Ann. Funct. Anal. 7 (2016), no. 4, 678–685
http://dx.doi.org/10.1215/20088752-3661179
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

GATEAUX DERIVATIVE OF THE NORM IN $\mathcal{K}(X;Y)$

PAWEŁ WÓJCIK

Communicated by G. Androulakis

Abstract. In this article, we consider the φ-Gateaux derivative of the norm in spaces of compact operators in such a way as to extend the Kečkić theorem. Our main result determines the φ-Gateaux derivative of the $\mathcal{K}(X;Y)$ norm.

1. Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a normed space, and let $x, y \in X$. The *directional derivative of the norm at x in the y-direction* is defined by

$$D(x,y) := \lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}, \quad x,y \in X.$$

Convexity of the norm yields that the above definition is meaningful. The norm derivative is important in approximation theory and in the geometry of Banach spaces. In [6], the concept of φ-Gateaux derivatives was developed in order to substitute the usual concept of Gateaux derivatives at points which are not smooth. Let $\varphi \in [0, 2\pi)$, or let $\varphi \in \{0, -\pi\}$, if the space X is over $\mathbb{R}$. The *φ-Gateaux derivative of the norm at x in the φ, y-direction* is defined by

$$D_\varphi(x,y) := \lim_{t\to 0^+} \frac{\|x+te^{i\varphi}y\| - \|x\|}{t}, \quad x,y \in X.$$

It is a straightforward verification to show that

$$D_\varphi(x,y) = D(x, e^{i\varphi}y), \quad x,y \in X. \tag{1.1}$$

Received Mar. 31, 2016; Accepted Jul. 11, 2016.
2010 *Mathematics Subject Classification.* Primary 46B20; Secondary 47L05, 46G05.
Keywords. space of compact operator, Gateaux derivative, dual space, adjoint operator, extreme point.

Given a normed space X and a Banach space Y, both over the same field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), we write $\mathcal{K}(X;Y)$ for the space of all compact operators from X into Y. For $A \in \mathcal{K}(X;Y)$, put $\mathcal{M}(A) := \{y \in S_X : \|Ay\| = \|A\|\}$. It is known that in this case $\mathcal{M}(A^*) \neq \emptyset$. But if X is reflexive, then $\mathcal{M}(A) \neq \emptyset$. Kečkić [6] proved the following theorem.

Theorem 1.1 ([6, Theorem 2.6]). *Let H be a complex Hilbert space, $A, B \in \mathcal{K}(H)$. Then*

$$D_\varphi(A,B) = \max\big\{D_\varphi(Ay, By) : y \in \mathcal{M}(A)\big\}.$$

Similar investigations have been carried out by Kečkić in $\mathcal{B}(H)$ (see [7]). In the present article, we will generalize Theorem 1.1 (see Theorems 2.2 and 2.3). The method of proof presented here is different from that of [6] and [7]. Furthermore, our proofs include both real and complex cases. The unit sphere of X is denoted by S_X. Fix $x \in X \setminus \{0\}$. We consider the set $J(x)$ defined as follows:

$$J(x) := \big\{x^* \in X^* : \|x^*\| = 1, x^*(x) = \|x\|\big\}. \tag{1.2}$$

It is easy to check that the set $J(x)$ is convex and closed and that $J(x) \subset S_{X^*}$. By the Hahn–Banach theorem, we get $J(x) \neq \emptyset$ for all $x \in X \setminus \{0\}$.

The next result is from the geometry of the normed spaces. While it may be known to some, we present it here for the reader's convenience.

Theorem 1.2. *Let X be a normed space. Then one has the representation*

$$D(x,y) = \sup\big\{\operatorname{Re} x^*(y) : x^* \in J(x)\big\} \quad \textit{for all } x, y \in X. \tag{1.3}$$

So, in particular,

$$\forall_{x^* \in J(x)} \quad \operatorname{Re} x^*(y) \leq D(x,y). \tag{1.4}$$

Let X be a normed space over $\mathbb{K}$. If the norm is generated by an inner product $\langle \cdot \mid \cdot \rangle$, we consider the standard orthogonality relation: $x \perp y :\Leftrightarrow \langle x \mid y \rangle = 0$. In the general case, there are several notions of orthogonality, with one of the most outstanding ones being the definition introduced by Birkhoff [3, p. 170] (see also James [5, p. 265]):

$$x \perp_{\mathrm{B}} y \quad :\Leftrightarrow \quad \forall_{\lambda \in \mathbb{K}} \quad \|x\| \leq \|x + \lambda y\|.$$

A well-known theorem of Singer [10] will be useful in the next section.

Theorem 1.3 ([10, p. 170]). *Let X be a normed linear space, let F be an n-dimensional subspace of X, and let $x \in X \setminus F$. The following statements are equivalent.*

(a) *First, $x \perp_{\mathrm{B}} F$.*
(b) *Second, there exist h extremal points $\varphi_1, \dots, \varphi_h$ of S_{X^*}, where $1 \leq h \leq n+1$ if the scalars are real and $1 \leq h \leq 2n+1$ if the scalars are complex and h numbers $\lambda_1, \dots, \lambda_h > 0$ with $\sum_{j=1}^h \lambda_j = 1$, such that*

$$\forall_{y \in F} \quad \sum_{j=1}^h \lambda_j \varphi_j(y) = 0 \qquad \textit{and} \qquad \forall_{j=1,\dots,h} \quad \varphi_j(x) = \|x\|.$$

A useful tool in our approach in the next section is a theorem of Collins and Ruess [4] (see also [9]) which characterizes the extremal points of the unit sphere in $\mathcal{K}(X;Y)^*$ in terms of extremal points of the unit spheres in X^{**} and Y^*. By $\mathrm{Ext}(W)$ we denote the set of all extremal points of a given set W. By the Krein–Milman theorem, the closed unit ball B_{X^*} has many extreme points. In particular, $\mathrm{Ext}(S_{X^*}) \neq \emptyset$, $\mathrm{Ext}(S_{X^{**}}) \neq \emptyset$.

Theorem 1.4 ([4, Theorem 2.2], [9, Theorem 1])**.** *If X and Y are Banach spaces, then*

$$\mathrm{Ext}(S_{\mathcal{K}(X;Y)^*}) = \big\{x^{**} \otimes y^* \in \mathcal{K}(X;Y)^* : x^{**} \in \mathrm{Ext}(S_{X^{**}}), y^* \in \mathrm{Ext}(S_{Y^*})\big\},$$

where $x^{**} \otimes y^* \colon \mathcal{K}(X;Y) \to \mathbb{K}$, $(x^{**} \otimes y^*)(T) := x^{**}(T^* y^*)$ for every $T \in \mathcal{K}(X;Y)$.

2. Main results

It will be assumed that all Banach spaces are over $\mathbb{K}$. We will extend Theorem 1.1 in this section. But first we need to prove the following lemma.

Lemma 2.1. *Suppose that $A, B \in \mathcal{K}(X;Y)$. Then*

$$A \perp_{\mathrm{B}} B \Rightarrow \exists_{h\in\{2,3\}} \exists_{\lambda_1,\dots,\lambda_h\in[0,1]} \exists_{y_1^*,\dots,y_h^*\in\mathcal{M}(A^*)\cap\mathrm{Ext}(S_{Y^*})} \exists_{x_k^{**}\in J(A^*y_k^*)\cap\mathrm{Ext}(S_{X^{**}})} :$$
$$\sum_{k=1}^{h} \lambda_k x_k^{**}(B^* y_k^*) = 0 \qquad \textit{and} \qquad \sum_{k=1}^{h} \lambda_k = 1.$$

Proof. Suppose that $A \perp_{\mathrm{B}} B$. Then $A^* \perp_{\mathrm{B}} B^*$. Clearly, $\dim(\mathrm{span}\{B^*\}) = 1$. Applying Theorem 1.3, we obtain

$$\sum_{k=1}^{h} \lambda_k \varphi_k(B^*) = 0 \qquad \text{and} \qquad \varphi_k(A^*) = \|A^*\| \qquad \text{and} \qquad \sum_{k=1}^{h} \lambda_k = 1 \tag{2.1}$$

for some $h \in \{2,3\}$, $\lambda_1, \dots, \lambda_h \in [0,1]$ and for some $\varphi_1, \dots, \varphi_h \in \mathrm{Ext}(S_{\mathcal{K}(X;Y)^*})$.

By Theorem 1.4, we have $\varphi_k = x_k^{**} \otimes y_k^*$ for some $x_k^{**} \in \mathrm{Ext}(S_{X^{**}})$, $y_k^* \in \mathrm{Ext}(S_{Y^*})$. Now the condition (2.1) becomes

$$\sum_{k=1}^{h} \lambda_k x_k^{**}(B^* y_k^*) = 0 \qquad \text{and} \qquad x_k^{**}(A^* y_k^*) = \|A^*\| \qquad \text{and} \qquad \sum_{k=1}^{h} \lambda_k = 1.$$

Since $x_k^{**}(A^* y_k^*) = \|A^*\|$ and $\|x_k^{**}\| = 1$, we also have $\|A^* y_k^*\| = \|A^*\|$. Thus we obtain $y_k^* \in \mathcal{M}(A^*)$ and $x_k^{**} \in J(A^* y_k^*)$, which completes the proof. □

Now, we are ready to present a generalization of Theorem 1.1. We prove the main result of this paper.

Theorem 2.2. *Suppose that $A, B \in \mathcal{K}(X;Y)$ and that $A \neq 0$. Then*

$$D_\varphi(A,B) = \sup\big\{D_\varphi(A^* y^*, B^* y^*) : y^* \in \mathcal{M}(A^*) \cap \mathrm{Ext}(S_{Y^*})\big\}. \tag{2.2}$$

Proof. First, we show that

$$D(A,B)=\sup\big\{D(A^*y^*,B^*y^*): y^*\in\mathcal{M}(A^*)\cap\operatorname{Ext}(S_{Y^*})\big\}. \tag{2.3}$$

It is easy to check that $D(A,B)=D(A^*,B^*)$. Indeed,

$$D(A,B)=\lim_{t\to 0^+}\frac{\|A+tB\|-\|A\|}{t}=\lim_{t\to 0^+}\frac{\|A^*+tB^*\|-\|A^*\|}{t}=D(A^*,B^*).$$

Therefore, we may compute $D(A^*,B^*)$ instead of $D(A,B)$. Fix $t\in(0,+\infty)$. Fix $y^*\in\mathcal{M}(A^*)\cap\operatorname{Ext}(S_{Y^*})$ to obtain

$$\begin{aligned}\frac{\|A^*y^*+tB^*y^*\|-\|A^*y^*\|}{t}&=\frac{\|A^*y^*+tB^*y^*\|-\|A^*\|}{t}\\&\le\frac{\|A^*+tB^*\|-\|A^*\|}{t}.\end{aligned} \tag{2.4}$$

Since t was arbitrarily chosen from the interval $(0,+\infty)$, letting $t\to 0^+$ in (2.4) we obtain

$$D(A^*y^*,B^*y^*)\le D(A^*,B^*).$$

Since y^* was arbitrarily chosen from the set $\mathcal{M}(A^*)\cap\operatorname{Ext}(S_{Y^*})$, we get

$$\sup\big\{D(A^*y^*,B^*y^*): t\in\mathcal{M}(A^*)\cap\operatorname{Ext}(S_{Y^*})\big\}\le D(A^*,B^*).$$

Now we prove the converse inequality. It follows from the above inequality that

$$\begin{aligned}D(A^*,B^*)\ &\ge\ \sup\big\{D(A^*y^*,B^*y^*): y^*\in\mathcal{M}(A^*)\cap\operatorname{Ext}(S_{Y^*})\big\}\\&\overset{(1.4)}{\ge}\ \sup\big\{\sup\big\{\operatorname{Re}x^{**}(B^*y^*): x^{**}\in J(A^*y^*)\big\}:\\&\qquad y^*\in\mathcal{M}(A^*)\cap\operatorname{Ext}(S_{Y^*})\big\}\\&=:\ \beta.\end{aligned} \tag{2.5}$$

So it suffices to show that $D(A^*,B^*)\le\beta$. It follows from (2.5) that

$$\forall_{y^*\in\mathcal{M}(A^*)\cap\operatorname{Ext}(S_{Y^*})}\forall_{x^{**}\in J(A^*y^*)}\qquad \operatorname{Re}x^{**}(B^*y^*)\le\beta. \tag{2.6}$$

Fix $f\in J(A^*)$. Then by (1.2), $f\in\mathcal{K}(Y^*;X^*)^*$, $\|f\|=1$, and $f(A^*)=\|A^*\|$. Note in particular that $f\colon\mathcal{K}(Y^*;X^*)\to\mathbb{K}$. Let us define $\alpha:=-\frac{f(B^*)}{f(A^*)}=-\frac{f(B^*)}{\|A\|}$. Then

$$f(\alpha A^*+B^*)=0,$$

whence, for all λ in $\mathbb{K}$,

$$\begin{aligned}\|A^*\|&=f(A^*)=f(A^*)+\lambda 0=f(A^*)+\lambda f(\alpha A^*+B^*)\\&=f\big(A^*+\lambda(\alpha A^*+B^*)\big)\le\big\|A^*+\lambda(\alpha A^*+B^*)\big\|.\end{aligned}$$

That means that $A^*\perp_{\mathrm{B}}\alpha A^*+B^*$, which implies also that $A\perp_{\mathrm{B}}\alpha A+B$. Using Lemma 2.1, we obtain

$$\sum_{k=1}^{h}\lambda_k x_k^{**}\big(\alpha A^*(y_k^*)+B^*(y_k^*)\big)=0,\qquad \sum_{k=1}^{h}\lambda_k=1 \tag{2.7}$$

for some $h \in \{2, 3\}$, $y_k^* \in \mathcal{M}(A^*) \cap \operatorname{Ext}(S_{Y^*})$, $x_k^{**} \in J(A^* y_k^*) \cap \operatorname{Ext}(S_{X^{**}})$, and for some $\lambda_1, \dots, \lambda_h \in [0, 1]$. It follows from (2.7) that

$$\begin{aligned} 0 &= \sum_{k=1}^{h} \lambda_k x_k^{**}\big(\alpha A^*(y_k^*) + B^*(y_k^*)\big) \\ &= \alpha \sum_{k=1}^{h} \lambda_k x_k^{**}\big(A^*(y_k^*)\big) + \sum_{k=1}^{h} \lambda_k x_k^{**}\big(B^*(y_k^*)\big) \\ &= -\frac{f(B^*)}{\|A^*\|} \sum_{k=1}^{h} \lambda_k \|A^*\| + \sum_{k=1}^{h} \lambda_k x_k^{**}\big(B^*(y_k^*)\big) \\ &= -\frac{f(B^*)}{\|A^*\|} \|A^*\| \sum_{k=1}^{h} \lambda_k + \sum_{k=1}^{h} \lambda_k x_k^{**}\big(B^*(y_k^*)\big) \\ &= -f(B^*) + \sum_{k=1}^{h} \lambda_k x_k^{**}\big(B^*(y_k^*)\big). \end{aligned}$$

That means that $f(B^*) = \sum_{k=1}^{h} \lambda_k x_k^{**}(B^*(y_k^*))$, which also implies that

$$\begin{aligned} \operatorname{Re} f(B^*) &= \sum_{k=1}^{h} \lambda_k \operatorname{Re} x_k^{**}\big(B^*(y_k^*)\big) \\ &\overset{(2.6)}{\leq} \sum_{k=1}^{h} \lambda_k \beta = \beta. \end{aligned}$$

Since f was arbitrarily chosen from the set $J(A^*)$, we get

$$\sup\big\{\operatorname{Re} f(B^*) : f \in J(A^*)\big\} \leq \beta. \tag{2.8}$$

Combining (1.3) and (2.8), we immediately get $D(A^*, B^*) \leq \beta$. The proof of the equality (2.3) is complete. Next we show (2.2). Finally, we deduce from (1.1) that

$$\begin{aligned} D_\varphi(A, B) &= D(A, e^{i\varphi} B) \\ &\overset{(2.3)}{=} \sup\big\{D(Ay^*, e^{i\varphi} By^*) : y^* \in \mathcal{M}(A^*) \cap \operatorname{Ext}(S_{Y^*})\big\} \\ &= \sup\big\{D_\varphi(Ay^*, By^*) : y^* \in \mathcal{M}(A^*) \cap \operatorname{Ext}(S_{Y^*})\big\}. \end{aligned}$$

The proof of Theorem 2.2 is complete. □

Theorem 2.2 can be strengthened as follows.

Theorem 2.3. *Let Y be a reflexive Banach space. Suppose that $A, B \in \mathcal{K}(X; Y)$ and $A \neq 0$. Then*

$$D_\varphi(A, B) = \max\big\{D_\varphi(A^* y^*, B^* y^*) : y^* \in \mathcal{M}(A^*)\big\}.$$

Proof. Bearing in mind the above proof and (1.1), we may prove only that

$$D(A, B) = \max\big\{D(A^* y^*, B^* y^*) : y^* \in \mathcal{M}(A^*)\big\}.$$

In a way similar to the proof of Theorem 2.2, we obtain an inequality

$$\sup\{D(A^*y^*, B^*y^*) : y^* \in \mathcal{M}(A^*)\} \le D(A,B). \tag{2.9}$$

By Theorems 1.2 and 2.2, let us choose sequences $(y_n^*)_{n\in\mathbb{N}} \subset \mathcal{M}(A^*)$, $x_n^{**} \in J(A^*y_n^*)$ such that

$$\operatorname{Re} x_n^{**}(B^*y_n^*) \longrightarrow D(A,B). \tag{2.10}$$

The closed unit ball $B_{X^{**}}$ is weak*-compact. By reflexivity of Y^*, the closed unit ball B_{Y^*} is weak-compact. Therefore, without loss of generality, we may assume that there are an element y_o^* in B_{Y^*}, a functional $x_o^{**} \in B_{X^{**}}$, and subsequences $(y_{n_k}^*)_{k\in\mathbb{N}} \subset B_{Y^*}$, $(x_{n_k}^{**})_{k\in\mathbb{N}} \subset B_{X^{**}}$ such that

$$y_{n_k}^* \xrightarrow{w} y_o^*, \qquad x_{n_k}^{**} \xrightarrow{w^*} x_o^{**}.$$

Since A^*, B^* are compact operators, then A^*, B^* are completely continuous. That means that $A^*y_{n_k}^* \longrightarrow A^*y_o^*$ and $B^*y_{n_k}^* \longrightarrow B^*y_o^*$. Now the condition (2.10) becomes

$$\operatorname{Re} x_o^{**}(B^*y_o^*) = D(A,B). \tag{2.11}$$

Then by a straightforward computation, we can prove that $x_o^{**} \in J(A^*y_o^*)$, $y_o^* \in \mathcal{M}(A^*)$. Finally, we prove that the supremum in (2.9) is attained. Indeed, we have

$$\begin{aligned} D(A,B) &\overset{(2.11)}{=} \operatorname{Re} x_o^{**}(B^*y_o^*) \overset{(1.4)}{\le} D(A^*y_o^*, B^*y_o^*) \\ &\le \sup\{D(A^*y^*, B^*y^*) : y^* \in \mathcal{M}(A^*)\} \overset{(2.9)}{\le} D(A,B). \end{aligned}$$

Therefore $D(A,B) = D(A^*y_o^*, B^*y_o^*) = \sup\{D(A^*y^*, B^*y^*) : y^* \in \mathcal{M}(A^*)\}$, so we can write max instead of sup. The proof of Theorem 2.3 is complete. □

If X and Y are Banach spaces and $A \in \mathcal{K}(X;Y)$, then: $A^{**}|_X = A$. If X is reflexive, then X^{**} is identified with X. Moreover, $A^{**}|_X$ is identified with A. In this case, $\mathcal{M}(A) \ne \emptyset$ for each A in $\mathcal{K}(X;Y)$. Clearly, $D_\varphi(A^*, B^*) = D_\varphi(A,B)$. Combining these facts with our Theorems 2.2 and 2.3, we obtain the following corollary.

Theorem 2.4. *Let X be a reflexive Banach space, and let $A, B \in \mathcal{K}(X;Y)$. Then*

$$\begin{aligned} D_\varphi(A,B) &= \sup\{D_\varphi(Ay, By) : y \in \mathcal{M}(A) \cap \operatorname{Ext}(S_X)\} \\ &= \max\{D_\varphi(Ay, By) : y \in \mathcal{M}(A)\}. \end{aligned}$$

3. Remarks

Let X be a complex normed space. The mappings D, D_φ are continuous with respect to the second variable. Fix $x, y \in X$, and note that, due to (1.1), a mapping $[0, 2\pi) \ni \varphi \to D_\varphi(x,y) \in \mathbb{R}$ is also continuous.

The functions D, D_φ characterize the Birkhoff orthogonality in the following sense. If $x, y \in X$, then it is well known that

$$x \perp_{\mathrm{B}} y \quad \Leftrightarrow \quad \inf\{D_\varphi(x,y) : \varphi \in [0, 2\pi)\} \ge 0.$$

As a consequence, we give a characterization of orthogonality in the sense of Birkhoff in the space $\mathcal{K}(X;Y)$.

Theorem 3.1. *Let X, Y be reflexive Banach spaces over $\mathbb{C}$. Suppose that $A, B \in \mathcal{K}(X;Y)$ and $A \neq 0$. Then the following conditions are equivalent:*

(a) $A \perp_B B$,
(b) $\inf\{\sup\{D_\varphi(Ay, By) : y \in \mathcal{M}(A) \cap \mathrm{Ext}(S_X)\} : \varphi \in [0, 2\pi)\} \geq 0$,
(c) $\inf\{\max\{D_\varphi(Ay, By) : y \in \mathcal{M}(A)\} : \varphi \in [0, 2\pi)\} \geq 0$,
(d) $\min\{\max\{D_\varphi(Ay, By) : y \in \mathcal{M}(A)\} : \varphi \in [0, 2\pi)\} \geq 0$.

Proof. The equivalence between (a), (b), and (c) follows from Theorem 2.4. Obviously (d) $\Rightarrow$ (c). We prove the implication (c) $\Rightarrow$ (d). Note that a mapping $[0, 2\pi) \ni \varphi \to D_\varphi(A, B) \in \mathbb{R}$ is continuous. It is easy to see that a set $\mathbb{T} := \{e^{i\varphi} \in \mathbb{C} : \varphi \in [0, 2\pi)\}$ is compact. Then we define a mapping $\gamma \colon \mathbb{T} \to \mathbb{R}$ by

$$\gamma(e^{i\varphi}) := D(A, e^{i\varphi}B) = D_\varphi(A, B) = \max\big\{D_\varphi(Ay, By) : y \in \mathcal{M}(A)\big\}.$$

The mapping γ is continuous, so γ attains its minimum. Therefore, we can write min instead of inf. □

Remark 3.2. If $X = Y$ is a Hilbert space, it is possible to expand Theorem 3.1. Namely, $A \perp_B B$ if and only if there is $x \in X$ such that $\|x\| = 1$, $\|Ax\| = \|A\|$, and $Ax \perp_B Bx$. It is known as the *Bhatia–Šemrl property* (see, e.g., [2], [6], [7]). However, in the absence of an inner product, this is impossible (see [1], [8]).

In fact, condition (d) in Theorem 3.1 is equivalent to the Bhatia–Šemrl property in Hilbert spaces, but not in Banach spaces! This makes this theorem interesting even in the framework of finite-dimensional normed spaces, since condition (d) in Theorem 3.1 is, probably, the closest condition to the Bhatia–Šemrl property that can be obtained.

References

1. C. Benítez, M. Fernández, and M. L. Soriano, *Orthogonality of matrices*, Linear Algebra Appl. **422** (2007), no. 1, 155–163. Zbl 1125.15026. MR2299002. DOI 10.1016/j.laa.2006.09.018. 684
2. R. Bhatia and P. Šemrl, *Orthogonality of matrices and some distance problems*, Linear Algebra Appl. **287** (1999), no. 1–3, 77–85. Zbl 0937.15023. MR1662861. DOI 10.1016/S0024-3795(98)10134-9. 684
3. G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), no. 2, 169–172. Zbl 0012.30604. MR1545873. DOI 10.1215/S0012-7094-35-00115-6. 679
4. H. S. Collins and W. Ruess, *Weak compactness in spaces of compact operators and of vector-valued functions*, Pacific J. Math. **106** (1983), no. 1, 45–71. Zbl 0488.46057. MR0694671. 680
5. R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), no. 2, 265–292. Zbl 0037.08001. MR0021241. 679
6. D. J. Kečkić, *Orthogonality in $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$ spaces and normal derivations*, J. Operator Theory **51** (2004), no. 1, 89–104. Zbl 1068.46024. MR2055806. 678, 679, 684
7. D. J. Kečkić, *Gateaux derivative of $\mathcal{B}(H)$ norm*, Proc. Amer. Math. Soc. **133** (2005), no. 7, 2061–2067. Zbl 1066.46036. MR2137872. DOI 10.1090/S0002-9939-05-07746-4. 679, 684
8. C.-K. Li and H. Schneider, *Orthogonality of matrices*, Linear Algebra Appl. **347** (2002), no. 1–3, 115–122. Zbl 1003.15028. MR1899885. DOI 10.1016/S0024-3795(01)00530-4. 684
9. A. Lima and G. Olsen, *Extreme points in duals of complex operator spaces*, Proc. Amer. Math. Soc. **94** (1985), no. 3, 437–440. Zbl 0581.47029. MR0787889. DOI 10.2307/2045230. 680

10. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Grundlehren Math. Wiss. **171**, Springer, Berlin, 1970. Zbl 0197.38601. MR0270044. 679

Institute of Mathematics, Pedagogical University of Cracow, Podchorążych 2, 30-084 Kraków, Poland.
E-mail address: pwojcik@up.krakow.pl