Ann. Funct. Anal. 8 (2017), no. 1, 124–132
http://dx.doi.org/10.1215/20088752-3764566
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

A GRÜSS TYPE OPERATOR INEQUALITY

T. BOTTAZZI[1] and C. CONDE[1,2*]

Communicated by T. Yamazaki

Abstract. In 2001, Renaud obtained a Grüss type operator inequality involving the usual trace functional. In this article, we give a refinement of that result, and we answer positively Renaud's open problem.

1. Introduction

In 1935, Grüss [6] obtained the following inequality: if f, g are integrable real functions on $[a, b]$ and there exist real constants $\alpha, \beta, \gamma, \delta$ such that $\alpha \leq f(x) \leq \beta, \gamma \leq g(x) \leq \delta$ for all $x \in [a, b]$, then

$$\Big|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx\Big| \leq \frac{1}{4}(\beta-\alpha)(\delta-\gamma),$$

and the inequality is sharp in the sense that the constant $1/4$ cannot be replaced by a smaller constant. This inequality has been investigated, applied, and generalized by many mathematicians, including Banić, Bourin, Matharu, Moslehian, Ilišević, Renaud, and Varošanec, among others, in different areas of mathematics (see [8] and the references within).

In this work, $\mathcal{H}$ denotes a (complex, separable) Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $(\mathbb{B}(\mathcal{H}), \|\cdot\|)$ be the C^*-algebra of all bounded linear operators acting on $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with the uniform norm. We denote by Id the identity operator, and for any $A \in \mathbb{B}(\mathcal{H})$, we consider A^* its adjoint and $|A| = (A^*A)^{1/2}$ the absolute

Received May 5, 2016; Accepted Jul. 25, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 39B05; Secondary 39B42, 47B10, 47A12, 47A30.
Keywords. Grüss inequality, variance, trace inequality, distance formula.

value of A. For $A \in \mathbb{B}(\mathcal{H})$, we use $R(A), N(A)$, respectively, to denote the range and kernel of A.

By $\mathbb{B}(\mathcal{H})^+$, we denote the cone of positive operators of $\mathbb{B}(\mathcal{H})$; that is, $\mathbb{B}(\mathcal{H})^+ := \{T \in \mathbb{B}(\mathcal{H}) : \langle Th, h\rangle \geq 0 \ \forall h \in \mathcal{H}\}$. In the case when $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. For each $T \in \mathbb{B}(\mathcal{H})$, we denote its spectrum by $\sigma(T)$; that is, $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \operatorname{Id} \text{ is not invertible}\}$ and a complex number $\lambda \in \mathbb{C}$ is said to be in the approximate point spectrum of the operator T, and we denote by $\sigma_{\mathrm{ap}}(T)$ if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T - \lambda)x_n \to 0$.

For each operator T, we consider

$$r(T) = \sup\bigl\{|\lambda| : \lambda \in \sigma(T)\bigr\} \text{ spectral radius of } T,$$
$$W(T) = \bigl\{\langle Th, h\rangle : \|h\| = 1\bigr\} \text{ numerical range of } T,$$

and

$$w(T) = \sup\bigl\{|\lambda| : \lambda \in W(T)\bigr\} \text{ numerical radius of } T.$$

Recall, for all $T \in \mathbb{B}(\mathcal{H})$, that $r(T) \leq w(T) \leq \|T\| \leq 2w(T)$, $\sigma(T) \subseteq \overline{W(T)}$, and by the Toeplitz–Hausdorff theorem, $W(T)$ is convex.

Renaud [11] gave a bounded linear operator analogue of the Grüss inequality by replacing integrable functions by operators and the integration by a trace function as follows. Let $A, T \in \mathbb{B}(\mathcal{H})$, and suppose that $W(A)$ and $W(T)$ are contained in disks of radii R_A and R_T, respectively. Then, for any positive trace class operator, P with $\operatorname{tr}(P) = 1$ holds

$$\bigl|\operatorname{tr}(PAT) - \operatorname{tr}(PA)\operatorname{tr}(PT)\bigr| \leq 4R_AR_T, \tag{1.1}$$

and if A and T are normal (i.e., $T^*T = TT^*$), the constant 4 can be replaced by 1. We can easily see that, if $A = \alpha \operatorname{Id}$ or $T = \beta \operatorname{Id}$ with $\alpha, \beta \in \mathbb{C}$, then the left-hand side is equal to zero. In the same article, Renaud proposed the following open problem: to characterize $k(A, T)$ such that

$$\bigl|\operatorname{tr}(PAT) - \operatorname{tr}(PA)\operatorname{tr}(PT)\bigr| \leq k(A, T)R_AR_T \tag{1.2}$$

with $1 \leq k(A, T) \leq 4$, in particular, whether it depends on A and T separately, (i.e., whether we can write $k(A, T) = h(A)h(T)$), where $h(A), h(T)$ are suitably defined constants.

In this paper we give a positive answer to the open problem proposed by Renaud, and we obtain an explicit formula for $k(A, T) = h(A)h(T)$. Also, we generalize the inequality (1.1) for normal to transloid operators.

2. Preliminaries

Let us begin with the notation and necessary definitions. The set of compact operators in $\mathcal{H}$ is denoted by $B_0(\mathcal{H})$. If $T \in B_0(\mathcal{H})$, then we denote by $\{s_n(T)\}$ the sequence of singular values of T, that is, the eigenvalues of $|T|$ (decreasingly ordered). The notion of unitary invariant norms can be defined also for operators on Hilbert spaces, a norm $|\!|\!| \cdot |\!|\!|$ that satisfies the invariance property $|\!|\!| UXV |\!|\!| = |\!|\!| X |\!|\!|$ for a pair of unitary operators U, V. Recall that each unitarily invariant norm is defined on a natural subclass $\mathcal{J}_{|\!|\!|\cdot|\!|\!|}$ of $B_0(\mathcal{H})$ called the *norm ideal associated* *with the norm* $|||\cdot|||$. There is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on norm ideals of operators. More precisely, if $|||\cdot|||$ is a unitarily invariant norm, then there is a unique symmetric gauge function g such that $|||T||| = g(\{s_n(T)\})$ for any $T \in \mathcal{J}_{|||\cdot|||}$. If $\dim R(T) = 1$, then $|||T||| = s_1(T)g(e_1) = g(e_1)\|T\|$. By convention, we assume that $g(e_1) = 1$. If $x, y \in \mathcal{H}$, then we denote $x \otimes y$ the rank 1 operator defined on $\mathcal{H}$ by $(x \otimes y)(z) = \langle z, y\rangle x$. Then $\|x \otimes y\| = \|x\|\|y\| = |||x \otimes y|||$.

The best-known examples of unitary invariant norms are called *Schatten p-norms*. For $1 \leq p < \infty$, let

$$\|T\|_p^p = \sum_n s_n(T)^p = \operatorname{tr}|T|^p,$$

and let

$$B_p(\mathcal{H}) = \{T \in \mathcal{H} : \|T\|_p < \infty\},$$

called the *p-Schatten class* of $\mathbb{B}(\mathcal{H})$. This is the subset of compact operators with singular values in l_p. The positive operators with trace 1 are called *density operators* (or states), and we denote this set by $\mathcal{S}(\mathcal{H})$. The ideal $B_2(\mathcal{H})$ is called the *Hilbert–Schmidt class*, and it is a Hilbert space with the inner product $\langle S, T\rangle_2 = \operatorname{tr}(ST^*)$. (For a reference on the theory of norm ideals and their associated unitarily invariant norms, see [5].)

An operator $A \in \mathbb{B}(\mathcal{H})$ is called *normaloid* if $r(A) = \|A\| = \omega(A)$. If $A - \mu \operatorname{Id}$ is normaloid for all $\mu \in \mathbb{C}$, then A is called *transloid*.

Finally, for $A, T \in \mathbb{B}(\mathcal{H})$ and $P \in \mathcal{S}(\mathcal{H})$, we introduce the following notation:

$$V_P(A, T) = \operatorname{tr}(PAT) - \operatorname{tr}(PA)\operatorname{tr}(PT).$$

In the particular case $T = A^*$, we get the variance of A with respect to P. More precisely, Audenaert in [1] considered the following notion: given $A, P \in \mathcal{M}_n$, $P \geq 0, \operatorname{tr}(P) = 1$, the variance of A with respect to the matrix P is

$$V_P(A) = \operatorname{tr}\big(|A|^2 P\big) - \big|\operatorname{tr}(AP)\big|^2 = V_P(A, A^*).$$

Note that $V_P(A - \lambda \operatorname{Id}) = V_P(A)$. Furthermore, he showed that, if $A \in \mathcal{M}_n$, then

$$\max\big\{\operatorname{tr}\big(|A|^2 P\big) - \big|\operatorname{tr}(AP)\big|^2 : P \in \mathcal{M}_n^+, \operatorname{tr}(P) = 1\big\} = \operatorname{dist}(A, \mathbb{C}\operatorname{Id})^2, \tag{2.1}$$

and the maximization over P on the left-hand side can be restricted to density matrices of rank 1.

3. Distance formulas and Renaud's inequality

Let A and T be linear bounded operators acting on $\mathcal{H}$; the vector-function $A - \lambda T$ is known as the *pencil* generated by A and T. Evidently, there is at least one complex number λ_0 such that

$$\|A - \lambda_0 T\| = \inf_{\lambda \in \mathbb{C}} \|A - \lambda T\|.$$

The number λ_0 is unique if $0 \notin \sigma_{\mathrm{ap}}(T)$ (or, equivalently, if $\inf\{\|Tx\| : \|x\| = 1\} > 0$). Different authors, following [12], called this unique number the *center of mass* of A with respect to T, and we denote it by $c(A,T)$. When $T=\mathrm{Id}$, we write $c(A)$. Following Paul, for $A,T\in\mathbb{B}(\mathcal{H})$ such that $0\notin\sigma_{\mathrm{ap}}(T)$, we consider

$$M_T(A)=\sup_{\|x\|=1}\Big[\|Ax\|^2-\frac{|\langle Ax,Tx\rangle|^2}{\langle Tx,Tx\rangle}\Big]^{1/2}=\sup_{\|x\|=1}\Big\|Ax-\frac{\langle Ax,Tx\rangle}{\langle Tx,Tx\rangle}Tx\Big\|. \qquad (3.1)$$

In [9], Paul proved that $M_T(A)=\mathrm{dist}(A,\mathbb{C}T)$. The unique minimizer is characterized by the following conditions: there exists a sequence of unit vectors $\{x_n\}$ such that

$$\big\|(A-\lambda_0T)x_n\big\|\to\|A-\lambda_0T\|\qquad\text{and}\qquad\big\langle(A-\lambda_0T)x_n,x_n\big\rangle\to 0.$$

In [4], Gevorgyan proved that

$$c(A,T)=\lim_{n\to\infty}\frac{\langle Ay_n,Ty_n\rangle}{\langle Ty_n,Ty_n\rangle}, \qquad (3.2)$$

where $\{y_n\}$ is a sequence of unit vectors which approximate the supremum in (3.1). In the particular case that $T=\mathrm{Id}$ and A is a Hermitian operator, then it is easy to see that

$$\min_{\lambda\in\mathbb{C}}\|A-\lambda\,\mathrm{Id}\,\|=\frac{\lambda_{\max}(A)-\lambda_{\min}(A)}{2}, \qquad (3.3)$$

where $\lambda_{\max}(A)$ (resp., $\lambda_{\max}(A)$) denotes the maximum (resp., minimum) eigenvalue of A. Observe that the minimum is

$$c(A)=\frac{\lambda_{\max}(A)+\lambda_{\min}(A)}{2}.$$

We recall other formulas that express the distance from A to the one-dimensional subspace $\mathbb{C}T$. Then

$$\mathrm{dist}(A,\mathbb{C}T)=\sup\big\{\big|\langle Ax,y\rangle\big|:\|x\|=\|y\|=1,\langle Tx,y\rangle=0\big\} \qquad (3.4)$$

if $A,T\in\mathbb{B}(\mathcal{H})$ and $0\notin\sigma_{\mathrm{ap}}(T)$. In the particular case where $T=\mathrm{Id}$, we get

$$\begin{aligned}\mathrm{dist}(A,\mathbb{C}\,\mathrm{Id})&=\frac{1}{2}\sup\big\{\|AX-XA\|:X\in\mathbb{B}(\mathcal{H}),\|X\|=1\big\}\\&=\sup\big\{\big\|(\mathrm{Id}-Q)AQ\big\|:Q\text{ is a rank one projection}\big\}\\&=\sup\big\{\big\|(\mathrm{Id}-Q)AQ\big\|:Q\text{ is a projection}\big\}.\end{aligned} \qquad (3.5)$$

In the following statement we present a new proof of the relation between the variance of A with respect to P and the distance from A to the unidimensional subspace $\mathbb{C}\,\mathrm{Id}$.

Proposition 3.1. *Let $A\in\mathbb{B}(\mathcal{H})$, and let $P\in\mathcal{S}(\mathcal{H})$. Then*

$$\begin{aligned}\mathrm{tr}\big(|A|^2P\big)-\big|\mathrm{tr}(AP)\big|^2&=\|AP^{1/2}\|_2^2-\big|\langle AP^{1/2},P^{1/2}\rangle_2\big|^2\\&=\big\|AP^{1/2}-\langle AP^{1/2},P^{1/2}\rangle_2P^{1/2}\big\|_2^2\\&=\min_{\lambda\in\mathbb{C}}\|AP^{1/2}-\lambda P^{1/2}\|_2^2\le\min_{\lambda\in\mathbb{C}}\|A-\lambda\,\mathrm{Id}\,\|.\end{aligned}$$

Proof. These inequalities are simple consequences from the following general statement for any Hilbert space $\mathcal{H}$: let $x, y \in \mathcal{H}$ with $y \neq 0$; then

$$\inf_{\lambda\in\mathbb{C}} \|x-\lambda y\|^2 = \frac{\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2}{\|y\|^2}. \qquad \square$$

The following statement is an extension of Audenaert's formula to infinite dimension.

Remark 3.2. We show that the equality (2.1) holds in the infinite-dimensional context; that is, for $A \in \mathbb{B}(\mathcal{H})$, we have

$$\sup\bigl\{\bigl[\operatorname{tr}\bigl(|A|^2P\bigr) - \bigl|\operatorname{tr}(AP)\bigr|^2\bigr]^{1/2} : P \in \mathcal{S}(\mathcal{H})\bigr\} = \operatorname{dist}(A, \mathbb{C}\,\mathrm{Id}). \tag{3.6}$$

First, we obtain this equality from Prasanna's result in [10]. Indeed, note that

$$\begin{aligned}\operatorname{dist}(A,\mathbb{C}\,\mathrm{Id})^2 &= \sup_{\|x\|=1} \|Ax\|^2 - \bigl|\langle Ax,x\rangle\bigr|^2 \\ &\le \sup\bigl\{\operatorname{tr}\bigl(|A|^2P\bigr) - \bigl|\operatorname{tr}(AP)\bigr|^2 : P\in\mathcal{S}(\mathcal{H})\bigr\} \\ &\le \operatorname{dist}(A,\mathbb{C}\,\mathrm{Id})^2.\end{aligned}$$

On the other hand, another way to prove (3.6) is to reduce the problem to a finite dimension and use the classical Audenaert formula. Now we give the idea of this proof.

For the sake of clarity, we denote

$$m := \min_{\lambda\in\mathbb{C}} \|A-\lambda\,\mathrm{Id}\,\|$$

and

$$M := \sup\bigl\{\bigl[\operatorname{tr}\bigl(|A|^2P\bigr) - \bigl|\operatorname{tr}(AP)\bigr|^2\bigr]^{1/2} : P\in\mathcal{S}(\mathcal{H})\bigr\}.$$

By Proposition 3.1, we have that $M \le m$. Suppose by contradiction that $M < m$. Then there exists $\epsilon > 0$ such that

$$M < \|A-\lambda\,\mathrm{Id}\,\| - \epsilon \tag{3.7}$$

for any $\lambda \in \mathbb{C}$. By the equality (3.2), we have that $c(A) \in \overline{W(A)}$, and then $|c(A)| \le w(A)$. As any closed ball in the complex plane is a compact set, we can find $\lambda_1, \dots, \lambda_m \in \mathcal{H}$ such that

$$B\bigl(0,\omega(A)\bigr) \subseteq \bigcup_{j=1}^{m} \Bigl\{\lambda\in\mathbb{C} : |\lambda-\lambda_j| < \frac{\epsilon}{2}\Bigr\}.$$

Now, we choose unit vectors $h_1, \dots, h_m \in \mathcal{H}$ with the following property: $\|(A-\lambda_j\,\mathrm{Id})h_j\| > \|A-\lambda_j\,\mathrm{Id}\,\| - \frac{\epsilon}{2}$. Let $\mathcal{H}' = \operatorname{span}\{h_1,\dots,h_m, Ah_1,\dots,Ah_m\}$ and $n = \dim\mathcal{H}'$. Applying (2.1) to the compressions of A and Id, respectively, we get

$$\begin{aligned}\operatorname{dist}(A',\mathbb{C}\,\mathrm{Id}_n) &= \max\bigl\{\bigl[\operatorname{tr}\bigl(|A'|^2P'\bigr) - \bigl|\operatorname{tr}(A'P')\bigr|^2\bigr]^{1/2} : P'\in\mathcal{M}_n^+, \operatorname{tr}(P')=1\bigr\} \\ &= M'.\end{aligned} \tag{3.8}$$

One easily verifies that, if $\lambda \in B(0, \omega(A))$, then there exists $j \in \{1, \dots, m\}$ such that

$$\begin{aligned}\|A' - \lambda \operatorname{Id}_n\| &> \|A' - \lambda_j \operatorname{Id}_n\| - \frac{\epsilon}{2} \geq \big\|(A' - \lambda_j \operatorname{Id}_n)h_j\big\| - \frac{\epsilon}{2} \\ &= \big\|(A - \lambda_j \operatorname{Id})h_j\big\| - \frac{\epsilon}{2} > \|A - \lambda_j \operatorname{Id}\| - \epsilon. \end{aligned} \tag{3.9}$$

Thus, combining (3.7) and (3.9), we get

$$\min_{\lambda \in \mathbb{C}} \|A' - \lambda \operatorname{Id}_n\| > M \geq M', \tag{3.10}$$

and we have here a contradiction with (3.8), and therefore $m = M$.

The following two results give upper bounds for $V_P(A, T)$.

Lemma 3.3. *Let $A, T \in \mathbb{B}(\mathcal{H})$, and let $P \in \mathcal{S}(\mathcal{H})$. Then, for any $\lambda, \mu \in \mathbb{C}$ holds*

$$\big|V_P(A, T)\big| \leq \|A - \lambda \operatorname{Id}\|\|T - \mu \operatorname{Id}\| - \big|\operatorname{tr}\big(P(A - \lambda \operatorname{Id})\big)\operatorname{tr}\big(P(T - \mu \operatorname{Id})\big)\big|.$$

Proof. Define the following semi-inner product for $X, Y \in \mathbb{B}(\mathcal{H})$ and $P \in \mathcal{S}(\mathcal{H})$:

$$(X, Y)_{2,P} = \langle P^{1/2}X, P^{1/2}Y\rangle_2.$$

Following the proof given by Dragomir in [3, Theorem 2], we obtain for any $E \in \mathbb{B}(\mathcal{H})$ such that $(E, E)_{2,P} = 1$:

$$\begin{aligned}\big|(X, Y)_{2,P} - (X, E)_{2,P}(E, Y)_{2,P}\big| &\leq (X, X)_{2,P}^{1/2}(Y, Y)_{2,P}^{1/2} - \big|(X, E)_{2,P}(E, Y)_{2,P}\big|. \\ &= (X, X)_{2,P}^{1/2}(Y, Y)_{2,P}^{1/2} - G_E(X, Y).\end{aligned}$$

Since $(\operatorname{Id}, \operatorname{Id})_{2,P} = 1$, we have

$$\begin{aligned}&\big|V_P(A, T)\big| \\ &\quad = \big|V_P(A - \lambda \operatorname{Id}, T - \mu \operatorname{Id})\big| \\ &\quad = \big|\big(A - \lambda \operatorname{Id}, (T - \mu \operatorname{Id})^*\big)_{2,P} - (A - \lambda \operatorname{Id}, \operatorname{Id})_{2,P}\big(\operatorname{Id}, (T - \mu \operatorname{Id})^*\big)_{2,P}\big| \\ &\quad \leq (A - \lambda \operatorname{Id}, A - \lambda \operatorname{Id})_{2,P}^{1/2}(T^* - \bar{\mu} \operatorname{Id}, T^* - \bar{\mu} \operatorname{Id})_{2,P}^{1/2} - G_{\operatorname{Id}}(A - \lambda \operatorname{Id}, T^* - \bar{\mu} \operatorname{Id}) \\ &\quad = \operatorname{tr}\big(P\big|(A - \lambda \operatorname{Id})^*\big|^2\big)^{1/2}\operatorname{tr}\big(P|T - \mu \operatorname{Id}|^2\big)^{1/2} - G_{\operatorname{Id}}(A - \lambda \operatorname{Id}, T^* - \bar{\mu} \operatorname{Id}) \\ &\quad \leq \big\|\big|(A - \lambda \operatorname{Id})^*\big|^2\big\|^{1/2}\big\||T - \mu \operatorname{Id}|^2\big\|^{1/2} - \big|\operatorname{tr}\big(P(A - \lambda \operatorname{Id})\big)\operatorname{tr}\big(P(T - \mu \operatorname{Id})\big)\big| \\ &\quad = \|A - \lambda \operatorname{Id}\|\|T - \mu \operatorname{Id}\| - \big|\operatorname{tr}\big(P(A - \lambda \operatorname{Id})\big)\operatorname{tr}\big(P(T - \mu \operatorname{Id})\big)\big|. \qquad \square\end{aligned}$$

Proposition 3.4. *Let $A, T \in \mathbb{B}(\mathcal{H})$, and let $P \in \mathcal{S}(\mathcal{H})$. Then*

$$\begin{aligned}\big|V_P(A, T)\big| &\leq \sup_{\widetilde{P} \in \mathcal{S}(\mathcal{H})} \big|\operatorname{tr}(\widetilde{P}AT) - \operatorname{tr}(\widetilde{P}A)\operatorname{tr}(\widetilde{P}T)\big| \\ &\leq \operatorname{dist}(A, \mathbb{C} \operatorname{Id})\operatorname{dist}(T, \mathbb{C} \operatorname{Id}).\end{aligned} \tag{3.11}$$

Proof. By Lemma 3.3, we have

$$\big|V_P(A, T)\big| \leq \|A - \lambda \operatorname{Id}\|\|T - \mu \operatorname{Id}\| - \big|\operatorname{tr}\big(P(A - \lambda \operatorname{Id})\big)\operatorname{tr}\big(P(T - \mu \operatorname{Id})\big)\big|$$

for $A, T \in \mathbb{B}(\mathcal{H})$, $P \in \mathcal{S}(\mathcal{H})$, and any $\lambda, \mu \in \mathbb{C}$. Therefore,

$$\begin{aligned}\left|V_P(A,T)\right| &\leq \sup_{\widetilde{P}\in\mathcal{S}(\mathcal{H})} \left|\operatorname{tr}(\widetilde{P}AT) - \operatorname{tr}(\widetilde{P}A)\operatorname{tr}(\widetilde{P}T)\right| \\ &\leq \operatorname{dist}(A, \mathbb{C}\operatorname{Id})\operatorname{dist}(T, \mathbb{C}\operatorname{Id}). \qquad \square\end{aligned}$$

Remark 3.5. If we define $V_P : \mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H}) \to \mathbb{C}$, $V_P(A,T) := \operatorname{tr}(PAT) - \operatorname{tr}(PA)\operatorname{tr}(PT)$, then V_P is a bilinear function, and by (3.11) a continuous mapping with $\|V_P\| \leq 1$.

Now, we give a new proof and a refinement of (1.1).

Proposition 3.6. *Let $A, T \in \mathbb{B}(\mathcal{H})$, and we suppose that $W(A), W(T)$ are contained in closed disk $D(\lambda_0, R_A), D(\mu_0, R_T)$, respectively. Then, for any $P \in \mathcal{S}(\mathcal{H})$,*

$$\begin{aligned}\left|\operatorname{tr}(PAT) - \operatorname{tr}(PA)\operatorname{tr}(PT)\right| &\leq \sup_{\widetilde{P}\in\mathcal{S}(\mathcal{H})} \left|\operatorname{tr}(\widetilde{P}AT) - \operatorname{tr}(\widetilde{P}A)\operatorname{tr}(\widetilde{P}T)\right| \\ &\leq \operatorname{dist}(A, \mathbb{C}\operatorname{Id})\operatorname{dist}(T, \mathbb{C}\operatorname{Id}) \\ &\leq \|A - \lambda_0 \operatorname{Id}\|\|T - \mu_0 \operatorname{Id}\| \\ &\leq 4w(A - \lambda_0 \operatorname{Id})w(T - \mu_0 \operatorname{Id}) \\ &\leq 4R_A R_T. \qquad (3.12)\end{aligned}$$

In particular, if A and T are normal operators, then we have

$$\begin{aligned}\left|\operatorname{tr}(PAT) - \operatorname{tr}(PA)\operatorname{tr}(PT)\right| &\leq \sup_{\widetilde{P}\in\mathcal{S}(\mathcal{H})} \left|\operatorname{tr}(\widetilde{P}AT) - \operatorname{tr}(\widetilde{P}A)\operatorname{tr}(\widetilde{P}T)\right| \\ &\leq \operatorname{dist}(A, \mathbb{C}\operatorname{Id})\operatorname{dist}(T, \mathbb{C}\operatorname{Id}) = r_A r_T, \qquad (3.13)\end{aligned}$$

where r_S denotes the radius of the unique smallest disc containing $\sigma(S)$ for any $S \in \mathbb{B}(\mathcal{H})$.

Proof. The inequalities are consequences of (3.11). In the last inequality, we use the fact that $W(A - \lambda_0 \operatorname{Id}) \subset D(0, R_A)$ and $W(T - \mu_0 \operatorname{Id}) \subset D(0, R_T)$, respectively. On the other hand, Björck and Thomée [2] have shown that, for a normal operator A,

$$\operatorname{dist}(A, \mathbb{C}\operatorname{Id}) = \sup_{\|x\|=1} \left(\|Ax\|^2 - \left|\langle Ax, x\rangle\right|^2\right)^{1/2} = r_A, \qquad (3.14)$$

and this completes the proof. $\square$

Remark 3.7. From (3.13), if we let A be a positive invertible operator, $T = A^{-1}$ and $P = x \otimes x$ with $x \in \mathcal{H}$ with $\|x\| = 1$, then

$$\begin{aligned}\left|\operatorname{tr}(PAT) - \operatorname{tr}(PA)\operatorname{tr}(PT)\right| &= \left|1 - \langle Ax, x\rangle\langle A^{-1}x, x\rangle\right| \\ &\leq \operatorname{dist}(A, \mathbb{C}\operatorname{Id})\operatorname{dist}(A^{-1}, \mathbb{C}\operatorname{Id}) = r_A r_{A^{-1}};\end{aligned}$$

that is, we obtain the Kantorovich inequality for an operator A acting on an infinite-dimensional Hilbert space $\mathcal{H}$ with $0 < m \leq A \leq M$.

In 1972, Istratescu [7] generalized the equality (3.14) to the transloid class operators. Then we have the following statement.

Proposition 3.8. *Let $A, T \in \mathbb{B}(\mathcal{H})$ with A and T transloid operators. Then*

$$\begin{aligned}\big|\operatorname{tr}(PAT) - \operatorname{tr}(PA)\operatorname{tr}(PT)\big| &\le \sup_{\widetilde{P}\in\mathcal{S}(\mathcal{H})} \big|\operatorname{tr}(\widetilde{P}AT) - \operatorname{tr}(\widetilde{P}A)\operatorname{tr}(\widetilde{P}T)\big| \\ &\le \operatorname{dist}(A, \mathbb{C}\operatorname{Id})\operatorname{dist}(T, \mathbb{C}\operatorname{Id}) = r_A r_T. \qquad (3.15)\end{aligned}$$

Proof. It follows from the same arguments in the proof of inequality (3.13). □

The previous proposition generalizes Renaud's result for normal operators since the classes of transloid and normal operators are related by the inclusion as follows:

$$\text{normal} \subseteq \text{quasinormal} \subseteq \text{subnormal} \subseteq \text{hyponormal} \subseteq \text{transloid},$$

where at least the first inclusion is proper.

In the following statement, we obtain a parametric refinement of (1.1).

Theorem 3.9. *Let $A, T \in \mathbb{B}(\mathcal{H})$ with $A, T \notin \mathbb{C}\operatorname{Id}$, and suppose that $W(A), W(T)$ are contained in the closed disk $D(\lambda_0, R_A)$ and $D(\mu_0, R_T)$, respectively. Thus, for any $P \in \mathcal{S}(\mathcal{H})$, we get*

$$\begin{aligned}\big|\operatorname{tr}(PAT) - \operatorname{tr}(PA)\operatorname{tr}(PT)\big| &\le \sup_{\widetilde{P}\in\mathcal{S}(\mathcal{H})} \big|\operatorname{tr}(\widetilde{P}AT) - \operatorname{tr}(\widetilde{P}A)\operatorname{tr}(\widetilde{P}T)\big| \\ &\le \operatorname{dist}(A, \mathbb{C}\operatorname{Id})\operatorname{dist}(T, \mathbb{C}\operatorname{Id}) \\ &\le h_\lambda(A)h_\mu(T)\omega(A - \lambda_0\operatorname{Id})\omega(T - \mu_0\operatorname{Id}) \\ &\le h_\lambda(A)h_\mu(T)R_A R_T, \qquad (3.16)\end{aligned}$$

where

$$h_\lambda(A) = 2(1-\lambda) + \lambda\frac{\|A - c(A)\operatorname{Id}\|}{w(A - \lambda_0\operatorname{Id})}, \qquad h_\mu(T) = 2(1-\mu) + \mu\frac{\|T - c(T)\operatorname{Id}\|}{w(T - \mu_0\operatorname{Id})},$$

and $1 \le h_\lambda(A)h_\mu(T) \le 4$ for any $\lambda, \mu \in [0, 1]$.

Proof. Let $\lambda \in [0, 1]$. Then

$$\begin{aligned}\big\|A - c(A)\operatorname{Id}\big\| &\le \lambda\big\|A - c(A)\operatorname{Id}\big\| + (1-\lambda)\|A - \lambda_0\operatorname{Id}\| \\ &\le \lambda\big\|A - c(A)\operatorname{Id}\big\| + 2(1-\lambda)w(A - \lambda_0\operatorname{Id}) \\ &= w(A - \lambda_0\operatorname{Id})\Big(2(1-\lambda) + \lambda\frac{\|A - c(A)\operatorname{Id}\|}{w(A - \lambda_0\operatorname{Id})}\Big) \\ &= w(A - \lambda_0\operatorname{Id})h_\lambda(A),\end{aligned}$$

where $1 \le h_\lambda(A) \le 2$ since $\|A - c(A)\operatorname{Id}\| \le \|A - \lambda_0\operatorname{Id}\| \le 2w(A - \lambda_0\operatorname{Id})$. This inequality completes the proof. □

Note that the previous result gives a positive answer to Renaud's open question (1.2).

Corollary 3.10. *Under the same notation as in Theorem 3.9, if $A-\lambda_0\,\mathrm{Id}$ and $T-\mu_0\,\mathrm{Id}$ are normaloid operators, then, for any $\lambda,\mu\in[0,1]$,*

$$\begin{aligned}\left|\operatorname{tr}(PAT)-\operatorname{tr}(PA)\operatorname{tr}(PT)\right| &\le \sup_{\widetilde{P}\in\mathcal{S}(\mathcal{H})}\left|\operatorname{tr}(\widetilde{P}AT)-\operatorname{tr}(\widetilde{P}A)\operatorname{tr}(\widetilde{P}T)\right|\\ &\le \operatorname{dist}(A,\mathbb{C}\,\mathrm{Id})\operatorname{dist}(T,\mathbb{C}\,\mathrm{Id})\\ &\le (2-\lambda)(2-\mu)\omega(A-\lambda_0\,\mathrm{Id})\omega(T-\mu_0\,\mathrm{Id})\\ &\le (2-\lambda)(2-\mu)R_AR_T.\end{aligned}$$

References

1. K. Audenaert, *Variance bounds, with an application to norm bounds for commutators*, Linear Algebra Appl. **432** (2010), no. 5, 1126–1143. Zbl 1194.60020. MR2577614. DOI 10.1016/j.laa.2009.10.022. 126
2. G. Björck and V. Thomée, *A property of bounded normal operators in Hilbert space*, Ark. Mat. **4** (1963), 551–555. Zbl 0194.15103. MR0149308. 130
3. S. Dragomir, *Some refinements of Schwarz inequality*, Suppozionul de Matematică şi Aplica ţii, Polytechnical Institute Timişoara, Romania, **1–2**, (1985), 13–16. Zbl 0594.46018. 129
4. L. Gevorgyan, *On minimal norm of a linear operator pencil*, Dokl. Nats. Akad. Nauk Armen. **110** (2010), no. 2, 97–104. MR2724943. 127
5. I. Gohberg and M. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monogr. **18**, Amer. Math. Soc., Providence, R.I., 1969. Zbl 0181.13504. MR0246142. 126
6. G. Grüss, *Über das Maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx$, Math. Z., **39** (1935), 215–226. Zbl 0010.01602. MR1545499. DOI 10.1007/BF01201355. 124
7. V. Istratescu, *On a class of normaloid operators*, Math. Z. **124** (1972), 199–202. Zbl 0216.16703. MR0291854. 130
8. J. S. Matharu and M. S. Moslehian, *Grüss inequality for some types of positive linear maps*, J. Operator Theory **73** (2015), no. 1, 265-278. Zbl 06465642. MR3322766. DOI 10.7900/jot.2013nov20.2040. 124
9. K. Paul, *Translatable radii of an operator in the direction of another operator*, Sci. Math. **2** (1999), no. 1, 119–122. Zbl 0952.47032. MR1688391. 127
10. S. Prasanna, *The norm of a derivation and the Björck-Thomeé-Istrăţescu theorem*, Math. Japon. **26** (1981), no. 5, 585–588. Zbl 0475.47007. MR0636597. 128
11. P. Renaud, *A matrix formulation of Grüss inequality*, Linear Algebra Appl. **335** (2001), 95–100. Zbl 0982.15025. MR1850816. DOI 10.1016/S0024-3795(01)00278-6. 125
12. J. Stampfli, *The norm of a derivation*, Pacific J. Math. **33** (1970), 737–747. Zbl 0197.10501. MR0265952. 126

[1]Instituto Argentino de Matemática "Alberto P. Calderón", Saavedra 15, 3° piso, (C1083ACA) Buenos Aires, Argentina.

E-mail address: tpbottaz@ungs.edu.ar; cconde@ungs.edu.ar

[2]Instituto de Ciencias, Universidad Nacional de General Sarmiento, J. M. Gutierrez 1150, (B1613GSX) Los Polvorines, Argentina.

E-mail address: cconde@ungs.edu.ar