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GENERALIZED SHIFT-INVARIANT SYSTEMS AND APPROXIMATELY DUAL FRAMES

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Abstract. Dual pairs of frames yield a procedure for obtaining perfect reconstruction of elements in the underlying Hilbert space in terms of superpositions of the frame elements. However, practical constraints often force us to apply sequences that do not exactly form dual frames. In this article, we consider the important case of generalized shift-invariant systems and provide various ways of estimating the deviation from perfect reconstruction that occur when the systems do not form dual frames. The deviation from being dual frames will be measured either in terms of a perturbation condition or in terms of the deviation from equality in the duality conditions.

1. Introduction

Frame theory is a tool to obtain expansions of elements in a Hilbert space in terms of "convenient building blocks." In fact, if two sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ form a pair of dual frames for $\mathcal{H}$, then each $f \in \mathcal{H}$ has a representation

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k. \tag{1.1}$$

In signal processing terms, this is expressed by saying that dual pairs of frames lead to *perfect reconstruction*. However, practical constraints will often force us to deal with systems that do not lead to perfect reconstruction, for example,

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sequences $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ that are not exactly dual frames. The purpose of this article is to derive estimates for the corresponding deviation from equality in (1.1) in the important case where $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are generalized shift-invariant systems in $L^2(\mathbb{R})$. The estimate will be formulated in terms of the operator norm

$$\Big\| I - \sum_{k=1}^\infty \langle \cdot, g_k\rangle f_k \Big\| = \sup_{\|f\|=1} \Big\| f - \sum_{k=1}^\infty \langle f, g_k\rangle f_k \Big\|,$$

and the deviation from $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ being dual frames will be measured in terms of a perturbation condition or in terms of the deviation from the duality conditions (see Section 2 for details).

The article is organized as follows. In the rest of the Introduction we provide the necessary information about frame theory and approximately dual frames. Section 2 discusses the generalized shift-invariant systems. The new results concerning approximate dual frames are stated and proved in Section 3.

In order to set the stage for the discussion to follow, recall that a sequence $\{f_k\}_{k=1}^\infty$ in a Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{k=1}^\infty \big|\langle f, f_k\rangle\big|^2 \le B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1.2}$$

The sequence $\{f_k\}_{k=1}^\infty$ is a Bessel sequence if at least the upper condition in (1.2) holds; and two frames $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are dual frames if (1.1) holds. Note that if $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are Bessel sequences and (1.1) holds, then $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are automatically frames.

Given any Bessel sequence $\{f_k\}_{k=1}^\infty$, one can define a bounded operator $T : \ell^2(\mathbb{N}) \to \mathcal{H}$ by $T\{c_k\}_{k=1}^\infty := \sum c_k f_k$; the operator T is called the *synthesis operator* or *preframe operator*. It is easy to see that the adjoint operator is given by $T^* : \mathcal{H} \to \ell^2(\mathbb{N}), T^*f = \{\langle f, f_k\rangle\}_{k=1}^\infty$. Denoting the synthesis operators for two Bessel sequences $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ by T (resp., U), it is clear that $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are dual frames if and only if

$$TU^* = I. \tag{1.3}$$

(For more information on frames, we refer the reader to [1], [6], and [7].) In case $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are not precisely a pair of dual frames, it is important to be able to measure the deviation from perfect reconstruction; for example, it would be preferable to obtain an estimate of the form

$$\Big\| f - \sum_{k=1}^\infty \langle f, g_k\rangle f_k \Big\| \le \epsilon \|f\|, \quad \forall f \in \mathcal{H} \tag{1.4}$$

for a small value of $\epsilon > 0$. For the case where $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are Bessel sequences, this idea has been formalized in [3] by saying that $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ form *approximately dual frames* if (1.4) holds for some $\epsilon < 1$. Further applications appear in [9] by Feichtinger, Grybos, and Onchis, and [10] by Feichtinger, Onchis, and Wiesmeyr; the latter work contains a discussion of wavelet frames and approximation on subspaces, but not with the exact concept of approximately dual frames as defined in [3]. The results in the current article will provide various ways of estimating ϵ for the important class of generalized shift-invariant systems, to be introduced next.

2. Preliminaries on GSI-systems

Generalized shift-invariant systems (GSI-systems) were introduced by Hernández, Labate, and Weiss [13] and Ron and Shen [20] as a general framework for considering Gabor systems, shift-invariant systems, and wavelet systems simultaneously. We return to a more detailed description of these systems in Section 3. Considering the translation operator

$$T_y : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad T_y f(x) = f(x-y), \quad x, y \in \mathbb{R},$$

the formal definition of a GSI-system is as follows.

Definition 2.1. A *generalized shift-invariant system* in $L^2(\mathbb{R})$ is a collection of functions of the form $\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}, j\in J}$, where $\{\phi_j\}_{j\in J} \subset L^2(\mathbb{R})$ and $\{c_j\}_{j\in J}$ is a countable collection of positive numbers.

For the purpose of this article, we need to be able to verify the Bessel condition for a GSI-system, and to characterize dual frames with the GSI-structure. In this section, we will provide the necessary background information on this.

The following result from [4, Theorem 3.1] provides a convenient sufficient condition for a GSI-system to be a Bessel sequence; it is a generalization of a result in [17]. It is formulated in terms of the Fourier transform, on $L^1(\mathbb{R})$ defined by $\widehat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}\,dx$, and extended to $L^2(\mathbb{R})$ in the usual way.

Lemma 2.2. *Given a GSI-system $\{T_{c_j k}\phi_j\}_{j\in J, k\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, assume that*

$$B := \sup_{\gamma\in\mathbb{R}} \sum_{j\in J}\sum_{k\in\mathbb{Z}} \frac{1}{c_j}\left|\widehat{\phi_j}(\gamma)\widehat{\phi_j}(\gamma - c_j^{-1}k)\right| < \infty. \tag{2.1}$$

Then $\{T_{c_j k}\phi_j\}_{j\in J, k\in\mathbb{Z}}$ is a Bessel sequence with bound B.

For generalized shift-invariant systems, perfect reconstruction has been characterized in terms of a number of equations. In order to state the duality conditions, we need certain technical conditions. Let

$$\mathcal{D} := \left\{ f \in L^2(\mathbb{R}) \mid \operatorname{supp} \widehat{f} \text{ is compact and } \widehat{f} \in L^\infty(\mathbb{R}) \right\}.$$

It is clear that $\mathcal{D}$ is a dense subspace of $L^2(\mathbb{R})$.

The duality conditions are valid under certain very mild technical conditions, stated next. The *local integrability condition* (LIC) was introduced by Hernández, Labate, and Weiss in [13]; the weaker *α-local integrability condition* (α-LIC) appeared in [14] by Jakobsen and Lemvig.

Definition 2.3. Consider two GSI-systems $\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}, j\in J}$ and $\{T_{c_j k}\widetilde{\phi}_j\}_{k\in\mathbb{Z}, j\in J}$.

(i) If

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\frac{1}{c_j}\int_{\operatorname{supp}\widehat{f}}\left|\widehat{f}(\gamma+c_j^{-1}m)\widehat{\phi}_j(\gamma)\right|^2 d\gamma<\infty \tag{2.2}$$

for all $f\in\mathcal{D}$, we say that $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ satisfies the LIC condition.

(ii) We say that $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$ satisfy the *dual α-LIC condition* if

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\frac{1}{c_j}\int_{-\infty}^{\infty}\left|\widehat{f}(\gamma)\widehat{f}(\gamma+c_j^{-1}m)\widehat{\phi}_j(\gamma)\widehat{\widetilde{\phi}}_j(\gamma+c_j^{-1}m)\right| d\gamma<\infty, \tag{2.3}$$

for all $f\in\mathcal{D}$.

We say that $\{T_{c_jk}\phi_j\}_{j\in\mathbb{Z}}$ satisfies the α-LIC condition if (2.3) holds with $\phi_j=\widetilde{\phi}_j$.

Finally, in order to formulate the duality conditions, we need to consider a certain reindexing of the GSI-systems. Given a GSI-system $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$, let

$$\Lambda:=\{c_j^{-1}n : j\in J, n\in\mathbb{Z}\}, \tag{2.4}$$

and, for $\alpha\in\Lambda$, let

$$J_\alpha:=\{j\in J : \exists n\in\mathbb{Z}\text{ such that }\alpha=c_j^{-1}n\}. \tag{2.5}$$

In [13], Hernández, Labate, and Weiss characterized duality for two GSI-systems satisfying the LIC condition. Jakobsen and Lemvig proved in [14] that the same result holds under the weaker dual α-LIC condition.

Proposition 2.4. *Assume that the GSI-systems $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$ are Bessel sequences and that they satisfy the dual α-LIC condition. Then $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$ are dual frames if and only if*

$$\sum_{j\in J_\alpha}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma+\alpha)=\delta_{\alpha,0},\quad \textit{almost every } \gamma\in\mathbb{R} \tag{2.6}$$

for all $\alpha\in\Lambda$.

Note that (2.6) is equivalent to the equations

$$\begin{cases}\sum_{j\in J}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma)-1=0,\\ \sum_{j\in J_\alpha}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma+\alpha)=0,\quad \alpha\in\Lambda\setminus\{0\}.\end{cases} \tag{2.7}$$

The formulation (2.7) is more convenient for our purpose. In fact, for GSI-systems $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$, we will show that we can estimate the deviation from perfect reconstruction by the deviation from equality in (2.7).

3. Approximately dual GSI-frames

We will now derive various ways of estimating the deviation from perfect reconstruction for a pair of GSI-systems. The first result (to be stated in Theorem 3.3) will measure the deviation from the given systems being dual frames directly in terms of the deviation from equality in the duality conditions (2.7). We begin with a few technical lemmas.

Lemma 3.1. *Assume that* $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ *and* $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$ *satisfy the dual* α*-LIC condition. Then*

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\frac{1}{c_j}\int_{-\infty}^{\infty}\left|\widehat{f}(\gamma)\widehat{g}(\gamma+c_j^{-1}m)\widehat{\phi}_j(\gamma)\widehat{\widetilde{\phi}}_j(\gamma+c_j^{-1}m)\right|d\gamma<\infty \tag{3.1}$$

for all $f,g\in\mathcal{D}$.

Proof. Define the function κ via $\widehat{\kappa}(\gamma)=\max(|\widehat{f}(\gamma)|,|\widehat{g}(\gamma)|)$; then $\kappa\in\mathcal{D}$ and

$$\left|\widehat{f}(\gamma)\widehat{g}(\gamma+c_j^{-1}m)\right|\leq\left|\widehat{\kappa}(\gamma)\widehat{\kappa}(\gamma+c_j^{-1}m)\right|.$$

Thus, the expression (3.1) is finite by the dual α-LIC condition applied on the function κ. □

The following lemma is a variant of a result in [13, Proposition 2.4], which is a key step in the proof of Proposition 2.4. The modifications consist in the use of the dual α-LIC condition instead of the stronger LIC-condition used in [13]; also, in [13] the functions f and g below were identical, while it is essential for our purpose that they are allowed to be different functions.

Lemma 3.2. *Assume that the GSI-systems* $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ *and* $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$ *satisfy the dual* α*-LIC-condition. Then for* $f,g\in\mathcal{D}$*, the function*

$$\omega(y):=\sum_{j\in J}\sum_{k\in\mathbb{Z}}\langle T_yf,T_{c_jk}\phi_j\rangle\langle T_{c_jk}\widetilde{\phi}_j,T_yg\rangle \tag{3.2}$$

is continuous, and

$$\omega(y)=\sum_{\alpha\in\Lambda}\Big(\int_{-\infty}^{\infty}\widehat{f}(\gamma)\overline{\widehat{g}(\gamma+\alpha)}\sum_{j\in J_\alpha}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma+\alpha)\Big)e^{2\pi i\alpha\cdot y} \tag{3.3}$$

pointwise for all $y\in\mathbb{R}$.

Proof. We will refer to [13] for the parts of the proof that are unaffected by the mentioned changes, and focus on the parts where the dual α-LIC condition is used. First, for any $f\in\mathcal{D}$, the arguments in [13] show that $\langle f,T_{c_jk}\phi_j\rangle$ is the $(-k)$th Fourier coefficient of the 1-periodic function

$$F_j(\mu)=\frac{1}{c_j}\sum_{n\in\mathbb{Z}}\widehat{f}\big(c_j^{-1}(\mu+n)\big)\overline{\widehat{\phi}_j\big(c_j^{-1}(\mu+n)\big)}. \tag{3.4}$$

Using Parseval's equation and elementary manipulations on the sums (see [13] or [1]), it follows that for $j\in J$, the function

$$\omega_j(y):=\sum_{k\in\mathbb{Z}}\langle T_yf,T_{c_jk}\phi_j\rangle\langle T_{c_jk}\widetilde{\phi},T_yg\rangle$$

is continuous and equals a trigonometric polynomial,

$$\omega_j(y) = \sum_{m\in\mathbb{Z}} c_{m,j} e^{2\pi i c_j^{-1} m y},$$

where the Fourier coefficients are

$$c_{m,j} = \frac{1}{c_j}\int_{-\infty}^{\infty} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma + c_j^{-1}m)}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma + c_j^{-1}m)\,d\gamma. \tag{3.5}$$

Thus, in order to show that the function ω is continuous, it is enough to show that

$$\sum_{j\in J}\sum_{m\in\mathbb{Z}} |c_{m,j}| < \infty; \tag{3.6}$$

this is an easy application of Lemma 3.1. □

The following result measures the deviation from exact reconstruction in terms of the deviation from equality in the duality conditions in (2.6). It generalizes a result from [3, Theorem 5.2]; we discuss this in more detail in Example 3.5.

Theorem 3.3. *Assume that the GSI-systems* $\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}, j\in J}$ *and* $\{T_{c_j k}\widetilde{\phi}_j\}_{k\in\mathbb{Z}, j\in J}$ *are Bessel sequences and that they satisfy the dual* α*-LIC-condition for all* $f \in \mathcal{D}$*; denote the associated preframe operators by* T *(resp.,* U*). Then*

$$\begin{aligned}\|I - UT^*\| \le {}& \Big\|\sum_{j\in J}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma) - 1\Big\|_\infty \\ &+ \sum_{\alpha\in\Lambda\setminus\{0\}}\Big\|\sum_{j\in J_\alpha}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma+\alpha)\Big\|_\infty.\end{aligned} \tag{3.7}$$

Proof. Note that in terms of the function ω in (3.2), we have $\omega(0) = \langle UT^*f, g\rangle$. Using (3.3) in Lemma 3.2 with $y = 0$, we see that for $f, g \in \mathcal{D}$,

$$\begin{aligned}\big\langle (UT^*f - f), g\big\rangle &= \sum_{\alpha\in\Lambda}\Big(\int_{-\infty}^{\infty}\widehat{f}(\gamma)\overline{\widehat{g}(\gamma+\alpha)}\sum_{j\in J_\alpha}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma+\alpha)\Big) - \langle\widehat{f},\widehat{g}\rangle \\ &= \int_{-\infty}^{\infty}\widehat{f}(\gamma)\overline{\widehat{g}(\gamma)}\Big(\sum_{j\in J}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma) - 1\Big)\,d\gamma \\ &\quad + \sum_{\alpha\in\Lambda\setminus\{0\}}\int_{-\infty}^{\infty}\widehat{f}(\gamma)\overline{\widehat{g}(\gamma+\alpha)}\sum_{j\in J_\alpha}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma+\alpha)\,d\gamma.\end{aligned}$$

It follows that

$$\begin{aligned}&\big|\big\langle (UT^*f - f), g\big\rangle\big| \\ &\quad\le \int_{-\infty}^{\infty}\big|\widehat{f}(\gamma)\overline{\widehat{g}(\gamma)}\big|\Big|\sum_{j\in J}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma) - 1\Big|\,d\gamma \\ &\qquad + \sum_{\alpha\in\Lambda\setminus\{0\}}\int_{-\infty}^{\infty}\big|\widehat{f}(\gamma)\overline{\widehat{g}(\gamma+\alpha)}\big|\Big|\sum_{j\in J_\alpha}\frac{1}{c_j}\overline{\widehat{\phi}_j(\gamma)}\widehat{\widetilde{\phi}}_j(\gamma+\alpha)\Big|\,d\gamma\end{aligned}$$

$$\leq \Big\| \sum_{j \in J} \frac{1}{c_j} \overline{\widehat{\phi}_j(\gamma)} \widehat{\widetilde{\phi}}_j(\gamma) - 1 \Big\|_\infty \|f\|_2 \|g\|_2$$

$$+ \sum_{\alpha \in \Lambda \setminus \{0\}} \Big\| \sum_{j \in J_\alpha} \frac{1}{c_j} \overline{\widehat{\phi}_j(\gamma)} \widehat{\widetilde{\phi}}_j(\gamma + \alpha) \Big\|_\infty \|f\|_2 \|g\|_2.$$

Since $\mathcal{D}$ is dense in $L^2(\mathbb{R})$, it follows that

$$\begin{aligned} \|UT^* f - f\|_2 &= \sup_{\|g\|=1} \big| \big\langle (UT^* f - f), g \big\rangle \big| \\ &\leq \Big(\Big\| \sum_{j \in J} \frac{1}{c_j} \overline{\widehat{\phi}_j(\gamma)} \widehat{\widetilde{\phi}}_j(\gamma) - 1 \Big\|_\infty \\ &\qquad + \sum_{\alpha \in \Lambda \setminus \{0\}} \Big\| \sum_{j \in J_\alpha} \frac{1}{c_j} \overline{\widehat{\phi}_j(\gamma)} \widehat{\widetilde{\phi}}_j(\gamma + \alpha) \Big\|_\infty \Big) \|f\|_2, \end{aligned}$$

as desired. □

We will now derive some consequences of Theorem 3.3. First, let us consider a *shift-invariant system*, that is, a collection of functions $\{T_{ck}\phi_j\}_{k \in \mathbb{Z}, j \in J}$, where $c > 0$ and $\{\phi_j\}_{j \in J}$ is a countable collection of functions in $L^2(\mathbb{R})$; this corresponds to a GSI-system where the parameters c_j are independent of $j \in J$. (The frame analysis of such systems was pioneered by Ron and Shen [19] and Janssen [15].) For a shift-invariant system, the index sets Λ and J_α take the form $\Lambda = c^{-1}\mathbb{Z}$, $J_\alpha = J$. Furthermore, the LIC is automatically satisfied if $\{T_{ck}\phi_j\}_{k \in \mathbb{Z}, j \in J}$ is a Bessel sequence (see [14]). Thus, we obtain the following explicit version of Theorem 3.3.

Corollary 3.4. *Assume that the shift-invariant systems $\{T_{ck}\phi_j\}_{k \in \mathbb{Z}, j \in J}$ and $\{T_{ck}\widetilde{\phi}_j\}_{k \in \mathbb{Z}, j \in J}$ are Bessel sequences, and denote the associated preframe operators by T (resp., U). Then*

$$\|I - UT^*\| \leq \frac{1}{c} \Big(\Big\| \sum_{j \in J} \overline{\widehat{\phi}_j(\gamma)} \widehat{\widetilde{\phi}}_j(\gamma) - c \Big\|_\infty + \sum_{n \in \mathbb{Z} \setminus \{0\}} \Big\| \sum_{j \in J} \overline{\widehat{\phi}_j(\gamma)} \widehat{\widetilde{\phi}}_j(\gamma + n/c) \Big\|_\infty \Big). \tag{3.8}$$

Let us consider a concrete case, namely, the Gabor systems. For $b \in \mathbb{R}$, define the *modulation operator* on $L^2(\mathbb{R})$ by $E_b f(x) = e^{2\pi i b x} f(x)$.

Example 3.5. Given $a, b > 0$, the *Gabor system* generated by a function $g \in L^2(\mathbb{R})$ is given by

$$\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} = \big\{ e^{2\pi i m b x} g(x - na) \big\}_{m,n \in \mathbb{Z}}.$$

Note that $E_{mb}T_{na}g(x) = e^{2\pi i m n a b} T_{na}E_{mb}g(x)$. It follows that two Gabor frames $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_{na}\widetilde{g}\}_{m,n \in \mathbb{Z}}$ are dual frames if and only if the shift-invariant systems $\{T_{na}E_{mb}g\}_{m,n \in \mathbb{Z}}$ and $\{T_{na}E_{mb}\widetilde{g}\}_{m,n \in \mathbb{Z}}$ are dual frames; furthermore, denoting the preframe operators for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_{na}\widetilde{g}\}_{m,n \in \mathbb{Z}}$

by V (resp., W) and the preframe operators for $\{T_{na}E_{mb}g\}_{m,n\in\mathbb{Z}}$ and $\{T_{na}E_{mb}\widetilde{g}\}_{m,n\in\mathbb{Z}}$ by T (resp., U), we have

$$WV^*f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}\widetilde{g} = \sum_{m,n\in\mathbb{Z}} \langle f, T_{na}E_{mb}g\rangle T_{na}E_{mb}\widetilde{g} = UT^*f.$$

With $\widehat{E_{mb}g}(\gamma) = T_{mb}\widehat{g}(\gamma) = \widehat{g}(\gamma - b)$, Corollary 3.4 now yields the estimate

$$\begin{aligned} \|I - WV^*\| \le \frac{1}{a}\Big(&\Big\|\sum_{m\in\mathbb{Z}} \overline{\widehat{g}(\gamma - mb)}\widehat{\widetilde{g}}(\gamma - mb) - a\Big\|_\infty \\ &+ \sum_{n\in\mathbb{Z}\setminus\{0\}} \Big\|\sum_{m\in\mathbb{Z}} \overline{\widehat{g}(\gamma - mb)}\widehat{\widetilde{g}}(\gamma - mb - n/a)\Big\|_\infty\Big). \end{aligned} \tag{3.9}$$

Note that for periodicity reasons, it is enough to take the L^∞-norm in (3.9) over $\gamma \in [0, b[$. Now it is easy to estimate (3.9), for example, by imposing certain decay conditions on the functions $g, \widetilde{g}$. In fact, estimates of terms like those in (3.9) are standard in frame theory (see, e.g., [1], [6], [7]).

In the particular case of Gabor frames, a similar result was obtained in the time domain in [3], but by applying the stronger conditions that the functions $g, \widetilde{g}$ belong to the Wiener space.

The next example shows that Theorem 3.3 is not appropriate for application to wavelet systems. This will motivate the analysis to follow, which will lead to an alternative method of estimating the deviation from perfect reconstruction.

Example 3.6. Let the scaling operator on $L^2(\mathbb{R})$ be given by $Df(x) := 2^{1/2}f(2x)$. The *wavelet system* generated by a function $\psi \in L^2(\mathbb{R})$ is the collection of functions

$$\{D^jT_k\psi\}_{j,k\in\mathbb{Z}} = \big\{2^{j/2}\psi(2^jx - k)\big\}_{j,k\in\mathbb{Z}}.$$

A wavelet system $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ is a GSI-system. In fact,

$$\{D^jT_k\psi\}_{j,k\in\mathbb{Z}} = \{T_{2^{-j}k}D^j\psi\}_{j,k\in\mathbb{Z}} = \{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J},$$

where $c_j = 2^{-j}, \phi_j = D^j\psi, J = \mathbb{Z}$. Note that the set Λ in (2.4) can be written as

$$\Lambda = \{2^jn \mid j, n \in \mathbb{Z}\} = \{2^km \mid k \in \mathbb{Z}, m \text{ odd}\};$$

and, given $\alpha \in \Lambda$ on the form $\alpha = 2^km$ where $k \in \mathbb{Z}$ and m is odd,

$$J_\alpha = \{j \in \mathbb{Z} \mid \exists n \in \mathbb{Z} \text{ such that } 2^km = 2^jn\} = \{\dots, k-1, k\}. \tag{3.10}$$

The duality equations (2.7) take the form

$$\begin{cases} \sum_{j\in\mathbb{Z}} \overline{\widehat{\psi}(2^{-j}\gamma)}\widehat{\widetilde{\psi}}(2^{-j}\gamma) - 1 = 0, \\ \sum_{j=-\infty}^{k} \overline{\widehat{\psi}(2^{-j}\gamma)}\widehat{\widetilde{\psi}}(2^{-j}(\gamma + 2^km)) = 0, \quad k \in \mathbb{Z}, m \text{ odd}. \end{cases} \tag{3.11}$$

For $m, k \in \mathbb{Z}$, consider the function

$$\theta_{m,k}(\gamma) := \sum_{j=-\infty}^{k} \overline{\widehat{\psi}(2^{-j}\gamma)}\widehat{\widetilde{\psi}}\big(2^{-j}(\gamma + 2^km)\big);$$

then

$$\begin{aligned}\theta_{m,k+1}(\gamma) &= \sum_{j=-\infty}^{k+1} \overline{\widehat{\psi}(2^{-j}\gamma)}\widehat{\widetilde{\psi}}\big(2^{-j}(\gamma+2^{k+1}m)\big) \\ &= \sum_{j=-\infty}^{k+1} \overline{\widehat{\psi}(2^{-(j-1)}\gamma/2)}\widehat{\widetilde{\psi}}\big(2^{-(j-1)}(\gamma/2+2^{k}m)\big) \\ &= \sum_{j=-\infty}^{k} \overline{\widehat{\psi}(2^{-j}\gamma/2)}\widehat{\widetilde{\psi}}\big(2^{-j}(\gamma/2+2^{k}m)\big) = \theta_{m,k}(\gamma/2).\end{aligned}$$

In particular, this shows that $\|\theta_{m,k}\|_\infty$ is independent of $k \in \mathbb{Z}$; in other words, the second set of equations in (3.11) holds if and only if the equation holds for $k = 0$; that is, if and only if

$$\sum_{j=-\infty}^{0} \overline{\widehat{\psi}(2^{-j}\gamma)}\widehat{\widetilde{\psi}}\big(2^{-j}(\gamma+m)\big) = 0, \quad \forall m \text{ odd}.$$

If just one of the α-equations in (2.7) does not hold, then $\|\theta_{m,k}\|_\infty \neq 0$ for some m, k; since this term appears infinitely often in the infinite sum on the right-hand side of (3.7), the estimate is not useful. The conclusion is that for wavelet systems, the deviation from equality in the duality equations seldom gives a useful estimate for the deviation from perfect reconstruction.

Motivated by the negative outcome in 3.6, we will now derive an alternative method for estimating the deviation from perfect reconstruction. Here we consider again two GSI-systems $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$, but now we will measure the deviation from perfect reconstruction in terms of how much the two systems deviate from a pair of dual frames $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}\widetilde{g}_j\}_{k\in\mathbb{Z},j\in J}$, measured via the Bessel condition in Lemma 2.2.

Theorem 3.7. *Assume that the GSI-systems* $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ *and* $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$ *are Bessel sequences, with preframe operators* T *and* U*. Furthermore, let* $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ *and* $\{T_{c_jk}\widetilde{g}_j\}_{k\in\mathbb{Z},j\in J}$ *be a pair of dual frames for* $L^2(\mathbb{R})$*, with Bessel bounds* B_g *(resp.,* $B_{\widetilde{g}}$*). Finally, let*

$$B_{g-\phi} := \sup_{\gamma\in\mathbb{R}} \sum_{j\in J}\sum_{k\in\mathbb{Z}} \frac{1}{c_j}\big|(\widehat{g_j}-\widehat{\phi_j})(\gamma)(\widehat{g_j}-\widehat{\phi_j})(\gamma - c_j^{-1}k)\big|$$

and

$$B_{\widetilde{g}-\widetilde{\phi}} := \sup_{\gamma\in\mathbb{R}} \sum_{j\in J}\sum_{k\in\mathbb{Z}} \frac{1}{c_j}\big|(\widehat{\widetilde{g}_j}-\widehat{\widetilde{\phi}_j})(\gamma)(\widehat{\widetilde{g}_j}-\widehat{\widetilde{\phi}_j})(\gamma - c_j^{-1}k)\big|.$$

Then

$$\|I - UT^*\| \leq B_{\widetilde{g}}^{1/2}B_{g-\phi}^{1/2} + B_{\widetilde{g}-\widetilde{\phi}}^{1/2}(B_{g-\phi}^{1/2} + B_g^{1/2}). \tag{3.12}$$

Proof. Denote the preframe operators for $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}\widetilde{g}_j\}_{k\in\mathbb{Z},j\in J}$ by V and W, respectively. Then for $f\in L^2(\mathbb{R})$,

$$\begin{aligned} \|f-UT^*f\| &= \|WV^*f-UT^*f\| \\ &= \big\|W(V^*-T^*)f+(W-U)T^*f\big\| \\ &\le \big(\|W\|\|V-T\|+\|W-U\|\|T\|\big)\|f\|. \end{aligned} \tag{3.13}$$

We now estimate the terms in (3.13). Clearly $\|W\|\le B_{\widetilde{g}}^{1/2}$. Furthermore, $V-T$ is the preframe operator for the GTI-system $\{T_{c_jk}(g_j-\phi_j)\}_{k\in\mathbb{Z},j\in J}$; thus Lemma 2.2 implies that $\|V-T\|\le B_{g-\phi}^{1/2}$. Similarly, $\|W-U\|\le B_{\widetilde{g}-\widetilde{\phi}}^{1/2}$. Using the fact that $\|T\|\le\|T-V\|+\|V\|\le B_{g-\phi}^{1/2}+B_g^{1/2}$, we finally arrive at the estimate (3.12). □

Theorem 3.7 has recently been used to construct approximately dual frames of Gabor frames generated by the Gaussian (see [2] for details). A further consequence of the analysis in [2] is that certain scalings of the B-splines B_N converge towards the Gaussian whenever $N\to\infty$, in the sense that the Bessel bound for any Gabor system generated by the difference between the scaled B-splines and the Gaussian tends to zero. In particular, the result implies that, for any choice of translation parameter $a>0$ and modulation parameter $b>0$ such that $ab<1$, the Gabor system generated by the scaled B-splines generates frames whenever the order of the B-spline is sufficiently high. This result is rather surprising in view of the many known obstructions to the frame property for B-splines (see [8], [11], [12], [16], [18]). We also note that the arguments used in the proof are of a general nature, which allows for a similar formulation for general frames in Hilbert spaces (see [5]).

The next result is a consequence of Theorem 3.7 and its proof. Actually, the result highlights the key idea behind all the results in the article. Indeed, we will consider two dual GSI frames $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}\widetilde{g}_j\}_{k\in\mathbb{Z},j\in J}$ and estimate the deviation from perfect reconstruction that occurs when the windows—typically due to practical constraints—are "truncated." Due to the nature of the GSI-conditions, the truncation will take place in the Fourier domain.

Corollary 3.8. *Assume that* $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ *and* $\{T_{c_jk}\widetilde{g}_j\}_{k\in\mathbb{Z},j\in J}$ *are dual frames and that*

$$B_g:=\sup_{\gamma\in\mathbb{R}}\sum_{j\in J}\sum_{k\in\mathbb{Z}}\frac{1}{c_j}\big|\widehat{g}_j(\gamma)\widehat{g}_j(\gamma-c_j^{-1}k)\big|<\infty$$

and

$$B_{\widetilde{g}}:=\sup_{\gamma\in\mathbb{R}}\sum_{j\in J}\sum_{k\in\mathbb{Z}}\frac{1}{c_j}\big|\widehat{\widetilde{g}}_j(\gamma)\widehat{\widetilde{g}}_j(\gamma-c_j^{-1}k)\big|<\infty.$$

Given two collections of compact sets $\{S_j\}_{j\in J},\{\widetilde{S}_j\}_{j\in J}\subset\mathbb{R}$, *define the functions* ϕ_j *and* $\widetilde{\phi}_j$ *by*

$$\widehat{\phi}_j:=\widehat{g}_j\chi_{S_j},\qquad \widehat{\widetilde{\phi}}_j:=\widehat{\widetilde{g}}_j\chi_{\widetilde{S}_j},\quad j\in J. \tag{3.14}$$

Finally, denote the preframe operators for the GSI systems $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ *and* $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$ *by* T *(resp.,* U*). Then*

$$\|I-UT^*\| \leq B_{\widetilde{g}}^{1/2}\Big(\sup_{\gamma\in\mathbb{R}}\sum_{j\in J}\sum_{k\in\mathbb{Z}}\frac{1}{c_j}\big|(\widehat{g}_j\chi_{\mathbb{R}\setminus S_j})(\gamma)(\widehat{g}_j\chi_{\mathbb{R}\setminus S_j})(\gamma-c_j^{-1}k)\big|\Big)^{1/2}$$
$$+B_g^{1/2}\Big(\sup_{\gamma\in\mathbb{R}}\sum_{j\in J}\sum_{k\in\mathbb{Z}}\frac{1}{c_j}\big|(\widehat{\widetilde{g}}_j\chi_{\mathbb{R}\setminus\widetilde{S_j}})(\gamma)(\widehat{\widetilde{g}}_j\chi_{\mathbb{R}\setminus\widetilde{S_j}})(\gamma-c_j^{-1}k)\big|\Big)^{1/2}.$$

Proof. The definition of the functions ϕ_j immediately shows that

$$\sum_{j\in J}\sum_{k\in\mathbb{Z}}\frac{1}{c_j}\big|\widehat{\phi}_j(\gamma)\widehat{\phi}_j(\gamma-c_j^{-1}k)\big| \leq \sum_{j\in J}\sum_{k\in\mathbb{Z}}\frac{1}{c_j}\big|\widehat{g}_j(\gamma)\widehat{g}_j(\gamma-c_j^{-1}k)\big| \leq B_g \quad (3.15)$$

for almost every $\gamma\in\mathbb{R}$; thus $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$ is a Bessel sequence. By the symmetry in the conditions, $\{T_{c_jk}\widetilde{\phi}_j\}_{k\in\mathbb{Z},j\in J}$ is also a Bessel sequence.

In order to obtain the desired estimate on $\|I-UT^*\|$, we now refer to (3.13). As before, $\|W\|\leq B_{\widetilde{g}}^{1/2}$. Furthermore, by (3.15), B_g is a Bessel bound for $\{T_{c_jk}\phi_j\}_{k\in\mathbb{Z},j\in J}$, so $\|T\|\leq B_g$. The rest follows from the estimates on $\|V-T\|$ and $\|W-U\|$ in the proof of Theorem 3.7. □

It is clear from the proof of Corollary 3.8 that the same idea can be used "the opposite way around." That is, if we know that certain GSI-systems generated by compactly supported truncated versions $g_j,\widetilde{g_j}$ of some functions $\varphi_j,\widetilde{\varphi_j}$ generate dual frames or approximately dual frames, then Bessel estimates will yield information about how far the GSI-systems generated by $\varphi_j,\widetilde{\varphi_j}$ are from yielding perfect reconstruction. We leave the derivations of concrete statements to the interested reader.

In the concrete case of wavelet systems, we will now derive a completely explicit version of Corollary 3.8. All terms will be formulated via Bessel conditions that can be estimated by standard techniques in frame theory (see [1], [6], [7]). We will see that it is important that the "cut-of" determined by the sets S_j in Corollary 3.8 is allowed to depend on j.

Example 3.9. Consider a dual pair of frames $\{D^jT_kg\}_{j,k\in\mathbb{Z}},\{D^jT_k\widetilde{g}\}_{j,k\in\mathbb{Z}}$. As we have seen in Example 3.6, they correspond to GSI-systems $\{T_{c_jk}g_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_{c_jk}\widetilde{g_j}\}_{k\in\mathbb{Z},j\in J}$ with $c_j=2^{-j}$, $g_j=D^jg$, $\widetilde{g_j}=D^j\widetilde{g}$, and $J=\mathbb{Z}$. Now,

$$\widehat{g_j}(\gamma)=D^{-j}\widehat{g}(\gamma).$$

Thus, it is natural to consider the functions ϕ_j defined by

$$\widehat{\phi_j}(\gamma)=D^{-j}(\widehat{g}\chi_{[-N,N]})(\gamma)=\widehat{g_j}(\gamma)\chi_{[-2^jN,2^jN]})(\gamma)$$

for some $N\in\mathbb{N}$; this corresponds exactly to (3.14) with $S_j=[-2^jN,2^jN]$. Alternatively, denoting the Fourier transform by $\mathcal{F}$ and using the convolution, to be denoted by $*$, we have

$$\phi_j=\mathcal{F}^{-1}D^{-j}(\widehat{g}\chi_{[-N,N]})=D^j\mathcal{F}^{-1}(\widehat{g}\chi_{[-N,N]})=D^j(g*\mathcal{F}^{-1}\chi_{[-N,N]});$$

thus, the GSI-system $\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z},j\in J}$ in fact equals the wavelet system $\{D^j T_k \phi\}_{j,k\in\mathbb{Z}}$ with $\phi = g * \mathcal{F}^{-1}\chi_{[-N,N]}$.

Clearly, we also define the function $\widetilde{\phi}$ by $\widetilde{\phi} = \widetilde{g} * \mathcal{F}^{-1}\chi_{[-N,N]}$. Now, letting T and U denote the preframe operators for the wavelet systems $\{D^j T_k \phi\}_{j,k\in\mathbb{Z}}$ and $\{D^j T_k \widetilde{\phi}\}_{j,k\in\mathbb{Z}}$, the estimate in Example 3.6 takes the form

$$\begin{aligned}
&\|I - UT^*\| \\
&\quad \le B_{\widetilde{g}}^{1/2}\Big(\sup_{\gamma\in[1,2]}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\big|(\widehat{g}\chi_{\mathbb{R}\setminus[-N,N]})(2^{-j}\gamma)(\widehat{g}\chi_{\mathbb{R}\setminus[-N,N]})(2^{-j}\gamma - k)\big|\Big)^{1/2} \\
&\qquad + B_{g}^{1/2}\Big(\sup_{\gamma\in[1,2]}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\big|(\widehat{\widetilde{g}}\chi_{\mathbb{R}\setminus[-N,N]})(2^{-j}\gamma)(\widehat{\widetilde{g}}\chi_{\mathbb{R}\setminus[-N,N]})(2^{-j}\gamma - k)\big|\Big)^{1/2}.
\end{aligned}$$

The terms appearing in parentheses can be estimated exactly as in the standard calculations for the Bessel bound of a wavelet system (see [1], [6], [7]).

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