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AN INEQUALITY FOR EXPECTATION OF MEANS OF POSITIVE RANDOM VARIABLES

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ABSTRACT. Suppose that X, Y are positive random variables and m is a numerical (commutative) mean. We prove that the inequality $\mathrm{E}(m(X,Y)) \le m(\mathrm{E}(X),\mathrm{E}(Y))$ holds if and only if the mean is generated by a concave function. With due changes we also prove that the same inequality holds for all operator means in the Kubo–Ando setting. The case of the harmonic mean was proved by C. R. Rao and B. L. S. Prakasa Rao.

1. Introduction and preliminaries

Let x, y be positive real numbers. The arithmetic, geometric, harmonic, and logarithmic means are defined by

$$m_a(x,y) = \frac{x+y}{2}, \qquad m_g(x,y) = \sqrt{xy},$$
$$m_h(x,y) = \frac{2}{x^{-1}+y^{-1}}, \qquad m_l(x,y) = \frac{x-y}{\log x - \log y}.$$

Suppose $X, Y : \Omega \to (0,+\infty)$ are positive random variables. Linearity of the expectation operator trivially implies

$$\mathrm{E}\big(m_a(X,Y)\big) = m_a\big(\mathrm{E}(X),\mathrm{E}(Y)\big).$$

On the other hand, the Cauchy–Schwarz inequality implies

$$\mathrm{E}\big(m_g(X,Y)\big) \le m_g\big(\mathrm{E}(X),\mathrm{E}(Y)\big).$$

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Working on a result by Fisher on ancillary statistics, Rao [11], [12] obtained the following proposition by an application of Hölder's inequality together with the harmonic-geometric mean inequality.

Proposition 1.1. *We have*

$$\mathrm{E}\big(m_h(X,Y)\big) \le m_h\big(\mathrm{E}(X),\mathrm{E}(Y)\big). \tag{1.1}$$

It is natural to ask about the generality of this result. For example, does it hold also for the logarithmic mean? To properly answer this question, it is better to choose one of the many axiomatic approaches to the notion of a mean.

In Section 2 we recall the notion of *perspective* of a function, and in Section 3 we recall that a mean of pairs of positive numbers may be represented as the perspective of a certain representing function. In Section 4 we prove that inequality (1.1) holds for a mean m_f if and only if the representing function f is concave.

Once this is done it becomes natural to address the analog question in the noncommutative setting. A positive answer to the case of the matrix harmonic mean was given by Prakasa Rao in [10] and by C. R. Rao in [13]. But also in this case the inequality holds in a much wider generality. In Section 5 we recall the notion of noncommutative perspectives and some of their properties, while in Section 6 we describe the subclass of Kubo–Ando operator means. In Section 7 we show that inequality (1.1) holds true also in the noncommutative case. This follows from the fact that operator means are generated by operator monotone functions; indeed, operator monotonicity of a function defined in the positive half-line implies operator concavity [6, Corollary 2.2], rendering the noncommutative setting completely different from the commutative counterpart.

In Section 8 we consider the random matrix case which to some extent encompasses the previous results.

2. Perspective of a function: commutative case

Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set, and let $g\colon K \to \mathbb{R}$ be a function. We consider the set

$$L = \big\{(x,t) \;\big|\; t>0, t^{-1}x \in K\big\}.$$

Definition 2.1. The perspective $\mathcal{P}_g$ of g is the function $\mathcal{P}_g\colon L \to \mathbb{R}$ defined by setting

$$\mathcal{P}_g(x,t) = tg(t^{-1}x), \quad (x,t) \in L.$$

The following classical result is well known.

Proposition 2.2. *The perspective $\mathcal{P}_g$ of a convex function g is convex.*

Example 2.3. Consider the convex function

$$g(x) = x\log x, \quad x>0$$

with limit $g(0)=0$ and set $K=(0,\infty)$. Then the perspective is the relative entropy

$$\mathcal{P}_g(x,t) = x\log x - x\log t$$

for $x,t>0$.

Notice that the perspective of a concave function is concave.

3. Means for positive numbers

We use the notation $\mathbb{R}_+ = (0, +\infty)$.

Definition 3.1. A bivariate *mean* [9] is a function $m\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

(1) $m(x,x) = x$.
(2) $m(x,y) = m(y,x)$.
(3) $x < y \Rightarrow x < m(x,y) < y$.
(4) $x < x'$ and $y < y' \Rightarrow m(x,y) < m(x',y')$.
(5) m is continuous.
(6) m is positively homogeneous; that is, $m(tx,ty) = t \cdot m(x,y)$ for $t > 0$.

We use the notation $\mathcal{M}_{\rm num}$ for the set of bivariate means described above.

Definition 3.2. Let $\mathcal{F}_{\rm num}$ denote the class of functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

(1) f is continuous.
(2) f is monotone increasing.
(3) $f(1) = 1$.
(4) $tf(t^{-1}) = f(t)$ for $t > 0$.

The following result is straightforward.

Proposition 3.3. *There is a bijection between $\mathcal{M}_{\rm num}$ and $\mathcal{F}_{\rm num}$ given by the formulas*

$$m_f(x,y) = yf(y^{-1}x) \qquad \text{and} \qquad f_m(t) = m(1,t)$$

for positive numbers x, y, and t.

3.1. Some examples of means.

The functions in Table 1 are all concave and even operator concave.

However, there exist nonconcave functions in $\mathcal{F}_{\rm num}$. Consider, for example, the function

$$g(x) = \frac{1}{4}\begin{cases} x+3 & 0 \le x \le 1, \\ 3x+1 & x \ge 1. \end{cases}$$

This piecewise affine function is convex and belongs to $\mathcal{F}_{\rm num}$.

4. The main result: commutative case

Theorem 4.1. *Take a function $f \in \mathcal{F}_{\rm num}$. The inequality*

$$\mathrm{E}\big(m_f(X,Y)\big) \le m_f\big(\mathrm{E}(X), \mathrm{E}(Y)\big) \tag{4.1}$$

holds for arbitrary positive random variables X and Y if and only if f is concave.

Proof. Suppose inequality (4.1) holds for a function f. Take $\Omega = \{1,2\}$ as a state space with probabilities p and $1-p$, and let Y be the constant function 1. We set $X(1) = x_1$ and $X(2) = x_2$ for given $x_1, x_2 > 0$. We then have $\mathrm{E}(Y) = 1$, and thus

$$m_f\big(\mathrm{E}(X), \mathrm{E}(Y)\big) = \mathrm{E}(Y) f\Big(\frac{\mathrm{E}(X)}{\mathrm{E}(Y)}\Big) = f\big(px_1 + (1-p)x_2\big).$$

TABLE 1.

Name	function	mean
arithmetic	$\frac{1+x}{2}$	$\frac{x+y}{2}$
WYD, $\beta \in (0,1)$	$\frac{x^\beta + x^{1-\beta}}{2}$	$\frac{x^\beta y^{1-\beta} + x^{1-\beta} y^\beta}{2}$
geometric	$\sqrt{x}$	$\sqrt{xy}$
harmonic	$\frac{2x}{x+1}$	$\frac{2}{x^{-1}+y^{-1}}$
logarithmic	$\frac{x-1}{\log x}$	$\frac{x-y}{\log x - \log y}$

We also have

$$m_f(X,Y)(1) = Y(1) f\Big(\frac{X(1)}{Y(1)}\Big) = f(x_1)$$

and

$$m_f(X,Y)(2) = Y(2) f\Big(\frac{X(2)}{Y(2)}\Big) = f(x_2).$$

Therefore,

$$\begin{aligned} pf(x_1) + (1-p)f(x_2) &= \mathrm{E}\big(m_f(X,Y)\big) \le m_f\big(\mathrm{E}(X), \mathrm{E}(Y)\big) \\ &= f\big(px_1 + (1-p)x_2\big), \end{aligned}$$

implying that f is concave.

Suppose on the other hand that f is concave, and consider two positive random variables X and Y. We only have to prove the theorem under the assumption that X and Y are simple random variables (finite linear combinations of indicator functions). The general case then follows since any positive random variable is a pointwise increasing limit of simple random variables. The (different) values of X are denoted by $x_1, \dots, x_n$ with associated (marginal or unconditional) probabilities $p_1, \dots, p_n$. The (different) values of Y are denoted by $y_1, \dots, y_m$ with associated (marginal or unconditional) probabilities $q_1, \dots, q_m$.

The stochastic variable $m_f(X,Y)$ takes the values $m_f(x_i, y_j)$ with probabilities $P(X = x_i, Y = y_j)$ for $i = 1, \dots, n$ and $j = 1, \dots, m$ (possibly counted with multiplicity). The mean m_f is the perspective of f, and thus concave by Proposition 2.2. We may therefore apply Jensen's inequality and obtain

$$\begin{aligned} \mathrm{E}\big(m_f(X,Y)\big) &= \sum_{i=1}^{n} \sum_{j=1}^{m} P(X = x_i, Y = y_j) m_f(x_i, y_j) \\ &\le m_f\Big(\sum_{i=1}^{n} \sum_{j=1}^{m} P(X = x_i, Y = y_j)(x_i, y_j)\Big) \end{aligned}$$

$$= m_f\Big(\sum_{i=1}^{n}\sum_{j=1}^{m} P(X = x_i, Y = y_j)x_i,$$

$$\sum_{j=1}^{m}\sum_{i=1}^{n} P(X = x_i, Y = y_j)y_j\Big),$$

where we interchanged the summations in the second argument of m_f. Since the sums of the joint probabilities

$$\sum_{j=1}^{m} P(X = x_i, Y = y_j) = p_i \qquad \text{and} \qquad \sum_{i=1}^{n} P(X = x_i, Y = y_j) = q_j,$$

we obtain

$$\mathrm{E}\big(m_f(X,Y)\big) \le m_f\Big(\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{m} q_j y_j\Big) = m_f\big(\mathrm{E}(X), \mathrm{E}(Y)\big),$$

which is the desired inequality (4.1). □

5. Noncommutative perspective

For the basic results of this section we refer to [2], [1], and [3]. Let f be a function defined in the open positive half-line. In Section 2 we recalled the perspective of f as the function of two variables $\mathcal{P}_f(t,s) = sf(s^{-1}t)$, where $t,s > 0$. Depending on the application, we may also consider the function $(t,s) \to \mathcal{P}_f(s,t)$ and denote this as the perspective of f.

If A and B are commuting positive definite matrices, then the matrix $\mathcal{P}_f(A,B)$ is well defined by the functional calculus, and it coincides with $Bf(B^{-1}A)$. However, even if A and B do not commute, by choosing an appropriate ordering, one may define the perspective.

Definition 5.1. Let f be a function defined in the open positive half-line. The (noncommutative) perspective $\mathcal{P}_f$ of f is then defined by setting

$$\mathcal{P}_f(A,B) = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}$$

for positive definite operators A and B.

For the following basic result confer [2, Theorem 2.2], [3, Theorem 1.1], and [1, Theorem 2.2].

Theorem 5.2. *The (noncommutative) perspective $\mathcal{P}_f$ is convex if and only if f is operator convex.*

Let $f\colon (0,\infty) \to \mathbf{R}$ be a convex function. Since the perspective $\mathcal{P}_f$ is both convex and positively homogenous, we obtain the inequality

$$\mathcal{P}_f\Big(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i y_i\Big) \le \sum_{i=1}^{n} \lambda_i \mathcal{P}_f(x_i, y_i)$$

for tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of positive numbers and positive numbers $\lambda_1, \ldots, \lambda_n$. By setting all $\lambda_i = 1$, this entails the inequality

$$\mathcal{P}_f(\operatorname{Tr} A, \operatorname{Tr} B) \le \operatorname{Tr} \mathcal{P}_f(A, B)$$

for commuting positive definite matrices A and B.

The transformer inequality for the noncommutative perspective of an operator convex function is essentially proved in [5, Theorem 2.2]. Since the perspective of an operator convex function is a convex regular operator map, the statement also follows from [7, Lemma 2.1].

Proposition 5.3 (the transformer inequality)**.** *Let $f\colon (0,\infty) \to \mathbb{R}$ be an operator convex function. The noncommutative perspective $\mathcal{P}_f$ satisfies the inequality*

$$\mathcal{P}_f(C^*AC, C^*BC) \le C^*\mathcal{P}_f(A, B)C$$

for every contraction C and positive definite operators A and B.

Notice that by homogeneity we obtain

$$\mathcal{P}_f(C^*AC, C^*BC) \le C^*\mathcal{P}_f(A, B)C$$

for any operator C. In particular, if C is invertible, then we have

$$\mathcal{P}_f(A, B) \le (C^*)^{-1}\mathcal{P}_f(C^*AC, C^*BC)C^{-1} \le \mathcal{P}_f(A, B).$$

Hence there is equality, and thus

$$C^*\mathcal{P}_f(A, B)C = \mathcal{P}_f(C^*AC, C^*BC). \tag{5.1}$$

Proposition 5.4. *Let $\mathcal{P}_f$ be the noncommutative perspective of an operator convex function $f\colon (0,\infty) \to \mathbb{R}$, and let $c_1, \ldots, c_n$ be operators on a Hilbert space $\mathcal{H}$ such that $c_1^*c_1 + \cdots + c_n^*c_n = 1$. Then*

$$\mathcal{P}_f\Bigl(\sum_{i=1}^n c_i^*A_ic_i, \sum_{i=1}^n c_i^*B_ic_i\Bigr) \le \sum_{i=1}^n c_i^*\mathcal{P}_f(A_i, B_i)c_i$$

for positive definite operators $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ acting on $\mathcal{H}$.

Proof. The perspective $\mathcal{P}_f$ is a convex regular operator map of two variables (see [5], [3], and [7]). The statement thus follows from Jensen's inequality for convex regular operator maps (see [7, Theorem 2.2]). □

6. Operator means in the sense of Kubo–Ando

The celebrated Kubo–Ando theory of matrix means (see [8], [9], [4]) may today be considered as part of the theory of perspectives of positive operator concave functions. This setting is simpler than the general theory of perspectives since a positive operator concave function necessarily is increasing, while a positive operator convex function may not necessarily be monotonic.

Definition 6.1. A bivariate *mean* for pairs of positive operators is a function

$$(A, B) \to m(A, B)$$

defined in and with values in positive definite operators on a Hilbert space and satisfying, mutatis mutandis, conditions (1) to (5) in Definition 3.1. In addition, the *transformer inequality*

$$C^* m(A, B) C \leq m(C^* AC, C^* BC)$$

holds for positive definite A, B and arbitrary C.

Notice that the transformer inequality replaces (6) in Definition 3.1. We denote by $\mathcal{M}_{\mathrm{op}}$ the set of matrix means.

Example 6.2. The arithmetic, geometric, and harmonic (matrix) means are defined respectively by setting

$$\begin{aligned} A \nabla B &= \frac{1}{2}(A + B), \\ A \# B &= A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}, \\ A ! B &= 2(A^{-1} + B^{-1})^{-1}. \end{aligned}$$

We recall that a function $f\colon (0, \infty) \to \mathbb{R}$ is said to be *operator monotone (increasing)* if

$$A \leq B \quad \Rightarrow \quad f(A) \leq f(B)$$

for positive definite operators on an arbitrary Hilbert space. An operator monotone function f is said to be *symmetric* if $f(t) = tf(t^{-1})$ for $t > 0$ and *normalized* if $f(1) = 1$.

Definition 6.3. $\mathcal{F}_{\mathrm{op}}$ is the class of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(1) f is operator monotone increasing,
(2) $tf(t^{-1}) = f(t)t > 0$,
(3) $f(1) = 1$.

The fundamental result, due to Kubo and Ando, is the following.

Theorem 6.4. *There is bijection between* $\mathcal{M}_{\mathrm{op}}$ *and* $\mathcal{F}_{\mathrm{op}}$ *given by the formula*

$$m_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

Remark 6.5. All the functions in $\mathcal{F}_{\mathrm{op}}$ are (operator) concave, making the operator case quite different from the numerical one.

If ρ is a density matrix and A is self-adjoint, then the expectation of A in the state ρ is defined by setting $\mathrm{E}_\rho(A) = \mathrm{Tr}(\rho A)$.

7. The main result: noncommutative case

Theorem 7.1. *Take $f \in \mathcal{F}_{\mathrm{op}}$. Then*

$$\mathrm{E}_\rho\big(m_f(A,B)\big) \le m_f\big(\mathrm{E}_\rho(A),\mathrm{E}_\rho(B)\big). \tag{7.1}$$

Proof. Consider a spectral resolution

$$\rho = \sum_{i=1}^n \lambda_i e_i$$

of the density matrix ρ in terms of one-dimensional orthogonal eigenprojections $e_1,\dots,e_n$ with corresponding eigenvalues $\lambda_1,\dots,\lambda_n$ counted with multiplicity. By setting $c_i = \lambda_i^{1/2} e_i$ for $i=1,\dots,n$, we obtain

$$\mathrm{E}_\rho(A) = \mathrm{Tr}\,\rho A = \mathrm{Tr}\sum_{i=1}^n c_i^* A c_i$$

for any operator A. By using the transformer inequality, we obtain

$$\begin{aligned}\mathrm{E}_\rho\big(m_f(A,B)\big) &= \mathrm{Tr}\sum_{i=1}^n c_i^* m_f(A,B) c_i\\ &\le \mathrm{Tr}\, m_f\Big(\sum_{i=1}^n c_i^* A c_i, \sum_{i=1}^n c_i^* B c_i\Big)\\ &\le m_f\Big(\mathrm{Tr}\sum_{i=1}^n c_i^* A c_i, \mathrm{Tr}\sum_{i=1}^n c_i^* B c_i\Big)\\ &= m_f\big(\mathrm{E}_\rho(A),\mathrm{E}_\rho(B)\big),\end{aligned}$$

where in the second inequality we used that the operators

$$\sum_{i=1}^n c_i^* A c_i \qquad \text{and} \qquad \sum_{i=1}^n c_i^* B c_i$$

are commuting. ☐

8. The random matrix case

Let $(\Omega,\mathcal{F},P)$ be a probability space. A map $X\colon \Omega\to M_n$ is called a *random matrix*. We may write

$$X = (X_{i,j})_{i,j=1}^n \colon \Omega \to M_n,$$

and say that X is a positive definite random matrix if

$$X(\omega) = \big(X_{i,j}(\omega)\big)_{i,j=1}^n$$

is positive definite for P-almost all $\omega\in\Omega$. We may readily consider other types of definiteness for random matrices.

Definition 8.1. A positive semidefinite random matrix $\rho\colon \Omega\to M_n$ is called a *random density matrix* if $\mathrm{Tr}\,\rho = 1$ for P-almost all $\omega\in\Omega$.

Let X and ρ be random matrices on the probability space $(\Omega, \mathcal{F}, P)$, and suppose that ρ is a random density matrix. We introduce the pointwise expectation $\mathrm{E}_\rho(X)$ by setting

$$(\mathrm{E}_\rho X)(\omega) = \operatorname{Tr} \rho(\omega) X(\omega), \quad \omega \in \Omega.$$

The pointwise expectation $\mathrm{E}_\rho(X)$ is a random variable with mean

$$\mathrm{E}\bigl(\mathrm{E}_\rho(X)\bigr) = \int_\Omega \operatorname{Tr} \rho(\omega) X(\omega)\, dP(\omega).$$

If ρ is a constant density matrix, then

$$\mathrm{E}\bigl(\mathrm{E}_\rho(X)\bigr) = \operatorname{Tr} \rho \int_\Omega X(\omega)\, dP(\omega) = \operatorname{Tr} \rho \mathrm{E}(X) = \mathrm{E}_\rho\bigl(\mathrm{E}(X)\bigr),$$

where $\mathrm{E}(X)$ is the constant matrix with entries

$$\mathrm{E}(X)_{i,j} = \int_\Omega X_{i,j}(\omega)\, dP(\omega), \quad i, j = 1, \ldots, n.$$

Theorem 8.2. *Let X and Y be positive definite random matrices on a probability space $(\Omega, \mathcal{F}, P)$. For $f \in \mathcal{F}_{\mathrm{op}}$, we obtain the inequality*

$$\mathrm{E}\mathrm{E}_\rho\bigl(m_f(X, Y)\bigr) \le m_f\bigl(\mathrm{E}\mathrm{E}_\rho(X), \mathrm{E}\mathrm{E}_\rho(Y)\bigr)$$

for each random density matrix ρ on $(\Omega, \mathcal{F}, P)$.

Proof. The matrices $X(\omega)$, $Y(\omega)$, and $\rho(\omega)$ are positive definite, and $\rho(\omega)$ has a unit trace for almost all $\omega \in \Omega$. The inequality between random variables

$$\mathrm{E}_{\rho(\omega)}\Bigl(m_f\bigl(X(\omega), Y(\omega)\bigr)\Bigr) \le m_f\Bigl(\mathrm{E}_{\rho(\omega)}\bigl(X(\omega)\bigr), \mathrm{E}_{\rho(\omega)}\bigl(Y(\omega)\bigr)\Bigr)$$

is therefore valid by our noncommutative inequality in Theorem 7.1. In particular, by taking the mean on both sides, we obtain

$$\begin{aligned}\mathrm{E}\mathrm{E}_\rho\bigl(m_f(X, Y)\bigr) &\le \mathrm{E}\Bigl(m_f\bigl(\mathrm{E}_\rho(X), \mathrm{E}_\rho(Y)\bigr)\Bigr)\\ &\le m_f\bigl(\mathrm{E}\mathrm{E}_\rho(X), \mathrm{E}\mathrm{E}_\rho(Y)\bigr),\end{aligned}$$

where in the last inequality we used the commutative inequality in Theorem 4.1. □

Notice that Theorem 8.2 reduces to the noncommutative inequality when Ω is a one-point space, and reduces to the commutative inequality when $n = 1$. If ρ is a constant matrix, then the order of E and E_ρ in the inequality may be reversed.

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