Ann. Funct. Anal. 7 (2016), no. 4, 622–635
http://dx.doi.org/10.1215/20088752-3660864
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ON THE ARAKI–LIEB–THIRRING INEQUALITY IN THE SEMIFINITE VON NEUMANN ALGEBRA

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Communicated by J.-C. Bourin

Abstract. This paper extends a recent matrix trace inequality of Bourin–Lee to semifinite von Neumann algebras. This provides a generalization of the Lieb–Thirring-type inequality in von Neumann algebras due to Kosaki. Some new inequalities, even in the matrix case, are also given for the Heinz means.

1. Introduction

Let $\mathbb{M}_n$ be the space of $n \times n$ complex matrices. The Lieb–Thirring inequality [17] states that, for $0 \leq A, B \in \mathbb{M}_n$ and $p \geq 1$,

$$\operatorname{Tr}\big((B^{\frac{1}{2}}AB^{\frac{1}{2}})^p\big) \leq \operatorname{Tr}(A^pB^p).$$

Let A and B be positive self-adjoint operators on a Hilbert space, and let f be any increasing continuous function on $[0,\infty)$ such that $f(0)=0$ and $t \to f(e^t)$ is a convex function. Araki [2] shows a refinement of the Lieb–Thirring inequality as follows:

$$\operatorname{Tr} f\big((B^{\frac{1}{2}}AB^{\frac{1}{2}})^q\big) \leq \operatorname{Tr} f(B^{\frac{q}{2}}A^qB^{\frac{q}{2}}) \quad \text{for all } q \geq 1. \tag{1.1}$$

Here the condition $f(0)=0$ ensures that the trace is well defined; that is, $\infty - \infty$ does not occur. Recently, Bourin and Lee [6] proved that if f is increasing and $t \to f(e^t)$ is convex, then the inequality

$$\operatorname{Tr} f\big((BZ^*AZB)^q\big) \leq \operatorname{Tr} f(B^qZ^*A^qZB^q) \quad \text{for all } q \geq 1 \tag{1.2}$$

Copyright 2016 by the Tusi Mathematical Research Group.
Received Dec. 15, 2015; Accepted May 5, 2016.
2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 46L52.
Keywords. Araki–Lieb–Thirring inequality, von Neumann algebra, τ-measurable operator.

holds for contraction $Z \in \mathbb{M}_n$ and $0 \le A, B \in \mathbb{M}_n$. For further results about the Lieb–Thirring inequality, the reader is referred to [6] and [14]. In 1992, Kosaki [16] proved the Araki–Lieb–Thirring inequality (1.1) for τ-compact operators associated with the semifinite von Neumann algebra $\mathcal{M}$.

By adapting the techniques in [4] and [12], we obtain some inequalities which are related to the Araki–Lieb–Thirring inequality. In particular, we show that the inequalities (1.1) and (1.2) hold for τ-measurable operators associated with the semifinite von Neumann algebra $\mathcal{M}$. We will conclude this paper with a series of submajorization inequalities which are related to the Heinz means.

2. Preliminaries

Unless stated otherwise, $\mathcal{M}$ will always denote a semifinite von Neumann algebra acting on the Hilbert space $\mathcal{H}$ with a normal semifinite faithful trace τ. We refer to [20] and [21] for noncommutative integration. We denote the identity of $\mathcal{M}$ by 1. A closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x) \subseteq \mathcal{H}$ is said to be *affiliated with* $\mathcal{M}$ if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, then we define its distribution function by $\lambda_s(x) = \tau(e_s^{\perp}(|x|))$ and x will be called τ-*measurable* if and only if $\lambda_s(x) < \infty$ for some $s > 0$, where $e_s^{\perp}(|x|) = e_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s, ∞). If $\mathcal{M}$ is a finite von Neumann algebra and x is affiliated with $\mathcal{M}$, then x is automatically τ-measurable. The decreasing rearrangement of x is defined by $\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \le t\}, t > 0$. We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $t \to \lambda_t(x)$ and $t \to \mu_t(x)$, respectively (see [12] for basic properties and detailed information on decreasing rearrangement of x).

The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closures of the algebraic sum and product. The measure topology in $L_0(\mathcal{M})$ is the vector space topology defined via the neighborhood base $\{N(\varepsilon, \delta) : \varepsilon, \delta > 0\}$, where $N(\varepsilon, \delta) = \{x \in L_0(\mathcal{M}) : \tau(e_{(\varepsilon,\infty)}(|x|)) \le \delta\}$ and $e_{(\varepsilon,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (ε, ∞). With respect to the measure topology, $L_0(\mathcal{M})$ is a complete topological $*$-algebra. As usual, we denote by $\|\cdot\|$ $(= \|\cdot\|_\infty)$ the usual operator norm.

Let L_0 be the set of all Lebesgue-measurable functions on $(0, \infty)$. For $f \in L_0$ we define its nonincreasing rearrangement as

$$f^*(t) = \inf\bigl\{s > 0 : d_f(s) = m\bigl\{r : \bigl|f(r)\bigr| > s\bigr\} \le t\bigr\}, \quad t > 0,$$

where m denotes the Lebesgue measure on $(0, \infty)$. By a *symmetric Banach space* on $(0, \infty)$ we mean a Banach lattice E of measurable functions on $(0, \infty)$ satisfying the following properties: (a) E contains a simple function; (b) if $f \in L_0$ and $g \in E$ with $f^* \le g^*$, then $f \in E$ and $\|f\|_E \le \|g\|_E$. It is called *fully symmetric* if, in addition, for $f \in L_0$ and $g \in E$ with

$$\int_0^t f^*(s)\,ds \le \int_0^t g^*(s)\,ds, \quad t > 0,$$

we have $f \in E$ and $\|f\|_E \leq \|g\|_E$. The norm on E is said to be a *σ-Fatou norm* if

$$0 \leq f_i \uparrow f, \quad f_i, f \in E \Rightarrow \|f_i\|_E \uparrow \|f\|_E. \tag{2.1}$$

If the norm on E is a σ-Fatou norm, then the natural embedding of E into the second associate space (Köthe bidual) $E^{\times\times}$ is an isometry. Consequently, a symmetric Banach space which has a σ-Fatou norm is automatically fully symmetric. Let E be a symmetric Banach space on $(0,\infty)$. For $0 < r < \infty$, $E^{(r)}$ will denote the quasi-Banach spaces defined by

$$E^{(r)} := \{g \in L_0 : |g|^r \in E\} \qquad \text{and} \qquad \|g\|_{E^{(r)}} = \big\||g|^r\big\|_E^{\frac{1}{r}}.$$

As is shown in [18], if E is a symmetric Banach space, then $E^{(r)}$ is a symmetric quasi-Banach space.

Let E be a symmetric Banach space on $(0,\infty)$. We define

$$E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \mu(x) \in E\} \qquad \text{and} \qquad \|x\|_{E(\mathcal{M})} = \big\|\mu(x)\big\|_E.$$

Then $(E(\mathcal{M}), \|\cdot\|_{E(\mathcal{M})})$ is a noncommutative symmetric Banach space (see [22]). If $E = L^p$, then $E(\mathcal{M})$ is the usual noncommutative L_p spaces $L^p(\mathcal{M})$. For $0 < r < \infty$, we define

$$E(\mathcal{M})^{(r)} = \{x \in L_0(\mathcal{M}) : |x|^r \in E(\mathcal{M})\} \qquad \text{and} \qquad \|x\|_{E(\mathcal{M})^{(r)}} = \big\||x|^r\big\|_{E(\mathcal{M})}^{\frac{1}{r}}.$$

As is shown in Proposition 3.1 of [9], if E is a symmetric Banach space, then $E^{(r)}(\mathcal{M}) = E(\mathcal{M})^{(r)}$, where $E^{(r)}(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \mu(x) \in E^{(r)}\}$ and $\|x\|_{E^{(r)}(\mathcal{M})} = \|\mu(x)\|_{E^{(r)}}$. It is well known that $E(\mathcal{M})^{(r)}$ is a noncommutative symmetric quasi-Banach space (see [9], [22]). Let $0 < r_0, r_1, r < \infty$ with $\frac{1}{r} = \frac{1}{r_0} + \frac{1}{r_1}$. Then the Hölder inequality on $E(\mathcal{M})^{(r)}$ is that

$$\|xy\|_{E(\mathcal{M})^{(r)}} \leq \|x\|_{E(\mathcal{M})^{(r_0)}} \|y\|_{E(\mathcal{M})^{(r_1)}} \tag{2.2}$$

holds for all $x \in E(\mathcal{M})^{(r_0)}$ and $y \in E(\mathcal{M})^{(r_1)}$ (see inequality (1.3) in [8, p. 492]). Further details for commutative and noncommutative symmetric Banach spaces may be found in [9], [8], [18], and [22].

The following well-known basic facts are needed in our proofs, and we list them for the reader's convenience. If $x \in L_0(\mathcal{M})$ and $y, z \in \mathcal{M}$, then

$$\mu(yxz) \leq \|y\|\mu(x)\|z\|, \tag{2.3}$$

$$\mu(x) = \mu_t(x^*) = \mu(|x|), \tag{2.4}$$

and

$$f(\mu(x)) = \mu(f(|x|)), \tag{2.5}$$

where f is a continuous increasing function on $[0,\infty)$ such that $f(0) = 0$ (see Lemma 2.5 in [12]). On the other hand, it follows from Theorem 2.2 in [7] that

$$\int_0^t \mu_s(xy)\,ds \leq \int_0^t \mu_s(x)\mu_s(y)\,ds, \quad t > 0 \tag{2.6}$$

holds for $x, y \in L_0(\mathcal{M})$.

If $x \in L_0(\mathcal{M})$ satisfies a "Lorentz space"-type condition of the form

$$x \in \mathcal{M} \text{ or } \mu_t(x) \leq Ct^{-\alpha}, \quad C, \alpha > 0, t > 0,$$

then we may define

$$\Lambda_t(x) = \exp\Big(\int_0^t \log \mu_s(x)\, ds\Big), \quad t > 0.$$

Let $x \in \mathcal{M}$. From the definition of $\Lambda_t(x)$ and the properties of $\mu_t(x)$, we obtain

$$\Lambda_t(x) = \Lambda_t(x^*) = \Lambda_t\big(|x|\big), \quad t > 0 \tag{2.7}$$

and

$$\Lambda_t(x^\alpha) = \Lambda_t(x)^\alpha, \quad t > 0, \text{if } \alpha > 0 \text{ and } x > 0. \tag{2.8}$$

Moreover, it follows from Theorem 2.3 in [7] (or Theorem 2 in [19]) that

$$\Lambda_t(xy) \leq \Lambda_t(x)\Lambda_t(y), \quad t > 0 \tag{2.9}$$

holds for all $x, y \in \mathcal{M}$.

If $x, y \in L_0(\mathcal{M})$, then we say that x is *submajorized* by y and we write $x \precsim y$ if and only if

$$\int_0^t \mu_s(x)\, ds \leq \int_0^t \mu_s(y)\, ds \quad \text{for all } t \geq 0.$$

The set $\mathcal{K}$ of all τ-compact operators is defined by

$$\mathcal{K} = \big\{x \in L_0(\mathcal{M}) : \lim_{t\to\infty} \mu_t(x) = 0\big\}.$$

For $0 \leq x, y \in \mathcal{K}$, and $p \geq 1$, it was shown by Kosaki (see [16, Theorem 2]) that

$$f\big(|xy|^p\big) \precsim f\big(|x^p y^p|\big), \tag{2.10}$$

where f is a continuous increasing function on $[0, \infty)$ such that $f(0) = 0$ and $t \to f(e^t)$ is convex. In what follows we will prove that (2.10) holds for all $0 \leq x, y \in L_0(\mathcal{M})$. Our idea of the proof follows the one given in [16].

Lemma 2.1. *Let $x, y \in \mathcal{M}$. If the product xy is self-adjoint, then we have*

$$\Lambda_t(xy) \leq \Lambda_t(yx), \quad t > 0.$$

Proof. If $x, y \in \mathcal{M}$ satisfy that $\lim_{t\to\infty} \mu_t(x) = \lim_{t\to\infty} \mu_t(y) = 0$, then the result follows from Remark 1 in [16]. The general case can be done similarly to the proof of Remark 1 in [16]. For convenience, we give its proof.

If xy is self-adjoint, then it follows from (2.7), (2.8), and (2.9) that

$$\begin{aligned}\Lambda_t(xy)^{2n} &= \Lambda_t\big(|xy|^{2n}\big) = \Lambda_t\big((xy)^{2n}\big)\\ &= \Lambda_t\big(x(yx)^{2n-1}y\big) \leq \Lambda_t(x)\Lambda_t(yx)^{2n-1}\Lambda_t(y), \quad t > 0\end{aligned}$$

holds for each $n \in \mathbb{N}^+$. Taking the $2n$th roots of the both sides and then letting $n \to \infty$, we obtain the desired result. □

Lemma 2.2. *Let $0 \le x, y \in \mathcal{M}$ and $n \in \mathbb{N}^+$. Then*

$$\Lambda_t\big(|xy|^{2^n}\big) \le \Lambda_t(x^{2^n} y^{2^n}), \quad t > 0.$$

Moreover, we have

$$\Lambda_t(x^{\frac{1}{2^n}} y^{\frac{1}{2^n}}) \le \Lambda_t(xy)^{\frac{1}{2^n}}, \quad t > 0.$$

Proof. By Lemma 2.1, we have

$$\Lambda_t\big(|xy|^2\big) = \Lambda_t\big((xy)^* xy\big) = \Lambda_t(yx^2y) \le \Lambda_t(x^2y^2), \quad t > 0.$$

Therefore, (2.7) and (2.8) imply that

$$\begin{aligned}\Lambda_t\big(|xy|^{2^n}\big) &= \Lambda_t\big(|xy|\big)^{2^n} = \Lambda_t\big(|xy|^2\big)^{2^{n-1}} \\ &\le \Lambda_t(x^2y^2)^{2^{n-1}} = \Lambda_t\big(|x^2y^2|^{2^{n-1}}\big), \quad t > 0.\end{aligned}$$

Repeating this argument, we deduce

$$\Lambda_t\big(|xy|^{2^n}\big) \le \Lambda_t(x^{2^n} y^{2^n}), \quad t > 0.$$

Replacing x, y by $x^{\frac{1}{2^n}}, y^{\frac{1}{2^n}}$, respectively, we get

$$\Lambda_t(x^{\frac{1}{2^n}} y^{\frac{1}{2^n}}) \le \Lambda_t(xy)^{\frac{1}{2^n}}, \quad t > 0. \qquad \square$$

Lemma 2.3. *Let $p \ge 1$, and let $f : [0, \infty) \to [0, \infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex. If $0 \le x, y \in \mathcal{M}$, then we have*

$$f\big(|xy|^p\big) \precsim f\big(|x^p y^p|\big). \tag{2.11}$$

Proof. Since $0 \le x, y \in \mathcal{M}$, for every $t > 0$, we have

$$\int_0^t \mu_s\big(f\big(|x^p y^p|\big)\big)\, ds \le \int_0^t f\big(\|x^p y^p\|\big)\, ds = tf\big(\|x^p y^p\|\big) < \infty. \tag{2.12}$$

If the inequality $\Lambda_t(x^p y^p) \le \Lambda_t(xy)^p, t > 0$ is valid for $p = \alpha$ and $p = \beta$, $0 < \alpha < \beta \le 1$, then so is the case for $p = \frac{\alpha+\beta}{2}$. Indeed, by the assumption and Lemma 2.1, we deduce

$$\begin{aligned}\Lambda_t(x^{\frac{\alpha+\beta}{2}} y^{\frac{\alpha+\beta}{2}})^2 &= \Lambda_t(y^{\frac{\alpha+\beta}{2}} x^{\alpha+\beta} y^{\frac{\alpha+\beta}{2}}) \\ &= \Lambda_t\big(y^{\frac{\beta-\alpha}{2}} (y^\alpha x^{\alpha+\beta} y^\alpha) y^{\frac{\beta-\alpha}{2}}\big) \\ &\le \Lambda_t\big((y^\alpha x^{\alpha+\beta} y^\alpha) y^{\beta-\alpha}\big) \\ &\le \Lambda_t(y^\alpha x^\alpha) \Lambda_t(x^\beta y^\beta) \\ &= \Lambda_t(x^\alpha y^\alpha) \Lambda_t(x^\beta y^\beta) \le \Lambda_t(xy)^{\alpha+\beta}, \quad t > 0.\end{aligned}$$

This means that $\Lambda_t(x^{\frac{\alpha+\beta}{2}} y^{\frac{\alpha+\beta}{2}}) \le \Lambda_t(xy)^{\frac{\alpha+\beta}{2}}, t > 0$; thus, Lemma 2.2 implies that the inequality $\Lambda_t(x^p y^p) \le \Lambda_t(xy)^p, t > 0$, is valid for p in the dense subset in $(0, 1]$.

Replacing x^p, y^p by x, y, respectively, we observe that $\Lambda_t(xy)^p \le \Lambda_t(x^p y^p), t > 0$ is valid for p in the dense subset in $[1, \infty)$. Combining this with Corollary 4.2 in [11], we get that

$$\int_0^t f\big(\mu_s\big(|xy|^p\big)\big)\, ds \le \int_0^t f\big(\mu_s(x^p y^p)\big)\, ds, \quad t > 0 \tag{2.13}$$

holds for p in the dense subset in $[1,\infty)$. For any $p \in [1,\infty)$, we choose a sequence $\{p_n\}$ such that (2.13) is valid and $p_n \to p$. Since the map: $x \to \mu_s(x)$ is norm continuous, the standard argument on norm convergence and dominated convergence theorem and the fact (2.12) show that

$$\int_0^t f\big(\mu_s(x^p y^p)\big)\,ds = \lim_{n\to\infty}\int_0^t f\big(\mu_s(x^{p_n} y^{p_n})\big)\,ds$$

and

$$\int_0^t f\big(\mu_s\big(|xy|^p\big)\big)\,ds = \lim_{n\to\infty}\int_0^t f\big(\mu_s\big(|xy|^{p_n}\big)\big)\,ds.$$

This means that (2.13) is valid for all $p \in [1,\infty)$. It follows from (2.5) that $f(|xy|^p) \precsim f(|x^p y^p|), p \geq 1$, is valid for $0 \leq x, y \in \mathcal{M}$. □

Let $0 \leq x, y \in L_0(\mathcal{M})$. Then Corollary 3.6 in [5] tells us that

$$\mu(xy) = \mu(yx). \tag{2.14}$$

Based on Lemma 2.3 and (2.14), we obtain the generalizations of inequality (2.10) for τ-measurable operators associated with the semifinite von Neumann algebra $\mathcal{M}$.

Proposition 2.4. *Let $p \geq 1$, and let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex. For $0 \leq x, y \in L_0(\mathcal{M})$, we have*

$$f\big(|xy|^p\big) \precsim f\big(|x^p y^p|\big). \tag{2.15}$$

Proof. If $\int_0^{t_0} \mu_s(f(|x^p y^p|))\,ds = \infty$ holds for some $t_0 > 0$, then it is clear that

$$\int_0^t \mu_s\big(f\big(|xy|^p\big)\big)\,ds \leq \int_0^t \mu_s\big(f\big(|x^p y^p|\big)\big)\,ds, \quad t > t_0.$$

Therefore, without loss of generality, we suppose that $0 \leq x, y \in L_0(\mathcal{M})$ satisfy

$$\int_0^t \mu_s\big(f\big(|x^p y^p|\big)\big)\,ds < \infty \quad \text{for all } t > 0. \tag{2.16}$$

Let $x = \int_0^\infty \lambda\,de_\lambda$ and $y = \int_0^\infty \lambda\,df_\lambda$ be the spectral decompositions. We write $x_n = \int_0^n \lambda\,de_\lambda$ and $y_n = \int_0^n \lambda\,df_\lambda$. Then $x_n, y_n \in \mathcal{M}$. By Lemma 2.6 in [12], we deduce that

$$\mu(x - x_n) = \mu\big(x e_{(n,\infty)}(x)\big) \leq \mu(x)\chi_{[0,\tau(e_{(n,\infty)}(x))]}.$$

From Proposition 21 of [21, Chapter I], we have $\lim_{n\to\infty} \tau(e_{(n,\infty)}(x)) = 0$, which means that $\lim_{n\to\infty} \mu_t(x - x_n) = 0$. By Lemma 3.1 in [12], we obtain $x_n^{2p} \uparrow x^{2p}$, $n \to \infty$ in measure topology. Therefore, Lemma 3.4 in [12] implies that

$$\mu(y_m^p x_n^{2p} y_m^p) \uparrow \mu(y_m^p x_n^{2p} y_m^p), \quad n \to \infty.$$

Fixing $m \in \mathbb{N}^+$, from inequality (2.13) and the monotone convergence theorem and the fact (2.16), we have

$$\begin{aligned}\int_0^t f\big(\mu_s\big(|xy_m|^p\big)\big)\,ds &= \int_0^t f\big(\mu_s(y_m x^2 y_m)^{\frac{p}{2}}\big)\,ds\\ &= \sup_n \int_0^t f\big(\mu_s(y_m x_n^2 y_m)^{\frac{p}{2}}\big)\,ds\\ &= \sup_n \int_0^t f\big(\mu_s\big(|x_n y_m|^p\big)\big)\,ds \le \sup_n \int_0^t f\big(\mu_s(x_n^p y_m^p)\big)\,ds\\ &= \sup_n \int_0^t f\big(\mu_s(y_m^p x_n^{2p} y_m^p)^{\frac{1}{2}}\big)\,ds = \int_0^t f\big(\mu_s(y_m^p x^{2p} y_m^p)^{\frac{1}{2}}\big)\,ds\\ &= \int_0^t f\big(\mu_s(x^p y_m^p)\big)\,ds, \quad t>0.\end{aligned}$$

By (2.14), (2.4), and (2.5), we obtain

$$\mu\big(|xy_m|^p\big) = \mu(xy_m)^p = \mu(y_m x)^p = \mu\big(|y_m x|^p\big)$$

and $\mu(x^p y_m^p) = \mu(y_m^p x^p)$; hence, we can use the monotone convergence as in the preceding argument and get that

$$\int_0^t f\big(\mu_s\big(|xy|^p\big)\big)\,ds \le \int_0^t f\big(\mu_s(x^p y^p)\big)\,ds, \quad t>0$$

holds for $0 \le x, y \in L_0(\mathcal{M})$. Combining this with (2.5), we obtain that $f(|xy|^p) \precsim f(|x^p y^p|), p \ge 1$ is valid for $0 \le x, y \in L_0(\mathcal{M})$. □

We conclude this section with a series of submajorization inequalities that lead to refinement of the submajorization inequality in Proposition 2.4.

Lemma 2.5. *Let $0 \le x, z \in L_0(\mathcal{M})$, and let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex.*

(1) *If $1 \le p < \infty$, then $f((xzx)^p) \precsim f(x^p z^p x^p)$.*
(2) *If $0 < p \le 1$, then $f(x^p z^p x^p) \precsim f((xzx)^p)$.*

Proof. The proof can be done similarly to the proof of Lemma 3.1 in [4] by using Proposition 2.4. For convenience, we give its proof.

(1) Since $g(t) = f(t^2)$ is a continuous increasing function on $[0,\infty)$ such that $g(0) = 0$ and $t \to g(e^t) = f(e^{2t})$ is convex, by (2.4), (2.5), and Proposition 2.4, we have

$$\begin{aligned}\int_0^t \mu_s\big(f\big((xzx)^p\big)\big)\,ds &= \int_0^t f\big(\mu_s\big(|z^{\frac{1}{2}}x|^{2p}\big)\big)\,ds = \int_0^t f\big(\mu_s\big(|z^{\frac{1}{2}}x|^p\big)^2\big)\,ds\\ &\le \int_0^t f\big(\mu_s\big(|z^{\frac{p}{2}}x^p|\big)^2\big)\,ds = \int_0^t f\big(\mu_s(x^p z^p x^p)\big)\,ds, \quad t>0.\end{aligned}$$

(2) Since $g(t) = f(t^{2p})$ is a continuous increasing function on $[0,\infty)$ such that $g(0) = 0$ and $t \to g(e^t) = f(e^{2t})$ is convex, by (2.4), (2.5), and Proposition 2.4,

we obtain

$$\begin{aligned}\int_0^t f\big(\mu_s(x^p z^p x^p)\big)\,ds &= \int_0^t f\Big(\mu_s\big((x^p z^p x^p)^{\frac{1}{p}}\big)^p\Big)\,ds = \int_0^t f\Big(\mu_s\big(|z^{\frac{p}{2}} x^p|^{\frac{1}{p}}\big)^{2p}\Big)\,ds \\ &\le \int_0^t f\Big(\mu_s\big(|z^{\frac{1}{2}} x|\big)^{2p}\Big)\,ds \\ &= \int_0^t f\big(\mu_s(xzx)^p\big)\,ds, \quad t > 0. \end{aligned}$$

□

Lemma 2.6. *Let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex. Let $x, z \in L_0(\mathcal{M})$, and let $z \ge 0$.*

(1) *If $1 \le p < \infty$, then $f((xzx^*)^p) \precsim f(|x|^p z^p |x|^p)$.*
(2) *If $0 < p \le 1$, then $f(|x|^p z^p |x|^p) \precsim f((xzx^*)^p)$.*

Proof. The proof is similar to the proof of Theorem 3.2 in [4]. The details are omitted. □

Lemma 2.7. *Let $p \ge 1$, and let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex. Let $x, z \in L_0(\mathcal{M})$, and let z be self-adjoint. Then*

$$f\big(|xzx^*|^p\big) \precsim f\big(|x|^p |z|^p |x|^p\big).$$

Proof. By slightly modifying the proof of Theorem 3.6 in [4], we can prove this corollary and omit the details. □

Remark 2.8. If $\mathcal{M}$ is a finite von Neumann algebra, then the results of Lemmas 2.5–2.7 are contained in [4].

3. Main results

Let f be a nonnegative operator monotone function on $[0,\infty)$, and let $\mathcal{M}$ be a semifinite von Neumann algebra. If $y \in \mathcal{M}$ is a contraction and $x \in L_0(\mathcal{M})$, then

$$y^* f(x) y \le f(y^* x y). \tag{3.1}$$

This result is proved in [13] when x is a bounded linear operator, in [15] for finite trace, and in [3] for the general case as above.

The following lemma plays a central role in our investigation.

Lemma 3.1. *Let $0 \le x, z \in L_0(\mathcal{M})$, and let $y \in \mathcal{M}$ be a contraction. Let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex.*

(1) *If $1 \le p < \infty$, then $f((xy^* zyx)^p) \precsim f(x^p y^* z^p y x^p)$.*
(2) *If $0 < p \le 1$, then $f(x^p y^* z^p y x^p) \precsim f((xy^* zyx)^p)$.*

Proof. (1) For $p \ge 1$, we put $g(t) = f(t^p)$. Then g is a continuous increasing function on $[0,\infty)$ such that $g(0) = 0$ and $t \to g(e^t) = f(e^{pt})$ is convex. Let $y \in \mathcal{M}$ be a contraction. It follows from Lemma 3.1.1 of [3] (i.e., inequality (3.1)) that $y^*zy \le (y^*z^py)^{\frac{1}{p}}$ for $p \ge 1$. Hence $\mu(x(y^*z^py)^{\frac{1}{p}}x) \ge \mu(xy^*zyx)$. Following (2.5), we obtain

$$\mu\big(g\big(x(y^*z^py)^{\frac{1}{p}}x\big)\big) = g\big(\mu\big(x(y^*z^py)^{\frac{1}{p}}x\big)\big) \ge g\big(\mu(xy^*zyx)\big) = \mu\big(g(xy^*zyx)\big).$$

By Lemma 2.5, we deduce

$$\begin{aligned} f(x^py^*z^pyx^p) &= g\big((x^py^*z^pyx^p)^{\frac{1}{p}}\big) \\ &\succsim g\big(x(y^*z^py)^{\frac{1}{p}}x\big) \\ &\ge g(xy^*zyx) = f\big((xy^*zyx)^p\big). \end{aligned}$$

(2) A similar discussion to the proof of (1) shows that

$$f\big((xy^*zyx)^p\big) \succsim f\big(x^p(y^*zy)^px^p\big) \ge f(x^py^*z^pyx^p). \qquad \square$$

Now, using Lemma 3.1, we get our first main result of this paper.

Proposition 3.2. *Let $0 \le x, z \in L_0(\mathcal{M})$, and let $y \in \mathcal{M}$ be a contraction. Let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex. Then for $0 < p \le q < \infty$, we have*

(1) $f((x^py^*z^pyx^p)^{\frac{1}{p}}) \precsim f((x^qy^*z^qyx^q)^{\frac{1}{q}})$;

(2) $f((x^{\frac{1}{q}}y^*z^{\frac{1}{q}}yx^{\frac{1}{q}})^q) \precsim f((x^{\frac{1}{p}}y^*z^{\frac{1}{p}}yx^{\frac{1}{p}})^p)$.

Proof. (1) We put $g(t) = f(t^{\frac{1}{q}})$. Then g is a continuous increasing function on $[0,\infty)$ such that $g(0) = 0$ and $t \to g(e^t) = f(e^{\frac{1}{q}t})$ is convex. By Lemma 3.1, we obtain

$$f\big((x^py^*z^pyx^p)^{\frac{1}{p}}\big) = g\big((x^py^*z^pyx^p)^{\frac{q}{p}}\big) \precsim g(x^qy^*z^qyx^q) = f\big((x^qy^*z^qyx^q)^{\frac{1}{q}}\big).$$

(2) The proof can be done similarly to (1). The details are omitted. $\square$

Corollary 3.3. *Let E be a fully symmetric Banach space, and let $0 \le x, z \in E(\mathcal{M})^{(3)}$, and $y \in \mathcal{M}$ be a contraction. Then*

(1) *$g(p) = \||x^py^*z^pyx^p|^{\frac{1}{p}}\|_{E(\mathcal{M})}$ is an increasing function on $(0,\infty)$;*

(2) *$h(p) = \||x^{\frac{1}{p}}y^*z^{\frac{1}{p}}yx^{\frac{1}{p}}|^p\|_{E(\mathcal{M})}$ is a decreasing function on $(0,\infty)$.*

Proof. (1) Let $0 \le x, z \in E(\mathcal{M})^{(3)}$, and let $y \in \mathcal{M}$ be a contraction. By (2.2) and (2.3), we have

$$\begin{aligned} \big\||x^py^*z^pyx^p|^{\frac{1}{p}}\big\|^p_{E(\mathcal{M})} &= \|x^py^*z^pyx^p\|_{E(\mathcal{M})^{(\frac{1}{p})}} \\ &\le \|x^p\|^2_{E(\mathcal{M})^{(\frac{3}{p})}}\|y^*z^py\|_{E(\mathcal{M})^{(\frac{3}{p})}} \\ &\le \|x^3\|^{\frac{2p}{3}}_{E(\mathcal{M})}\|z^3\|^{\frac{p}{3}}_{E(\mathcal{M})} \\ &= \|x\|^{2p}_{E(\mathcal{M})^{(3)}}\|z\|^p_{E(\mathcal{M})^{(3)}} < \infty. \end{aligned}$$

This implies that $g(p) < \infty$ for all $p \in (0,\infty)$. Hence the result follows immediately from Proposition 3.2.

(2) The proof is similar to the proof of (1). $\square$

Remark 3.4. The following results can be done similarly to Corollary 3.3 by using Proposition 2.4.

Let $0 \le x, z \in L_0(\mathcal{M})$, and let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex. Then, for $0 < p \le q < \infty$, we have

(1) $f((x^p z^p)^{\frac{1}{p}}) \precsim f((x^q z^q)^{\frac{1}{q}})$—if E is a fully symmetric Banach space and $0 \le x, z \in E(\mathcal{M})^{(2)}$, then $g(p) = \||x^p z^p|^{\frac{1}{p}}\|_{E(\mathcal{M})}$ is an increasing function on $(0,\infty)$;

(2) $f((x^{\frac{1}{q}} z^{\frac{1}{q}})^q) \precsim f((x^{\frac{1}{p}} z^{\frac{1}{p}})^p)$—if E is a fully symmetric Banach space and $0 \le x, z \in E(\mathcal{M})^{(2)}$, then $h(p) = \||x^{\frac{1}{p}} z^{\frac{1}{p}}|^p\|_{E(\mathcal{M})}$ is a decreasing function on $(0,\infty)$.

Now, using Lemma 3.1, we get the other main result of this paper.

Proposition 3.5. *Let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex, and let $x, z \in L_0(\mathcal{M})$, and let $z \ge 0$.*

(1) *For any contraction $y \in \mathcal{M}$ and $0 < p \le 1$, we have*

$$f\big(|x|^p y^* z^p y |x|^p\big) \precsim f\big((x^* y^* z y x)^p\big).$$

(2) *For any contraction $y \in \mathcal{M}$ and $1 \le p < \infty$, we have*

$$f\big((x^* y^* z y x)^p\big) \precsim f\big(|x|^p y^* z^p y |x|^p\big).$$

Proof. By slightly modifying the proof of Lemma 2.6 (or Theorem 3.2 in [4]), we can prove this proposition and omit the details. □

Corollary 3.6. *Let E be a fully symmetric Banach space, and let $y \in \mathcal{M}$ be a contraction.*

(1) *If $0 < p \le 1$ and $x, z \in E(\mathcal{M})^{(3p)}$ and $z \ge 0$, then*

$$\big\||x|^p y^* z^p y |x|^p\big\|_{E(\mathcal{M})} \le \big\|(x^* y^* z y x)^p\big\|_{E(\mathcal{M})}.$$

(2) *If $1 \le p < \infty$ and $x, z \in E(\mathcal{M})^{(3p)}$ and $z \ge 0$, then*

$$\big\|(x^* y^* z y x)^p\big\|_{E(\mathcal{M})} \le \big\||x|^p y^* z^p y |x|^p\big\|_{E(\mathcal{M})}.$$

Proof. The result can be done similarly to the proof of Corollary 3.3 by using Proposition 3.5. □

In view of the result in Proposition 3.5, we obtain another refinement of the first inequality in Lemma 3.1.

Proposition 3.7. *Let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex, and let $x, z \in L_0(\mathcal{M})$ and z be self-adjoint. For any contraction $y \in \mathcal{M}$ and $1 \le p < \infty$, we have*

$$f\big(|x^* y^* z y x|^p\big) \precsim f\big(|x|^p y^* |z|^p y |x|^p\big).$$

Proof. The proof can be done similarly to the proof of Lemma 2.7 (or Theorem 3.6 in [4]) by using Proposition 3.5. The details are omitted. □

Corollary 3.8. *Let $0 \le x, z, y \in L_0(\mathcal{M})$, and let $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $f(0) = 0$ and $t \to f(e^t)$ is convex.*

(1) *If $1 \le p < \infty$, then $f(\frac{(x(y+z)x)^p}{2^p}) \precsim f(\frac{x^p(y^p+z^p)x^p}{2})$.*
(2) *If $0 < p \le 1$, then $f(\frac{x^p(y^p+z^p)x^p}{2}) \precsim f(\frac{(x(y+z)x)^p}{2^p})$.*

Proof. Let $X = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix}, Y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Then

$$(XY^*ZYX)^p = \begin{pmatrix} \frac{(x(y+z)x)^p}{2^p} & 0 \\ 0 & 0 \end{pmatrix}, \qquad X^pY^*Z^pYX^p = \begin{pmatrix} \frac{x^p(y^p+z^p)x^p}{2} & 0 \\ 0 & 0 \end{pmatrix}.$$

According to Lemma 3.1, we obtain

$$f\big((XY^*ZYX)^p\big) \precsim f(X^pY^*Z^pYX^p), \quad 1 \le p < \infty$$

and

$$f(X^pY^*Z^pYX^p) \precsim f\big((XY^*ZYX)^p\big), \quad 0 < p \le 1.$$

This completes the proof. ∎

If we replace x by 1 in Corollary 3.8, then

$$(y+z)^p \precsim 2^{p-1}(y^p + z^p) \tag{3.2}$$

for $0 \le z, y \in L_0(\mathcal{M})$, and $p \ge 1$.

The following result is a special case of Theorem 5.3 in [10]. Let $0 \le x_1, x_2 \in L_0(\mathcal{M})$. If $f : [0,\infty) \to [0,\infty)$ is any nonnegative concave function, then

$$f(x_1 + x_2) \precsim f(x_1) + f(x_2). \tag{3.3}$$

Based on (3.2) and (3.3), we have the following result, which is related to the Heinz means.

Corollary 3.9. *Let $0 \le x, y \in L_0(\mathcal{M})$, and let $0 \le p \le 1$. Then*

$$\mu(x^py^{1-p} + y^px^{1-p}) \precsim 2^{|\frac{1}{2}-p|}\mu(x+y).$$

Proof. Since the result is obvious when $p = 0, 1$, we only need to prove it for $0 < p < 1$. Let us first assume $0 < p \le \frac{1}{2}$. We write $r = \frac{1}{p}$ and $q = \frac{1}{1-p}$. Then $1 \le q \le 2 \le r$ and $\frac{1}{r} + \frac{1}{q} = 1$. Let $A = \begin{pmatrix} x^p & y^p \\ 0 & 0 \end{pmatrix}$, and let $B = \begin{pmatrix} y^{1-p} & 0 \\ x^{1-p} & 0 \end{pmatrix}$. According to (2.4), (2.5), (2.6), and the usual Hölder inequality, we obtain

$$\begin{aligned}
&\int_0^t \mu_s(x^py^{1-p} + y^px^{1-p})\,ds \\
&\quad= \int_0^t \mu_s(AB)\,ds \le \int_0^t \mu_s(A)\mu_s(B)\,ds \\
&\quad= \int_0^t \mu_s(AA^*)^{\frac{1}{2}}\mu_s(B^*B)^{\frac{1}{2}}\,ds \\
&\quad= \int_0^t \mu_s(x^{2p} + y^{2p})^{\frac{1}{2}}\mu_s(x^{2(1-p)} + y^{2(1-p)})^{\frac{1}{2}}\,ds \\
&\quad\le \Big(\int_0^t \mu_s(x^{2p} + y^{2p})^{\frac{r}{2}}\,ds\Big)^{\frac{1}{r}}\Big(\int_0^t \mu_s(x^{2(1-p)} + y^{2(1-p)})^{\frac{q}{2}}\,ds\Big)^{\frac{1}{q}}
\end{aligned}$$

for all $t > 0$. Since $\frac{q}{2} \le 1$, by inequality (3.3), we get

$$\Big(\int_0^t \mu_s(x^{2(1-p)} + y^{2(1-p)})^{\frac{q}{2}}\, ds\Big)^{\frac{1}{q}} \le \Big(\int_0^t \mu_s(x^{q(1-p)} + y^{q(1-p)})\, ds\Big)^{\frac{1}{q}} = \Big(\int_0^t \mu_s(x+y)\, ds\Big)^{1-p}, \quad t > 0.$$

Note that $\frac{r}{2} \ge 1$. It follows from inequality (3.2) that

$$\Big(\int_0^t \mu_s(x^{2p} + y^{2p})^{\frac{r}{2}}\, ds\Big)^{\frac{1}{r}} \le \Big(2^{\frac{p}{2}-1}\int_0^t \mu_s(x^{rp} + y^{rp})\, ds\Big)^{\frac{1}{r}} = 2^{\frac{1}{2}-p}\Big(\int_0^t \mu_s(x+y)\, ds\Big)^{p}, \quad t > 0.$$

Hence, for $0 \le p \le \frac{1}{2}$, we have

$$\int_0^t \mu_s(x^p y^{1-p} + y^p x^{1-p})\, ds \le 2^{\frac{1}{2}-p}\int_0^t \mu_s(x+y)\, ds, \quad t > 0. \tag{3.4}$$

Now we suppose that $\frac{1}{2} \le p < 1$. If we replace p by $1-p$ and interchange x and y in (3.4), we obtain

$$\begin{aligned}\int_0^t \mu_s(x^p y^{1-p} + y^p x^{1-p})\, ds &= \int_0^t \mu_s\big((x^p y^{1-p} + y^p x^{1-p})^*\big)\, ds \\ &= \int_0^t \mu_s(y^{1-p}x^p + x^{1-p}y^p)\, ds \\ &\le 2^{p-\frac{1}{2}}\int_0^t \mu_s(x+y)\, ds, \quad t > 0.\end{aligned}$$

This completes the proof. □

The matrix version of Corollary 3.9 appears in [1]. The result in Corollary 3.9 can be extended to normal measurable operators.

Corollary 3.10. *Let x and y be normal in $L_0(\mathcal{M})$, and let $p, q \ge 0$ with $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$\mu(x^p y^q + y^p x^q) \precsim 2^{|\frac{1}{2}-\frac{1}{p}|}\mu\big(|x|^{p+q} + |y|^{p+q}\big).$$

Proof. Without loss of generality, we may assume that $p \le q$. Then $1 \le p \le 2 \le q$. Let $A = \left(\begin{smallmatrix} x^p & y^p \\ 0 & 0\end{smallmatrix}\right)$, and let $B = \left(\begin{smallmatrix} y^q & 0 \\ x^q & 0\end{smallmatrix}\right)$. Since x and y are normal, then $|A^*| = |A|$, and hence

$$\begin{aligned}\mu(AA^*) &= \mu\left(\begin{pmatrix} x^p & y^p \\ 0 & 0\end{pmatrix}\begin{pmatrix} (x^p)^* & 0 \\ (y^p)^* & 0\end{pmatrix}\right) \\ &= \mu\left(\begin{pmatrix} |x^*|^{2p} + |y^*|^{2p} & 0 \\ 0 & 0\end{pmatrix}\right) \\ &= \mu\left(\begin{pmatrix} |x|^{2p} + |y|^{2p} & 0 \\ 0 & 0\end{pmatrix}\right) \\ &= \mu\big(|x|^{2p} + |y|^{2p}\big).\end{aligned}$$

Similarly, $\mu(B^*B) = \mu(|x|^{2q}+|y|^{2q})$. According to (2.4), (2.5), and (2.6), we obtain

$$\begin{aligned}\int_0^t \mu_s(x^p y^q + y^p x^q)\,ds &= \int_0^t \mu_s(AB)\,ds \le \int_0^t \mu_s(A)\mu_s(B)\,ds \\ &= \int_0^t \mu_s(AA^*)^{\frac{1}{2}}\mu_s(B^*B)^{\frac{1}{2}}\,ds \\ &= \int_0^t \mu_s\big(|x|^{2p}+|y|^{2p}\big)^{\frac{1}{2}}\mu_s\big(|x|^{2q}+|y|^{2q}\big)^{\frac{1}{2}}\,ds, \quad t>0.\end{aligned}$$

Then the proof can be done similarly to Corollary 3.9. The details are omitted. □

The matrix version of Corollary 3.10 appears in [1].

Corollary 3.11. *Let $p, q \ge 0$ with $\frac{1}{p}+\frac{1}{q}=1$, and let E be a fully symmetric Banach space. If x and y are normal in $E(\mathcal{M})^{(p+q)}$, then*

$$\|x^p y^q + y^p x^q\|_{E(\mathcal{M})} \le 2^{|\frac{1}{2}-\frac{1}{p}|}\big\||x|^{p+q}+|y|^{p+q}\big\|_{E(\mathcal{M})}.$$

Proof. The result follows immediately from Corollary 3.10.

Remark 3.12. All the results in this section, in the matrix case and the type I_∞ case, are contained in Bourin and Lee's paper [6], except Corollaries 3.9–3.11 on the Heinz means.

Acknowledgments. The author is grateful to the editor and the anonymous referee for making helpful comments and suggestions which have been incorporated into this version of the paper. This research was supported in part by the National Natural Science Foundation of China no. 11401507 and the Natural Science Foundation of Xinjiang University (Starting Fund for Doctors, grant no. BS150202).

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