ψ over ξ, what can be said about the sizes of X and Y?

This problem has gained some interest in the literature. Burkholder [4] showed that if X is a nonnegative submartingale and we have the domination $X_0 \geq |Y_0|$, $|\phi_s| \geq |\zeta_s|$ and $\psi_s \geq |\xi_s|$ for all s, then

$$\lambda \mathbb{P}(Y^* \geq \lambda) \leq 3\|X\|_1$$

for any $\lambda > 0$ and

$$\|Y\|_p \leq \max\big\{(p-1)^{-1}, 2p-1\big\}\|X\|_p, \quad 1 < p < \infty$$

(see also [5] for more general inequalities under the assumption of strong differential subordination). Here we have used the notation $X^* = \sup_{t\geq 0}|X_t|$ and $\|X\|_p = \sup_t \|X_t\|_p$ for $p \geq 1$. Furthermore, both inequalities are sharp. These results have been strengthened by Choi [6] and [7], who showed that if $\alpha \geq 0$ is a fixed number, X is a nonnegative submartingale and, in addition,

$$|X_0| \geq |Y_0|, \qquad |\phi_s| \geq |\zeta_s|, \qquad \text{and} \qquad \alpha\psi_s \geq |\xi_s| \quad \text{for all } s, \tag{1.2}$$

then

$$\lambda \mathbb{P}(Y^* \geq \lambda) \leq (\alpha + 2)\|X\|_1$$

for any $\lambda > 0$ and

$$\|Y\|_p \leq \max\big\{(p-1)^{-1}, (\alpha+1)p - 1\big\}\|X\|_p, \quad 1 < p < \infty.$$

Again, the constants $\alpha + 2$ and $\max\{(p-1)^{-1}, (\alpha+1)p-1\}$ are optimal. There is a natural question about the validity of the above estimates without the assumption on the sign of X. The purpose of the present paper is to answer this question and, as an application, to establish some related results for subharmonic functions on open subsets of $\mathbb{R}^n$.

In fact, we study this problem under a weaker assumption. For any semimartingales X and Y, we say that Y is *differentially subordinate* to X if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t (see Bañuelos and Wang [1] or Wang [14] for discussion). Here $[X,X]$ denotes the quadratic variance process of X (see, e.g., Dellacherie and Meyer [10]). This type of domination implies many interesting inequalities if X and Y are martingales or local martingales (see [14]). However, it turns out to be too weak for our purposes. We work under the assumption of *α-strong differential subordination* (α-subordination in short), introduced by Wang [14] in the particular case $\alpha = 1$, and by Osękowski [13] for general $\alpha \geq 0$. The definition is the following. Let X be an adapted submartingale, let Y be an adapted semimartingale, and write the Doob-Meyer decompositions

$$X = X_0 + M + C, \qquad Y = Y_0 + N + D, \tag{1.3}$$

where M, N are local martingale parts and C, D are finite variation processes. In general, the decompositions may not be unique; however, we assume that C is predictable, and this determines the first of them. Let α be a fixed nonnegative number. We say that Y is α-subordinate to X if Y is differentially subordinate to X and there is a decomposition (1.3) for Y such that the process $(\alpha C_t - |D|_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t. Here $|D|_t$ denotes the total variation of D on the interval $[0,t]$. Two observations are in order. First, in the setting of Itō processes described in (1.1), if $|X_0| \geq |Y_0|$, $|\phi_s| \geq |\zeta_s|$, and $\alpha\psi_s \geq |\xi_s|$ for all s, then, obviously, Y is α-subordinate to X. Second, the above domination extends to the case when Y takes values in a certain separable Hilbert space $\mathcal{H}$ (which can be assumed to be ℓ^2): one applies the Doob-Meyer decomposition for each coordinate of Y and then rewrites the definition of α-subordination with $[Y,Y] = \sum_{j=1}^{\infty}[Y^j, Y^j]$ and $|D| = \sum_{j=1}^{\infty} |D^j|$.

Now we are ready to state one of the main results of the paper.

THEOREM 1.1

Let $\alpha \geq 0$ be fixed. Suppose that X is a submartingale and Y is an $\mathcal{H}$-valued semimartingale which is α-subordinate to X. Then

$$\sup_{\lambda>0} \lambda\mathbb{P}(Y^* \geq \lambda) \leq C_\alpha \|X\|_1, \tag{1.4}$$

where

$$C_\alpha = \begin{cases} (\alpha+1)[1+(\alpha+1)^{1/\alpha}] & \text{if } \alpha \geq 1, \\ 6 & \text{if } \alpha \leq 1. \end{cases}$$

The constant is the best possible. It is already the best possible if $\mathcal{H} = \mathbb{R}$, and we restrict ourselves to the class of Itō processes (1.1) satisfying (1.2).

This theorem generalizes the following result of Hammack [11]. Suppose that X is a submartingale and Y is an Itō integral of H with respect to X, where H is a predictable process with values in the unit ball of $\mathcal{H}$. Then

$$\sup_{\lambda>0} \lambda\mathbb{P}(Y^* \geq \lambda) \leq 6\|X\|_1,$$

and the inequality is sharp. This is an immediate consequence of our result stated above since Y is 1-subordinate to X. Indeed, as the decomposition of Y we take $Y_t = Y_0 + \int_{0+}^t H_s\,dM_s + \int_{0+}^t H_s\,dC_s$, where M, C come from (1.3), and we observe that

$$[X,X]_t - [Y,Y]_t = \int_0^t (1-|H_s|^2)\,d[X,X]_s \qquad \text{and} \qquad C_t - |D_t| = \int_0^t (1-|H_s|)\,dC_s$$

for all $t \geq 0$.

To establish Theorem 1.1, we deal with the following stronger statement.

THEOREM 1.2

Under the assumptions of Theorem 1.1, we have

$$\sup_{\lambda>0} \lambda \mathbb{P}(Y^* \geq \lambda) \leq K_\alpha \|X^+\|_1 - (C_\alpha - K_\alpha)\mathbb{E}X_0, \tag{1.5}$$

where

$$K_\alpha = \begin{cases} (\alpha+1)^{1+1/\alpha} & \text{if } \alpha \geq 1, \\ 4 & \text{if } \alpha \leq 1. \end{cases}$$

The inequality is sharp. In consequence, if the submartingale X starts from zero, then

$$\sup_{\lambda>0} \lambda \mathbb{P}(Y^* \geq \lambda) \leq K_\alpha \|X\|_1. \tag{1.6}$$

This inequality is also sharp.

Concerning the moment inequalities, we have the following negative result.

THEOREM 1.3

Let $1 \leq p < \infty$ and $\beta > 0$. Then there is a nontrivial pair (X, Y) of Itō processes as in (1.1) such that

(i) $X_0 = Y_0 = 0$,
(ii) *X is a submartingale, Y is a martingale,*
(iii) $|\phi_s| = |\zeta_s|$ *for all $s > 0$*

and

$$\|Y\|_1 \geq \beta \|X\|_p.$$

In other words, moment inequalities fail to hold even under the most restrictive zero-domination.

A few words about the proof and the organization of the paper. The proof of (1.5) is based on Burkholder's method: the inequality follows if one constructs a certain special function and exploits its properties. We do this in Section 2. Section 3 concerns the sharpness of the estimate, and we also prove Theorem 1.3 there. In the final part of the paper we present an application: a weak-type inequality for smooth functions on Euclidean domains.

2. Proof of (1.5)

Let α be a fixed nonnegative number, and let ν be a positive integer. Consider the following subsets of $\mathbb{R} \times \mathbb{R}^\nu$. If $\alpha \geq 1$, then

$$D_1^\alpha = \big\{(x,y) : \alpha|x| + |y| \geq 1,\ x \leq 0\big\},$$
$$D_2^\alpha = \big\{(x,y) : |x| + |y| \geq 1,\ x \geq 0\big\},$$
$$D_3^\alpha = (\mathbb{R} \times \mathbb{R}^\nu) \setminus (D_1^\alpha \cup D_2^\alpha).$$

If $\alpha \in [0,1)$, then let $D_i^\alpha = D_i^1$ for $i = 1,2,3$. The proof rests on the special functions $U_\alpha : \mathbb{R} \times \mathbb{R}^\nu \to \mathbb{R}$ given as follows. If $\alpha \geq 1$, then

$$U_\alpha(x,y) = \begin{cases} 1 - K_\alpha x^+ & \text{if } (x,y) \in D_1^\alpha \cup D_2^\alpha, \\ 1 - (\alpha x - |y| + 1)(\alpha x + \alpha|y| + 1)^{1/\alpha} & \text{if } (x,y) \in D_3^\alpha, \end{cases} \tag{2.1}$$

and $U_\alpha(x,y) = U_1(x,y)$ for $\alpha \in [0,1)$.

LEMMA 2.1

The functions U_α enjoy the following.

(i) *We have the majorization*

$$U_\alpha(x,y) \geq 1_{D_1^\alpha \cup D_2^\alpha}(x,y) - K_\alpha x^+.$$

(ii) *If $(x,y) \in D_3^\alpha$, then*

$$U_{\alpha x}(x,y) + \alpha |U_{\alpha y}(x,y)| \leq 0. \tag{2.2}$$

(iii) *If $(x,y) \in D_3^\alpha$ and $|y| \neq 0$, then for any $h \in \mathbb{R}$, $k \in \mathbb{R}^\nu$,*

$$\begin{aligned} &U_{\alpha xx}(x,y)h^2 + 2\big(U_{\alpha xy}(x,y)h, k\big) + \big(kU_{\alpha yy}(x,y), k\big) \\ &\quad \leq c_\alpha(x,y)(|k|^2 - h^2), \end{aligned} \tag{2.3}$$

where $c_\alpha(x,y) = (\alpha+1)(\alpha x + \alpha|y| + 1)^{1/\alpha - 1} \geq 0$ for $\alpha \geq 1$, and $c_\alpha(x,y) = 2$ for $\alpha \in [0,1)$.

(iv) *If $(x,y) \in D_3^\alpha$, then for any $h \in \mathbb{R}$, $k \in \mathbb{R}^\nu$ satisfying $|k| \leq |h|$, we have*

$$U_\alpha(x+h, y+k) \leq U_\alpha(x,y) + U_{\alpha x}(x,y)h + \big(U_{\alpha y}(x,y), k\big). \tag{2.4}$$

(v) *Assume that $(x,y) \in \mathbb{R} \times \mathbb{R}^\nu$ satisfies $|y| \leq |x|$. Then $U_\alpha(x,y) \leq -(\alpha+1)x$ for $\alpha \geq 1$ and $U_\alpha(x,y) \leq -2x$ for $\alpha \in [0,1)$.*

Proof

It is easy to see that we may restrict ourselves to the case $\alpha \geq 1$.

(i) We only need to prove the majorization on D_3^α. Then the inequality takes the form

$$1 - (\alpha x - |y| + 1)(\alpha x + \alpha|y| + 1)^{1/\alpha} \geq -K_\alpha x^+.$$

For a fixed x, the left-hand side increases as $|y|$ increases. Hence it suffices to show the estimate for $y = 0$: $1 - (\alpha x + 1)^{1+1/\alpha} \geq -K_\alpha x^+$. This is evident for $x \leq 0$ (then $\alpha x + 1 \leq 1$), while for $x \geq 0$ we use the fact that the function $F(x) = 1 - (\alpha x + 1)^{1+1/\alpha}$ is concave and lies above the linear $G(x) = -K_\alpha x$ on $[0,1]$ since $F(0) = G(0)$ and $F(1) = G(1) + 1 > G(1)$.

Before we proceed, let us mention the following easy consequence, which is used below. By the fact that U_α is continuous and $1_{D_1^\alpha \cup D_2^\alpha}$ is upper semicontinuous, we see that for any $\eta > 1$ there is $R = R(\eta) > 0$ such that $R(\eta) \to 0$ as $\eta \downarrow 1$ and

$$U_\alpha(x,y) \geq 1_{D_1^\alpha \cup D_2^\alpha}(\eta x, \eta y) - K_\alpha x^+ - R(\eta). \tag{2.5}$$

(ii) A direct computation shows that

$$
\begin{aligned}
U_{\alpha x}(x,y) &= -(\alpha+1)(\alpha x+\alpha|y|+1)^{1/\alpha-1}[\alpha x+1+(\alpha-1)|y|],\\
U_{\alpha y}(x,y) &= (\alpha+1)(\alpha x+\alpha|y|+1)^{1/\alpha-1}y,
\end{aligned}
\tag{2.6}
$$

so

$$U_{\alpha x}(x,y)+\alpha|U_{\alpha y}(x,y)| = -(\alpha+1)(\alpha x+\alpha|y|+1)^{1/\alpha-1}(\alpha x+1-|y|)\le 0.$$

(iii) A little calculation leads to

$$U_{\alpha xx}(x,y)h^2+2\big(U_{\alpha xy}(x,y)h,k\big)+\big(kU_{\alpha yy}(x,y),k\big)=I_1+I_2,$$

where

$$
\begin{aligned}
I_1 &= (\alpha+1)(\alpha x+\alpha|y|+1)^{1/\alpha-1}(|k|^2-h^2),\\
I_2 &= (\alpha+1)(1-\alpha)|y|(\alpha x+\alpha|y|+1)^{1/\alpha-2}[h+(y,k)/|y|]^2\le 0.
\end{aligned}
$$

This proves the claim.

(iv) If $h=0$, the bound is trivial. Suppose, then, that $h\neq 0$, and consider a function $G:\mathbb{R}\to\mathbb{R}$ given by $G(t)=U_\alpha(x+t,y+tk/h)$. Let $t_0=\sup\{t:(x+t,y+tk/h)\in D_1^\alpha\}<0$ and $t_1=\inf\{t:(x+t,y+tk/h\}\in D_2^\alpha\}>0$. We have that G is continuous, equal to 1 on $(-\infty,t_0]$, and linear on $[t_1,\infty)$. In addition, G is concave on (t_0,t_1); this is guaranteed by (iii) and the assumption $|k|\le|h|$. Thus rewriting (2.4) in the form $G(h)\le G(0)+G'(0)h$, we see that it suffices to prove that $G'(0)\le 0$ and $G'(0)\ge G'(t_1+)=-K_\alpha$. By (ii), we have

$$G'(0)\le U_{\alpha x}(x,y)+|U_{\alpha y}(x,y)|\cdot\frac{|k|}{h}\le U_{\alpha x}(x,y)+\alpha|U_{\alpha y}(x,y)|\le 0.$$

Furthermore, using $(y,k)/h\ge -|y|$ and the estimate $x+|y|\le 1$ coming from the definition of D_3^α,

$$
\begin{aligned}
G'(0) &= -(\alpha+1)(\alpha x+\alpha|y|+1)^{1/\alpha-1}\left[\alpha x+1+(\alpha-1)|y|-\frac{(y,k)}{h}\right]\\
&\ge -(\alpha+1)(\alpha x+\alpha|y|+1)^{1/\alpha}\\
&\ge -(\alpha+1)^{1/\alpha+1}=-K_\alpha.
\end{aligned}
$$

(v) By (iv), we have

$$U_\alpha(x,y)\le U_\alpha(0,0)+U_{\alpha x}(0,0)x+\big(U_{\alpha y}(0,0),y\big)=-(\alpha+1)x.$$

This completes the proof. □

For any semimartingale X there exists a unique continuous local martingale part X^c of X satisfying

$$[X,X]_t=|X_0|^2+[X^c,X^c]_t+\sum_{0<s\le t}|\triangle X_s|^2$$

for all $t\ge 0$. (Here $\triangle X_s=X_s-X_{s-}$ is the jump of X at time $s>0$.) Furthermore, $[X^c,X^c]=[X,X]^c$, the pathwise continuous part of $[X,X]$. We need [14, Lemma 1], which can be stated as follows.

LEMMA 2.2
If X and Y are semimartingales, then Y is differentially subordinate to X if and only if Y^c is differentially subordinate to X^c, $|Y_0| \le |X_0|$, and for any $s > 0$ we have $|\triangle Y_s| \le |\triangle X_s|$.

Now we turn to the proofs of the announced estimates.

Proof of (1.5)
Let us start with some reductions. First, we may assume that $\|X^+\|_1 < \infty$; otherwise, there is nothing to prove. Second, by homogeneity, it suffices to prove that

$$\mathbb{P}(Y^* \ge 1) \le K_\alpha \|X^+\|_1 - (C_\alpha - K_\alpha)\mathbb{E}X_0. \tag{2.7}$$

The third observation is that we may restrict ourselves to the case $\alpha \ge 1$: indeed, if X, Y satisfy the assumptions of Theorem 1.1 with some $\alpha < 1$, then they satisfy the assumptions for $\alpha = 1$ as well, and $C_\alpha = C_1$, $K_\alpha = K_1$ for $\alpha \in [0,1)$. The next step is to reduce (1.4) to the case of finite-dimensional Hilbert spaces $\mathcal{H}$. To do this, we observe that we may take $\mathcal{H}$ to be equal to ℓ^2. For a fixed positive integer ν, the truncated process

$$Y_t^{(\nu)} = (Y_t^1, Y_t^2, \ldots, Y_t^\nu, 0, 0, \ldots)$$

is α-subordinate to X and, in addition, for any $\delta < 1$, we have $\mathbb{P}(Y^* \ge 1) \le \lim_{\nu\to\infty} \mathbb{P}(Y^{(\nu)*} \ge \delta)$. Thus having established (2.7) for finite-dimensional $\mathcal{H}$, we may write

$$\delta\mathbb{P}(Y^* \ge 1) \le K_\alpha \|X^+\|_1 - (C_\alpha - K_\alpha)\mathbb{E}X_0,$$

and it suffices to let $\delta \uparrow 1$ to obtain (2.7) in full generality. Therefore, from now on, $\mathcal{H} = \mathbb{R}^\nu$ for some positive integer ν.

The main tool in the proof is the Itō formula. However, we are not allowed to apply it to the function U_α since it is not sufficiently smooth. Therefore, we need to use some extra approximation arguments. Fix a number $\eta > 1$, and introduce the stopping time $\tau = \inf\{t : (X_t, Y_t) \notin D_3^\alpha/\eta\}$ (here $D_3^\alpha/\eta = \{(x,y) \in \mathbb{R}\times\mathbb{R}^\nu : (\eta x, \eta y) \in D_3^\alpha\}$). Suppose that $\delta > 0$ satisfies

$$\operatorname{dist}(D_1^\alpha \cup D_2^\alpha, D_3^\alpha/\eta) > \delta, \tag{2.8}$$

and consider a C^∞-function $g : \mathbb{R}\times\mathbb{R}^\nu \to [0,\infty)$, supported on the ball of center $(0,0) \in \mathbb{R}\times\mathbb{R}^\nu$ and radius δ, satisfying $\int_{\mathbb{R}\times\mathbb{R}^\nu} g = 1$. Introduce a function $U_\alpha^\delta : \mathbb{R}\times\mathbb{R}^\nu \to \mathbb{R}$, given by the convolution

$$U_\alpha^\delta(x,y) = \int_{\mathbb{R}\times\mathbb{R}^\nu} U_\alpha(x-u, y-v) g(u,v)\, du\, dv.$$

Observe that by (2.8), if $(x,y) \in D_3^\alpha/\eta$, then for all (u,v) lying in the support of g we have $(x-u, y-v) \in D_3^\alpha$. Consequently, for these (x,y), the function U_α^δ enjoys the properties described in Lemma 2.1(ii), (iii), (iv). In (iii), we replace

$c_\alpha(x,y)$ by

$$c_\alpha^\delta(x,y)=\int_{\mathbb{R}\times\mathbb{R}^\nu} c_\alpha(x-u,y-v)g^\delta(u,v)\,du\,dv\geq 0.$$

Indeed, note that U_α is of class C^1 in D_3^α (see (2.6)), so the properties follow from the integration.

The function U_α^δ is of class C^∞, so we may apply Itō's formula and obtain

$$U_\alpha^\delta(X_{\tau\wedge t},Y_{\tau\wedge t})=U_\alpha^\delta(X_0,Y_0)+I_1+I_2/2+I_3+I_4, \tag{2.9}$$

where (recall M, N, C, D given by (1.3) with the decomposition of Y coming from the α-subordination)

$$\begin{aligned}
I_1&=\int_{0+}^{\tau\wedge t} U_{\alpha x}^\delta(X_{s-},Y_{s-})\,dM_s+\int_{0+}^{\tau\wedge t} U_{\alpha y}^\delta(X_{s-},Y_{s-})\,dN_s,\\
I_2&=\int_{0+}^{\tau\wedge t} U_{\alpha xx}^\delta(X_{s-},Y_{s-})\,d[M^c,M^c]+2\sum_{i=1}^{\nu}\int_{0+}^{\tau\wedge t} U_{\alpha xy_i}^\delta(X_{s-},Y_{s-})\,d[M^c,N^{ic}]\\
&\quad+\sum_{i,j=1}^{\nu}\int_{0+}^{\tau\wedge t} U_{\alpha y_iy_j}^\delta(X_{s-},Y_{s-})\,d[N^{ic},N^{jc}]_s,\\
I_3&=\int_{0+}^{\tau\wedge t} U_{\alpha x}^\delta(X_{s-},Y_{s-})\,dC_s+\int_{0+}^{\tau\wedge t} U_{\alpha y}^\delta(X_{s-},Y_{s-})\,dD_s,\\
I_4&=\sum_{0<s\leq\tau\wedge t}\big[U_\alpha^\delta(X_s,Y_s)-U_\alpha^\delta(X_{s-},Y_{s-})\\
&\quad-U_{\alpha x}^\delta(X_{s-},Y_{s-})\Delta X_s-\big(U_{\alpha y}^\delta(X_{s-},Y_{s-}),\Delta Y_s\big)\big].
\end{aligned}$$

Now let us look at the terms in (2.9). We have $\mathbb{E}I_1=0$ by the properties of stochastic integrals. Furthermore, I_2 is nonpositive. To see this, we proceed as in [14]. We approximate the integrals by appropriate Riemann sums and apply (2.3) to the function U_α^δ (which is permitted since $(X_{s-},Y_{s-})\in D_3^\alpha/\lambda$). This yields

$$I_2\leq c_\alpha^\delta(x,y)\big(-[X^c,X^c]_{\tau\wedge t}+[Y^c,Y^c]_{\tau\wedge t}-(-[X^c,X^c]_0+[Y^c,Y^c]_0)\big)\leq 0$$

due to the differential subordination of Y^c to X^c. To deal with I_3, note that by α-subordination, and then by (2.2),

$$\begin{aligned}
I_3&\leq\int_{0+}^{\tau\wedge t} U_{\alpha x}^\delta(X_{s-},Y_{s-})\,dC_s+\int_{0+}^{\tau\wedge t} |U_{\alpha y}^\delta(X_{s-},Y_{s-})|d|D_s|\\
&\leq\int_{0+}^{\tau\wedge t} U_{\alpha x}^\delta(X_{s-},Y_{s-})\,dC_s+\int_{0+}^{\tau\wedge t} \alpha|U_{\alpha y}^\delta(X_{s-},Y_{s-})|\,dC_s\leq 0.
\end{aligned}$$

Finally, $I_4\leq 0$ due to Lemma 2.1(iv): here we use the inequality $|\triangle Y_s|\leq|\triangle X_s|$ coming from the differential subordination. Thus we have shown that

$$\mathbb{E}U_\alpha^\delta(X_{\tau\wedge t},Y_{\tau\wedge t})\leq\mathbb{E}U_\alpha^\delta(X_0,Y_0). \tag{2.10}$$

Now, note that $|U_\alpha(x,y)| \le L + K_\alpha x^+$ for some absolute constant L, which implies that $|U_\alpha^\delta(x,y)| \le L + K_\alpha(x^+ + \delta)$. Moreover, U_α is continuous; thus, letting $\delta \to 0$ in (2.10) and using Lebesgue's dominated convergence theorem, one obtains

$$\mathbb{E}U_\alpha(X_{\tau\wedge t}, Y_{\tau\wedge t}) \le \mathbb{E}U_\alpha(X_0, Y_0) \le -(\alpha+1)\mathbb{E}X_0.$$

Here in the last passage we have exploited Lemma 2.1(v) together with the fact that $|Y_0| \le |X_0|$. Combining this with (2.5), we get

$$\mathbb{P}\big((X_{\tau\wedge t}, Y_{\tau\wedge t}) \notin D_3^\alpha/\eta\big) \le K_\alpha \mathbb{E}X_{\tau\wedge t}^+ - (\alpha+1)\mathbb{E}X_0 + R(\eta).$$

Now $\{Y^* \ge 1\} \subseteq \{\tau < \infty\} = \bigcup_t \{(X_{\tau\wedge t}, Y_{\tau\wedge t}) \notin D_3^\alpha/\eta\}$, so

$$\begin{aligned}\mathbb{P}(Y^* \ge 1) &\le K_\alpha \sup_t \mathbb{E}X_{\tau\wedge t}^+ - (\alpha+1)\mathbb{E}X_0 + R(\eta)\\ &\le K_\alpha \sup_t \mathbb{E}X_t^+ - (\alpha+1)\mathbb{E}X_0 + R(\eta)\end{aligned}$$

by Doob's optional sampling theorem. (The process $(X_t^+)_{t\ge 0}$ is a submartingale.) Letting $\eta \downarrow 1$ completes the proof of (1.5). □

3. Sharpness and lack of moment estimates

3.1. Sharpness

We construct examples of Itō processes X, Y, which will exhibit the optimality of the constants C_α, K_α in (1.4) and (1.6), respectively. This also proves that the estimate (1.5) is sharp.

The construction consists of two parts. The first step is to find, for any $\varepsilon > 0$, an appropriate pair (F, G) of Itō processes starting from zero such that

$$\mathbb{P}(G^* \ge 1) = 1 \quad \text{and} \quad \|F_\infty\|_1 \le K_\alpha^{-1} + \varepsilon$$

and another pair (F, G) of Itō processes, satisfying $F_0 = -G_0 \equiv -C_\alpha^{-1}$,

$$\mathbb{P}(G^* \ge 1) = 1 \quad \text{and} \quad \|F_\infty\|_1 \le C_\alpha^{-1} + \varepsilon.$$

Here, as usual, F_∞ denotes the pointwise limit of F_t as $t \to \infty$. Next, in the second part, we modify these pairs so that the above conditions are satisfied, but with $\|F_\infty\|_1$ replaced by $\|F\|_1$. This immediately yields the claim.

Part I. We present a unified construction which produces both pairs (F, G) mentioned above. Assume first that $\alpha \ge 1$, let $x_0 \in \{-C_\alpha^{-1}, 0\}$, and pick a large positive integer N. Set $\delta = 1/(2N)$, and let $(B_t)_{t\ge 0}$ be a one-dimensional Brownian motion started at x_0. For $n = 1, 2, \ldots, N$, let

$$\ell_n = \frac{-1 + 2(n-1)\delta}{\alpha+1}, \qquad r_n = (2n-1)\delta,$$

and put $\ell_{N+1} = 0$, $r_{N+1} = 2$. Introduce the stopping times $\tau_i = \tau_i(\alpha)$, $0 \le i \le N+1$, as follows: $\tau_0 \equiv 0$ and, by induction,

$$\tau_n = \inf\big\{t > \tau_{n-1} : B_t \le \ell_n \text{ or } B_t \ge r_n\big\}, \quad n = 1, 2, \ldots, N+1.$$

Note that the sequence (ℓ_n) is increasing; hence, if $B_{\tau_k}=\ell_k$ for some k, then $\tau_k=\tau_{k+1}=\cdots=\tau_{N+1}$. We are ready to introduce Itō processes $F=(F_t)_{t\geq 0}$ and $G=(G_t)_{t\geq 0}$. Let $F_0\equiv -G_0\equiv x_0$, let

$$dF_t = 1_{\{t\leq \tau_{N+1}\}}\,dB_t + 1_{\{\tau_{N+1}<t\leq \tau_{N+1}-B_{\tau_{N+1}}\}}\,dt,$$

and let

$$dG_t=\Big(\sum_{n=1}^{N+1}(-1)^n 1_{\{\tau_{n-1}<t\leq\tau_n\}}\Big)dB_t+\alpha\operatorname{sgn}(G_{\tau_{N+1}})1_{\{\tau_{N+1}<t\leq \tau_{N+1}-B_{\tau_{N+1}}\}}\,dt.$$

Clearly, F is a submartingale which dominates G in a sense described in (1.2). For a better understanding of these two processes, it is convenient to look at the properties of $(F_t,G_t)_{t\geq 0}$ at two stages: for $t\leq\tau_{N+1}$, where it has martingale behavior, and for $t>\tau_{N+1}$, where F is nondecreasing. The pair starts from $(x_0,-x_0)$, and for $t\in(\tau_{n-1},\tau_n]$, $n\leq N$, it moves along the line of slope $(-1)^n$ until it reaches the set $\{(x,y):-\alpha x+|y|=1\}$ or $G_{\tau_n}=(-1)^n\delta$. If the first possibility occurs, we have $\tau_n=\tau_{N+1}$; in the second case, the move continues and the slope switches to $(-1)^{n+1}$. On $t\in(\tau_N,\tau_{N+1}]$ the behavior is similar, but here we stop the move if F reaches zero or 2. One easily checks that at the end of the first stage, $(F_{\tau_{N+1}},|G_{\tau_{N+1}}|)=(2,1)$ (i.e., when $B_{\tau_n}=r_n$ for all $n=1,2,\ldots,N+1$) or $-\alpha F_{\tau_{N+1}}+|G_{\tau_{N+1}}|=1$ (i.e., when $B_{\tau_n}=\ell_n$ for some n). Now, in the first case, the pair stops ultimately: we have $F_{\tau_{N+1}}=B_{\tau_{N+1}}=2$, so the event $\{\tau_{N+1}<t\leq\tau_{N+1}-B_{\tau_{N+1}}\}$ is empty. If the second possibility occurs, then $(F_{\tau_{N+1}+t},|G_{\tau_{N+1}+t}|)=(F_{\tau_{N+1}}+t,|G_{\tau_{N+1}+t}|+\alpha t)$ for $t\in[0,-F_{\tau_{N+1}}]$, and then the pair stops. We see that $\tau:=\tau_{N+1}+1$ can be regarded as the terminal stopping time of the pair (F,G): we have $dF_t=dG_t=0$ for $t\geq\tau$.

In the case $\alpha\in[0,1)$, the construction is similar. Let $\tau_j=\tau_j(1)$, $j=0,1,2,\ldots,$ let $N+1$ be the stopping times coming from the case $\alpha=1$, and let $\tau_{N+2}=\inf\{t>\tau_{N+1}:B_t\leq -2 \text{ or } B_t\geq 0\}$. The pair (F,G) is given by $F_0=-G_0\equiv x_0$ and

$$dF_t = 1_{\{t\leq \tau_{N+2}\}}\,dB_t + 1_{\{\tau_{N+2}<t\leq \tau_{N+2}-B_{\tau_{N+2}}\}}\,dt,$$

$$dG_t=\Big(\sum_{n=1}^{N+1}(-1)^n 1_{\{\tau_{n-1}<t\leq\tau_n\}}\Big)\,dB_t+\operatorname{sgn}(G_{\tau_{N+1}})1_{\{\tau_{N+1}<t\leq \tau_{N+2}\}}\,dB_t.$$

Therefore, comparing to the case $\alpha\geq 1$, we see that the second stage splits into two steps: a martingale move of (F,G) along the line $-x+y=1$ or $x+y=-1$ on the interval $[\tau_{N+1},\tau_{N+2}]$ and the second step, for $t\geq\tau_{N+2}$, when F is nondecreasing. We see that G is a martingale which is differentially subordinate to F; hence, G is α-subordinate to F for any $\alpha\geq 0$. We define the terminal stopping time by $\tau:=\tau_{N+2}+2$.

Now we prove the aforementioned bounds for F and G.

LEMMA 3.1

We have $G^*\geq 1$ almost surely and $\|F\|_1\leq 2$. Furthermore, for any $\varepsilon>0$ there *is N such that*

$$\|F_\infty\|_1 = \|F_\tau\|_1 \le \big(1 + (\alpha+1)x_0\big)(1+\alpha)^{-(\alpha+1)/\alpha} + \varepsilon.$$

REMARK 3.1

Note that $(1+(\alpha+1)x_0)(1+\alpha)^{-(\alpha+1)/\alpha}$ is equal to C_α^{-1} or K_α^{-1} (resp., depending on whether $x_0 = -C_\alpha^{-1}$ or $x_0 = 0$).

Proof of Lemma 3.1

The first two properties are obvious: we have $|G_\tau| = 1$ and $|F_t| \le 2$ for any $t \ge 0$. We prove the third condition only for $\alpha \ge 1$; for the remaining α the calculations can be performed in a similar manner. Note that $F_\tau \in \{0,2\}$ and $F_\tau = 2$ if and only if $\tau_1 < \tau_2 < \cdots < \tau_N$ and $F_{\tau_{N+1}} = 2$; that is, $B_{\tau_n} = r_n$ for all $n = 1,2,\dots,N+1$. For convenience, let $r_0 = x_0$, and note that by the definition of τ_n and elementary properties of Brownian motion, we may write the following:

$$\begin{aligned}
\mathbb{P}(F_\tau = 2) &= \prod_{n=1}^{N+1} \frac{r_{n-1} - \ell_n}{r_n - \ell_n} \\
&= \frac{r_0 - \ell_1}{r_1 - \ell_1} \cdot \frac{r_N - \ell_{N+1}}{r_{N+1} - \ell_{N+1}} \prod_{n=2}^{N} \frac{r_{n-1} - \ell_n}{r_n - \ell_n} \\
&= \frac{x_0 + (\alpha+1)^{-1}}{\delta + (\alpha+1)^{-1}} \cdot \frac{1-\delta}{2} \cdot \prod_{n=2}^{N} \Big(1 - \frac{2\delta(\alpha+1)}{1 + \delta[(2n-1)\alpha + 1]}\Big) \\
&\le \frac{(1 + x_0(\alpha+1))(1-\delta)}{2(1+\delta(\alpha+1))} \exp\Big[-2\delta(\alpha+1) \sum_{n=2}^{N} \big(1 + \delta[(2n-1)\alpha + 1]\big)^{-1}\Big] \\
&\le \frac{(1 + x_0(\alpha+1))(1-\delta)}{2(1+\delta(\alpha+1))} \Big(\frac{1 + \delta + (2N+1)\delta\alpha}{1 + \delta + 5\delta\alpha}\Big)^{-(\alpha+1)/\alpha}.
\end{aligned}$$

Here in the first inequality we have used the elementary bound $1 - x \le e^{-x}$ and in the second estimate we have exploited the fact that

$$\begin{aligned}
2\delta \sum_{n=2}^{N} \big(1 + \delta[1 + (2n-1)\alpha]\big)^{-1} &\ge \int_{5\delta}^{(2N+1)\delta} (1 + \delta + \alpha x)^{-1}\, dx \\
&= \frac{1}{\alpha} \log \frac{1 + \delta + (2N+1)\delta\alpha}{1 + \delta + 5\delta\alpha}.
\end{aligned}$$

The claim follows. Recall that $\delta = (2N)^{-1}$, so letting $N \to \infty$ implies that the above upper bound for $\mathbb{P}(F_\tau = 2)$ converges to

$$\frac{(\alpha+1)x_0 + 1}{2}(1+\alpha)^{-(\alpha+1)/\alpha},$$

as needed. $\square$

Part II. Note that there is no hope for the equality $\|F_\infty\|_1 = \|F\|_1$ since the submartingale F takes negative values. Thus we need some additional modification of the pair to ensure that the first moment of the dominating process is arbitrarily close to $\|F_\tau\|_1$. The main idea is to work on small portions of the probability space, using the appropriate copy of (F, G) on each portion. To be more precise, let $\varepsilon > 0$ be given and fixed. For the sake of convenience, we split the reasoning into four steps.

Step 1: An auxiliary parameter K. By Lemma 3.1, there are N and $K > 0$ such that

$$\|F_t\|_1 \le \big(1 + (\alpha+1)x_0\big)(1+\alpha)^{-(\alpha+1)/\alpha} + 2\varepsilon$$

whenever $t \ge K$.

Step 2: Time-shifted copies of (F, G). For $j = 0, 1, 2, \ldots$, let (F^j, G^j) be a pair given by the above construction but with $(B_t)_{t\ge 0}$ replaced by the time-shifted Brownian motion

$$B_t^j = \begin{cases} x_0 & \text{if } t \le Kj, \\ x_0 + B_t - B_{Kj} & \text{if } t > Kj. \end{cases}$$

Then $(F_t^j, G_t^j) = (x_0, -x_0)$ for $t \le Kj$ and

$$\big((F^j_{Kj+t}, G^j_{Kj+t})\big)_{t\ge 0} \quad \text{has the same distribution as } (F, G). \tag{3.1}$$

Furthermore, F^j, G^j are Itō processes with respect to the original Brownian motion B, and F^j dominates G^j in the sense of (1.2).

Step 3: Definition of (X, Y). Fix a positive integer k, and consider a random variable η independent of B, with the distribution $\mathbb{P}(\eta = j) = 1/k$ for $j = 0, 1, 2, \ldots, k-1$. This random variable splits Ω into k parts $\{\eta = 0\}$, $\{\eta = 1\}, \ldots, \{\eta = k-1\}$. We define

$$(X_t, Y_t) = (F_t^j, G_t^j) \quad \text{on } \{\eta = j\}$$

for $t \ge 0$ and $j = 0, 1, 2, \ldots, k-1$. Then, by Step 2, both X and Y are Itō processes with respect to B, and the domination (1.2) is satisfied.

Step 4: Final calculations. Observe that

$$\mathbb{P}(Y^* \ge 1) = \frac{1}{k}\sum_{j=0}^{k-1} \mathbb{P}(G^{j*} \ge 1) = 1$$

and, for any $t \ge 0$,

$$\|X_t\|_1 = \frac{1}{k}\sum_{j=0}^{k-1} \|F_t^j\|_1.$$

Now, if $t \le Kj$, then $F_t^j = x_0$, so $\|F_t^j\|_1 = -x_0$, and hence,

$$\|F_t^j\|_1 \le \big(1 + (\alpha+1)x_0\big)(1+\alpha)^{-(\alpha+1)/\alpha} + 2\varepsilon.$$

If $t \in (Kj, Kj+K)$, then $\|F_t^j\|_1 = \|F_{t-Kj}\|_1 \le 2$ in virtue of Lemma 3.1. Finally, if $t \ge Kj + K$, then by Step 1,

$$\|F_t^j\|_1 = \|F_{t-Kj}\|_1 \le \big(1 + (\alpha+1)x_0\big)(1+\alpha)^{-(\alpha+1)/\alpha} + 2\varepsilon.$$

In consequence, we obtain

$$\sup_{t\geq 0}\|X_t\|_1 \leq \frac{k-1}{k}\left[\big(1+(\alpha+1)x_0\big)(1+\alpha)^{-(\alpha+1)/\alpha}+2\varepsilon\right]+\frac{2}{k}$$
$$< \big(1+(\alpha+1)x_0\big)(1+\alpha)^{-(\alpha+1)/\alpha}+3\varepsilon,$$

provided k is sufficiently large. This completes the proof of the sharpness.

3.2. Lack of moment inequalities

The argumentation is similar to that in the previous subsection. Let B be a Brownian motion starting from zero, let $\tau_0=\inf\{t>0:|B_t|=1\}$, and by induction, let

$$\tau_n=\inf\big\{t>\tau_{n-1}:B_t=-n-1 \text{ or } B_t\geq 0\big\}, \quad n=1,2,\ldots.$$

Now, for a fixed positive integer N, let $F_0=G_0\equiv 0$, and let

$$dF_t = 1_{\{t\leq\tau_{2N-1}\}}\,dB_t + 1_{\{\tau_{2N-1}<t\leq\tau_{2N-1}-B_{\tau_{2N-1}}\}}\,dt,$$
$$dG_t=\Big(\sum_{n=1}^{2N-1}(-1)^n 1_{\{\tau_{n-1}<t\leq\tau_n\}}\Big)\,dB_t.$$

The processes F, G satisfy the conditions (i), (ii), and (iii) described in Theorem 1.3. In addition, if we set $\tau=\inf\{t>\tau_0:F_t\geq 0\}$, we have

$$\|F_\tau\|_p=\frac{1}{2},\qquad \|G_\tau\|_1\geq\sum_{k=1}^{2N-1}\frac{1}{2(k+1)}. \tag{3.2}$$

The equality is trivial: $F_\tau=1$ on the set $\{B_{\tau_0}=1\}$ (which has probability $1/2$), and $F_\tau=0$ on the complement of this event. To prove the inequality for $\|G_\tau\|_1$, observe that if $k=1,2,\ldots,2N-1$, then

$$|G_\tau|=\Big|\sum_{n=1}^{k}(-1)^k(B_{\tau_n}-B_{\tau_{n-1}})\Big|=2\Big\lfloor\frac{k+1}{2}\Big\rfloor$$

on the set $\{\tau=\tau_k>\tau_{k-1}\}$. Therefore, since

$$\{\tau=\tau_k>\tau_{k-1}\}=\{B_{\tau_0}=-1,B_{\tau_1}=-2,\ldots,B_{\tau_{k-1}}=-k,B_{\tau_k}=0\},$$

we obtain

$$\|G_\tau\|_1\geq\sum_{k=1}^{2N-1}2\Big\lfloor\frac{k+1}{2}\Big\rfloor\mathbb{P}(\tau=\tau_k>\tau_{k-1})$$
$$=\sum_{k=1}^{2N-1}2\Big\lfloor\frac{k+1}{2}\Big\rfloor\cdot\frac{1}{2k(k+1)},$$

which yields the desired estimate.

Thus, for any β one can choose N such that $\|G\|_1=\|G_\tau\|_1>\beta\|F_\tau\|_p$. However, as before, this does not give the claim since $\|F\|_p>\|F_\tau\|_p$. Therefore, the pair (F,G) must be modified; this is done exactly in the same manner as previously, using small portions of the probability space and appropriate copies of (F,G). The details are left to the reader.

4. Inequality for smooth functions

As an application of Theorems 1.1 and 1.2, we present a weak-type estimate for α-subordinate smooth functions on Euclidean domains. Suppose that Ω is an open subset of $\mathbb{R}^n$, n being a fixed positive integer, such that $0 \in \Omega$. Let $\overline{\Omega}$ be a bounded subdomain of Ω with $0 \in \overline{\Omega}$ and $\partial\overline{\Omega} \subset \Omega$. Denote by μ the harmonic measure on $\partial\overline{\Omega}$ with respect to zero. Consider two real-valued C^2-functions u, v on Ω. Following [2], we say that v is differentially subordinate to u if

$$|\nabla v(x)| \leq |\nabla u(x)| \quad \text{for } x \in \Omega.$$

Furthermore, for $\alpha \geq 0$, the function v is α-subordinate to u if it is differentially subordinate to u and, in addition,

$$|\Delta v(x)| \leq \alpha |\Delta u(x)| \quad \text{for } x \in \Omega$$

(see [5], [9]). The inequalities comparing the sizes of u and v under the assumption of (strong) differential subordination were studied by a number of authors (see, e.g., [1]–[3], [5], [8], [9], [13]). Our contribution in this direction is described in the following result.

THEOREM 4.1

Let $\alpha \geq 0$, and suppose that u is subharmonic, v is α-subordinate to u, and $|v(0)| \leq |u(0)|$. Then

$$\sup_{\lambda>0} \lambda\mu\big(|v(x)| \geq \lambda\big) \leq C_\alpha \int_{\partial\overline{\Omega}} |u(x)|\, d\mu(x) \tag{4.1}$$

and

$$\sup_{\lambda>0} \lambda\mu\big(|v(x)| \geq \lambda\big) \leq K_\alpha \int_{\partial\overline{\Omega}} u(x)^+\, d\mu(x) - (C_\alpha - K_\alpha)u(0). \tag{4.2}$$

Proof

Consider n-dimensional Brownian motion W starting from zero, and let τ denote the exit time of $\overline{\Omega}$: $\tau = \inf\{t : W_t \notin \overline{\Omega}\}$. Introduce the processes

$$X = (X_t)_{t\geq 0} = \big(u(W_{\tau\wedge t})\big)_{t\geq 0}, \qquad Y = (Y_t)_{t\geq 0} = \big(v(W_{\tau\wedge t})\big)_{t\geq 0},$$

and apply Itō's formula: for any $t \geq 0$,

$$X_t = u(0) + \int_0^t \nabla u(W_{\tau\wedge s})\, dW_s + \frac{1}{2}\int_0^t \triangle u(W_{\tau\wedge s})\, ds = X_0 + M_t + C_t,$$

$$Y_t = v(0) + \int_0^t \nabla v(W_{\tau\wedge s})\, dW_s + \frac{1}{2}\int_0^t \triangle v(W_{\tau\wedge s})\, ds = Y_0 + N_t + D_t.$$

Since

$$[M,M]_t - [N,N]_t = |u(0)|^2 - |v(0)|^2 + \int_0^t \big(|\nabla u(W_{\tau\wedge s})|^2 - |\nabla v(W_{\tau\wedge s})|^2\big)\, ds$$

and

$$\alpha C_t - |D|_t = \frac{1}{2}\int_0^t \big(\alpha\triangle u(W_{\tau\wedge s}) - |\triangle v(W_{\tau\wedge s})|\big)\, ds,$$

we see that α-subordination of the functions u and v implies that Y is α-subordinate to X. Since $\mu(|v(x)| \geq \lambda) \leq \mathbb{P}(Y^* \geq \lambda)$ and $\|X^+\|_1 = \int_{\partial\overline{\Omega}} u(x)^+ \, d\mu(x)$, we see that (1.5) implies (4.2) and this, in turn, yields (4.1). $\square$

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