The canonical module of a Cox ring

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To Professor Jun-ichi Nishimura on the occasion of his sixtieth birthday

Abstract In this paper, we describe the graded canonical module of a Noetherian multisection ring of a normal projective variety. In particular, in the case of the Cox ring, we prove that the graded canonical module is a graded free module of rank one with the shift of degree K_X. We give two kinds of proofs. The first one utilizes the equivariant twisted inverse functor developed by the first author. The second proof is down-to-earth, which avoids the twisted inverse functor, but some additional assumptions are required in this proof.

1. Introduction

Cox [1] studied the total coordinate ring (Cox ring) of toric varieties. Cox rings of normal projective varieties have become such important and interesting objects that many mathematicians try to prove (in)finite generation or to study their ring-theoretic properties (generators, relations, syzygies, homological properties, etc.).

In this paper, we describe the graded canonical module of a Noetherian multisection ring of a normal projective variety in Theorem 1.2. Using this result, we give a necessary and sufficient condition for the canonical module to be a free module (see Corollary 1.3). Since the Cox ring is a unique factorization domain as in [2], its graded canonical module is a free module. We prove that the graded canonical module of the Cox ring of a normal projective variety X is a graded free module of rank one with the shift of degree K_X (see Corollary 1.5). Many mathematicians study syzygies of the Cox ring. One of the purposes of our study is to contribute to their study.

We give two kinds of proofs for Theorem 1.2 in this paper. In Section 3, we prove Theorem 1.2 by using the equivariant twisted inverse functor developed in [6]. The second proof is given in Section 4. The second proof is down-to-earth, which avoids the twisted inverse functor, but some additional assumptions are required in this proof.

From now on, we give precise definitions, and state our main theorem and corollaries.

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DEFINITION 1.1

Let X be a d-dimensional normal projective variety over a field k. Let $D_1, \dots, D_s$ be Weil divisors on X. We define a ring $R(X; D_1, \dots, D_s)$ to be

$$\bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i\Big)\Big) t_1^{n_1}\cdots t_s^{n_s} \subset k(X)[t_1^{\pm 1}, \dots, t_s^{\pm 1}].$$

For a Weil divisor F on X, we set

$$M_F = \bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i + F\Big)\Big) t_1^{n_1}\cdots t_s^{n_s} \subset k(X)[t_1^{\pm 1}, \dots, t_s^{\pm 1}];$$

that is, M_F is a $\mathbb{Z}^s$-graded $R(X; D_1, \dots, D_s)$-module such that

$$[M_F]_{(n_1,\dots,n_s)} = H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i + F\Big)\Big) t_1^{n_1}\cdots t_s^{n_s}.$$

In this paper, for a normal variety X we denote by $\mathrm{Cl}(X)$ the class group of X, and for a Weil divisor F on X, we denote by $\overline{F}$ the class of F in $\mathrm{Cl}(X)$.

In the case where $\mathrm{Cl}(X)$ is freely generated by $\overline{D_1}, \dots, \overline{D_s}$, the ring $R(X; D_1, \dots, D_s)$ is called the *Cox ring* of X, and it is denoted by $\mathrm{Cox}(X)$. Recall that $\mathrm{Cox}(X)$ is uniquely determined by X up to isomorphisms, that is, independent of the choice of $D_1, \dots, D_s$.

THEOREM 1.2

Let X be a normal projective variety over a field k such that $H^0(X, \mathcal{O}_X) = k$. Assume that $D_1, \dots, D_s$ are Weil divisors on X which satisfy the following three conditions.

(1) *$\overline{D_1}, \dots, \overline{D_s}$ are linearly independent over $\mathbb{Z}$ in the divisor class group $\mathrm{Cl}(X)$.*

(2) *The lattice $\mathbb{Z}D_1 + \cdots + \mathbb{Z}D_s$ contains an ample Cartier divisor.*

(3) *The ring $R(X; D_1, \dots, D_s)$ is Noetherian.*

Then, $R(X; D_1, \dots, D_s)$ is a local $\mathbb{Z}^s$-graded k-domain (see Definition 2.1), and we have an isomorphism

$$\omega_{R(X;D_1,\dots,D_s)} \simeq M_{K_X}$$

of $\mathbb{Z}^s$-graded $R(X; D_1, \dots, D_s)$-modules.

Let X be a normal projective variety, and let $D_1, \dots, D_s$ be Weil divisors on X. Then, $R(X; D_1, \dots, D_s)$ is a Krull domain as in [2]. (In particular, if $R(X; D_1, \dots, D_s)$ is Noetherian, then it is a Noetherian integrally closed domain.) Therefore, the divisor class group of $R(X; D_1, \dots, D_s)$ can be defined. If Theorem 1.2(2) is satisfied, then we have an exact sequence

$$(1.1) \qquad 0 \longrightarrow \mathbb{Z}\overline{D_1} + \cdots + \mathbb{Z}\overline{D_s} \longrightarrow \mathrm{Cl}(X) \xrightarrow{p} \mathrm{Cl}\big(R(X; D_1, \dots, D_s)\big) \longrightarrow 0$$

defined by $p(\overline{F}) = \overline{M_F}$ for each Weil divisor F, where $\overline{M_F}$ stands for the isomorphism class containing M_F. We construct this map p explicitly in Remark 4.4.

Theorem 1.2 says that

$$p(\overline{K_X}) = \overline{\omega_{R(X;D_1,\dots,D_s)}}$$

if (1), (2), and (3) in Theorem 1.2 are satisfied. In particular, we have the following corollary.

COROLLARY 1.3

Suppose that the assumptions in Theorem 1.2 are satisfied.

Then, $\omega_{R(X;D_1,\dots,D_s)}$ is a free $R(X;D_1,\dots,D_s)$-module if and only if

$$\overline{K_X} \in \mathbb{Z}\overline{D_1} + \cdots + \mathbb{Z}\overline{D_s}$$

in $\mathrm{Cl}(X)$.

We give some examples (Example 5.1, Example 5.2) in Section 5.

REMARK 1.4

Suppose that the assumptions in Theorem 1.2 are satisfied. Put $d = \dim X$. Let $F_1, \dots, F_r$ be linearly independent elements in $\mathbb{Z}D_1 + \cdots + \mathbb{Z}D_s$. Put

$$L_1 = \mathbb{Z}D_1 + \cdots + \mathbb{Z}D_s \supset L_2 = \mathbb{Z}F_1 + \cdots + \mathbb{Z}F_r.$$

We think that $R(X;D_1,\dots,D_s)$ is graded by L_1. We have

$$R(X;F_1,\dots,F_r) = R(X;D_1,\dots,D_s)|_{L_2}.$$

By Theorem 1.2,

$$\omega_{R(X;F_1,\dots,F_r)} = \omega_{R(X;D_1,\dots,D_s)}|_{L_2} \tag{1.2}$$

if the lattice L_2 contains an ample Cartier divisor.

If we take the graded dual on both sides of (1.2), we know that

$$H^{d+r}_{\mathfrak{m}_2}\big(R(X;F_1,\dots,F_r)\big) = H^{d+s}_{\mathfrak{m}_1}\big(R(X;D_1,\dots,D_s)\big)|_{L_2}, \tag{1.3}$$

where $\mathfrak{m}_1$ (resp., $\mathfrak{m}_2$) is the unique maximal homogeneous ideal of $R(X;D_1,\dots,D_s)$ (resp., $R(X;F_1,\dots,F_r)$). Here note that the dimension of $R(X;D_1,\dots,D_s)$ (resp., $R(X;F_1,\dots,F_r)$) is equal to $d+s$ (resp., $d+r$). We obtain the equalities

$$H^{d+s}_{\mathfrak{m}_1}\big(R(X;D_1,\dots,D_s)\big) = H^{d+1}_{S_+}\big(R(X;D_1,\dots,D_s)\big),$$
$$H^{d+r}_{\mathfrak{m}_2}\big(R(X;F_1,\dots,F_r)\big) = H^{d+1}_{S_+}\big(R(X;F_1,\dots,F_r)\big)$$

in Remark 4.5. The equality (1.3) also follows from the above equalities.

In Example 5.3, we give an example where the equality (1.2) is not satisfied if we remove the assumption that L_2 contains an ample Cartier divisor.

Suppose that X is a normal projective variety with a finitely generated free divisor class group. We may think that the Cox ring is graded by $\mathrm{Cl}(X)$. By the exact sequence (1.1), $\mathrm{Cox}(X)$ is a unique factorization domain as in [2]. Therefore, the canonical module $\omega_{\mathrm{Cox}(X)}$ is a $\mathrm{Cl}(X)$-graded free $\mathrm{Cox}(X)$-module of rank one if $\mathrm{Cox}(X)$ is a Noetherian ring. By Theorem 1.2, we can determine the degree of the homogeneous generator of $\omega_{\mathrm{Cox}(X)}$ as follows.

COROLLARY 1.5
Let X be a normal projective variety over a field. Assume that $\mathrm{Cl}(X)$ is a finitely generated free abelian group and the Cox ring of X is Noetherian.

Then, the canonical module of the Cox ring is a rank one free $\mathrm{Cl}(X)$-graded module whose generator is of degree $-\overline{K_X} \in \mathrm{Cl}(X)$.

Suppose that $D_1, \dots, D_s$ are $\mathbb{Q}$-divisors. Assume (1), (2), and (3) in Theorem 1.2. Let H_1 be the set of all the closed subvarieties of X of codimension one. Set

$$D_i = \sum_{V \in H_1} \frac{p_{i,V}}{q_{i,V}} V,$$

where the $p_{i,V}$'s and $q_{i,V}$'s are integers such that $(p_{i,V}, q_{i,V}) = 1$ and $q_{i,V} > 0$ for each V. (If $p_{i,V} = 0$, then $q_{i,V} = 1$.) Then, as in [11, Theorem 2.8], we obtain the following corollary.

COROLLARY 1.6
With the notation as above,

$$\omega_{R(X;D_1,\dots,D_s)} \simeq \bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s} H^0\Bigl(X, \mathcal{O}_X\Bigl(\sum_i n_i D_i + K_X + \sum_{V\in H_1} \frac{q_V - 1}{q_V} V\Bigr)\Bigr),$$

where q_V is the least common multiple of $q_{1,V}, \dots, q_{s,V}$ for each $V \in H_1$.

2. The canonical module of a local $\mathbb{Z}^s$-graded k-domain

DEFINITION 2.1
Let k be a field. R is called a *local $\mathbb{Z}^s$-graded k-domain* if the following conditions are satisfied:

- $R = \bigoplus_{\underline{a}\in\mathbb{Z}} R_{\underline{a}}$ is a Noetherian $\mathbb{Z}^s$-graded domain;
- $R_{\underline{0}} = k$;
- suppose that $\mathfrak{m}$ is the ideal of R generated by all the homogeneous elements of R of degree different from $\underline{0}$; then $\mathfrak{m} \neq R$.

Assume that R is a local $\mathbb{Z}^s$-graded k-domain. We remark that the ideal $\mathfrak{m}$ as above is the unique maximal homogeneous ideal of R. Further, R is of finite type over k. Therefore, the height of $\mathfrak{m}$ coincides with the dimension of R.

For a $\mathbb{Z}^s$-graded module M over a local $\mathbb{Z}^s$-graded k-domain R, ${}^*\mathrm{Hom}_k(M,k)$ denotes the graded dual of M; that is, ${}^*\mathrm{Hom}_k(M,k)$ is a $\mathbb{Z}^s$-graded R-module such that ${}^*\mathrm{Hom}_k(M,k)_{\underline{a}} = \mathrm{Hom}_k(M_{-\underline{a}}, k)$.

DEFINITION 2.2
Let R be a local $\mathbb{Z}^s$-graded k-domain with the maximal homogeneous ideal $\mathfrak{m}$.

Then,

$$ {}^*\mathrm{Hom}_k\bigl(H^{\dim R}_{\mathfrak{m}}(R), k\bigr) $$

is called the *canonical module* of R and denoted by ω_R. Here $H^{\dim R}_{\mathfrak{m}}(R)$ is the $(\dim R)$th local cohomology group, and it has a natural structure of a $\mathbb{Z}^s$-graded R-module.

We emphasize that ω_R has a structure of a $\mathbb{Z}^s$-graded R-module. We refer the reader to [5] and [9] for the general theory of $\mathbb{Z}^s$-graded rings.

LEMMA 2.3

Let X be a normal projective variety over a field k such that $H^0(X, \mathcal{O}_X) = k$. Assume that $D_1, \ldots, D_s$ are Weil divisors on X which satisfy assumptions (1) and (3) in Theorem 1.2.

Then, $R(X; D_1, \ldots, D_s)$ is a local $\mathbb{Z}^s$-graded k-domain.

Proof

We denote the ring $R(X; D_1, \ldots, D_s)$ simply by R.

To prove that R is a local $\mathbb{Z}^s$-graded k-domain, we need to show that the ideal $\mathfrak{m}$ is not equal to R, where $\mathfrak{m}$ is the ideal of R generated by all the homogeneous elements of R of degree different from $\underline{0}$. Assume the contrary. Then, there exists $(n_1, \ldots, n_s) \neq (0, \ldots, 0)$ such that

$$ H^0\Bigl(X, \mathcal{O}_X\Bigl(\sum_i n_i D_i\Bigr)\Bigr) \neq 0 \qquad \text{and} \qquad H^0\Bigl(X, \mathcal{O}_X\Bigl(-\sum_i n_i D_i\Bigr)\Bigr) \neq 0. $$

Then, there exists an effective Weil divisor F_1 that is linearly equivalent to $\sum_i n_i D_i$. Since $\overline{D_1}, \ldots, \overline{D_s}$ are linearly independent over $\mathbb{Z}$ in the divisor class group $\mathrm{Cl}(X)$ by our assumption, $F_1 \neq 0$. In the same way, there exists a nonzero effective Weil divisor F_2 that is linearly equivalent to $-\sum_i n_i D_i$. Then, the nonzero effective Weil divisor $F_1 + F_2$ is linearly equivalent to 0. This is a contradiction. $\square$

REMARK 2.4

Suppose that assumptions (2) and (3) in Theorem 1.2 are satisfied. Set $R = R(X; D_1, \ldots, D_s)$.

Suppose that $D = a_1 D_1 + \cdots + a_s D_s$ is an ample Cartier divisor, where $a_1, \ldots, a_s \in \mathbb{Z}$. Set

$$ S = \bigoplus_{n \geq 0} H^0\bigl(X, \mathcal{O}_X(nD)\bigr) t^n \subset k(X)[t]. $$

We have a ring homomorphism $S \to R$ defined by $f t^n \mapsto f(t_1^{a_1} \cdots t_s^{a_s})^n$ for $f \in H^0(X, \mathcal{O}_X(nD))$. We think of S as a subring of R by this ring homomorphism. Here $X = \mathrm{Proj}(S)$. Set $X' = X \setminus \mathrm{Sing}(X)$, and set $D'_i = D_i|_{X'}$ for $i = 1, \ldots, s$.

Replacing D by nD for a sufficiently large n (if necessary), we may assume that there exist $f_1, \dots, f_\ell \in H^0(X, \mathcal{O}_X(D))$ which satisfy the following two conditions:

(1) $X' = \bigcup_{j=1}^{\ell} D_+(f_j t)$;

(2) D'_i is a principal divisor on $D_+(f_j t)$ for any i and j.

Let $(f_j t_1^{a_1} \cdots t_s^{a_s} \mid j)R$ be the ideal of R generated by $\{f_j t_1^{a_1} \cdots t_s^{a_s} \mid j = 1, \dots, \ell\}$. Then, we can show that

(2.1) the height of the ideal $(f_j t_1^{a_1} \cdots t_s^{a_s} \mid j)R$ is bigger than or equal to 2

as follows. Let H_1 be the set of closed subvarieties of X of codimension one. For $V \in H_1$, we set

$$P_V = \bigoplus_{(n_1,\dots,n_s) \in \mathbb{Z}^s} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i - V\Big)\Big) t_1^{n_1} \cdots t_s^{n_s} \subset R, \tag{2.2}$$

that is, $P_V = M_{-V}$. Then, as in [2, p. 632],

$$\{P_V \mid V \in H_1\}$$

coincides with the set of all the height one homogeneous prime ideals of R. Here recall that $P_V \cap S$ is equal to the defining ideal of V in the ring S. Since $V_+((f_j t \mid j)S)$ coincides with $\mathrm{Sing}(X)$, the ideal $(f_j t_1^{a_1} \cdots t_s^{a_s} \mid j)R$ is not contained in any height one homogeneous prime ideal of R. Recall here that $\mathrm{Sing}(X)$ is a closed subset of X of codimension bigger than or equal to 2.

Thus, we know that the height of the ideal $(f_j t_1^{a_1} \cdots t_s^{a_s} \mid j)R$ is bigger than or equal to two.

3. A proof of the main theorem using the twisted inverse functor

We prove Theorem 1.2 in this section using the twisted inverse functor. We prove

$$\omega_{R(X;D_1,\dots,D_s)} \simeq M_{K_X}$$

without assuming Theorem 1.2(1). If we remove assumption (1) in Theorem 1.2, the ring $R(X; D_1, \dots, D_s)$ may not be a local $\mathbb{Z}^s$-graded k-domain. In this case, we cannot define $\omega_{R(X;D_1,\dots,D_s)}$ by using local cohomologies as in Definition 2.2. In this section, we give an alternative definition of $\omega_{R(X;D_1,\dots,D_s)}$ using the twisted inverse functor as in Definition 3.1. Of course, both definitions coincide in the case where $R(X; D_1, \dots, D_s)$ is a local $\mathbb{Z}^s$-graded k-domain as in Remark 3.4.

DEFINITION 3.1

Let k be a field, let G be a finite-type group scheme over k, and let $f : X \to \operatorname{Spec} k$ be a G-scheme which is separated of finite type. Then the dualizing complex $\mathbb{I}_X$ of X is defined to be $f^! \mathcal{O}_{\operatorname{Spec} k}$, where $f^!$ denotes the (equivariant) twisted inverse functor $f^! : D_{\mathrm{Lqc}}(G, \operatorname{Spec} k) \to D_{\mathrm{Lqc}}(G, X)$ (see [6, (20.5), Chapter 29]). Assume that X is nonempty and connected. Then we define $s := \inf\{i \mid H^i(\mathbb{I}_X) \neq 0\}$ and $\omega_X := H^s(\mathbb{I}_X)$. We call ω_X the (G-equivariant) canonical sheaf of X. If, moreover, $X = \operatorname{Spec} A$ is affine, then $\Gamma(X, \omega_X)$ is denoted by ω_A and is called the (G-equivariant) canonical module of A.

REMARK 3.2

Note that $\mathbb{I}_X \in D^b_{\mathrm{Coh}}(G, X)$ and $\omega_X \in \mathrm{Coh}(G, X)$. Note also that if we forget the G-action, then $\mathbb{I}_X$ as an object of $D^b(X)$ is the dualizing complex of X and ω_X is the canonical sheaf of X in the usual sense. If X is Cohen-Macaulay, then $\omega_X[-s] \cong \mathbb{I}_X$ in $D(G, X)$, where $s = \inf\{i \mid H^i(\mathbb{I}_X) \neq 0\}$. (We use this fact freely later for nonsingular varieties.) If X is equidimensional and U is a G-stable open subset, then $\omega_X|_U \cong \omega_U$. In general, for a quasi-compact separated G-morphism $g : U \to X$, the canonical map $u : \omega_X \to g_* g^* \omega_X$ is $(G, \mathcal{O}_X)$-linear. If X is normal, g is the inclusion from a G-stable open subset U, and the codimension of $X \setminus U$ in X is at least 2, then u is an isomorphism of coherent $(G, \mathcal{O}_X)$-modules. So we have $\omega_X \cong g_* \omega_U$ this case.

REMARK 3.3

Let k be a field, let H be a finite-type k-group scheme, and let $G := \mathbb{G}_m \times H$. A G-algebra R is nothing but a $\mathbb{Z}$-graded k-algebra $R = \bigoplus_{i \in \mathbb{Z}} R_i$, which is also an H-algebra such that each R_i is an H-submodule of R, where the $\mathbb{Z}$-grading is given by the $\mathbb{G}_m$-action. Let R be a positively graded G-algebra. That is, $R = \bigoplus_{i \in \mathbb{Z}} R_i$ is a G-algebra such that $R_i = 0$ for $i < 0$ and $R_0 = k$. Assume that R is a finitely generated domain.

Let $d = \dim R$, and let $\Theta = \operatorname{Spec} R$. Let θ be the unique G-stable closed point of Θ corresponding to the unique graded maximal ideal $\mathfrak{m}$ of R. Then $\omega_\Theta = H^{-d}(\mathbb{I}_\Theta)$. By the equivariant local duality [8, (4.18)], for any $\mathbb{F} \in D_{\mathrm{Coh}}(G, \Theta)$,

$$\underline{H}^i_\theta(\mathbb{F}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_\Theta}\big(\underline{\mathrm{Ext}}^{-i}_{\mathcal{O}_\Theta}(\mathbb{F}, \mathbb{I}_\Theta), \mathcal{E}\big),$$

where $\underline{\mathrm{Hom}}_{\mathcal{O}_\Theta}$ denotes the sheaf-Hom in the category of $(G, \mathcal{O}_\Theta)$-modules (see [6, (2.24), Chapter 29]) and $\underline{\mathrm{Ext}}^{-i}_{\mathcal{O}_\Theta}$ its right-derived functor. $\underline{H}^i_\theta$ denotes the equivariant local cohomology (see [7, (7.7)]), and $\mathcal{E} = \underline{H}^0_\theta(\mathbb{I}_\Theta)$ is the G-sheaf of Matlis of the G-local G-scheme (Θ, θ) (see [8]). Letting $\mathbb{F} = \mathcal{O}_\Theta$ and $i = d$,

$$\underline{H}^d_\theta(\mathcal{O}_\Theta) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_\Theta}(\omega_\Theta, \mathcal{E}).$$

In other words,

$$H^d_{\mathfrak{m}}(R) \cong \mathrm{Hom}_R(\omega_R, E),$$

where $E = H^0(\Theta, \mathcal{E})$. By [8, Lemma 5.4, Remark 5.6], it is easy to see that $(?)^\vee = \mathrm{Hom}_R(?, E)$ is equivalent to ${}^*\mathrm{Hom}_k(?, k)$ as a functor from the full subcategory of the category of (G, R)-modules consisting of modules whose degree i component is a finite-dimensional k-vector space for each $i \in \mathbb{Z}$ (the $\mathbb{Z}$-grading is given by the $\mathbb{G}_m$-action) to itself, and $(?)^\vee(?)^\vee$ is equivalent to the identity functor. Thus, $(H^d_{\mathfrak{m}}(R))^\vee \cong \omega_R$.

Let $H = \mathbb{G}_m^s$. Let $\varphi : X \to Y$ be an affine H-morphism between H-schemes. Assume that the action of H on Y is trivial. Then $\varphi_* \mathcal{O}_X$ is a quasi-coherent

$(H, \mathcal{O}_Y)$-algebra in the sense that $\varphi_*\mathcal{O}_X$ is both a quasi-coherent $(H, \mathcal{O}_Y)$-module and an $\mathcal{O}_Y$-algebra, and the unit map $\mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ and the product $\varphi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \varphi_*\mathcal{O}_X \to \varphi_*\mathcal{O}_X$ are $(H, \mathcal{O}_Y)$-linear. This is equivalent to saying that $\varphi_*\mathcal{O}_X$ is a $\mathbb{Z}^s$-graded $\mathcal{O}_Y$-algebra. Conversely, for a given trivial H-scheme Y and a quasi-coherent $(H, \mathcal{O}_Y)$-algebra (i.e., a $\mathbb{Z}^s$-graded $\mathcal{O}_Y$-algebra) $\mathcal{A}$, letting $X := \operatorname{Spec}_Y \mathcal{A}$ and letting $\varphi : X \to Y$ be the canonical map, $\varphi : X \to Y$ is an affine H-morphism such that $\varphi_*\mathcal{O}_X \cong \mathcal{A}$. In this case, a quasi-coherent $(H, \mathcal{O}_X)$-module $\mathcal{M}$ yields a graded $\mathcal{A}$-module $\varphi_*\mathcal{M}$, and conversely, a graded $\mathcal{A}$-module $\mathcal{N}$ determines a quasi-coherent $(H, \mathcal{O}_X)$-module $\mathcal{M}$ such that $\varphi_*\mathcal{M} \cong \mathcal{N}$ uniquely, up to isomorphisms.

REMARK 3.4
Let $H = \mathbb{G}_m^s$, and let $G = \mathbb{G}_m \times H = \mathbb{G}_m^{s+1}$.

Let R be a $\mathbb{Z}^s$-graded k-algebra. Then R is an H-algebra. Assume, moreover, that R is a local $\mathbb{Z}^s$-graded k-domain (see Definition 2.1).

Then the convex polyhedral cone in $\mathbb{R}^s$ generated by $\{\underline{a} \in \mathbb{Z}^s \mid R_{\underline{a}} \neq 0\}$ does not contain a line. So there is a linear function $\varphi : \mathbb{Z}^s \to \mathbb{Z}$ such that $R = \bigoplus_{i \in \mathbb{Z}} R_i$ is a positively graded G-algebra; that is, $R_0 = k$ and $R_i = 0$ for $i < 0$, and each R_i is an H-submodule of R, where $R_i = \bigoplus_{\varphi(\underline{a}) = i} R_{\underline{a}}$. Note that R is assumed to be a finitely generated domain.

As in Remark 3.3, $(H^d_{\mathfrak{m}}(R))^\vee \cong \omega_R$ as (G, R)-modules. In particular, they are isomorphic as (H, R)-modules or $\mathbb{Z}^s$-graded R-modules.

This shows that our new definition of ω_R as a graded R-module is consistent with Definition 2.2.

We assume (2) and (3) in Theorem 1.2. Set $R = R(X; D_1, \ldots, D_s)$. Take an ample Cartier divisor D, a subring S, and $f_1, \ldots, f_\ell \in H^0(X, \mathcal{O}_X(D))$ as in Remark 2.4.

We set

$$Y = \operatorname{Spec}_{X'}\Bigl(\bigoplus_{(n_1,\ldots,n_s) \in \mathbb{N}_0{}^s} \mathcal{O}_{X'}\Bigl(\sum_i n_i D'_i\Bigr) t_1^{n_1} \cdots t_s^{n_s}\Bigr),$$

where $\mathbb{N}_0$ denotes the set of all nonnegative integers. Let $\pi : Y \to X'$ be the structure morphism. Consider the open subscheme

$$Z = \operatorname{Spec}_{X'}\Bigl(\bigoplus_{(n_1,\ldots,n_s) \in \mathbb{Z}^s} \mathcal{O}_{X'}\Bigl(\sum_i n_i D'_i\Bigr) t_1^{n_1} \cdots t_s^{n_s}\Bigr) \tag{3.1}$$

of Y. Let $i : Z \to Y$ be the open immersion.

For each affine open subset U of X', we have a ring homomorphism

$$R \longrightarrow \bigoplus_{(n_1,\ldots,n_s) \in \mathbb{Z}^s} H^0\Bigl(U, \mathcal{O}_{X'}\Bigl(\sum_i n_i D'_i\Bigr)\Bigr) t_1^{n_1} \cdots t_s^{n_s}$$

induced by the restriction to the open set U of X'. Therefore, we have a natural morphism $j : Z \to \operatorname{Spec}(R)$:

$$
\begin{array}{ccc}
 & Z & \xrightarrow{i} & Y \\
(3.2) & \downarrow j & & \downarrow \pi \\
 & \operatorname{Spec}(R) \setminus V\big((f_j t_1^{a_1} \cdots t_s^{a_s} \bigm| j)R\big) \subset \operatorname{Spec}(R) & & X'
\end{array}
$$

LEMMA 3.5

The morphism $j : Z \to \operatorname{Spec}(R)$ *coincides with the open immersion* $\operatorname{Spec}(R) \setminus V((f_j t_1^{a_1} \cdots t_s^{a_s} \mid j)R) \subset \operatorname{Spec}(R)$.

Proof

There exist α_{ij}'s in $k(X)^\times$ such that

$$R[(f_j t_1^{a_1} \cdots t_s^{a_s})^{-1}] = \Gamma\big(D_+(f_j t), \mathcal{O}_X\big)[(\alpha_{1j} t_1)^{\pm 1}, \ldots, (\alpha_{sj} t_s)^{\pm 1}]$$

for each j. The ring corresponding to the affine open set $(\pi i)^{-1}(D_+(f_j t))$ just coincides with the above ring. Thus, we obtain

$$Z = \operatorname{Spec}(R) \setminus V\big((f_j t_1^{a_1} \cdots t_s^{a_s} \bigm| j)R\big). \qquad \square$$

The group $(\mathbb{G}_m)^s$ acts on all the schemes in diagram (3.2). The action on X' is trivial. All the morphisms in diagram (3.2) are compatible with the group action.

As in Definition 3.1, we can define the canonical sheaves with group action for the schemes in diagram (3.2). That is, the canonical sheaves for all the schemes in diagram (3.2) are $\mathbb{Z}^s$-graded.

Then, we have $\omega_R|_Z = \omega_Z$ by compatibility with open immersions (see Remark 3.2). Since the height of $(f_j t_1^{a_1} \cdots t_s^{a_s} \mid j)R$ is bigger than 1 as in Remark 2.4, we have an isomorphism

$$H^0(Z, \omega_Z) = \omega_R$$

which is compatible with the group action (see Remark 3.2).

On the other hand, $\pi : Y \to X'$ is a smooth morphism of relative dimension s that is compatible with the group action. Then, the sheaf of differentials $\Omega_{Y/X'}$ naturally has a group action; that is, it has a structure of a $\mathbb{Z}^s$-graded sheaf. We have

$$\bigwedge^s \Omega_{Y/X'} \simeq \pi^* \mathcal{O}_{X'}\Big(\sum_i D'_i\Big)(-1, \ldots, -1),$$

where $(-1, \ldots, -1)$ denotes the shift of degree.

Then, by [6, Theorem 28.11], we have

$$
\begin{aligned}
\omega_Y &= \bigwedge^s \Omega_{Y/X'} \otimes_{\mathcal{O}_Y} \pi^* \omega_{X'} \\
&\simeq \pi^* \mathcal{O}_{X'}\Big(\sum_i D'_i\Big)(-1, \ldots, -1) \otimes_{\mathcal{O}_Y} \pi^* \mathcal{O}_{X'}(K_{X'}) \\
&= \pi^* \mathcal{O}_{X'}\Big(\sum_i D'_i + K_{X'}\Big)(-1, \ldots, -1).
\end{aligned}
$$

So we have

$$\omega_Z = i^*\omega_Y = (\pi i)^*\mathcal{O}_{X'}\Big(\sum_i D_i' + K_{X'}\Big)(-1,\dots,-1).$$

Then,

$$\begin{aligned}\omega_R &= H^0(Z,\omega_Z)\\ &= H^0\big(X',(\pi i)_*\omega_Z\big)\\ &= H^0\Big(X',(\pi i)_*(\pi i)^*\mathcal{O}_{X'}\Big(\sum_i D_i' + K_{X'}\Big)(-1,\dots,-1)\Big).\end{aligned}$$

By the equivariant projection formula ([6, Lemma 26.4]), we have

$$\begin{aligned}&(\pi i)_*(\pi i)^*\mathcal{O}_{X'}\Big(\sum_i D_i' + K_{X'}\Big)(-1,\dots,-1)\\ &\simeq \Big(\mathcal{O}_{X'}\Big(\sum_i D_i' + K_{X'}\Big)\otimes_{\mathcal{O}_{X'}}(\pi i)_*\mathcal{O}_Z\Big)(-1,\dots,-1)\\ &= \Big(\mathcal{O}_{X'}\Big(\sum_i D_i' + K_{X'}\Big)\otimes_{\mathcal{O}_{X'}}\Big[\bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s}\mathcal{O}_{X'}\Big(\sum_i n_iD_i'\Big)\Big]\Big)(-1,\dots,-1)\\ &= \Big(\bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s}\mathcal{O}_{X'}\Big(\sum_i (n_i+1)D_i' + K_{X'}\Big)\Big)(-1,\dots,-1)\\ &= \bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s}\mathcal{O}_{X'}\Big(\sum_i n_iD_i' + K_{X'}\Big).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\omega_R &= \bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s} H^0\Big(X',\mathcal{O}_{X'}\Big(\sum_i n_iD_i' + K_{X'}\Big)\Big)\\ &= \bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s} H^0\Big(X,\mathcal{O}_{X}\Big(\sum_i n_iD_i + K_{X}\Big)\Big).\end{aligned}$$

We have completed the proof of Theorem 1.2. □

4. Another proof of Theorem 1.2

In this section, we give another proof for Theorem 1.2 without using the twisted inverse functor, and we have to assume that the scheme X' in Remark 2.4 is smooth over k. Note that it is automatically satisfied if the base field k is perfect.

The idea of this proof is based on [10, Lemma 13]. In the proof in this section, we have to assume that Theorem 1.2 is true if $s=1$.

REMARK 4.1

In the case where $s=1$, Theorem 1.2 is proved using Serre duality. Assume that $s=1$ and that a_1D_1 is a very ample Cartier divisor for some $a_1>0$. Put

$R=R(X;D_1)$. Then, we have

$$\begin{aligned}&\bigoplus_{n_1\in\mathbb{Z}} H^0\bigl(X,\mathcal{O}_X(n_1D_1+K_X)\bigr)=\bigoplus_{n_1\in\mathbb{Z}}\operatorname{Hom}_{\mathcal{O}_X}\bigl(\mathcal{O}_X(-n_1D_1),\omega_X\bigr)\\&=\bigoplus_{n_1\in\mathbb{Z}}\operatorname{Hom}_k\bigl(H^d(X,\mathcal{O}_X(-n_1D_1)),k\bigr)=\bigoplus_{n_1\in\mathbb{Z}}\operatorname{Hom}_k\bigl(H^{d+1}_{R_+}(R)_{-n_1},k\bigr)\\&={}^*\operatorname{Hom}_k\bigl(H^{d+1}_{R_+}(R),k\bigr)=\omega_R.\end{aligned}$$

First, we prove a basic fact on class groups as follows. The authors guess that it is well known; however, we do not know an adequate reference. Therefore, we give a proof here.

LEMMA 4.2

Let U be a normal scheme. Let $F_1,\dots,F_t$ be Cartier divisors on U. Put

$$W=\operatorname{Spec}_U\Bigl(\bigoplus_{m_1,\dots,m_t\in\mathbb{Z}}\mathcal{O}_U\Bigl(\sum_{j=1}^t m_jF_j\Bigr)\Bigr).$$

Let $\pi:W\to U$ be the structure morphism.

Then, the sequence

$$0\longrightarrow\mathbb{Z}\overline{F_1}+\cdots+\mathbb{Z}\overline{F_t}\longrightarrow\operatorname{Cl}(U)\xrightarrow{\pi^*}\operatorname{Cl}(W)\longrightarrow 0$$

is exact.

Proof

Recall that, since $\pi:W\to U$ is a flat morphism, the pullback map

$$\pi^*:\operatorname{Cl}(U)\longrightarrow\operatorname{Cl}(W)$$

is defined as in [3].

Put

$$\begin{aligned}E&=\operatorname{Spec}_U\Bigl(\bigoplus_{m_1,\dots,m_t\ge 0}\mathcal{O}_U\Bigl(\sum_{j=1}^t m_iF_i\Bigr)\Bigr)\\&=\operatorname{Spec}_U\bigl(\operatorname{Sym}_{\mathcal{O}_U}(\mathcal{O}_U(F_1)\oplus\cdots\oplus\mathcal{O}_U(F_t))\bigr),\\E_j&=\operatorname{Spec}_U\Bigl(\bigoplus_{m_j\ge 0}\mathcal{O}_U(m_jF_j)\Bigr)=\operatorname{Spec}_U\Bigl(\operatorname{Sym}_{\mathcal{O}_U}(\mathcal{O}_U(F_j))\Bigr)\end{aligned}$$

for $j=1,\dots,t$. Let

$$p:E\longrightarrow U,$$

$$p_j:E_j\longrightarrow U,$$

$$q_j:E\longrightarrow E_j$$

be the natural morphisms. Remark that $p = p_j q_j$ for any j. By [3, Theorem 3.3(a)], the flat pullback map

$$p^* : \mathrm{Cl}(U) \longrightarrow \mathrm{Cl}(E)$$

is an isomorphism. Let

$$s_j : U \longrightarrow E_j$$

be the zero section of the vector bundle $p_j : E_j \to U$. Then,

$$W = E \setminus \bigcup_j q_j^{-1}\bigl(s_j(U)\bigr).$$

By [3, Proposition 1.8], we have the following exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow \sum_j \mathbb{Z}\overline{q_j^{-1}\bigl(s_j(U)\bigr)} & \longrightarrow & \mathrm{Cl}(E) & \longrightarrow & \mathrm{Cl}(W) & \longrightarrow & 0 \\ & & \uparrow p^* & & & & \\ & & \mathrm{Cl}(U) & & & & \end{array}$$

Then $s_j(U)$ is linearly equivalent to $p_j^*(-F_j)$ by [3, Example 3.3.2]. Therefore, $q_j^{-1}(s_j(U))$ is linearly equivalent to $-p^*(F_j)$. Thus, we have the desired exact sequence. □

LEMMA 4.3

Suppose that Theorem 1.2(2) is satisfied.

Then, we have the following.

(1) *The set*

$$\{M_F \mid F \text{ is a Weil divisor on } X\}$$

just coincides with the set of divisorial fractional ideals which are $\mathbb{Z}^s$-graded $R(X; D_1, \ldots, D_s)$-submodules of $k(X)[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$.

(2) *For Weil divisors F_1 and F_2, F_1 is linearly equivalent to F_2 if and only if M_{F_1} is isomorphic to M_{F_2} as a $\mathbb{Z}^s$-graded module.*

(3) *Further, assume that $R(X; D_1, \ldots, D_s)$ is Noetherian. Any nonzero finitely generated $\mathbb{Z}^s$-graded reflexive $R(X; D_1, \ldots, D_s)$-module of rank one is isomorphic to M_G as a $\mathbb{Z}^s$-graded module for some Weil divisor G on X.*

Proof

Set $R = R(X; D_1, \ldots, D_s)$.

First, we prove (1). Let F be a Weil divisor on X. We prove that M_F is a divisorial fractional ideal.

Since there exists an ample Cartier divisor in $\mathbb{Z}D_1 + \cdots + \mathbb{Z}D_s$, we can find $at_1^{m_1} \cdots t_s^{m_s}$ such that $\mathrm{div}_X(a) + \sum_i m_i D_i - F$ is an effective divisor, where $\mathrm{div}_X(a)$ is the principal Weil divisor corresponding to $a \in k(X)^\times$. By definition, it is easy to check that, for $a \in k(X)^\times$,

$$(at_1^{m_1} \cdots t_s^{m_s}) M_F = M_{F - \mathrm{div}_X(a) - \sum_i m_i D_i}. \tag{4.1}$$

We have only to show that $M_{F - \mathrm{div}_X(a) - \sum_i m_i D_i}$ is a divisorial fractional ideal.

We define H_1 and P_V as in Remark 2.4. Set

$$F - \operatorname{div}_X(a) - \sum_i m_i D_i = -\sum_{j=1}^{u} \ell_j V_j,$$

where $V_1, \dots, V_u$ are distinct elements in H_1. By the argument as above, all of the ℓ_j's are positive integers.

For $V \in H_1$, we define R_V as in [2, p. 629]. (Here we use the symbol V instead of F in [2].) In this case, we obtain

$$R_{P_V} = (R_V)_{\alpha_V R_V}$$

as in [2, p. 632]. Therefore, for any $\ell > 0$, we have

$${P_V}^{(\ell)} = {P_V}^{\ell} R_{P_V} \cap R = {\alpha_V}^{\ell} (R_V)_{\alpha_V R_V} \cap R = M_{-\ell V}.$$

Then,

$$M_{-\sum_j \ell_j V_j} = \bigcap_j M_{-\ell_j V_j} = \bigcap_j {P_{V_j}}^{(\ell_j)}.$$

Since the P_{V_j}'s are homogeneous prime ideals of R of height one, $M_{-\sum_j \ell_j V_j}$ is a divisorial fractional ideal.

Conversely, let N be a divisorial fractional ideal that is a $\mathbb{Z}^s$-graded R-submodule of $k(X)[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$. Using (4.1), it is sufficient to prove that N coincides with M_F for some Weil divisor F in the case where $N \subset R$. Then, N coincides with an intersection of symbolic powers of homogeneous prime ideals of height one. Therefore, there exist $V_1, \dots, V_u \in H_1$ and positive integers $\ell_1, \dots, \ell_u$ such that

$$N = \bigcap_j {P_{V_j}}^{(\ell_j)} = \bigcap_j M_{-\ell_j V_j} = M_{-\sum_j \ell_j V_j}.$$

Assertions (2) and (3) are easily verified, so we omit the proofs. $\square$

REMARK 4.4

Assume (2) in Theorem 1.2. Put $R = R(X; D_1, \dots, D_s)$. The map p in (1.1) is constructed as follows.

Let $\operatorname{HDiv}(R)$ be the free abelian group generated by all the homogeneous prime divisors of $\operatorname{Spec}(R)$, that is,

$$\operatorname{HDiv}(R) = \bigoplus_{V \in H_1(X)} \mathbb{Z}[\operatorname{Spec}(R/P_V)],$$

where $[\operatorname{Spec}(R/P_V)]$ denotes the generator corresponding to the closed subscheme $\operatorname{Spec}(R/P_V)$. Here we recall that

$$\{P_V \mid V \in H_1(X)\}$$

coincides with the set of all the homogeneous prime ideals of R of height one as in [2, p. 632].

We define

$$\xi' : \operatorname{Div}(X) \to \operatorname{HDiv}(R)$$

by

$$\xi'\Big(\sum_{V\in H_1(X)} n_V V\Big) = \sum_{V\in H_1(X)} n_V[\mathrm{Spec}(R/P_V)].$$

(This map ξ' is equal to ξ in [2, p. 631] if we identify $[\mathrm{Spec}(R/P_V)]$ with P_V.) As in [2], ξ' induces the map

$$\varphi' : \mathrm{Cl}(X) \to \mathrm{A}^1\big(\mathrm{Spec}(R)\big)$$

satisfying

$$\varphi'\Big(\overline{\sum_{V\in H_1(X)} n_V V}\Big) = \overline{\sum_{V\in H_1(X)} n_V[\mathrm{Spec}(R/P_V)]},$$

where $\mathrm{A}^1(\mathrm{Spec}(R))$ denotes the Chow group of $\mathrm{Spec}(R)$ of codimension one. Then, we have the following exact sequence:

$$0 \longrightarrow \mathbb{Z}\overline{D_1} + \cdots + \mathbb{Z}\overline{D_s} \longrightarrow \mathrm{Cl}(X) \xrightarrow{\varphi'} \mathrm{A}^1\big(\mathrm{Spec}(R)\big) \longrightarrow 0.$$

Let $\mathrm{Cl}(R)$ be the set of isomorphism classes of divisorial fractional ideals of R. Then we have the isomorphism

$$i : \mathrm{A}^1\big(\mathrm{Spec}(R)\big) \longrightarrow \mathrm{Cl}(R)$$

defined by

$$i\Big(\overline{\sum_{V\in H_1(X)} n_V[\mathrm{Spec}(R/P_V)]}\Big) = -\sum_{V\in H_1(X)} n_V\overline{P_V} = \overline{M_{\sum_{V\in H_1(X)} n_V V}},$$

where $\overline{P_V}$ (resp., $\overline{M_{\sum_{V\in H_1(X)} n_V V}}$) denotes the isomorphism class containing P_V (resp., $M_{\sum_{V\in H_1(X)} n_V V}$).

We set $p = i\varphi'$. Then,

$$p : \mathrm{Cl}(X) \to \mathrm{Cl}(R)$$

is the map which satisfies $p(\overline{F}) = \overline{M_F}$ for each Weil divisor F on X, and we have the exact sequence (1.1).

We now start to prove Theorem 1.2 in the case where $s \geq 2$. Suppose that all the assumptions in Theorem 1.2 are satisfied. We assume that X' in Remark 2.4 is smooth over k.

Since ω_R is a finitely generated $\mathbb{Z}^s$-graded reflexive R-module of rank one, there exists a Weil divisor G such that ω_R is isomorphic to M_G as a $\mathbb{Z}^s$-graded module as in Lemma 4.3.

We want to show that G is linearly equivalent to K_X. Assume the contrary, that is, $\overline{G-K_X} \neq 0$ in $\mathrm{Cl}(X)$.

Then, we can choose $F_1, \ldots, F_s \in \mathbb{Z}D_1 + \cdots + \mathbb{Z}D_s$ satisfying the following three conditions:

(1) $\mathbb{Z}F_1 + \cdots + \mathbb{Z}F_s = \mathbb{Z}D_1 + \cdots + \mathbb{Z}D_s$;
(2) if $b_1F_1 + \cdots + b_sF_s$ is linearly equivalent to a nonzero effective divisor, then $b_s > 0$;

(3) $\overline{G-K_X} \notin \mathbb{Z}\overline{F_1}+\cdots+\mathbb{Z}\overline{F_{s-1}}$.

Here, recall that s is bigger than 1, and $F_1,\dots,F_s$ are linearly independent over $\mathbb{Z}$ by assumption (1) in Theorem 1.2.

We define a map

$$\varphi : \mathbb{Z}D_1+\cdots+\mathbb{Z}D_s \longrightarrow \mathbb{Z}$$

by

$$\varphi(b_1F_1+\cdots+b_sF_s)=b_s.$$

The kernel of φ is equal to $\mathbb{Z}F_1+\cdots+\mathbb{Z}F_{s-1}$.

We think that R is a $\mathbb{Z}$-graded ring by

$$\deg(at_1^{n_1}\cdots t_s^{n_s})=\varphi(n_1D_1+\cdots+n_sD_s).$$

Set $T=\mathrm{Proj}(R)$. Take an ample Cartier divisor $D=\sum_i a_iD_i$, a subring S, and $f_1,\dots,f_\ell$ as in Remark 2.4. Set $Q=T\setminus V_+((f_jt_1^{a_1}\cdots t_s^{a_s}\mid j)R)$. In this section, for an ℓ-dimensional normal algebraic variety W over k which is smooth over k in codimension one, we define

$$\omega_W=\Big(\bigwedge^{\ell}\Omega_{W/k}\Big)^{**}.$$

However, for R we define ω_R as in Definition 2.2.

Then, it is easy to see that Q coincides with

$$\mathrm{Spec}_{X'}\bigg(\bigoplus_{m_1,\dots,m_{s-1}\in\mathbb{Z}}\mathcal{O}_{X'}\Big(\sum_{j=1}^{s-1}m_jF_j'\Big)\bigg),$$

where $F_j'=F_j|_{X'}$. Since $X\setminus X'=\mathrm{Sing}(X)$, the natural restriction

$$\mathrm{Cl}(X)\longrightarrow \mathrm{Cl}(X') \tag{4.2}$$

is an isomorphism. Let $\pi:Q\to X'$ be the structure morphism.

Then, by Lemma 4.2, we have the following exact sequence:

$$0\longrightarrow \mathbb{Z}\overline{F_1'}+\cdots+\mathbb{Z}\overline{F_{s-1}'}\longrightarrow \mathrm{Cl}(X')\xrightarrow{\pi^*}\mathrm{Cl}(Q)\longrightarrow 0. \tag{4.3}$$

Since $\dim Q=d+s-1$, we have

$$\omega_T|_Q=\omega_Q=\bigwedge^{d+s-1}\Omega_{Q/k}.$$

Since X' is smooth over k and $\pi:Q\to X'$ is a smooth morphism, the sequence

$$0\longrightarrow \pi^*\Omega_{X'/k}\longrightarrow\Omega_{Q/k}\longrightarrow\Omega_{Q/X'}\longrightarrow 0$$

is exact. Therefore, we have

$$\begin{aligned}\omega_Q&=\bigwedge^{d+s-1}\Omega_{Q/k}\\&=\bigwedge^{d}\pi^*\Omega_{X'/k}\otimes_{\mathcal{O}_Q}\bigwedge^{s-1}\Omega_{Q/X'}\end{aligned}$$

$$
\begin{aligned}
(4.4) \qquad &= \pi^* \bigwedge^d \Omega_{X'/k} \otimes_{\mathcal{O}_Q} \pi^* \mathcal{O}_{X'}(F'_1 + \cdots + F'_{s-1}) \\
&= \pi^* \omega_{X'} \otimes_{\mathcal{O}_Q} \mathcal{O}_Q \\
&= \pi^* \mathcal{O}_{X'}(K_{X'})
\end{aligned}
$$

since $\pi^* \mathcal{O}_{X'}(F'_1 + \cdots + F'_{s-1}) \simeq \mathcal{O}_Q$ by the exact sequence (4.3).

On the other hand, we have

$$
\omega_T = \Bigg(\bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i + G\Big)\Big)\Bigg)^{\sim},
$$

where the right-hand side is the coherent $\mathcal{O}_T$-module associated with the $\mathbb{Z}$-graded module

$$
\bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i + G\Big)\Big),
$$

that is, the canonical module of the $\mathbb{Z}$-graded ring R. (Note that we used the fact that Theorem 1.2 is true if $s = 1$.) Therefore, we obtain

$$
\omega_Q = \omega_T|_Q = \Bigg(\bigoplus_{m_1,\dots,m_{s-1}\in\mathbb{Z}} \mathcal{O}_{X'}\Big(\sum_{j=1}^{s-1} m_j F'_j + G'\Big)\Bigg)^{\sim},
$$

where $G' = G|_{X'}$, and the right-hand side is a coherent $\mathcal{O}_Q$-module associated with

$$
\bigoplus_{m_1,\dots,m_{s-1}\in\mathbb{Z}} \mathcal{O}_{X'}\Big(\sum_{j=1}^{s-1} m_j F'_j + G'\Big),
$$

that is, a sheaf of modules over the sheaf of algebras

$$
\bigoplus_{m_1,\dots,m_{s-1}\in\mathbb{Z}} \mathcal{O}_{X'}\Big(\sum_{j=1}^{s-1} m_j F'_j\Big)
$$

on X'. Therefore, we have

$$
\omega_Q = \pi^* \mathcal{O}_{X'}(G'). \tag{4.5}
$$

Here for Weil divisors E_1 and E_2 on X', we know that

$$
\pi^* \mathcal{O}_{X'}(E_1) \simeq \pi^* \mathcal{O}_{X'}(E_2) \Longleftrightarrow \overline{E_1} \equiv \overline{E_2} \mod \mathbb{Z}\overline{F'_1} + \cdots + \mathbb{Z}\overline{F'_{s-1}}
$$

by the exact sequence (4.3).

Thus, by (4.4) and (4.5), we obtain

$$
\overline{G' - K_{X'}} \in \mathbb{Z}\overline{F'_1} + \cdots + \mathbb{Z}\overline{F'_{s-1}}.
$$

Since the restriction (4.2) is an isomorphism, we know that

$$
\overline{G - K_X} \in \mathbb{Z}\overline{F_1} + \cdots + \mathbb{Z}\overline{F_{s-1}}
$$

in $\mathrm{Cl}(X)$. It is a contradiction.

We have completed the proof of Theorem 1.2. $\square$

REMARK 4.5

Suppose that all the assumptions in Theorem 1.2 are satisfied. We choose D, S as in Remark 2.4. Put $d = \dim X$. Then, $\dim R = d + s$ as in [2]. Let $\mathfrak{m}$ be the unique maximal homogeneous ideal of R. Let S_+ be the maximal ideal of S generated by all the homogeneous elements of positive degree. Then, we have

$$\begin{aligned} H_{\mathfrak{m}}^{d+s}(R) &= {}^*\operatorname{Hom}_k(\omega_R, k) \\ &= \bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s} \operatorname{Hom}_k\Bigl(H^0\Bigl(X, \mathcal{O}_X\Bigl(-\sum_i n_i D_i + K_X\Bigr)\Bigr), k\Bigr) \\ &= \bigoplus_{(n_1,\dots,n_s)\in\mathbb{Z}^s} H^d\Bigl(X, \mathcal{O}_X\Bigl(\sum_i n_i D_i\Bigr)\Bigr) \\ &= H_{S_+}^{d+1}(R) \end{aligned}$$

by Serre duality.

Note that there are many examples such that

$$H_{\mathfrak{m}}^{d+s-1}(R) \neq H_{S_+}^{d}(R).$$

5. Some examples

In this section, we give some examples.

EXAMPLE 5.1

Let $B = k[x,y,z]$ be a weighted polynomial ring over a field k with $\deg(x) = a$, $\deg(y) = b$, and $\deg(z) = c$, where a, b, c are pairwise coprime positive integers.

Let P be the kernel of the k-algebra homomorphism $k[x,y,z] \longrightarrow k[t]$ defined by $x \mapsto t^a$, $y \mapsto t^b$, $z \mapsto t^c$.

Let $\pi : X \to \operatorname{Proj}(B)$ be the blowup at $V_+(P)$. Let A be an integral Weil divisor on X satisfying $\mathcal{O}_X(A) = \pi^*\mathcal{O}_{\operatorname{Proj}(B)}(1)$. Put $E = \pi^{-1}(V_+(P))$. In this case, $\operatorname{Cl}(X)$ is freely generated by $\overline{A}$ and $\overline{E}$. We have

$$K_X = E - (a+b+c)A.$$

Then, $R(X; -E, A)$ coincides with the extended symbolic Rees ring

$$R = k[x,y,z,t^{-1}, Pt, P^{(2)}t^2, P^{(3)}t^3, \dots] \subset k[x,y,z,t,t^{-1}].$$

R is a $\mathbb{Z}^2$-graded ring with $\deg(x) = (0,a)$, $\deg(y) = (0,b)$, $\deg(z) = (0,c)$, $\deg(t) = (1,0)$. We know that

$$\omega_R = R\bigl(-1, -(a+b+c)\bigr).$$

For positive integers α and β, we define

$$R^{(\alpha,\beta)} = \bigoplus_{m_1,m_2\in\mathbb{Z}} R_{(\alpha m_1, \beta m_2)}.$$

Here, we remark that $R^{(\alpha,\beta)} = R(X; -\alpha E, \beta A)$. Therefore, $\omega_{R^{(\alpha,\beta)}}$ is an $R^{(\alpha,\beta)}$-free module if and only if $\alpha = 1$ and $\beta | (a+b+c)$.

EXAMPLE 5.2

Let k be a field, and let $t_1, \dots, t_r$ be rational distinct closed points in $\mathbb{P}^n_k$. Let I_i be the defining ideal of t_i in the homogeneous coordinate ring $B = k[x_0, \dots, x_n]$. Recall that I_i is generated by linearly independent n-linear forms. Therefore, for any $\ell > 0$,

$$I_i^\ell = (I_i^\ell)^{\mathrm{sat}} = I_i^{(\ell)},$$

where $(I_i^\ell)^{\mathrm{sat}}$ is the *saturation* of the ideal I_i^ℓ, and $I_i^{(\ell)}$ is the ℓth symbolic power of I_i.

Let $m_1, \dots, m_r$ be positive integers. Put

$$I = I_1^{m_1} \cap \cdots \cap I_r^{m_r}.$$

Let $\pi : X \to \mathbb{P}^n_k$ be the blowup at $t_1, \dots, t_r$. Put $E_i = \pi^{-1}(t_i)$ for $i = 1, \dots, \ell$. Let A be a Weil divisor on X that satisfies $\mathcal{O}_X(A) = \pi^* \mathcal{O}_{\mathbb{P}^n_k}(1)$. Then, $\mathrm{Cl}(X)$ is freely generated by $\overline{A}$, $\overline{E_1}, \dots, \overline{E_r}$. In this case, we have

$$K_X = (n-1)(E_1 + \cdots + E_r) - (n+1)A.$$

Put

$$R = R(X; -m_1 E_1 - \cdots - m_r E_r, A).$$

Here, $-m_1 E_1 - \cdots - m_r E_r + mA$ is an ample Cartier divisor for $m \gg 0$. Then, R coincides with the extended symbolic Rees ring of I; that is,

$$R = B[t^{-1}, It, I^{(2)}t^2, I^{(3)}t^3, \dots] \subset B[t, t^{-1}],$$

where

$$I^{(\ell)} = I_1^{\ell m_1} \cap \cdots \cap I_r^{\ell m_r}.$$

Assume that R is Noetherian. Then, by Corollary 1.3, we know that

$$\begin{aligned} \omega_R \simeq R &\Longleftrightarrow \overline{K_X} \in \mathbb{Z}\overline{(-m_1 E_1 - \cdots - m_r E_r)} + \mathbb{Z}\overline{A} \\ &\Longleftrightarrow \begin{cases} m_1 = \cdots = m_r \\ m_1 \mid (n-1). \end{cases} \end{aligned}$$

EXAMPLE 5.3

Here, we give an example where the equality (1.2) is not satisfied if we remove the assumption that L_2 contains an ample Cartier divisor.

Set $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$, $A_1 = \mathbb{P}^1_k \times (1:0)$ and $A_2 = (1:0) \times \mathbb{P}^1_k$. Then, $\mathrm{Cl}(X)$ is freely generated by $\overline{A_1}$ and $\overline{A_2}$. We know that

$$\mathrm{Cox}(X) = R(X; A_1, A_2) = k[x_0, x_1, y_0, y_1],$$

where $\deg(x_0) = \deg(x_1) = (1,0)$ and $\deg(y_0) = \deg(y_1) = (0,1)$.

Then, we have

$$\omega_{R(X; A_1, A_2)} = R(X; A_1, A_2)(-2, -2)$$

since $K_X = -2A_1 - 2A_2$.

For $a, b \in \mathbb{Z}$, we define

$$S_{a,b} = R(X; aA_1 + bA_2)$$

and

$$L_{a,b} = \mathbb{Z}(aA_1 + bA_2) \subset \mathbb{Z}A_1 + \mathbb{Z}A_2.$$

Then, we have

$$S_{a,b} = R(X; A_1, A_2)|_{L_{a,b}}.$$

In this case, $aA_1 + bA_2$ is ample if and only if $a > 0$ and $b > 0$.

Therefore, if $a > 0$ and $b > 0$, then

$$\omega_{S_{a,b}} = \omega_{R(X;A_1,A_2)}|_{L_{a,b}}$$

by (1.2). However, we have

$$\omega_{R(X;A_1,A_2)}|_{L_{1,0}} = 0 \neq \omega_{S_{1,0}}.$$

In the case where $a > 0$ and $b > 0$, $S_{a,b}$ is the Segre product of $k[x_0, x_1]^{(a)}$ and $k[y_0, y_1]^{(b)}$. Here, $k[x_0, x_1]^{(a)}$ is the ath Veronese subring of $k[x_0, x_1]$, that is,

$$k[x_0, x_1]^{(a)} = \bigoplus_{n \geq 0} k[x_0, x_1]_{na}.$$

Then, we have

$$\omega_{S_{a,b}} = \bigoplus_{n > 0} (k[x_0, x_1]_{na-2} \otimes_k k[y_0, y_1]_{nb-2}) = \omega_{k[x_0,x_1]^{(a)}} \# \omega_{k[y_0,y_1]^{(b)}}$$

as in [4, Theorem (4.3.1)].

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