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MAXIMAL BANACH IDEALS OF LIPSCHITZ MAPS

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Abstract. There are known results showing a canonical association between Lipschitz cross-norms (norms on the Lipschitz tensor product of a metric space and a Banach space) and ideals of Lipschitz maps from a metric space to a dual Banach space. We extend this association, relating Lipschitz cross-norms to ideals of Lipschitz maps taking values in general Banach spaces. To do that, we prove a Lipschitz version of the representation theorem for maximal operator ideals. As a consequence, we obtain linear characterizations of some ideals of (nonlinear) Lipschitz maps between metric spaces.

Introduction

In the theory of Banach spaces, it is well understood that there is a deep connection between norms on tensor products and certain special families of linear maps. The reason is simple: if E and F are vector spaces, a linear functional on $E \otimes F$ can naturally be identified with a linear map from E to the dual of F. If E and F are furthermore Banach spaces, a norm on the tensor product $E \otimes F$ will naturally define a collection of bounded linear maps from E to F^*. With this association in mind, it is possible to develop parallel and interconnected theories between these two worlds: one of norms on tensor products, and one corresponding to these special families of linear maps. Requiring rather minimal conditions for the norm on the tensor product gives rise to collections of linear maps that are in fact ideals—that is, they are closed under composition—and so

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one often talks about the relationship between tensor norms and operator ideals (see [8] for a very comprehensive account).

In our previous works [1], [2], we have started an analogous program for studying ideals of Lipschitz maps from a metric space to a Banach space. In [1] we introduced the concept of a Lipschitz tensor product $X \boxtimes E$ between a metric space X and a Banach space E, whose dual can be canonically identified with a family of Lipschitz maps from X to E^*. In [2] we showed that, just as in the Banach space case, the associated family of maps $X \to E^*$ will satisfy an ideal property under rather minimal conditions for the norm on $X \boxtimes E$. Moreover, we showed that several known examples of ideals of Lipschitz maps (Lipschitz maps, Lipschitz p-summing maps, maps admitting a Lipschitz factorization through a subset of an L_p space) are associated to norms on Lipschitz tensor products in this canonical way.

However, there is something slightly unsatisfactory about this association. It makes perfect sense to consider, say, a Lipschitz p-summing map from a metric space to a general Banach space, but our association only covers the case of maps taking values on a dual Banach space. The purpose of the present paper is to fill this gap and study the relationship between norms on Lipschitz tensor products and ideals of Lipschitz maps, even in the case when the latter take values in a Banach space which is not a dual space. In order to do that, we prove a Lipschitz version of the representation theorem for maximal operator ideals (see [8, Section 17.5], originally due to H. P. Lotz [12]). In a nutshell, that result says that under appropriate conditions on the ideal of maps (or the tensor norm), having a duality between the ideal and the tensor norm at the finite-dimensional level extends to a general duality that covers the case of maps taking values in a Banach space which is not a dual space.

Let us now describe the contents of the paper. In Section 1, we introduce notation and state some preliminary results from [1] and [2]. In particular, we define the objects that we will be studying: norms on the Lipschitz tensor product (called *Lipschitz cross-norms*) and what we mean by ideals of Lipschitz maps (called *Lipschitz operator ideals*). Up to now, these concepts had only been considered for one fixed metric space and one fixed Banach space at a time. In Section 2, we define what we have called *generic* versions of them: variants that cover all possible spaces at once and not just a fixed pair. Section 3 shows that, under mild technical conditions, there is a canonical correspondence between the generic versions of Lipschitz cross-norms and Lipschitz operator ideals. In Section 4, we prove a couple of technical lemmas, Lipschitz versions of two of the "basic lemmas" of Defant and Floret [8, Section 13]. Section 5 contains the main result, the aforementioned representation theorem. As in the linear case, in essence it says that (under mild technical assumptions) once we have a generic Lipschitz cross-norm and a generic Lipschitz operator ideal, which are associated for finite metric spaces and finite-dimensional Banach spaces, that yields a very general association covering even the case of maps taking values in a Banach space which is not a dual space. Finally, in Section 6, we use the representation theorem to deduce some consequences in the more general case of ideals of Lipschitz maps between metric spaces. In particular, we obtain a general theorem that characterizes, by linear means, certain families of nonlinear maps between metric spaces.

1. Notation and preliminary results

Given two metric spaces (X, d_X) and (Y, d_Y), let us recall that a map $f\colon X \to Y$ is said to be *Lipschitz* if there exists a constant C such that $d_Y(f(x), f(y)) \leq C d_X(x, y)$ for all $x, y \in X$. The smallest such constant C will be denoted by $\operatorname{Lip}(f)$. Evidently,

$$\operatorname{Lip}(f) = \sup\Big\{\frac{d_Y(f(x), f(y))}{d_X(x, y)} \colon x, y \in X, x \neq y\Big\}.$$

A pointed metric space X is a metric space with a basepoint 0. As usual, $\mathbb{K}$ denotes the field of real or complex numbers. We will consider a normed space E over $\mathbb{K}$ as a pointed metric space with the distance defined by its norm and the zero vector as the base point. As is customary, B_E and S_E stand for the closed unit ball of E and the unit sphere of E, respectively.

Given two pointed metric spaces X and Y, we denote by $\operatorname{Lip}_0(X, Y)$ the set of all basepoint-preserving Lipschitz maps from X to Y. If E is a Banach space, then $\operatorname{Lip}_0(X, E)$ is a Banach space under the Lipschitz norm Lip. The elements of $\operatorname{Lip}_0(X, E)$ are known as *Lipschitz operators*. The space $\operatorname{Lip}_0(X, \mathbb{K})$ is called the *Lipschitz dual of* X and will be denoted by $X^{\#}$.

For two Banach spaces E and F, $\mathcal{L}(E, F)$ stands for the Banach space of all bounded linear operators from E to F endowed with the canonical norm of operators. In particular, the topological dual $\mathcal{L}(E, \mathbb{K})$ are denoted by E^*.

We now recall some concepts and results stated in [1] and [2]. Let X be a pointed metric space, and let E be a Banach space. The *Lipschitz tensor product* $X \boxtimes E$ is the linear span of all linear functionals $\delta_{(x,y)} \boxtimes e$ on $\operatorname{Lip}_0(X, E^*)$ of the form

$$(\delta_{(x,y)} \boxtimes e)(f) = \big\langle f(x) - f(y), e\big\rangle$$

for $(x, y) \in X^2$ and $e \in E$. A norm α on $X \boxtimes E$ is a *Lipschitz cross-norm* if

$$\alpha(\delta_{(x,y)} \boxtimes e) = d(x, y)\|e\|$$

for all $(x, y) \in X^2$ and $e \in E$. We denote by $X \boxtimes_\alpha E$ the linear space $X \boxtimes E$ with norm α, and we denote by $X \widehat{\boxtimes}_\alpha E$ the completion of $X \boxtimes_\alpha E$.

For $g \in X^{\#}$ and $\phi \in E^*$, consider the linear functional $g \boxtimes \phi$ on $X \boxtimes E$ defined by

$$(g \boxtimes \phi)\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big) = \sum_{i=1}^{n} \big(g(x_i) - g(y_i)\big)\langle \phi, e_i\rangle,$$

and consider, for $h \in \operatorname{Lip}_0(X, Y)$ and $T \in \mathcal{L}(E, F)$, the linear operator $h \boxtimes T$ from $X \boxtimes E$ to $Y \boxtimes F$ given by

$$(h \boxtimes T)\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big) = \sum_{i=1}^{n} \delta_{(h(x_i),h(y_i))} \boxtimes T(e_i).$$

A Lipschitz cross-norm α on $X \boxtimes E$ is called *dualizable (uniform)* if, for each $g \in X^{\#}$ and $\phi \in E^*$, we have $g \boxtimes \phi \in (X \boxtimes_\alpha E)^*$ and $\|g \boxtimes \phi\| \leq \mathrm{Lip}(g)\|\phi\|$ (resp., if, for each $h \in \mathrm{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, we have $h \boxtimes T \in \mathcal{L}(X \boxtimes_\alpha E, X \boxtimes_\alpha E)$ and $\|h \boxtimes T\| \leq \mathrm{Lip}(h)\|T\|$).

The *Lipschitz injective norm* ε and the *Lipschitz projective norm* π on $X \boxtimes E$ were introduced in [1]. For $1 \leq p \leq \infty$, we also have the norms d_p and w_p on $X \boxtimes E$ (see [2] for definitions). It is known that all those norms are uniform and dualizable Lipschitz cross-norms on $X \boxtimes E$. Moreover, ε is the least dualizable Lipschitz cross-norm on $X \boxtimes E$, and π is the greatest Lipschitz cross-norm on $X \boxtimes E$. In fact, a norm α on $X \boxtimes E$ is a dualizable Lipschitz cross-norm if and only if $\varepsilon \leq \alpha \leq \pi$.

Let α be a Lipschitz cross-norm on $X \boxtimes E$. A basepoint-preserving map $f \colon X \to E^*$ is said to be an *α-Lipschitz operator* if there exists a real constant $C \geq 0$ such that

$$\Big|\sum_{i=1}^{n} \big\langle f(x_i) - f(y_i), e_i \big\rangle\Big| \leq C\alpha\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big)$$

for all $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$. The infimum of such constants C is denoted by $\mathrm{Lip}_\alpha(f)$ and called the *α-Lipschitz norm of f*. Equipped with that norm, the Banach space of all α-Lipschitz operators from X into E^* is denoted by $\mathrm{Lip}_\alpha(X, E^*)$ and it is isometrically isomorphic to $(X \widehat{\boxtimes}_\alpha E)^*$ via the map $\Lambda_0 \colon \mathrm{Lip}_\alpha(X, E^*) \to (X \widehat{\boxtimes}_\alpha E)^*$ defined by

$$\Lambda_0(f)(u) = \sum_{i=1}^{n} \big\langle f(x_i) - f(y_i), e_i \big\rangle$$

for $f \in \mathrm{Lip}_\alpha(X, E^*)$ and $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$.

For $\alpha = \epsilon, \pi$, the space $\mathrm{Lip}_\alpha(X, E^*)$ can be identified with the space of Lipschitz Grothendieck-integral operators $\mathrm{Lip}_{0GI}(X, E^*)$ (see [3]) and the space of Lipschitz operators $\mathrm{Lip}_0(X, E^*)$. For $\alpha = d_p, w_p$ with $1 \leq p \leq \infty$, $\mathrm{Lip}_\alpha(X, E^*)$ coincides with the space $\Pi^L_{p'}(X, E^*)$ of Lipschitz p'-summing operators and the space $\Gamma^{\mathrm{Lip}}_{p'}(X, E^*)$ of Lipschitz operators admitting a Lipschitz factorization through a subset of an $L_{p'}$ space, where p' denotes the conjugate index of p.

For a pointed metric space X (a Banach space E), we denote by $\mathrm{MFIN}(X)$ (resp., $\mathrm{FIN}(E)$) the set of all finite subsets of X containing the basepoint (resp., the set of all finite-dimensional subspaces of E). For a Banach space E, we denote by $\mathrm{COFIN}(E)$ the set of all finite-codimensional closed subspaces of E. Given $L \in \mathrm{COFIN}(E)$, let $Q^E_L \colon E \to E/L$ be the canonical projection, and given $Y \subset X$, let $I^X_Y \colon Y \to X$ be the canonical injection. For a Banach space E, $\kappa_E \colon E \to E^{**}$ denotes the canonical injection.

2. Generic versions of Lipschitz cross-norm and Lipschitz operator Banach ideal

Definition 2.1. By a *generic Lipschitz cross-norm* α, we mean an assignment for each pointed metric space X and each Banach space E of a Lipschitz cross-norm

$\alpha(\cdot; X, E)$ on the Lipschitz tensor product $X \boxtimes E$ (sometimes denoted simply by α if the spaces are clear from the context) such that we have the following:

(i) α is dualizable (i.e., $\varepsilon \le \alpha \le \pi$);
(ii) α satisfies the metric mapping property: if $h \in \mathrm{Lip}_0(X_0, X_1)$ and $T \in \mathcal{L}(E_0, E_1)$, then $\|h \boxtimes T\colon X_0 \boxtimes_\alpha E_0 \to X_1 \boxtimes_\alpha E_1\| \le \mathrm{Lip}(h)\|T\|$.

If the assignment and the conditions above are given only for finite pointed metric spaces and finite-dimensional Banach spaces, we say that α is FIN-generic.

A generic Lipschitz cross-norm α is said to be *finitely generated* if

$$\alpha(u; X, E) = \inf\bigl\{\alpha(u; X_0, E_0)\colon X_0 \in \mathrm{MFIN}(X), E_0 \in \mathrm{FIN}(E), u \in X_0 \boxtimes E_0\bigr\}$$

for every pointed metric space X, every Banach space E, and every $u \in X \boxtimes E$.

Note that condition (ii) in Definition 2.1 is a generalization of uniformity, and note that all the Lipschitz cross-norms presented in Section 1 (namely, ε, π, d_p, and w_p) are in fact finitely generated generic Lipschitz cross-norms.

Given a FIN-generic Lipschitz cross-norm α, we can use the following procedure to extend it to a finitely generated generic Lipschitz cross-norm.

Lemma 2.2. *Let α be a* FIN*-generic Lipschitz cross-norm. For a pointed metric space X, a Banach space E, and $u \in X \boxtimes E$, define*

$$\theta(u; X, E) = \inf\bigl\{\alpha(u; X_0, E_0)\colon X_0 \in \mathrm{MFIN}(X), E_0 \in \mathrm{FIN}(E), u \in X_0 \boxtimes E_0\bigr\}.$$

Then θ is a finitely generated generic Lipschitz cross-norm. Moreover, θ and α coincide on $X \boxtimes E$ whenever X is a finite pointed metric space and E is a finite-dimensional Banach space.

Proof. First, let us show that θ is a norm on $X \boxtimes E$. It is clear that θ satisfies $\theta(\lambda u) = |\lambda|\theta(u)$ because so does α. Let $u_1, u_2 \in X \boxtimes E$. Take $X_1, X_2 \in \mathrm{MFIN}(X)$, $E_1, E_2 \in \mathrm{FIN}(E)$ such that $u_j \in X_j \boxtimes E_j$, $j = 1, 2$. Then, if $X_0 = X_1 \cup X_2$ and $E_0 = E_1 + E_2$, using the metric mapping property of α applied to the inclusions $X_j \to X_0$ and $E_j \to E_0$, $j = 1, 2$, we have

$$\begin{aligned}\alpha(u_1 + u_2; X_0, E_0) &\le \alpha(u_1; X_0, E_0) + \alpha(u_2; X_0, E_0)\\ &\le \alpha(u_1; X_1, E_1) + \alpha(u_2; X_2, E_2),\end{aligned}$$

and, by taking the infimum over all such X_j, E_j, we conclude that

$$\theta(u_1 + u_2; X, E) \le \theta(u_1; X, E) + \theta(u_2; X, E),$$

giving the triangle inequality. Since ε and π are finitely generated and $\varepsilon \le \alpha \le \pi$, it follows that $\varepsilon \le \theta \le \pi$, and thus θ is a dualizable Lipschitz cross-norm on $X \boxtimes E$.

Now let X, Y be pointed metric spaces, let E, F be Banach spaces, let $h \in \mathrm{Lip}_0(X, Y)$, and let $S \in \mathcal{L}(E, F)$. Let $u \in X \boxtimes E$. Given $X_0 \in \mathrm{MFIN}(X)$ and $E_0 \in \mathrm{FIN}(E)$ such that $u \in X_0 \boxtimes E_0$, note that $Y_0 := h(X_0) \in \mathrm{MFIN}(Y)$, $F_0 := S(E_0) \in \mathrm{FIN}(F)$, and $(h \boxtimes S)(u) \in Y_0 \boxtimes F_0$. From the metric mapping property for α, we infer that

$$\theta\bigl((h \boxtimes S)(u); Y, F\bigr) \le \alpha\bigl((h \boxtimes S)(u); Y_0, F_0\bigr) \le \mathrm{Lip}(h)\|S\|\alpha(u; X_0, E_0),$$

and, by taking the infimum over all such X_0, E_0, we conclude that θ has the metric mapping property.

Now assume that X is a finite pointed metric space, E is a finite-dimensional Banach space, and $u \in X \boxtimes E$. By the definition of θ, clearly $\theta(u; X, E) \leq \alpha(u; X, E)$. Whenever $X_0 \in \mathrm{MFIN}(X)$, $E_0 \in \mathrm{FIN}(E)$ are such that $u \in X_0 \boxtimes E_0$, the metric mapping property of α applied to the inclusion maps $X_0 \to X$ and $E_0 \to E$ shows that $\alpha(u; X, E) \leq \alpha(u; X_0, E_0)$, and so we conclude that $\theta(u; X, E) = \alpha(u; X, E)$. $\square$

Definition 2.3. By a *generic Lipschitz operator Banach ideal* A, we mean an assignment for each pointed metric space X and each Banach space E of a linear subspace $A(X, E)$ of $\mathrm{Lip}_0(X, E)$ equipped with a norm $\|\cdot\|_A$ with the following properties.

(i) The Lipschitz rank 1 operator $g \cdot e \colon x \mapsto g(x)e$ from X to E belongs to $A(X, E)$ for every $g \in X^{\#}$ and $e \in E$, and $\|g \cdot e\|_A \leq \mathrm{Lip}(g)\|e\|$.
(ii) $\mathrm{Lip} \leq \|\cdot\|_A$.
(iii) $(A(X, E), \|\cdot\|_A)$ is a Banach space.
(iv) The (strengthened) ideal property: If $f \in A(X, E)$, $h \in \mathrm{Lip}_0(Z, X)$, and $S \in \mathcal{L}(E, F)$, then the composition Sfh belongs to $A(Z, F)$ and

$$\|Sfh\|_A \leq \|S\|\|f\|_A \,\mathrm{Lip}(h).$$

If the assignment and the conditions above are given only for finite pointed metric spaces and finite-dimensional Banach spaces, we say that A is FIN-*generic*.

Note that Definition 2.3 is a combination of the definitions of the Lipschitz operator Banach ideal and the Lipschitz operator Banach space introduced in [2], but with a stronger ideal property and for a general Banach space instead of a dual one. Note also that the spaces Lip_0, Π_p^L, and Γ_p^{Lip} are examples of generic Lipschitz operator Banach ideals.

Given a FIN-generic Lipschitz operator Banach ideal A, there are several different ways of extending it to a generic Lipschitz operator Banach ideal. Here we consider the “largest” such extension.

Lemma 2.4. *Let A be a* FIN-*generic Lipschitz operator Banach ideal. For a pointed metric space X, a Banach space E, and $f \in \mathrm{Lip}_0(X, E)$, define*

$$\|f\|_{A^{\max}} = \sup\bigl\{\|Q_L^E \circ f \circ I_Y^X\|_A \colon Y \in \mathrm{MFIN}(X), L \in \mathrm{COFIN}(E)\bigr\}$$

and

$$A^{\max}(X, E) = \bigl\{f \in \mathrm{Lip}_0(X, E) \colon \|f\|_{A^{\max}} < \infty\bigr\}.$$

Then $(A^{\max}, \|\cdot\|_{A^{\max}})$ is a generic Lipschitz operator Banach ideal. In addition, $A^{\max}(X, E) = A(X, E)$ holds isometrically whenever X is a finite pointed metric space and E is a finite-dimensional Banach space.

Proof. Clearly, $A^{\max}(X, E)$ is a nonempty subset of $\mathrm{Lip}_0(X, E)$. Since $\|\cdot\|_A$ is a norm, it is immediate that $A^{\max}(X, E)$ is a linear subspace of $\mathrm{Lip}_0(X, E)$ and that $\|\cdot\|_{A^{\max}}$ is a norm. We now verify the conditions in the definition of a generic Lipschitz operator Banach ideal.

(i) Let $g \in X^{\#}$, and let $e \in E$. For every $Y \in \mathrm{MFIN}(X)$ and $L \in \mathrm{COFIN}(E)$, we have

$$\big\|Q_L^E \circ (g \cdot e) \circ I_Y^X\big\|_A = \big\|(g|_Y) \cdot (Q_L^E e)\big\|_A \leq \mathrm{Lip}(g|_Y)\|Q_L^E e\| \leq \mathrm{Lip}(g)\|e\|,$$

and so $g \cdot e$ belongs to $A^{\max}(X, E)$ and $\|g \cdot e\|_{A^{\max}} \leq \mathrm{Lip}(g)\|e\|$.

(ii) Note that, for any $f \in \mathrm{Lip}_0(X, E)$,

$$\mathrm{Lip}(f) = \sup\big\{\mathrm{Lip}(Q_L^E \circ f \circ I_Y^X) \colon Y \in \mathrm{MFIN}(X), L \in \mathrm{COFIN}(E)\big\},$$

from which it follows that $\mathrm{Lip} \leq \|\cdot\|_{A^{\max}}$.

(iii) Since we already know that $A^{\max}(X, E)$ is a normed space, it suffices to show that every absolutely convergent series $\sum f_n$ in $A^{\max}(X, E)$ is convergent. Since $\mathrm{Lip} \leq \|\cdot\|_{A^{\max}}$, the series $\sum_n f_n$ converges in $\mathrm{Lip}_0(X, E)$ to a limit $f \in \mathrm{Lip}_0(X, E)$. Fix $Y \in \mathrm{FIN}(X)$, and fix $L \in \mathrm{COFIN}(E)$. Since Y is finite and E/L is finite-dimensional, by [2, Corollary 6.6] there exists a dualizable Lipschitz cross-norm α on $Y \boxtimes (E/L)^*$ such that $A(Y, E/L) = (Y \boxtimes_\alpha (E/L)^*)^*$. Note that the series $\sum_n Q_L^E \circ f_n \circ I_Y^X$ converges pointwise to $Q_L^E \circ f \circ I_Y^X$, and so, for each $u \in Y \boxtimes (E/L)^*$, we have

$$\begin{aligned}\big|(Q_L^E \circ f \circ I_Y^X)(u)\big| &= \Big|\sum_{n=1}^{+\infty}(Q_L^E \circ f_n \circ I_Y^X)(u)\Big| \\ &\leq \sum_{n=1}^{+\infty}\big|(Q_L^E \circ f_n \circ I_Y^X)(u)\big| \leq \alpha(u)\sum_{n=1}^{+\infty}\|Q_L^E \circ f_n \circ I_Y^X\|_A.\end{aligned}$$

Then it follows that $\|Q_L^E \circ f \circ I_Y^X\|_A \leq \sum_{n=1}^{+\infty}\|f_n\|_{A^{\max}}$, and thus $f \in A^{\max}(X, E)$ and $\|f\|_{A^{\max}} \leq \sum_{n=1}^{+\infty}\|f_n\|_{A^{\max}}$. Now, applying the same argument to $f - \sum_{n=1}^{N} f_n$ yields

$$\Big\|f - \sum_{n=1}^{N} f_n\Big\|_{A^{\max}} = \Big\|\sum_{n=N+1}^{+\infty} f_n\Big\|_{A^{\max}} \leq \sum_{n=N+1}^{+\infty}\|f_n\|_{A^{\max}},$$

and so the series $\sum_n f_n$ converges to f in $A^{\max}(X, E)$.

(iv) Let $f \in A^{\max}(X, E)$, let $h \in \mathrm{Lip}_0(Z, X)$, and let $S \in \mathcal{L}(E, F)$. Fix $Y \in \mathrm{MFIN}(Z)$, and fix $L \in \mathrm{COFIN}(F)$. Let $K \subset E$ be the kernel of the map $Q_L^F \circ S$, and note that $K \in \mathrm{COFIN}(E)$ and that, by the universal property of quotients, there is a linear map $\tilde{S} \colon E/K \to F/L$ with $\|\tilde{S}\| \leq \|S\|$ such that $\tilde{S}Q_K^E = Q_L^F S$. Thus, noting that $h(Y) \in \mathrm{FIN}(X)$ and using the ideal property of A,

$$\begin{aligned}\|Q_L^F \circ Sfh \circ I_Y^Z\|_A &= \|\tilde{S}Q_K^E \circ f \circ I_{h(Y)}^X \circ hI_Y^Z\|_A \\ &\leq \|\tilde{S}\|\|Q_K^E \circ f \circ I_{h(Y)}^X\|_A \,\mathrm{Lip}(hI_Y^Z) \leq \|S\|\|f\|_{A^{\max}}\,\mathrm{Lip}(h).\end{aligned}$$

Now let X be a finite pointed metric space, let E be a finite-dimensional Banach space, and let $f \in \mathrm{Lip}_0(X, E)$. From the definition of $A^{\max}$ and the ideal property for A, it is clear that $\|f\|_{A^{\max}} \leq \|f\|_A$. But taking $Y = X$ and $L = \{0\}$ in the definition of $\|f\|_{A^{\max}}$ shows that

$$\|f\|_{A^{\max}} \geq \|Q_{\{0\}}^E \circ f \circ I_X^X\|_A = \|Q_{\{0\}}^E \circ f\|_A = \|f\|_A,$$

where the last equality follows from the fact that $Q_{\{0\}}^E$ is a bijective isometry. $\square$

Definition 2.5. We call $(A^{\max}, \|\cdot\|_{A^{\max}})$ the *maximal hull* of A, and we say that a generic Lipschitz operator Banach ideal A is *maximal* if $(A, \|\cdot\|_A) = (A^{\max}, \|\cdot\|_{A^{\max}})$.

Note that $A^{\max}$ is always a maximal generic Lipschitz operator Banach ideal. A first example of a maximal generic Lipschitz operator Banach ideal is given by the ideal Lip_0 of Lipschitz operators. Suppose that $f \in \mathrm{Lip}_0^{\max}(X, E)$ with norm at most C, and let x, y be distinct points in X. Note that $\|Q_L^E \circ f(x) - Q_L^E \circ f(y)\| \leq Cd(x, y)$ for every $L \in \mathrm{codim}(E)$. By taking L to be the kernel of a norm 1 functional in E^* which norms $f(x) - f(y) \in E$, we conclude that $\|f(x)-f(y)\| \leq Cd(x, y)$, and thus f is Lipschitz with norm at most C as required. A similar but slightly more involved argument shows that Π_p^L and Γ_p^{Lip} are also maximal generic Lipschitz operator Banach ideals based on the fact that, given finitely many vectors in E, one can find $L \in \mathrm{COFIN}(E)$ such that the quotient $E \to E/L$ preserves the norms of those vectors.

There are also generic Lipschitz operator Banach ideals that are not maximal, for example, the Lipschitz compact operators from [10] and the Lipschitz p-nuclear operators from [7] (the reader is referred to those papers for the definitions). Any Lipschitz operator belongs to the maximal hull of the ideal of Lipschitz compact operators since every Lipschitz operator with finite domain is Lipschitz-compact, but it is easy to find Lipschitz operators which are not Lipschitz-compact, and thus the ideal of Lipschitz compact operators is not maximal. Similarly, using [7, Theorem 2.1], the existence of a linear operator from a separable Banach space to a dual Banach space which is p-integral but not p-nuclear shows that the ideal of Lipschitz p-nuclear operators is not maximal.

3. The association between finitely generated generic Lipschitz cross-norms and maximal generic Lipschitz operator Banach ideals

The main idea we will exploit is that to every finitely generated generic Lipschitz cross-norm one can canonically associate a maximal generic Lipschitz operator Banach ideal, and vice versa.

Definition 3.1. We say that a FIN-generic Lipschitz cross-norm α and a FIN-generic Lipschitz operator Banach ideal A are *associated* and we write $A \sim \alpha$ if, for every finite pointed metric space X and every finite-dimensional Banach space E, the relation $A(X, E^*) = (X \boxtimes_\alpha E)^*$, or, equivalently, $A(X, E^*) = \mathrm{Lip}_\alpha(X, E^*)$, holds isometrically.

The key will be the following generalization of [2, Theorem 5.3], whose heart is the fact that the metric mapping property of α and the (strengthened) ideal property of Lip_α are equivalent as long as we restrict ourselves to finite metric spaces and finite-dimensional Banach spaces.

Proposition 3.2. *Suppose that, for every finite pointed metric space X and every finite-dimensional Banach space E, α is a norm on $X \boxtimes E$ and $A(X, E)$ is a linear subspace of $\mathrm{Lip}_0(X, E)$ equipped with a norm $\|\cdot\|_A$ so that $A(X, E) = (X \boxtimes_\alpha E^*)^*$*

holds isometrically. Then α is a FIN*-generic Lipschitz cross-norm if and only if A is a* FIN*-generic Lipschitz operator Banach ideal.*

Proof. Suppose that A is a FIN-generic Lipschitz operator Banach ideal. Let X be a finite pointed metric space, and let E be a finite-dimensional Banach space. By hypothesis, α is already a norm on $X \boxtimes E$. The condition $\mathrm{Lip} \leq \|\cdot\|_A$ implies that $\alpha \leq \pi$ on $X \boxtimes E$, whereas the fact that for every $g \in X^{\#}$ and $e \in E$ we have $\|g \cdot e\|_A \leq \mathrm{Lip}(g)\|e\|$ implies that $\varepsilon \leq \alpha$ on $X \boxtimes E$. Thus α is a dualizable Lipschitz cross-norm. A small modification of the arguments in the proof of [2, Theorem 5.3(i)] shows that α has the metric mapping property.

Now suppose that α is a FIN-generic Lipschitz cross-norm. By hypothesis, $\|\cdot\|_A$ is a complete norm on $A(X,E)$. Reversing the arguments above, the condition $\alpha \leq \pi$ implies that $\mathrm{Lip} \leq \|\cdot\|_A$ on $A(X,E)$, whereas the condition $\varepsilon \leq \alpha$ implies that, for every $g \in X^{\#}$ and $e \in E$, we have $\|g \cdot e\|_A \leq \mathrm{Lip}(g)\|e\|$. Finally, a small modification of the argument in the proof of [2, Theorem 5.3(i)] shows that A has the (strengthened) ideal property. □

More generally, we have the following two lemmas that give constructions allowing us to go back and forth between generic Lipschitz cross-norms and generic Lipschitz operator Banach ideals.

Lemma 3.3. *Let α be a* FIN*-generic Lipschitz cross-norm. For a pointed metric space X and a Banach space E, given $f \in \mathrm{Lip}_0(X,E)$, define*

$$\|f\|_A = \sup\bigl\{\mathrm{Lip}_\alpha(Q_L^E \circ f \circ I_Y^X) \colon Y \in \mathrm{MFIN}(X), L \in \mathrm{COFIN}(E)\bigr\}$$

and

$$A(X,E) = \bigl\{f \in \mathrm{Lip}_0(X,E) \colon \|f\|_A < \infty\bigr\}.$$

Then A is a maximal generic Lipschitz operator Banach ideal associated to α.

Proof. First, a word about the definition: note that since E/L is finite-dimensional, it is a dual space, and thus it makes sense to consider the Lip_α-norm of the mapping $Q_L^E \circ f \circ I_Y^X \colon Y \to E/L$. Since α is a FIN-generic Lipschitz cross-norm, Proposition 3.2 implies that Lip_α is a FIN-generic Lipschitz operator Banach ideal. Therefore, from Lemma 2.4, A is a maximal generic Lipschitz operator Banach ideal that agrees isometrically with Lip_α whenever the pointed metric space is finite and the Banach space is finite-dimensional, and so $A \sim \alpha$. □

Lemma 3.4. *Let A be a* FIN*-generic Lipschitz operator Banach ideal. For a pointed metric space X, a Banach space E, and $u \in X \boxtimes E$, define $\alpha(u; X, E)$ as*

$$\inf\bigl\{\sup\bigl\{|f(u)| \colon \|f \colon X_0 \to E_0^*\|_A \leq 1\bigr\} \colon \\ X_0 \in \mathrm{MFIN}(X), E_0 \in \mathrm{FIN}(E), u \in X_0 \boxtimes E_0\bigr\}.$$

Then α is a finitely generated generic Lipschitz cross-norm associated to A.

Proof. For every finite pointed metric space X and every finite-dimensional Banach space E, consider the norm $\alpha_0(\cdot; X, E)$ on $X \boxtimes E$ given by duality with A

$$\alpha_0(u; X, E) = \bigl\{\sup|f(u)| \colon \|f \colon X \to E^*\|_A \leq 1\bigr\}$$

so that $(X \boxtimes_{\alpha_0} E^*)^* = A(X, E)$. From Proposition 3.2, it follows that α_0 is a FIN-generic Lipschitz cross-norm. By definition, α is obtained from α_0 by means of the procedure in Lemma 2.2, which implies that α is a finitely generated generic Lipschitz cross-norm that agrees with α_0 on $X \boxtimes E$ whenever the pointed metric space X is finite and the Banach space E is finite-dimensional, and so $A \sim \alpha$. □

The previous two lemmas show the following.

(i) For every FIN-generic Lipschitz cross-norm there is a maximal generic Lipschitz operator Banach ideal A such that $A \sim \alpha$.
(ii) For every FIN-generic Lipschitz operator Banach ideal there is a finitely generated generic Lipschitz cross-norm α such that $A \sim \alpha$.

Since both finitely generated generic Lipschitz cross-norms and maximal generic Lipschitz operator Banach ideals are determined by their behavior on finite pointed metric spaces and finite-dimensional Banach spaces, these constructions show that the relation $\sim$ is a one-to-one correspondence between finitely generated generic Lipschitz cross-norms and maximal generic Lipschitz operator Banach ideals.

4. Two basic lemmas for finitely generated generic Lipschitz cross-norms

According to [8, Section 13], there are five lemmas that are basic for the understanding and use of tensor norms. Here we prove Lipschitz versions of the two we will need later. Every $\varphi \in (X \widehat{\boxtimes}_\pi E)^* = \mathrm{Lip}_0(X, E^*)$ has a canonical extension $\varphi^\wedge \in (X \widehat{\boxtimes}_\pi E^{**})^* = \mathrm{Lip}_0(X, E^{***})$, characterized by the relation $\langle \varphi^\wedge, \delta_{(x,y)} \boxtimes v^{**} \rangle = \langle v^{**}, L_\varphi(x) - L_\varphi(y) \rangle$, where $L_\varphi := \Lambda_0^{-1}(\varphi) \in \mathrm{Lip}_0(X, E^*)$ is the Lipschitz operator given by $\langle \Lambda_0^{-1}(\varphi)(x), e \rangle = \varphi(\delta_{(x,0)} \boxtimes e)$. The following lemma tells us what happens if in fact $\varphi \in (X \widehat{\boxtimes}_\alpha E)^*$ (cf. [8, Lemma 13.2]).

Lemma 4.1 (Extension lemma)**.** *Let $\varphi \in (X \widehat{\boxtimes}_\pi E)^*$, and let α be a finitely generated generic Lipschitz cross-norm. Then $\varphi \in (X \widehat{\boxtimes}_\alpha E)^*$ if and only if $\varphi^\wedge \in (X \widehat{\boxtimes}_\alpha E^{**})^*$. In this case, $\|\varphi\|_{(X \widehat{\boxtimes}_\alpha E)^*} = \|\varphi^\wedge\|_{(X \widehat{\boxtimes}_\alpha E^{**})^*}$.*

Proof. The metric mapping property implies that the canonical inclusion map $\mathrm{id}_X \boxtimes \kappa_E \colon X \boxtimes_\alpha E \to X \boxtimes_\alpha E^{**}$ is contractive, and hence $\|\varphi\|_{(X \widehat{\boxtimes}_\alpha E)^*} \leq \|\varphi^\wedge\|_{(X \widehat{\boxtimes}_\alpha E^{**})^*}$.

For the converse, take $u_0 \in X \boxtimes E^{**}$ and $X_0 \in \mathrm{MFIN}(X), E_0 \in \mathrm{FIN}(E^{**})$ such that $u_0 \in X_0 \boxtimes E_0$. By the principle of local reflexivity (even in a weak form as in [8, Section 6.5]), for every $\varepsilon > 0$ there exists $R \in \mathcal{L}(E_0, E)$ with $\|R\| \leq 1 + \varepsilon$ such that, for all $v^{**} \in E_0$ and $x, y \in X_0$, $\langle v^{**}, L_\varphi(x) - L_\varphi(y) \rangle = \langle L_\varphi(x) - L_\varphi(y), Rv^{**} \rangle$. This means that $\langle \varphi^\wedge, \delta_{(x,y)} \boxtimes v^{**} \rangle = \langle \varphi, (\mathrm{id}_X \boxtimes R)(\delta_{(x,y)} \boxtimes v^{**}) \rangle$, and therefore $\langle \varphi^\wedge, u_0 \rangle = \langle \varphi, (\mathrm{id}_X \boxtimes R)(u_0) \rangle$. Hence $|\langle \varphi^\wedge, u_0 \rangle| \leq \|\varphi\| \|R\| \alpha(u_0; X_0, E_0) \leq (1 + \varepsilon) \|\varphi\| \alpha(u_0; X_0, E_0)$, which implies the result since α is finitely generated. □

Lipschitz cross-norms generally do not respect subspaces, but the embedding into the bidual is respected when the Lipschitz cross-norm is finitely generated (cf. [8, Lemma 13.3]).

Lemma 4.2 (Embedding lemma)**.** *If α is a finitely generated generic Lipschitz cross-norm, then the mapping $\mathrm{id}_X \boxtimes \kappa_E \colon X \mathbin{\widehat{\boxtimes}_\alpha} E \to X \mathbin{\widehat{\boxtimes}_\alpha} E^{**}$ is an isometry for every pointed metric space X and every Banach space E.*

Proof. As pointed out above, the metric mapping property implies that $\alpha(u; X, E^{**}) \le \alpha(u; X, E)$ for any $u \in X \boxtimes E$ (in an abuse of notation, we are not writing the map $\mathrm{id}_X \boxtimes \kappa_E$). Now, by the extension lemma,

$$\begin{aligned}\alpha(u; X, E) &= \sup\bigl\{\bigl|\langle\varphi, u\rangle\bigr| \colon \varphi \in (X \mathbin{\widehat{\boxtimes}_\alpha} E)^*, \|\varphi\|_{(X \mathbin{\widehat{\boxtimes}_\alpha} E)^*} \le 1\bigr\}\\ &= \sup\bigl\{\bigl|\langle\varphi^\wedge, u\rangle\bigr| \colon \varphi \in (X \mathbin{\widehat{\boxtimes}_\alpha} E)^*, \|\varphi\|_{(X \mathbin{\widehat{\boxtimes}_\alpha} E)^*} \le 1\bigr\}\\ &\le \sup\bigl\{\bigl|\langle\psi, u\rangle\bigr| \colon \psi \in (X \mathbin{\widehat{\boxtimes}_\alpha} E^{**})^*, \|\psi\|_{(X \mathbin{\widehat{\boxtimes}_\alpha} E^{**})^*} \le 1\bigr\} = \alpha(u; X, E^{**}),\end{aligned}$$

giving the reverse inequality. □

5. The representation theorem

We are now ready to present the main result of this paper. Modulo technical assumptions, philosophically it is a combination of [2, Theorems 5.3 and 6.5]: Lipschitz cross-norms that are both uniform and dualizable give rise to a very satisfactory duality theory.

Theorem 5.1. *Let A be a maximal generic Lipschitz operator Banach ideal, and let α be a finitely generated generic Lipschitz cross-norm, which are both associated with each other. Then, for every pointed metric space X and every Banach space E, the relations*

$$A(X, E^*) = (X \mathbin{\widehat{\boxtimes}_\alpha} E)^*, \tag{5.1}$$

$$A(X, E) = (X \mathbin{\widehat{\boxtimes}_\alpha} E^*)^* \cap \mathrm{Lip}_0(X, E) \tag{5.2}$$

hold isometrically.

Proof. First, note the diagram

$$\begin{tikzcd} \varphi \in (X \mathbin{\widehat{\boxtimes}_\alpha} E)^* \arrow[r, hook] \arrow[d, hook] & (X \mathbin{\widehat{\boxtimes}_\pi} E)^* \arrow[r, equal] \arrow[d, hook] & \mathrm{Lip}_0(X, E^*) \\ \varphi^\wedge \in (X \mathbin{\widehat{\boxtimes}_\alpha} E^{**})^* \arrow[r, hook] & (X \mathbin{\widehat{\boxtimes}_\pi} E^{**})^*. & \end{tikzcd}$$

The vertical arrows are isometries thanks to the embedding lemma, whereas the horizontal arrows are continuous because $\alpha \le \pi$. By the extension lemma, (5.1) will follow from (5.2).

In order to prove (5.2), we need to show that, for $f \in \mathrm{Lip}_0(X, E)$, f belongs to $A(X, E)$ if and only if the associated linear map $\varphi_f \colon X \boxtimes_\alpha E^* \to \mathbb{K}$ is continuous; that is, there is $C > 0$ such that

$$\bigl|u(f)\bigr| \le C\alpha(u; X, E^*), \quad \forall u \in X \boxtimes E^*. \tag{5.3}$$

Since A is maximal, it is clear that $f \in A(X, E)$ with $\|f\|_A \le C$ if and only if

$$\|Q_L^E \circ f \circ I_Y^X\|_A \le C, \quad \forall Y \in \mathrm{MFIN}(X), \forall L \in \mathrm{COFIN}(E). \tag{5.4}$$

Denote by L^0 the annihilator of L. Since $A(Y, E/L) = (Y \boxtimes_\alpha (E/L)^*)^* = (Y \boxtimes_\alpha L^0)^*$, and as L^0 varies over all spaces in $\mathrm{FIN}(E^*)$ when L varies over all spaces in $\mathrm{COFIN}(E)$, (5.4) is equivalent to

$$\left|u(Q_L^E \circ f \circ I_Y^X)\right| \leq C\alpha(u; Y, L^0), \quad \forall u \in Y \boxtimes L^0 \tag{5.5}$$

whenever $Y \in \mathrm{MFIN}(X)$ and $L^0 \in \mathrm{FIN}(E^*)$. Now, for such an $u \in Y \boxtimes L^0$, note that since both I_Y^X and $(Q_L^E)^*$ are canonical injections,

$$u(Q_L^E \circ f \circ I_Y^X) = \big((Q_L^E)^* u\big)(f \circ I_Y^X) = u(f).$$

Therefore, (5.5) is equivalent to (5.3) because α is finitely generated, finishing the proof. □

We can now show that maximal generic Lipschitz operator Banach ideals can be thought of as those arising as Lip_α for a finitely generated generic Lipschitz cross-norm α.

Corollary 5.2. *Let A be a generic Lipschitz operator Banach ideal. Then A is maximal if and only if there exists a finitely generated generic Lipschitz cross-norm α such that, for every pointed metric space X and every Banach space E,*

$$A(X, E) = (X \widehat{\boxtimes}_\alpha E^*)^* \cap \mathrm{Lip}_0(X, E) \tag{5.6}$$

holds isometrically. In this case,

$$A(X, E^*) = (X \widehat{\boxtimes}_\alpha E)^* \tag{5.7}$$

also holds isometrically for every pointed metric space X and every Banach space E.

Proof. Suppose that A is maximal. Let α be the finitely generated generic Lipschitz cross-norm associated to A given by Lemma 3.4. By the representation theorem, (5.6) holds isometrically.

Now suppose that there is a finitely generated generic Lipschitz cross-norm α such that (5.6) holds isometrically. It follows from the proof of the representation theorem that (5.7) must also hold isometrically, and so, in particular, $A \sim \alpha$. Let $f \in \mathrm{Lip}_0(X, E)$. If $f \in A(X, E)$, then, by the ideal property and the definition of $A^{\max}$, it follows that $\|f\|_{A^{\max}} \leq \|f\|_A$. Now assume that $f \in A^{\max}(X, E)$ with $\|f\|_{A^{\max}} \leq c$. By definition of $A^{\max}$, this means that (5.4) holds. Following the proof of the representation theorem, this in turn implies (5.3), which means that $f \in A(X, E)$ with $\|f\|_A \leq c$ because of (5.6). □

Another consequence is that a maximal generic Lipschitz operator Banach ideal respects the canonical embeddings into the bidual.

Corollary 5.3. *A maximal generic Lipschitz operator Banach ideal A is* regular*, which means that, for every pointed metric space X, every Banach space E, and every $f \in \mathrm{Lip}_0(X, E)$, $f \in A(X, E)$ if and only if $\kappa_E \circ f \in A(X, E^{**})$; moreover,*

$$\|f \colon X \to E\|_A = \|\kappa_E \circ f \colon X \to E^{**}\|_A.$$

Proof. Let α be the finitely generated generic Lipschitz cross-norm given by Corollary 5.2. Notice then that $A(X,E) \to (X \widehat{\boxtimes}_\alpha E^*)^* = A(X,E^{**})$, where the arrow is an isometry. The desired result follows. $\square$

6. Lipschitz operator ideals between metric spaces

Some important classes of Lipschitz maps satisfying an ideal property, like the Lipschitz p-summing maps or the maps admitting a Lipschitz factorization through a subset of an L_p space, are actually defined for maps between metric spaces. Thus it might seem that we are losing something by insisting on having a Banach space as a codomain as it has been done so far in this paper and in previous works (see [4], [6], [1], [2]). Nevertheless, we show next that generic Lipschitz operator Banach ideals satisfying a slightly stronger ideal property can be canonically extended to an ideal of Lipschitz maps between metric spaces. Recall that $\mathscr{F}(X)$ denotes the Lipschitz-free Banach space of a pointed metric space X, and that $\delta_X\colon X \to \mathscr{F}(X)$ denotes the canonical embedding. For a Banach space E, the barycentric map $\beta_E\colon \mathscr{F}(E) \to E$ is a norm 1 linear operator with $\beta_E \circ \delta_E = \mathrm{id}_E$ (see [9]).

Definition 6.1. We will say that a generic Lipschitz operator Banach ideal A is *strong* if, whenever X, Z are pointed metric spaces, E and F are Banach spaces, $f \in A(X,E)$, $h \in \mathrm{Lip}_0(Z,X)$, and $g \in \mathrm{Lip}_0(E,F)$, the composition gfh belongs to $A(Z,F)$ and $\|gfh\|_A \le \mathrm{Lip}(g)\|f\|_A \,\mathrm{Lip}(h)$.

The ideals Lip_0, Π_p^L, and Γ_p^L are examples of strong generic Lipschitz Banach ideals. The next proposition characterizes strong generic Lipschitz operator Banach ideals.

Proposition 6.2. *Let A be a generic Lipschitz operator Banach ideal. Then A is strong if and only if, for every pointed metric space X, every Banach space E, and every map $f \in \mathrm{Lip}_0(X,E)$, $f \in A(X,E)$ if and only if $\delta_E \circ f \in A(X,\mathscr{F}(E))$, and with $\|f\|_A = \|\delta_E \circ f\|_A$.*

Proof. Suppose that A is strong. Let X, E, and f be as above. If $f \in A(X,E)$, then by the ideal property $\delta_E \circ f \in A(X,\mathscr{F}(E))$ and

$$\begin{aligned}\|\delta_E \circ f\|_A &\le \mathrm{Lip}(\delta_E)\|f\|_A = \|f\|_A = \|\beta_E \circ \delta_E \circ f\|_A \\ &\le \mathrm{Lip}(\beta_E)\|\delta_E \circ f\|_A = \|\delta_E \circ f\|_A,\end{aligned}$$

and so $\|f\|_A = \|\delta_E \circ f\|_A$. The same chain of inequalities shows that if $\delta_E \circ f$ is in A, then so is f and with the same norm.

For the converse implication, let X, Z be pointed metric spaces, let E and F be Banach spaces, let $f \in A(X,E)$, let $h \in \mathrm{Lip}_0(Z,X)$, and let $g \in \mathrm{Lip}_0(E,F)$. By [11, Lemma 3.1], there exists a unique bounded linear operator $\widehat{g}\colon \mathscr{F}(E) \to \mathscr{F}(F)$ such that $\widehat{g} \circ \delta_E = \delta_F \circ g$. Furthermore, $\|\widehat{g}\| = \mathrm{Lip}(g)$. Since $f \in A(X,E)$, then by hypothesis $\delta_E \circ f \in A(X,\mathscr{F}(E))$, and thus by the ideal property $\delta_E \circ f \circ h \in A(Z,\mathscr{F}(E))$. Using the ideal property of generic Lipschitz operator Banach ideals again, we have $\widehat{g} \circ \delta_E \circ f \circ h = \delta_F \circ g \circ f \circ h \in A(Z,\mathscr{F}(F))$. By the hypothesis, we get $g \circ f \circ h \in A(Z, F)$. Moreover,

$$\begin{aligned}\|g \circ f \circ h\|_A &= \|\delta_F \circ g \circ f \circ h\|_A = \|\widehat{g} \circ \delta_E \circ f \circ h\|_A \\ &\le \|\widehat{g}\|\|\delta_E \circ f \circ h\|_A = \mathrm{Lip}(g)\|f \circ h\|_A \\ &\le \mathrm{Lip}(g)\|f\|_A \,\mathrm{Lip}(h). \qquad \square\end{aligned}$$

In the next result, we define an extension of the notion of the Lipschitz operator Banach ideal, now having a metric space as a codomain for the maps. The arguments are almost the same as those used to prove Proposition 6.2.

Proposition 6.3. *Let A be a strong generic Lipschitz operator Banach ideal. For any pointed metric spaces X and Y and $f \in \mathrm{Lip}_0(X, Y)$, define $f \in \tilde{A}(X, Y)$ if and only if $\delta_Y \circ f \in A(X, \mathscr{F}(Y))$, and denote $\|f\|_{\tilde{A}} = \|\delta_Y \circ f\|_A$.*

- (i) *For any pointed metric space X and any Banach space E, $f \in A(X, E)$ if and only if $f \in \tilde{A}(X, E)$, and, moreover, $\|f\|_{\tilde{A}} = \|f\|_A$.*
- (ii) *If $f \in \tilde{A}(X, Y)$, $h \in \mathrm{Lip}_0(W, X)$, and $g \in \mathrm{Lip}_0(Y, Z)$, then the composition $g \circ f \circ h$ belongs to $\tilde{A}(W, Z)$ and $\|g \circ f \circ h\|_{\tilde{A}} \le \mathrm{Lip}(g)\|f\|_{\tilde{A}} \,\mathrm{Lip}(h)$.*

Proof. (i) If $f \in A(X, E)$, then $\delta_E \circ f \in A(X, \mathscr{F}(E))$ and $\|\delta_E \circ f\|_A \le \mathrm{Lip}(\delta_E)\|f\|_A = \|f\|_A$ by the ideal property. Now assume that $\delta_E \circ f \in A(X, \mathscr{F}(E))$. Note that $\beta_E \circ \delta_E \circ f = f$, and so, by the ideal property, $f \in A(X, E)$ and

$$\|f\|_A = \|\beta_E \circ \delta_E \circ f\|_A \le \|\beta_E\|\|\delta_E \circ f\|_A = \|f\|_{\tilde{A}}.$$

(ii) By [11, Lemma 3.1], there exists a unique bounded linear operator $\widehat{g}\colon \mathscr{F}(Y) \to \mathscr{F}(Z)$ such that $\widehat{g} \circ \delta_Y = \delta_Z \circ g$. Furthermore, $\|\widehat{g}\| = \mathrm{Lip}(g)$. By the ideal property, $\delta_Z \circ g \circ f \circ h \in A(W, \mathscr{F}(Z))$ and $\|g \circ f \circ h\|_{\tilde{A}} = \|\delta_Z \circ (gf) \circ h\|_A = \|\widehat{g} \circ (\delta_Y \circ f) \circ h\|_A \le \|\widehat{g}\|\|\delta_Y \circ f\|_A \,\mathrm{Lip}(h) = \mathrm{Lip}(g)\|f\|_{\tilde{A}} \,\mathrm{Lip}(h)$. $\square$

In an abuse of notation, given a strong generic Lipschitz operator Banach ideal, we will still denote by A its extension to metric spaces (instead of $\tilde{A}$). We keep the notation $\|f\|_A$, though when we leave the Banach space context this is no longer a norm. Nevertheless, it still denotes a quantitative property of the map f.

The following result is interesting because it characterizes a nonlinear property in terms of a linear one closely related to [4, Theorem 4.6] and [5, Theorem 4.4]. Of course, as always happens in this kind of situation, we have simplified the mapping but made the spaces more complicated (cf. [8, Theorem 17.15]).

Theorem 6.4. *Let A be a strong and maximal generic Lipschitz operator Banach ideal, and let α be the finitely generated generic Lipschitz cross-norm which is associated to A. For any pointed metric spaces X and Y, and $f \in \mathrm{Lip}_0(X, Y)$, the following are equivalent:*

- (i) *$f \in A(X, Y)$,*
- (ii) *for all Banach spaces G (or only $G = Y^{\#}$), $f \boxtimes \mathrm{id}_G\colon X \widehat{\boxtimes}_\alpha G \to Y \widehat{\boxtimes}_\pi G$ is continuous.*

In this case, $\|f\|_A = \|f \boxtimes \mathrm{id}_{Y^{\#}}\colon X \widehat{\boxtimes}_\alpha Y^{\#} \to Y \widehat{\boxtimes}_\pi Y^{\#}\| \ge \|f \boxtimes \mathrm{id}_G\colon X \widehat{\boxtimes}_\alpha G \to Y \widehat{\boxtimes}_\pi G\|$.

Proof. Suppose that $f \in A(X, Y)$, and let G be a Banach space. The boundedness of $f \boxtimes \mathrm{id}_G \colon X \widehat{\boxtimes}_\alpha G \to Y \widehat{\boxtimes}_\pi G$ will follow from the boundedness of the adjoint map

$$(f \boxtimes \mathrm{id}_G)^* \colon (Y \widehat{\boxtimes}_\pi G)^* = \mathrm{Lip}_0(Y, G^*) \to (X \widehat{\boxtimes}_\alpha G)^* = A(X, G^*).$$

Now, for $v \in G$, $x \in X$, and $h \in \mathrm{Lip}_0(Y, G^*)$,

$$\begin{aligned} \big\langle \big[(f \boxtimes \mathrm{id}_G)^* h\big](x), v \big\rangle &= (\delta_{(x,0)} \boxtimes v)\big((f \boxtimes \mathrm{id}_G)^* h\big) = \big((f \boxtimes \mathrm{id}_G)[\delta_{(x,0)} \boxtimes v]\big)(h) \\ &= (\delta_{(f(x),0)} \boxtimes v)(h) = \big\langle h\big(f(x)\big), v \big\rangle. \end{aligned}$$

Therefore, $(f \boxtimes \mathrm{id}_G)^*$ is given by $h \in \mathrm{Lip}_0(Y, G^*) \mapsto h \circ f \in A(X, G^*)$, which has norm at most $\|f\|_A$ because of the ideal property.

Now suppose that $f \boxtimes \mathrm{id}_{Y^\#} \colon X \widehat{\boxtimes}_\alpha Y^\# \to Y \widehat{\boxtimes}_\pi Y^\#$ has norm c. By definition, $f \in A(X, Y)$ if and only if $\delta_Y \circ f \in A(X, \mathscr{F}(Y))$ and with the same norm, which by the representation theorem is equivalent to having $\delta_Y \circ f$ define an element of $(X \widehat{\boxtimes}_\alpha \mathscr{F}(Y)^*)^* = (X \widehat{\boxtimes}_\alpha Y^\#)^*$. Therefore, we seek to prove that, given $u \in X \widehat{\boxtimes} Y^\#$, $|u(\delta_Y \circ f)| \leq c\alpha(u)$. Note that, for a given $u \in X \widehat{\boxtimes} Y^\#$, $u(f \boxtimes \mathrm{id}_{Y^\#})$ belongs to $Y \widehat{\boxtimes} Y^\#$. Since $\kappa_{\mathscr{F}(Y)} \circ \delta_Y \colon Y \to Y^{\#*}$, we may consider $[(f \boxtimes \mathrm{id}_{Y^\#})u](\kappa_{\mathscr{F}(Y)} \circ \delta_Y)$. Note that this is in fact just $u(f)$ since δ_Y, $\kappa_{\mathscr{F}(Y)}$, and $\mathrm{id}_{Y^\#}$ are inclusions. Therefore,

$$\begin{aligned} \big|u(f)\big| &= \big|\big[(f \boxtimes \mathrm{id}_{Y^\#})u\big](\kappa_{\mathscr{F}(Y)} \circ \delta_Y)\big| \\ &\leq \mathrm{Lip}(\kappa_{\mathscr{F}(Y)} \circ \delta_Y)\pi\big((f \boxtimes \mathrm{id}_{Y^\#})u\big) \leq c\alpha(u), \end{aligned}$$

and the conclusion follows. □

Remark 6.5. In the previous proof, when the codomain is a Banach space E (resp., F^*) in part (ii) it suffices to consider $G = E^*$ (resp., $G = F$).

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