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UNIFORM OPENNESS OF MULTIPLICATION IN ORLICZ SPACES

IBRAHIM AKBARBAGLU,[1] SAEID MAGHSOUDI,[2*] and IRAJ RAHMANI[2]

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Abstract. Let Φ and Ψ be Young functions, and let $L^{\Phi}(\Omega)$ and $L^{\Psi}(\Omega)$ be corresponding Orlicz spaces on a measure space (Ω, μ). Our aim in this paper is to prove that, under mild conditions on Φ and Ψ, the multiplication from $L^{\Phi}(\Omega) \times L^{\Psi}(\Omega)$ onto $L^{1}(\Omega)$ is uniformly open. This generalizes an interesting recent result due to M. Balcerzak, A. Majchrzycki, and F. Strobin in 2013.

1. Introduction and preliminaries

Let X be a normed space. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. Following [4], we call a mapping $f : X \to Y$ *uniformly open* if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X \quad B\big(f(x), \delta\big) \subseteq f\big(B(x, \varepsilon)\big).$$

It can be observed that the function $\sqrt[p]{x}$, $p > 1$, is an open function from $[0, \infty)$ into $[0, \infty)$ but not uniformly open.

There is an increasing interest in the study of concepts related to the openness of natural bilinear maps on certain function spaces (see, e.g., [4], [6]–[8], [13]; also, for a nice survey, see [5]). Part of the reason for this may be that the classical Banach open mapping principle is not true for bilinear maps. The classical open mapping principle has been generalized in some directions (see, e.g., [10]).

In [4], the authors show that multiplication from $L^p(X) \times L^q(X)$ onto $L^1(X)$ is an open mapping where (X, μ) is an arbitrary measure space and $1 \le p, q \le$

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*Corresponding author.
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$\infty, 1/p + 1/q = 1$. Recently, they proved a stronger result: the multiplication is uniformly open [3].

In this paper, we prove, among other things, that the multiplication is uniformly open in the more general setting of Orlicz spaces.

Before going further, let us, for the reader's convenience, recall some necessary definitions concerning Orlicz spaces.

Orlicz spaces are genuine generalizations of the usual L^p-spaces. We refer to the two excellent books [9] and [12] for more details; also, [11] and [14] provide some useful information on the subject.

A function $\Phi : \mathbb{R} \to [0, \infty]$ is called a *Young function* if Φ is convex, even, and left-continuous with $\Phi(0) = 0$; we also assume that Φ is neither identically zero nor identically infinite on $\mathbb{R}$. For any Young function Φ, we define

$$a_\Phi = \sup\{x \in \mathbb{R} : \Phi(x) = 0\} \qquad \text{and} \qquad b_\Phi = \sup\{x \in \mathbb{R} : \Phi(x) < \infty\}.$$

A Young function Φ is called *finite* if $b_\Phi = \infty$. The Young function Ψ defined by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\} \in [0, \infty]$$

for $y \in \mathbb{R}$ is called the *complementary function* to Φ, and (Φ, Ψ) is called a *complementary pair* of Young functions. We also need an inverse of the Young function Φ. For a Young function Φ and $y \in [0, \infty)$, let

$$\Phi^{-1}(y) = \sup\{x \geq 0 : \Phi(x) \leq y\}.$$

For all $x \geq 0$, $\Phi(\Phi^{-1}(x)) \leq x$, and if $\Phi(x) < \infty$, then $x \leq \Phi^{-1}(\Phi(x))$. Moreover, $\Phi(\Phi^{-1}(x)) = x$ for $x \in [0, \Phi(b_\Phi)]$, and $x = \Phi^{-1}(\Phi(x))$ for $x \in [a_\Phi, b_\Phi]$. Also, if (Φ, Ψ) is a complementary of Young functions, then by Proposition 2.1.1(ii) in [12] we have

$$x \leq \Phi^{-1}(x)\Psi^{-1}(x) \leq 2x \tag{1.1}$$

for all $x \geq 0$.

We say a Young function Φ is Δ_2-*regular* whenever there exist $k > 0$ and $x_0 \geq 0$ such that $\Phi(2x) \leq k\Phi(x)$ for all $x \geq x_0$ with possibly $x_0 > 0$ if $\mu(\Omega) < \infty$ and $x_0 = 0$ otherwise.

Let (Ω, Σ, μ) be a measure space, and let Φ be a Young function. By $L^0(\Omega)$ we denote the linear space of equivalence classes of Σ-measurable real-valued functions on Ω; that is, we identify any two functions that are equal μ-almost everywhere on Ω. For each $f \in L^0(\Omega)$, we define

$$\rho_\Phi(f) = \int_\Omega \Phi(|f(x)|)\, d\mu(x).$$

Given a Young function Φ, the *Orlicz space* $L^\Phi(\Omega)$ is defined by

$$L^\Phi(\Omega) = \{f \in L^0(\Omega) : \rho_\Phi(af) < \infty, \text{for some } a > 0\}.$$

Similarly, let

$$M^\Phi(\Omega) = \{f \in L^0(\Omega) : \rho_\Phi(af) < \infty, \text{for all } a > 0\}.$$

Let us recall that $M^\Phi(\Omega)$ is nontrivial if Φ is finite and $M^\Phi(\Omega) = L^\Phi(\Omega)$ if and only if Φ is Δ_2-regular.

The Orlicz space $L^\Phi(\Omega)$ is a Banach space under the norm $N_\Phi(\cdot)$ (called the *Luxemburg–Nakano norm*) defined for $f \in L^\Phi(\Omega)$ by

$$N_\Phi(f) = \inf\big\{k > 0 : \rho_\Phi(f/k) \le 1\big\}.$$

It is well known that

$$N_\Phi(f) \le 1 \quad \text{if and only if } \rho_\Phi(f) \le 1,$$

and if $0 < \mu(F) < \infty$, then

$$N_\Phi(\chi_F) = \Big[\Phi^{-1}\Big(\frac{1}{\mu(F)}\Big)\Big]^{-1}$$

(see [12, Corollary 3.4.7]). Here χ_A denotes the characteristic function of a set $A \in \Sigma$.

If Φ vanishes only at zero, then another well-known norm on $L^\Phi(\Omega)$ called the *Orlicz norm* is defined by

$$\|f\|_\Phi = \sup\Big\{\int_\Omega |fg|\, d\mu : N_\Psi(g) \le 1\Big\}.$$

It follows from Proposition 3.3.4 in [12] that the Orlicz norm is equivalent to the Luxemburg–Nakano norm; in fact, we have

$$N_\Phi(f) \le \|f\|_\Phi \le 2N_\Phi(f)$$

for all $f \in L^\Phi(\Omega)$. The open ball at the center f and of radius r in $L^\Phi(\Omega)$ is denoted by $B_\Phi(f, r)$, and its closure is denoted by $\overline{B_\Phi}(f, r)$.

Finally, as an elementary example of the Young function, we can consider $\Phi(x) = |x|^p/p$ for $p > 1$. Then $\Psi(x) = |x|^q/q$, where $1/p + 1/q = 1$. Using this function Φ, we recover the classical Lebesgue spaces; that is, $L^\Phi(\Omega) = L^p(\Omega)$. We denote the open ball at the center f and of radius r in $L^1(\Omega)$ by $B_1(f, r)$.

2. Main results

We start with a result on normed Riesz spaces. For notation and terminology concerning Riesz spaces we refer to [2].

Let X be a normed space. We equip the space $X \times X$ with the norm

$$\big\|(x, y)\big\| = \max\big\{\|x\|, \|y\|\big\} \quad \big((x, y) \in X \times X\big).$$

Proposition 2.1. *Suppose that (L, ρ) is a normed Riesz space. Then the addition, infimum, and supremum operations on (L, ρ) are uniformly open.*

Proof. It is easy to see that $B(a, r) + B(b, s) = B(a + b, r + s)$ for all $a, b \in L$ and real numbers $r, s > 0$; we see that the addition is a uniformly open map.

We show that the infimum is a uniformly open map, and the proof of the supremum operation is similar. To this end, we only need to prove that $B(a \wedge b, r) \subseteq B(a, r) \wedge B(b, r)$ for all $a, b \in L$ and every real number $r > 0$. Let $z \in B(a \wedge b, r)$, and put

$$z_1 := a + (z - a \wedge b), \qquad z_2 := b + (z - a \wedge b).$$

Clearly, $z_1 \in B(a,r)$ and $z_2 \in B(b,r)$. Since

$$\begin{aligned} z_1 \wedge z_2 &= \big(a + (z - a \wedge b)\big) \wedge \big(b + (z - a \wedge b)\big) \\ &= \big(z \vee (a - b + z)\big) \wedge \big(z \vee (b - a + z)\big) \\ &= z \vee \big((a - b + z) \wedge (b - a + z)\big) \end{aligned}$$

and $(a - b + z) \wedge (b - a + z) = z - |b - a| \leq z$, we conclude that $z_1 \wedge z_2 = z \vee (z - |b - a|) = z$, which proves our claim. □

To prove our main result, we need some lemmas. The first one that shows the multiplication is uniformly open from $\mathbb{R} \times \mathbb{R}$ onto $\mathbb{R}$ was proved in [3].

Lemma 2.2. *If $r, R > 0$ and $x, y \in \mathbb{R}$, then*

$$\Big(xy - \frac{rR}{4}, xy + \frac{rR}{4}\Big) \subseteq (x - r, x + r) \cdot (y - R, y + R).$$

We fix an abstract measure space (Ω, Σ, μ), where μ is a measure on the σ-algebra Σ of subsets of a point set Ω.

We formulate and prove the following lemma for the sake of completeness.

Lemma 2.3. *Let Φ be a Young function that satisfies the Δ_2-regular condition. Then for every nonzero function f in $L^\Phi(\Omega)$ and every $\epsilon > 0$, there exists a measurable subset A such that $\mu(A) < \infty$, $\sup\{|f(x)| : x \in A\} < \infty$, and $N_\Phi(f\chi_{\Omega\setminus A}) < \epsilon$.*

Proof. Let $f \in L^\Phi(\Omega)$ be an arbitrary nonzero function. For every $k \in \mathbb{N}$, put $A_k = \{x \in \Omega : 1/k < |f(x)| < k\}$. Then (A_k) is an increasing sequence of subsets from Σ and $\{x \in \Omega : 0 < |f(x)| < \infty\} = \bigcup_{k\in\mathbb{N}} A_k$. Since $f \in L^\Phi(\Omega)$, we find that $\mu(\{x \in \Omega : |f(x)| = \infty\}) = 0$ and for any $a > 0$, by the Beppo Levi theorem,

$$\int_\Omega \Phi\big(a\big|f(x)\big|\big)\, d\mu(x) = \int_{\bigcup_{k\in\mathbb{N}} A_k} \Phi\big(a\big|f(x)\big|\big)\, d\mu(x) = \lim_{k\to\infty} \int_{A_k} \Phi\big(a\big|f(x)\big|\big)\, d\mu(x).$$

This together with the assumption that Φ satisfies the Δ_2-regular condition implies that, for every $a > 0$, there exists a natural number n depending on a such that

$$\int_{\Omega\setminus A_n} \Phi\big(a\big|f(x)\big|\big)\, d\mu(x) = \int_{(\bigcup_{k\in\mathbb{N}} A_k)\setminus A_n} \Phi\big(a\big|f(x)\big|\big)\, d\mu(x) < 1.$$

Now, take $a > 1/\epsilon$; hence, for this $n = n(a)$, $\sup\{|f(x)| : x \in A_n\} < \infty$ and, also, since $a > 0$ can be chosen sufficiently large and $|f(x)| > 1/n$, we get $\mu(A_n) < \infty$ and $N_\Phi(f\chi_{\Omega\setminus A_n}) < 1/a < \epsilon$. Put $A = A_n$ to finish the proof. □

Following [3], we call a measurable function $f : \Omega \to \mathbb{R}$ *countably valued* if $f = \sum_{n\in\mathbb{N}} a_n\chi_{A_n}$ for a sequence (a_n) of real numbers and a sequence (A_n) of measurable subsets with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ that forms a partition of Ω.

We need another technical lemma. Here we adopt a technique that was introduced in [3] to our setting.

Lemma 2.4. *Let (Φ, Ψ) be a complementary pair of Young functions. Then, for every $\epsilon > 0$ and any countably valued functions $f \in L^\Phi(\Omega)$, $g \in L^\Psi(\Omega)$, and $h \in B_1(fg, \epsilon^2/4)$, we have $h \in \overline{B_\Phi}(f, \epsilon) \cdot \overline{B_\Psi}(g, \epsilon)$.*

Proof. Let $\epsilon > 0$, and let $f \in L^\Phi(\Omega)$, $g \in L^\Psi(\Omega)$, and $h \in L^1(\Omega)$ be countably valued functions such that $\|fg - h\|_1 < \epsilon^2/4$. Then, by definition, there exists a sequence (A_n) of measurable sets that form a partition of Ω and sequences of reals (x_n), (y_n), and (z_n) such that

$$f = \sum_{n\in\mathbb{N}} x_n\chi_{A_n}, \qquad g = \sum_{n\in\mathbb{N}} y_n\chi_{A_n}, \qquad \text{and} \qquad h = \sum_{n\in\mathbb{N}} z_n\chi_{A_n}.$$

Let $E = \{n \in \mathbb{N} : z_n = x_n y_n \text{ or } \mu(A_n) = 0\}$. For all $n \in E$, put

(i) $u_n = x_n$ and $v_n = y_n$ if $z_n = x_n y_n$;
(ii) $u_n = \sqrt{|z_n|}$ and $v_n = \sqrt{|z_n|}\,\mathrm{sgn}(z_n)$ if $\mu(A_n) = 0$ and $z_n \neq x_n y_n$.

We may assume that $E \neq \mathbb{N}$; otherwise, there is nothing to prove. Put

$$\eta = \|fg - h\|_1 = \sum_{n\notin E} |z_n - x_n y_n|\mu(A_n)$$

and, for every $n \notin E$,

$$\lambda_n = |z_n - x_n y_n|\mu(A_n)/\eta.$$

Then

$$\eta < \epsilon^2/4, \quad \lambda_n \in (0,1] \quad \text{for all } n \notin E, \qquad \text{and} \qquad \sum_{n\notin E} \lambda_n = 1.$$

Now, for every $k \notin E$, we have

$$|z_k - x_k y_k| = \frac{\lambda_k \eta}{\mu(A_k)} < \frac{\lambda_k}{\mu(A_n)}\frac{\epsilon^2}{4} \overset{(1.1)}{\leq} \frac{1}{4}\Big(\epsilon\Phi^{-1}\Big(\frac{\lambda_k}{\mu(A_k)}\Big)\Big)\Big(\epsilon\Psi^{-1}\Big(\frac{\lambda_k}{\mu(A_k)}\Big)\Big).$$

Applying Lemma 2.2 to $r_k = \epsilon\Phi^{-1}(\frac{\lambda_k}{\mu(A_k)})$ and $R_k = \epsilon\Psi^{-1}(\frac{\lambda_k}{\mu(A_k)})$, we can find real numbers u_k, v_k such that $z_k = u_k v_k$ with $|u_k - x_k| < r_k$ and $|v_k - y_k| < R_k$. Set $u = \sum_{n\in\mathbb{N}} u_n\chi_{A_n}$, and set $v = \sum_{n\in\mathbb{N}} v_n\chi_{A_n}$. Then $h = uv$, and

$$\int_\Omega \Phi\Big(\frac{|u(t) - f(t)|}{\epsilon}\Big)\, d\mu(t) = \sum_{n\in\mathbb{N}} \Phi\Big(\frac{|u_n - x_n|}{\epsilon}\Big)\mu(A_n) \leq \sum_{n\notin E} \lambda_n = 1.$$

Hence $N_\Phi(u - f) \leq \epsilon$. Also, in the same way we have $N_\Psi(v - g) \leq \epsilon$. □

Now we are in a position to present our main result, which generalizes, using a similar technique, the main theorems in [3] and [4].

Theorem 2.5. *Let (Ω, Σ, μ) be a measure space, and let (Φ, Ψ) be a complementary pair of Young functions. If Φ and Ψ both satisfy the Δ_2-regular condition, then, for any $\epsilon > 0$ and $(f, g) \in L^\Phi(\Omega) \times L^\Psi(\Omega)$, we have*

$$B_1\Big(fg, \frac{\epsilon^2}{4}\Big) \subseteq B_\Phi(f, \epsilon) \cdot B_\Psi(g, \epsilon).$$

In particular, the multiplication mapping from $L^\Phi(\Omega) \times L^\Psi(\Omega)$ onto $L^1(\Omega)$ is a uniformly open mapping.

Proof. Let $\epsilon > 0$, and let $(f, g) \in L^\Phi(\Omega) \times L^\Psi(\Omega)$. Take $h \in B_1(fg, \epsilon^2/4)$. First we consider the case that (Ω, Σ, μ) is a finite measure space and f, g, h are bounded functions. Let $M > 0$ be such that $\mu(\Omega) < M$ and $\sup_{x\in\Omega}\{|f(x)|, |g(x)|, |h(x)|\} < M$. Since $\sqrt{4\|h - fg\|_1} < \epsilon$, we can choose $\epsilon_1 > 0$ such that

$$\epsilon_1 + \sqrt{4\|h - fg\|_1 + 8\epsilon_1} < \epsilon. \tag{2.1}$$

Now let $\delta > 0$ be such that

$$\delta < \epsilon_1 \min\Big\{\Phi^{-1}\Big(\frac{1}{M}\Big), \Psi^{-1}\Big(\frac{1}{M}\Big), \frac{1}{2M^2}\Big\}. \tag{2.2}$$

Let $\widetilde{f}$ be a countably valued function defined by

$$\widetilde{f}(x) = \begin{cases} k\delta & \text{if } f(x) \in [k\delta, (k+1)\delta) \text{ and } k \in \mathbb{N} \cup \{0\}; \\ -k\delta & \text{if } f(x) \in [(-k-1)\delta, -k\delta) \text{ and } k \in \mathbb{N} \cup \{0\}. \end{cases}$$

Let $\widetilde{g}$ be a countably valued function associated with g in an analogous way. Then, for all $x \in \Omega$, we have

$$\big|f(x) - \widetilde{f}(x)\big| < \delta, \qquad \big|g(x) - \widetilde{g}(x)\big| < \delta, \qquad \big|\widetilde{f}(x)\big| < M, \qquad \big|\widetilde{g}(x)\big| < M,$$

and

$$\begin{aligned} \big|f(x)g(x) - \widetilde{f}(x)\widetilde{g}(x)\big| &\le \big|f(x)g(x) - \widetilde{f}(x)g(x)\big| + \big|\widetilde{f}(x)g(x) - \widetilde{f}(x)\widetilde{g}(x)\big| \\ &< M\delta + M\delta = 2M\delta. \end{aligned}$$

Hence, by (2.2), we obtain

$$\begin{aligned} &N_\Phi(f - \widetilde{f}) \le \frac{\delta}{\Phi^{-1}(\frac{1}{M})} < \epsilon_1, \qquad N_\Psi(g - \widetilde{g}) \le \frac{\delta}{\Psi^{-1}(\frac{1}{M})} < \epsilon_1, \\ &\|fg - \widetilde{f}\widetilde{g}\|_1 \le 2M^2\delta < \epsilon_1. \end{aligned} \tag{2.3}$$

Now, in view of (2.1), we can choose a positive number α such that

$$\alpha < 1, \tag{2.4}$$

$$(1-\alpha)M^2 < \epsilon_1, \tag{2.5}$$

$$\epsilon_1 + \frac{1-\alpha}{\alpha}M + \frac{1}{\alpha}\sqrt{4\|h - fg\|_1 + 8\epsilon_1} < \epsilon. \tag{2.6}$$

Let $a_n = \alpha^n M$ for all $n \in \mathbb{N} \cup \{0\}$, and note that $a_n \searrow 0$ because $\alpha \in (0, 1)$.

Define $\widetilde{h}$ in the following way:

$$\widetilde{h}(x) = \begin{cases} 0 & \text{if } h(x) = 0; \\ a_{n+1} & \text{if } h(x) \in [a_{n+1}, a_n) \text{ and } n \in \mathbb{N} \cup \{0\}; \\ -a_{n+1} & \text{if } h(x) \in (-a_n, -a_{n+1}] \text{ and } n \in \mathbb{N} \cup \{0\}. \end{cases}$$

Clearly, $\widetilde{h}$ is countably valued, bounded by M, and, for every $x \in \Omega$ with $h(x) \neq 0$,

$$1 \le \frac{h(x)}{\widetilde{h}(x)} \le \frac{a_n}{a_{n+1}} = \frac{1}{\alpha}. \tag{2.7}$$

Moreover, $|h(x) - \widetilde{h}(x)| < (1-\alpha)M$ for every $x \in \Omega$. Hence, by (2.3) and (2.5), we have

$$\begin{aligned} \|\widetilde{h} - \widetilde{f}\widetilde{g}\|_1 &\le \|\widetilde{h} - h\|_1 + \|h - fg\|_1 + \|fg - \widetilde{f}\widetilde{g}\|_1 \\ &< M^2(1-\alpha) + \|h - fg\|_1 + \epsilon_1 \\ &\le \|h - fg\|_1 + 2\epsilon_1. \end{aligned} \tag{2.8}$$

Let

$$\epsilon_2 = \sqrt{4\|h - fg\|_1 + 8\epsilon_1}. \tag{2.9}$$

Then, by (2.8), $\|\widetilde{h} - \widetilde{f}\widetilde{g}\|_1 < \epsilon_2^2/4$, and, by Lemma 2.4, there are functions $u \in L^\Phi(\Omega)$ and $v \in L^\Psi(\Omega)$ such that $\widetilde{h} = uv$, $u \in \overline{B_\Phi}(\widetilde{f}, \epsilon_2)$, and $v \in \overline{B_\Psi}(\widetilde{g}, \epsilon_2)$.

Now define a function $\theta : \Omega \to \mathbb{R}$ in the following way:

$$\theta(x) = \begin{cases} 1 & \text{if } \widetilde{h}(x) = 0; \\ \frac{h(x)}{\widetilde{h}(x)} & \text{if } \widetilde{h}(x) \neq 0. \end{cases}$$

By (2.7), $1 \le \theta(x) \le 1/\alpha$ for every $x \in \Omega$, and $h = \theta\widetilde{h} = (\theta u)v$. Finally, by (2.1), (2.3), (2.6), and (2.9), we have

$$N_\Psi(g - v) \le N_\Psi(g - \widetilde{g}) + N_\Psi(\widetilde{g} - v) < \epsilon_1 + \epsilon_2 < \epsilon$$

and

$$\begin{aligned} N_\Phi(f - \theta u) &\le N_\Phi(f - \widetilde{f}) + N_\Phi(\widetilde{f} - \theta u) \\ &\le N_\Phi(f - \widetilde{f}) + N_\Phi(\widetilde{f} - \theta\widetilde{f}) + N_\Phi(\theta\widetilde{f} - \theta u) \\ &\le \epsilon_1 + N_\Phi\big((\theta - 1)\widetilde{f}\big) + N_\Phi\big(\theta(\widetilde{f} - u)\big) \\ &\le \epsilon_1 + \Big(\frac{1}{\alpha} - 1\Big)N_\Phi(\widetilde{f}) + \frac{1}{\alpha}N_\Phi(\widetilde{f} - u) \\ &\le \epsilon_1 + \frac{1-\alpha}{\alpha}M + \frac{1}{\alpha}\epsilon_2 < \epsilon. \end{aligned}$$

This proves the first case.

Now we turn to the general case. As before, let $\epsilon > 0$ be arbitrary, and let $(f, g) \in L^\Phi(\Omega) \times L^\Psi(\Omega)$. Let $\delta \in (0, \epsilon)$ be such that

$$\|h - fg\|_1 < \frac{\delta^2}{4}, \tag{2.10}$$

and let $\gamma > 0$ be such that

$$\delta + 2\gamma < \epsilon. \tag{2.11}$$

Since Φ and Ψ both satisfy the Δ_2-regular condition, then, by Lemma 2.3, we obtain a measurable subset A such that $\mu(A) < \infty$, $f|_A$, $g|_A$, and $h|_A$ are bounded and

$$\begin{aligned} &N_\Phi(f\chi_{\Omega\setminus A}) < \gamma, \qquad N_\Psi(g\chi_{\Omega\setminus A}) < \gamma, \\ &N_\Phi\big(\Phi^{-1}(|h|)\chi_{\Omega\setminus A}\big) < \gamma, \qquad N_\Psi\big(\Psi^{-1}(|h|)\chi_{\Omega\setminus A}\big) < \gamma. \end{aligned} \tag{2.12}$$

Using (2.10) and the first part of the proof for the space $(A, \Sigma|_A, \mu|_A)$, we infer that there exist $u \in L^{\Phi}(A)$ and $v \in L^{\Psi}(A)$ such that $h(x) = u(x)v(x)$ for $x \in A$ and

$$N_{\Phi}\big((f-u)\chi_A\big) < \delta \qquad \text{and} \qquad N_{\Psi}\big((g-v)\chi_A\big) < \delta. \tag{2.13}$$

Additionally, define

$$u(x) = \Phi^{-1}\big(|h(x)|\big), \qquad v(x) = \frac{|h(x)|\,\mathrm{sgn}(h(x))}{\Phi^{-1}(|h(x)|)}\chi_B(x) \quad (x \notin A),$$

where $B = \{x \in \Omega : h(x) \neq 0\}$. Then $h = uv$, and by (2.11), (2.12), and (2.13), we have

$$N_{\Phi}(f-u) \leq N_{\Phi}\big((f-u)\chi_A\big) + N_{\Phi}(f\chi_{\Omega\setminus A}) + N_{\Phi}\big(\Phi^{-1}\big(|h|\big)\chi_{\Omega\setminus A}\big) < \delta + 2\gamma < \epsilon,$$

and, analogously,

$$\begin{aligned} N_{\Psi}(g-v) &\leq N_{\Psi}\big((g-v)\chi_A\big) + N_{\Psi}(g\chi_{\Omega\setminus A}) + N_{\Psi}(v\chi_{\Omega\setminus A}) \\ &\overset{(1.1)}{\leq} \delta + \gamma + N_{\Psi}\big(\Psi^{-1}\big(|h|\big)\chi_{\Omega\setminus A}\big) \\ &< \delta + 2\gamma < \epsilon. \end{aligned}$$

□

We finish this work with the following example.

Example 2.6. (a) Let Φ be a Young function, and let (Ω, μ) be a measure space. Equip $L^{\Phi}(\Omega)$ with the usual ordering $f \leq g$ if and only if $f(t) \leq g(t)$ μ-almost everywhere on Ω. Then the infimum and supremum from $L^{\Phi}(\Omega) \times L^{\Phi}(\Omega)$ into $L^{\Phi}(\Omega)$ are uniformly open by Proposition 2.1.

(b) Consider the natural numbers with the counting measure. Let Φ and its complementary function, Ψ, satisfy the Δ_2-regular condition. Then

$$\ell^{\Phi}(\mathbb{N}) := \big\{(x_n) : \Sigma_{n=1}^{\infty}\Phi\big(|x_n|\big) < \infty\big\}.$$

From Theorem 2.5 it follows that the multiplication from $\ell^{\Phi}(\mathbb{N}) \times \ell^{\Psi}(\mathbb{N})$ onto $\ell^{1}(\mathbb{N})$ is uniformly open.

(c) Consider the real line $\mathbb{R}$ with the Lebesgue measure. Let $\Phi_1(x) = |x|^{\alpha}(1 + |\log|x||)$ with $\alpha > 1$, and let $\Phi_2(x) = |x|^{\alpha}\ln^{\beta}(|x| + e)$ with $\beta \geq 1$ and $\alpha > 1$. By Corollary 2.3.4 in [12], for any $i = 1, 2$, Φ_i and its complementary function, Ψ_i, satisfy the Δ_2-regular condition. Then Theorem 2.5 shows that the multiplication from $L^{\Phi_i}(\mathbb{R}) \times L^{\Psi_i}(\mathbb{R})$ onto $L^1(\mathbb{R})$ is uniformly open. Note that, by Theorems 3.1 and 3.2 in [1], we see that, for any $p > 1$, $L^{\Phi_1}(\mathbb{R}) \triangle L^p(\mathbb{R}) \neq \emptyset$ and there is not any $p > \alpha$ such that $L^{\Phi_2}(\mathbb{R}) \subseteq L^p(\mathbb{R})$.

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References

1. I. Akbarbaglu and S. Maghsoudi, *Large structures in certain subsets of Orlicz spaces*, Linear Algebra Appl. **438** (2013), no. 11, 4363–4373. Zbl 1292.46017. MR3034537. DOI 10.1016/j.laa.2013.01.038. 550
2. C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics*, 2nd ed., Math. Surveys Monogr. **105**, Amer. Math. Soc., Providence, RI, 2003. Zbl 1043.46003. MR2011364. DOI 10.1090/surv/105. 545
3. M. Balcerzak, A. Majchrzycki, and F. Strobin, *Uniformly openness of multiplication in Banach spaces* L_p, preprint. `http://arxiv.org/abs/1309.3433`. 544, 546, 547
4. M. Balcerzak, A. Majchrzycki, and A. Wachowicz, *Openness of multiplication in some function spaces*, Taiwanese J. Math. **17** (2013), no. 3, 1115–1126. Zbl 1292.46009. MR3072279. DOI 10.11650/tjm.17.2013.2521. 543, 547
5. M. Balcerzak, F. Strobin, and A. Wachowicz, "Bilinear mappings: selected properties and problems" in *Traditional and Present-Day Topics in Real Analysis*, Univ. of Łódź Press, Łódź, 2013. Zbl 0652.7816. MR3183772. 543
6. M. Balcerzak, A. Wachowicz, and W. Wilczyński, *Multiplying balls in the space of continuous functions on* $[0, 1]$, Studia Math. **170** (2005), no. 2, 203–209. Zbl 1093.46023. MR2184105. DOI 10.4064/sm170-2-5. 543
7. A. Komisarski, *A connection between multiplication in* $C(X)$ *and the dimension of* X, Fund. Math. **189** (2006), no. 2, 149–154. Zbl 1093.54004. MR2214575. DOI 10.4064/fm189-2-4. 543
8. S. Kowalczyk, *On operations in* $C(X)$ *determined by continuous functions*, Acta Math. Hungar. **142** (2014), no. 1, 56–71. Zbl pre06270605. MR3158851. DOI 10.1007/s10474-013-0329-5. 543
9. M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961. Zbl 0095.09103. MR0126722. 544
10. R. Li, S. Zhong, and C. Swartz, *An open mapping theorem without continuity and linearity*, Topology Appl. **157** (2010), no. 13, 2086–2093. Zbl 1207.46005. MR2665232. DOI 10.1016/j.topol.2010.05.004. 543
11. J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. **1034**, Springer, Berlin, 1983. Zbl 0557.46020. MR0724434. 544
12. M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Pure Appl. Math. **146**, Dekker, New York, 1991. Zbl 0724.46032. MR1113700. DOI 10.1080/03601239109372748. 544, 545, 550
13. A. Wachowicz, *Multiplying balls in* $C^{(N)}[0, 1]$, Real Anal. Exchange **34** (2009), no. 2, 445–450. Zbl 1184.46051. MR2569197. 543
14. A. C. Zaanen, *Riesz Spaces, II*, North-Holland Math. Library **30**, North-Holland, Amsterdam, 1983. Zbl 0519.46001. MR0704021. 544

[1]Department of Mathematics, Farhangian University, Tabriz, Iran.
E-mail address: ibrahim.akbarbaglu@gmail.com

[2]Department of Mathematics, University of Zanjan, Zanjan 45371-38791, Iran.
E-mail address: s_maghsodi@znu.ac.ir; i_rahmani@znu.ac.ir