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WEIGHTED BACKWARD SHIFT OPERATORS WITH INVARIANT DISTRIBUTIONALLY SCRAMBLED SUBSETS

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Abstract. We obtain a sufficient condition to ensure that weighted backward shift operators on Köthe sequence spaces $\lambda_p(A)$ admit an invariant distributionally ε-scrambled subset for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$. In particular, every Devaney chaotic weighted backward shift operator on $\lambda_p(A)$ supports such a subset.

1. Introduction and preliminaries

Sharkovsky's amazing discovery (see [16]), as well as Li and Yorke's famous work which introduced the mathematical concept of "chaos" (see [7]), have provoked the recent rapid advancement of discrete chaos theory. An essential feature of chaos is the impossibility of predicting its long-term dynamics due to the exponential separation of any two nearby orbits.

A generalization of the concept of Li–Yorke chaos is *distributional chaos*, introduced by Schweizer and Smítal [15]. Let (X, T) be a dynamical system. For any pair of points $x, y \in X$, define *lower* and *upper distributional functions*, $\mathbf{R} \longrightarrow [0,1]$ generated by T, x and y, as follows:

$$F_{x,y}(t,T) = \liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[0,t)}\big(d\big(T^j(x)T^j(y)\big)\big),$$

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and

$$F^*_{x,y}(t,T) = \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[0,t)}\big(d\big(T^j(x), T^j(y)\big)\big),$$

respectively, where $\chi_A(\cdot)$ denotes the characteristic function of set A. Clearly, both functions $F_{x,y}$ and $F^*_{x,y}$ are nondecreasing. A dynamical system (X,T) is *distributionally ε-chaotic* for a given $\varepsilon > 0$ if there exists an uncountable subset $D \subset X$ such that for any pair of distinct points $x, y \in D$, one has $F^*_{x,y}(t,T) = 1$ for all $t > 0$ and $F_{x,y}(\varepsilon, T) = 0$. The set D is a *distributionally ε-scrambled set* and the pair (x,y) a *distributionally ε-chaotic pair*. If (X,T) is distributionally ε-chaotic for any given $0 < \varepsilon < \operatorname{diam} X$, then (X,T) is said to exhibit *maximal distributional chaos*.

Let (X,d) be a complete metric space, and let $T : X \longrightarrow X$ be continuous. For the space X, define another metric d_1 as

$$d_1(x,y) = \frac{d(x,y)}{1 + d(x,y)},$$

for any $x, y \in X$. Clearly, the topologies generated by these two metrics are the same. It is easy to see that if $d_1(x,y) \geq \varepsilon$ holds for some $0 < \varepsilon < 1$, then $d(x,y) \geq \varepsilon/(1-\varepsilon)$. Combining this with the definition of distributional ε-chaos, we have the following result.

Proposition 1.1.

(1) *If the dynamical system (X,T) is distributionally ε-chaotic for some $0 < \varepsilon < 1$ under the metric d_1, then (X,T) is distributionally $\varepsilon/(1-\varepsilon)$-chaotic under the metric d.*

(2) *If the dynamical system (X,T) is distributionally ε-chaotic for some $0 < \varepsilon < 1$ under the metric d, then (X,T) is distributionally $\varepsilon/2$-chaotic under the metric d_1.*

During the last ten years, many research works were devoted to the "chaotic behavior" of backward shift operators on Köthe sequence spaces, and more generally, Banach or Fréchet spaces (see, e.g., [1], [2], [5], [6], [8]–[11], [13], [17], [18]). One of the most significant features is *hypercyclicity*; that is, the existence of a vector $x \in X$ whose orbit $\operatorname{orb}(T,x) := \{x, T(x), T^2(x), \ldots\}$, under a continuous and linear operator $T : X \longrightarrow X$ acting on a topological vector space X, is dense in X. The study of hypercyclic operators on sequence spaces started in 1969 when Rolewicz [13] showed that the weighted backward shift $\lambda B : l^p \longrightarrow l^p$, $(x_1, x_2, \ldots) \longmapsto (\lambda x_2, \lambda x_3, \ldots)$, is hypercyclic if $|\lambda| > 1$. Salas [14] extended the study of backward shift operators on l^2 to weighted backward shift and bilateral weighted shift operators. Grosse-Erdmann [5] gave an excellent survey on hypercyclic operators in 2003. Characterizations for hypercyclicity and Devaney chaos under backward shift operators on Köthe sequence spaces can be found in [11]. Martínez-Giménez [8] obtained some sufficient conditions for the operator $f(B_w)$ to be chaotic in the sense of Devaney. Equivalent conditions for Li–Yorke chaotic operators and distributionally chaotic operators on Banach spaces (more generally, Fréchet spaces) were obtained by Bernardes et al. in [1], [2]. In

2009, Martínez-Giménez, Oprocha, and Peris [9] provided sufficient conditions for uniform distributional chaos under backward shift operators on Köthe sequence spaces. In [17], the current authors provided a set of characterizations for uniform Li–Yorke chaos and a sufficient condition for maximal distributional chaos under backward shift operators on Köthe sequence spaces, and we proved in [18] that the annihilation operator of a quantum harmonic oscillator which is a specific weighted backward shift operator on the Köthe sequence space $\lambda_2(A)$ with $a_{j,k} = (j+1)^{k/2}$ exhibits maximal distributional chaos. Very recently, it was shown in [10] that the mixing property is not sufficient for distributional chaos under a linear operator. For more recent results on linear chaos, we refer the reader to [6] and the references therein.

Concerning the invariance of chaos, Du [4] proved that an interval map is turbulent if and only if there is an invariant scrambled set. Later, Oprocha [12] extended this result and proved that exactly the same characterization is valid for distributional chaos. In 2013, Doleželová [3] showed that a compact system with a weak specification property, fixed point, and infinitely many mutually distinct periods has a dense Mycielski invariant distributionally scrambled set. Very recently, for the full shift (Σ_2, σ) on two symbols, the present authors in [19] constructed an invariant distributionally ε-scrambled set for any $0 < \varepsilon < \operatorname{diam} \Sigma_2$, in which each point is transitive but is not weakly almost periodic.

Inspired by the above work on the chaoticity and invariant scrambled subsets of various operators and by methods and results developed in [9], [12], [17], [18], and [19], we aim here to study conditions under which a dynamical system $(\lambda_p(A), B_w)$ admits an invariant distributionally ε- scrambled subset for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$.

2. Backward shift on the Köthe sequence space

Now, we give a detailed description of a dynamical system $(\lambda_p(A), B_w)$. An infinite matrix $A = (a_{j,k})_{j,k\in\mathbf{N}}$ is called a *Köthe matrix* if, for each $j \in \mathbf{N}$, there exists some $k \in \mathbf{N}$ with $a_{j,k} > 0$ and $0 \leq a_{j,k} \leq a_{j,k+1}$ for all $j, k \in \mathbf{N}$.

Consider the *unilateral backward shift*, also called the *one-sided left shift*,

$$B(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots),$$

on the Köthe sequence space $\lambda_p(A)$ which is determined by a Köthe matrix A, where, for $1 \leq p < +\infty$,

$$\lambda_p(A) := \Big\{x \in \mathbf{K}^{\mathbf{N}} : \|x\|_k := \Big(\sum_{j=1}^{+\infty} |x_j a_{j,k}|^p\Big)^{1/p} < +\infty, \forall k \in \mathbf{N}\Big\},$$

and for $p = 0$,

$$\lambda_0(A) := \big\{x \in \mathbf{K}^{\mathbf{N}} : \lim_{j\to+\infty} x_j a_{j,k} = 0, \|x\|_k := \sup_{j\in\mathbf{N}} |x_j a_{j,k}|, \forall k \in \mathbf{N}\big\}.$$

It is possible to define a complete metric on $\lambda_p(A)$ which is invariant by translation:

$$d(x,y)=\sum_{n=1}^{+\infty}\frac{1}{2^n}\frac{\|x-y\|_n}{1+\|x-y\|_n}.$$

The operator $B:\lambda_p(A)\longrightarrow\lambda_p(A)$ is continuous and well defined if and only if the following condition on the matrix A is satisfied:

$$\forall n\in\mathbf{N},\quad \exists m>n \text{ such that } \sup_{j\in\mathbf{N}}\left|\frac{a_{j,n}}{a_{j+1,m}}\right|<+\infty, \tag{2.1}$$

where in the case of $a_{j+1,m}=0$, we have $a_{j,n}=0$; therefore, we consider $\frac{0}{0}$ as 1. Here, we give a brief description of the Köthe sequence space. Given a sequence $\{w_i\}_{i\geq 2}$ of strictly positive scalars, we may consider its associated *weighted backward shift* $B_w:\lambda_p(A)\longrightarrow\lambda_p(A)$,

$$B_w(x_1,x_2,\ldots):=(w_2x_2,w_3x_3,\ldots).$$

According to (2.1), the operator B_w is continuous if and only if

$$\forall n\in\mathbf{N},\quad \exists m>n \text{ such that } \sup_{j\in\mathbf{N}}\left|w_{j+1}\frac{a_{j,n}}{a_{j+1,m}}\right|<+\infty. \tag{2.2}$$

Set

$$\mathscr{W}_1=1,\qquad \mathscr{W}_i=\frac{1}{w_2\cdots w_i},\qquad \mathscr{W}_i^k=w_i\cdots w_{i-k+1},\quad i>1,k<i.$$

Then it is easy to see that, for any $k\in\mathbf{N}$ and any $x=(x_1,x_2,\ldots)\in\lambda_p(A)$, we have

$$B_w^k(x)=(\mathscr{W}_{k+1}^k x_{k+1},\mathscr{W}_{k+2}^k x_{k+2},\ldots).$$

We recall that the *upper density* $\overline{\mathscr{D}}(A)$ of a set $A\subset\mathbf{N}$ is defined by

$$\overline{\mathscr{D}}(A)=\limsup_{n\to+\infty}\frac{|A\cap[1,n]|}{n},$$

where $|\cdot|$ means the cardinality of a set.

3. Main results

Theorem 3.1. *Let A be a Köthe matrix, let $\{w_i\}_{i\geq 2}$ be a weight sequence satisfying (2.2) above, and let $1\leq p<+\infty$ (or $p=0$). If there exists an increasing sequence $E\subset\mathbf{N}$ such that for any $n\in\mathbf{N}$, $\sum_{j\in E}|\mathscr{W}_j a_{j,n}|^p<+\infty$ (or $\lim_{E\ni j\to+\infty}|\mathscr{W}_j a_{j,n}|=0$) and $\overline{\mathscr{D}}(E)=1$, then $B_w:\lambda_p(A)\longrightarrow\lambda_p(A)$ has an invariant distributionally ε-scrambled subset for any $0<\varepsilon<\operatorname{diam}\lambda_p(A)$.*

Proof. Suppose that $1\leq p<+\infty$. The proof for the case in which $p=0$ is basically the same using the corresponding sup norm. By hypothesis, there exists an increasing sequence $\{m_n\}_{n\in\mathbf{N}}\subset\mathbf{N}$ such that

$$1=\overline{\mathscr{D}}(E)=\lim_{n\to+\infty}\frac{|E\cap[1,m_n]|}{m_n}.$$

Combining this with $\nu := (\nu_1, \nu_2, \ldots) = \sum_{j\in E} \mathscr{W}_j e_j \in \lambda_p(A)$, where $e_j = (x_1, x_2, x_3, \ldots)$ with

$$x_i = \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases}$$

it follows that there exists a subsequence $\{M_n\}_{n\in\mathbf{N}}$ of $\{m_n\}_{n\in\mathbf{N}}$ such that, for any $n \in \mathbf{N}$, $\sum_{j=M_n}^{+\infty} |\nu_j a_{j,n}|^p < 1/2^n$, and $M_{n+1} - M_n \geq 4^{M_n}$. Take $\widetilde{\nu} = (\widetilde{\nu}_1, \widetilde{\nu}_2, \ldots)$ to be

$$\widetilde{\nu}_j = \begin{cases} k\nu_j & M_k \leq j < M_{k+1}, k \in \mathbf{N}, \\ \nu_j & 1 \leq j < M_1. \end{cases}$$

Because

$$\begin{aligned} \sum_{j=M_k}^{+\infty} |\widetilde{\nu}_j a_{j,k}|^p &= \sum_{l\geq k} \sum_{j=M_l}^{M_{l+1}-1} |\widetilde{\nu}_j a_{j,k}|^p = \sum_{l\geq k} \sum_{j=M_l}^{M_{l+1}-1} l^p |\nu_j a_{j,k}|^p \\ &\leq \sum_{l\geq k} \sum_{j=M_l}^{M_{l+1}-1} l^p |\nu_j a_{j,l}|^p \leq \sum_{l\geq k} \frac{l^p}{2^l} < +\infty, \end{aligned}$$

we have $\widetilde{\nu} \in \lambda_p(A)$.

For any $k, n \in \mathbf{N}$, denote

$$\widetilde{\nu}[k] = B_w^k(\widetilde{\nu}) = (\mathscr{W}_{k+1}^k \widetilde{\nu}_{k+1}, \mathscr{W}_{k+2}^k \widetilde{\nu}_{k+2}, \ldots)$$

and

$$\widetilde{\nu}[k, n] = (\underbrace{0, 0, \ldots, 0}_{n}, \mathscr{W}_{k+n+1}^k \widetilde{\nu}_{k+n+1}, \mathscr{W}_{k+n+2}^k \widetilde{\nu}_{k+n+2}, \ldots).$$

Since $\lim_{n\to+\infty} d(0, \widetilde{\nu}[k, n]) = 0$, it follows that, for any $k, n, q \in \mathbf{N}$, there exists $\zeta(k, n, q) \in \mathbf{N}$ such that, for all $m \geq \zeta(k, n, q)$ and all $1 \leq j \leq n$, we have that $d(0, \widetilde{\nu}[k + j, m]) < 1/q$.

Applying mathematical induction, it is easy to show that there exists a subsequence $\{\widehat{M}_n\}_{n\in\mathbf{N}}$ of $\{M_n\}_{n\in\mathbf{N}}$ such that, for any $n \in \mathbf{N}$,

(i) $\widehat{M}_1 = M_1$,

(ii) $\widehat{M}_{2n} = \min\{M_{2i} : M_{2i} \geq \widehat{M}_{2n-1} + 2^{\widehat{M}_{2n-1}} + 2\zeta(\widehat{M}_{2n-1}, 2^{\widehat{M}_{2n-1}}, 2n), i \in \mathbf{N}\}$,

(iii) $\widehat{M}_{2n+1} = \min\{M_{2i+1} : M_{2i+1} > \widehat{M}_{2n}, i \in \mathbf{N}\}$.

Arrange all odd prime numbers by the natural order $<$, and denote them as $\mathrm{P}_1, \mathrm{P}_2, \ldots$. For any $n, l \in \mathbf{N}$, set

$$\begin{aligned} \mathscr{B}_{n,l}^0 = \Big\{ & j \in \mathbf{N} : \widehat{M}_{\mathrm{P}_n^l+1} + (2u)l \leq j < \widehat{M}_{\mathrm{P}_n^l+1} + (2u+1)l, \\ & 0 \leq 2u \leq \Big[\frac{\widehat{M}_{\mathrm{P}_n^l+2} - \widehat{M}_{\mathrm{P}_n^l+1}}{l}\Big] - 3 \Big\}, \\ \mathscr{B}_{n,l}^1 = \Big\{ & j \in \mathbf{N} : \widehat{M}_{\mathrm{P}_n^l+1} + (2u+1)l \leq j < \widehat{M}_{\mathrm{P}_n^l+1} + (2u+2)l, \\ & 1 \leq 2u+1 \leq \Big[\frac{\widehat{M}_{\mathrm{P}_n^l+2} - \widehat{M}_{\mathrm{P}_n^l+1}}{l}\Big] - 3 \Big\}, \end{aligned}$$

$$\begin{aligned}\mathscr{B}_{n,l} &= \mathscr{B}_{n,l}^0 \cup \mathscr{B}_{n,l}^1 \\ &= \Big\{ j \in \mathbf{N} : \widehat{M}_{\mathrm{P}_n^l+1} \le j < \widehat{M}_{\mathrm{P}_n^l+1} + l\Big(\Big[\frac{\widehat{M}_{\mathrm{P}_n^l+2} - \widehat{M}_{\mathrm{P}_n^l+1}}{l}\Big] - 2\Big)\Big\},\end{aligned}$$

and

$$\mathscr{C}_n = \{ j \in \mathbf{N} : \widehat{M}_{16^n} \le j < \widehat{M}_{16^n+1} \}.$$

It is not difficult to check that, for any $n, l \in \mathbf{N}$, there exists $\Xi_{n,l} \in \{0,1\}$ such that

$$|E \cap \mathscr{B}_{n,l}^{\Xi_{n,l}}| \ge \frac{1}{2}\big|E \cap (\mathscr{B}_{n,l}^0 \cup \mathscr{B}_{n,l}^1)\big| = \frac{1}{2}|E \cap \mathscr{B}_{n,l}|$$

and

$$\lim_{n \to +\infty} \frac{|E \cap \mathscr{C}_n|}{\widehat{M}_{16^n+1}} = \lim_{n \to +\infty} \frac{|E \cap \mathscr{B}_{n,l}|}{\widehat{M}_{\mathrm{P}_n^l+2}} = \overline{\mathscr{D}}(E) = 1.$$

Take $\overline{\nu} = (\overline{\nu}_1, \overline{\nu}_2, \ldots)$ with

$$\overline{\nu}_j = \begin{cases} \widetilde{\nu}_j & j \in \mathscr{C}_n, n \in \mathbf{N}, \\ \widetilde{\nu}_j & j \in \mathscr{B}_{n,l}^{\Xi_{n,l}}, n, l \in \mathbf{N}, \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$D = \bigcup_{n=0}^{+\infty} B_w^n\big(\{\alpha\overline{\nu} : \alpha \in (0,1)\}\big).$$

Clearly, $B_w(D) \subset D$ and D is uncountable. Given any two fixed points $a, b \in D$, with $a \ne b$, there exist $\alpha, \beta \in (0,1)$ and $p, q \in \mathbf{Z}^+$ such that $a = B^p(\alpha\overline{\nu})$ and $b = B^q(\beta\overline{\nu})$. Assume that $p \le q$. Set $\mathscr{D} = \{k \in \mathbf{N} : a_{j,k} = 0, \forall j \in \mathbf{N}\}$. Without loss of generality, assume that $\mathscr{D} = \emptyset$. This implies that there exists $j_0 \in \mathbf{N}$ such that $a_{j_0,1} > 0$. According to the definition of Köthe matrix, it is easy to see that, for each $k \in \mathbf{N}$,

$$a_{j_0,k} \ge a_{j_0,1} > 0. \tag{3.1}$$

Now, we assert that (a, b) is a distributionally ε-chaotic pair for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$.

First, observing that for any $\widehat{M}_{2n-1} \le j < \widehat{M}_{2n-1} + 2^{\widehat{M}_{2n-1}} - q$, the first $\zeta(\widehat{M}_{2n-1}, 2^{\widehat{M}_{2n-1}}, 2n)$ symbols of $B_w^{j+p}(\widetilde{\nu})$ and $B_w^{j+p}(\widetilde{\nu})$ are all equal to zero, we have

$$\begin{aligned} d\big(B_w^j(a), B_w^j(b)\big) &\le d\big(0, B_w^j(a)\big) + d\big(0, B_w^j(b)\big) \\ &= d\big(0, B_w^{j+p}(\alpha\overline{\nu})\big) + d\big(0, B_w^{j+q}(\beta\overline{\nu})\big) \\ &\le d\big(0, B_w^{j+p}(\widetilde{\nu})\big) + d\big(0, B_w^{j+q}(\widetilde{\nu})\big) \\ &\le \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}. \end{aligned}$$

This implies that for any $t > 0$ there exists $N \in \mathbf{N}$ such that, for any $n \geq N$ and any $\widehat{M}_{2n-1} \leq j < \widehat{M}_{2n-1} + 2^{\widehat{M}_{2n-1}} - q$, we have $d(B_w^j(a), B_w^j(b)) < t$. Consequently,

$$\begin{aligned}
F^*_{a,b}(t, B_w) &= \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[0,t)}\big(d\big(B_w^j(a), B_w^j(b)\big)\big) \\
&\geq \limsup_{n\to+\infty} \frac{1}{\widehat{M}_{2n-1} + 2^{\widehat{M}_{2n-1}}} \sum_{j=1}^{\widehat{M}_{2n-1}+2^{\widehat{M}_{2n-1}}} \chi_{[0,t)}\big(d\big(B_w^j(a), B_w^j(b)\big)\big) \\
&\geq \limsup_{n\to+\infty} \frac{2^{\widehat{M}_{2n-1}} - q}{\widehat{M}_{2n-1} + 2^{\widehat{M}_{2n-1}}} \\
&= 1.
\end{aligned}$$

Second, given any fixed $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$, there exists $M \in \mathbf{N}$ such that

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{M}{1+M} \geq \varepsilon. \tag{3.2}$$

To prove that $F_{a,b}(\varepsilon, B_w) = 0$, consider the two following cases.

Case 1: $p = q$ and $a \neq \beta$. For any $j > j_0$, denote

$$B_w^{j-j_0}(\overline{\nu}) = (\xi_1^{(j)}, \xi_2^{(j)}, \ldots).$$

Observe that, for any $n > j_0$ and any $j \in E \cap \mathscr{C}_n$,

$$|\xi_{j_0}^{(j)}| = |\mathscr{W}_j^{j-j_0} \overline{\nu}_j| \geq \Big|\frac{16^n}{w_2 \cdots w_{j_0}}\Big|.$$

Combining this with (3.1), it follows that, for any $k \in \mathbf{N}$,

$$\begin{aligned}
&\big\|B_w^{j-j_0}\big((\alpha-\beta)\overline{\nu}\big)\big\|_k \\
&\quad= \Big(\sum_{i=1}^{+\infty} \big|(\alpha-\beta)\xi_i^{(j)} a_{i,k}\big|^p\Big)^{1/p} \\
&\quad\geq \big|(\alpha-\beta)\xi_{j_0}^{(j)} a_{j_0,k}\big| \\
&\quad\geq \Big|(\alpha-\beta)\frac{16^n}{w_2 \ldots w_{j_0}} a_{j_0,1}\Big| \longrightarrow +\infty \quad (n \longrightarrow +\infty).
\end{aligned}$$

This with (3.2) leads to the fact that there exists some $N' > j_0$ such that, for any $n \geq N'$ and any $j \in E \cap \mathscr{C}_n$,

$$\begin{aligned}
d\big(B_w^{j-j_0-q}(a), B_w^{j-j_0-q}(b)\big) &= d\big(0, B_w^{j-j_0}\big((\alpha-\beta)\overline{\nu}\big)\big) \\
&= \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{\|B_w^{j-j_0}((\alpha-\beta)\overline{\nu})\|_k}{1 + \|B_w^{j-j_0}((\alpha-\beta)\overline{\nu})\|_k} \\
&\geq \varepsilon.
\end{aligned}$$

Then

$$
\begin{aligned}
1 &\geq \limsup_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{[\varepsilon,+\infty)}\big(d\big(B_w^j(a), B_w^j(b)\big)\big) \\
&\geq \limsup_{n\to+\infty} \frac{1}{\widehat{M}_{16^n+1}} \sum_{j=1}^{\widehat{M}_{16^n+1}} \chi_{[\varepsilon,+\infty)}\big(d\big(B_w^j(a), B_w^j(b)\big)\big) \\
&\geq \lim_{n\to+\infty} \frac{|E\cap \mathscr{C}_n|}{\widehat{M}_{16^n+1}} = 1. \qquad (3.3)
\end{aligned}
$$

Case 2: $p<q$. Note that, for any $n>j_0$ and any $j\in E\cap \mathscr{B}_{n,q-p}^{\Xi_{n,q-p}}$,

$$
|\xi_{j_0}^{(j)}| = |\mathscr{W}_j^{j-j_0}\overline{\nu}_j| \geq \Big|\frac{\mathrm{P}_n^{q-p}+1}{w_2\cdots w_{j_0}}\Big|,
$$

and

$$
\xi_{j_0}^{(j+(q-p))} = \mathscr{W}_{j+(q-p)}^{j-j_0+(q-p)}\overline{\nu}_{j+(q-p)} = 0.
$$

This together with (3.1) leads to the fact that, for any $k\in\mathbf{N}$,

$$
\begin{aligned}
\big\|B_w^{j-j_0-p}(a-b)\big\|_k &= \big\|B_w^{j-j_0}(\alpha\overline{\nu}) - B_w^{j-j_0+(q-p)}(\beta\overline{\nu})\big\|_k \\
&\geq \big|(\alpha\xi_{j_0}^{(j)} - \beta\xi_{j_0}^{(j+(q-p))})a_{j_0,k}\big| \\
&\geq \Big|\alpha\frac{\mathrm{P}_n^{q-p}+1}{w_2\cdots w_{j_0}}a_{j_0,1}\Big| \longrightarrow +\infty \quad (n\longrightarrow+\infty). \qquad (3.4)
\end{aligned}
$$

Take

$$
E_n = \big\{j\in\mathscr{B}_{n,q-p} : j+(q-p)\in E\cap\mathscr{B}_{n,q-p}^{\Xi_{n,q-p}}\big\}.
$$

It is easy to verify that, for any $n>j_0$ and any $j\in E_n$,

$$
\xi_{j_0}^{(j)} = \mathscr{W}_j^{j-j_0}\overline{\nu}_j = 0
$$

and

$$
|\xi_{j_0}^{(j+(q-p))}| = |\mathscr{W}_{j+(q-p)}^{j-j_0+(q-p)}\overline{\nu}_{j+(q-p)}| \geq \Big|\frac{\mathrm{P}_n^{q-p}+1}{w_2\cdots w_{j_0}}\Big|.
$$

Combining this with (3.1), in a similar way as in the proof of (3.4), it follows that for any $j\in E_n$ and any $k\in\mathbf{N}$,

$$
\big\|B_w^{j-j_0-p}(a-b)\big\|_k \geq \Big|\beta\frac{\mathrm{P}_n^{q-p}+1}{w_2\cdots w_{j_0}}a_{j_0,1}\Big|. \qquad (3.5)
$$

Applying (3.2), (3.4), and (3.5), we have that there exists $N''>j_0$ such that for any $n\geq N''$ and any $j\in E_n\cup(E\cap\mathscr{B}_{n,q-p}^{\Xi_{n,q-p}})$,

$$
d\big(B_w^{j-j_0-p}(a), B_w^{j-j_0-p}(b)\big) \geq \varepsilon.
$$

Noting that $E_n \cap (E \cap \mathscr{B}_{n,q-p}^{\Xi_{n,q-p}}) = \emptyset$, we have

$$\begin{aligned}
1 &\geq \limsup_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{[\varepsilon,+\infty)}\big(d\big(B_w^j(a), B_w^j(b)\big)\big) \\
&\geq \limsup_{n\to+\infty} \frac{1}{\widehat{M}_{\mathrm{P}_n^{q-p}+2}} \sum_{j=1}^{\widehat{M}_{\mathrm{P}_n^{q-p}+2}} \chi_{[\varepsilon,+\infty)}\big(d\big(B_w^j(a), B_w^j(b)\big)\big) \\
&\geq \limsup_{n\to+\infty} \frac{|E_n| + |E \cap \mathscr{B}_{n,q-p}^{\Xi_{n,q-p}}|}{\widehat{M}_{\mathrm{P}_n^{q-p}+2}} \\
&\geq \limsup_{n\to+\infty} \frac{2|E \cap \mathscr{B}_{n,q-p}^{\Xi_{n,q-p}}| - (q-p)}{\widehat{M}_{\mathrm{P}_n^{q-p}+2}} \\
&\geq \limsup_{n\to+\infty} \frac{|E \cap \mathscr{B}_{n,q-p}| - (q-p)}{\widehat{M}_{\mathrm{P}_n^{q-p}+2}} = 1.
\end{aligned} \tag{3.6}$$

Summing up (3.3) and (3.6), we obtain

$$F_{a,b}(\varepsilon, B_w) = 1 - \limsup_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{[\varepsilon,+\infty)}\big(d\big(B_w^j(a), B_w^j(b)\big)\big) = 0.$$

Hence, D is an invariant distributionally ε-scrambled set for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$. □

Remark 3.2.

(1) It is remarkable that [1, Corollary 27] proved that for a Fréchet sequence space, the same hypothesis of Theorem 3.1 ensures the existence of a dense uniformly distributionally scrambled submanifold.

(2) A stronger condition given in [11, Corollary 3.4] characterizing chaos in the sense of Devaney was

$$\sum_{j=1}^{+\infty} |\mathscr{W}_j a_{j,n}|^p < +\infty.$$

This condition implies the existence of invariant distributionally ε-scrambled sets for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$, since the hypothesis of Theorem 3.1 above is satisfied for $E = \mathbf{N}$.

Let B_w be a weighted backward shift operator defined on a weighted l^p-space $l^p(\{a_j\}_{j\in\mathbf{N}})$ formed by a sequence of strictly positive weights $\{a_j\}_{j\in\mathbf{N}}$, where, for $1 \leq p < +\infty$,

$$l^p\big(\{a_j\}_{j\in\mathbf{N}}\big) = \left\{x = (x_1, x_2, \dots) : \|x\| = \Big(\sum_{j=1}^{+\infty} |a_j x_j|^p\Big)^{1/p} < +\infty\right\}$$

and

$$l^0\big(\{a_j\}_{j\in\mathbf{N}}\big) = \big\{x = (x_1, x_2, \dots) : \lim_{j\to+\infty} a_j x_j = 0, \|x\| = \sup_{j\in\mathbf{N}} |a_j x_j|\big\}.$$

Combining Proposition 1.1 with Theorem 3.1 yields the following result.

Corollary 3.3. *If there exists an increasing sequence $E \subset \mathbf{N}$ such that $\overline{\mathscr{D}}(E) = 1$ and $\sum_{j\in E} |\mathscr{W}_j a_j|^p < +\infty$ (or $\lim_{E\ni j\to+\infty} |\mathscr{W}_j a_j| = 0$), then $B_w : l^p(\{a_j\}_{j\in\mathbf{N}}) \longrightarrow l^p(\{a_j\}_{j\in\mathbf{N}})$ has an invariant distributionally ε-scrambled subset for any $0 < \varepsilon < \operatorname{diam} l^p(\{a_j\}_{j\in\mathbf{N}})$.*

Example 3.4. Consider the subspace of $\mathscr{H} = L^2(-\infty, +\infty)$

$$\mathscr{H}_1 := \Big\{\phi \in \mathscr{H} : \phi = \sum_{n=0}^{+\infty} c_n\psi_n, \sum_{n=0}^{+\infty} |c_n|^2(n+1)^r < +\infty, \forall r > 0\Big\},$$

where

$$\psi_n(x) = \frac{e^{-x^2/2}}{\sqrt{\sqrt{\pi}2^n n!}}(-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2}, \quad n = 0, 1, \ldots .$$

Here, $\mathscr{H}_1$ is an infinite-dimensional Fréchet space with topology defined by the system of seminorms $p_r(\cdot)$ of the form

$$p_r(\phi) = p_r\Big(\sum_{n=0}^{+\infty} c_n\psi_n\Big) = \Big(\sum_{n=0}^{+\infty} |c_n|^2(n+1)^r\Big)^{1/2}.$$

This topology on $\mathscr{H}_1$ can be equivalently introduced by the metric

$$\rho(\phi, \psi) = \sum_{m=0}^{+\infty} \frac{1}{2^m}\frac{p_m(\phi-\psi)}{1 + p_m(\phi-\psi)}.$$

According to the basic properties of Hermite polynomials, the annihilation operator for a quantum harmonic oscillator $\hat{a} = \frac{1}{\sqrt{2}}(x + \frac{\mathrm{d}}{\mathrm{d}x}) : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ is given by

$$\hat{a}(\psi_n) = \frac{1}{\sqrt{2}}\Big(x + \frac{\mathrm{d}}{\mathrm{d}x}\Big)\psi_n = \sqrt{n}\psi_{n-1}.$$

It is not difficult to check that the system $(\mathscr{H}_1, \hat{a})$ can also be represented as the weighted backward shift operator

$$B_w(x_1, x_2, \ldots) := (\sqrt{2}x_2, \sqrt{3}x_3, \ldots)$$

defined on the Köthe sequence space $\lambda_2(A)$ with $a_{j,k} = (j+1)^{k/2}$. From Theorem 3.1, it follows that $\hat{a}$ has an invariant distributionally ε-scrambled set for any $0 < \varepsilon < 2$ under the metric ρ. This shows that the main results in [18] follow naturally.

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