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A PERTURBED EIGENVALUE PROBLEM ON GENERAL DOMAINS

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Abstract. The perturbed eigenvalue problem $-\Delta u - \Delta_p u = \lambda V(x)u$, with $p \in (1, N) \setminus \{2\}$ and V a weight function which takes nonnegative values and may have singular points, is studied in an Orlicz–Sobolev setting on general open sets from $\mathbb{R}^N$ with $N \geq 3$. The analysis of these problems leads to a full characterization of the set of parameters λ for which the problem possesses nontrivial solutions as being an unbounded open interval.

1. Introduction and main results

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 3$) be an open set, and let $V : \Omega \to [0, \infty)$ be a function which satisfies the hypotheses

$$\begin{cases} V \in L^1_{\text{loc}}(\Omega), V = V_1 + V_2, V_1 \in L^{N/2}(\Omega), \\ \lim_{|x|\to\infty} |x|^2 V_2(x) = 0, \lim_{x\to y} |x-y|^2 V_2(x) = 0 \text{ for any } y \in \overline{\Omega}. \end{cases} \tag{1.1}$$

For example, a weight function V which satisfies condition (1.1) could be $V(x) = |x|^{-2}(1 + |x|^2)^{-1}[\log(2 + 1/|x|^2)]^{-2/N}$. (Other examples can be found in [6, Section 3].)

In [6], Szulkin and Willem analyzed the eigenvalue problem

$$-\Delta u = \lambda V(x)u, \quad u \in \mathcal{D}_0^{1,2}(\Omega), \tag{1.2}$$

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where $\mathcal{D}_0^{1,2}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ under the L^2-norm of the gradient (see Section 2 for the accurate definition of $\mathcal{D}_0^{1,2}(\Omega)$). Using an elementary argument based on a simple minimization procedure, the existence of infinitely many eigenvalues of (1.2) was proved in [6, Theorems 2.2, 2.3].

Motivated by the results from [6], the goal of this paper is to give a complete description of the set of parameters λ for which the following perturbed eigenvalue problem has nontrivial solutions:

$$-\Delta u - \Delta_p u = \lambda V(x) u, \quad u \in \mathcal{D}_0^{1,\Phi_p}(\Omega), \tag{1.3}$$

where $p \in (1, N) \setminus \{2\}$ and $\Phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\Phi_p(t) := \frac{t^2}{2} + \frac{|t|^p}{p}$, and the Orlicz–Sobolev-type space $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ is obtained as the closure of $C_0^\infty(\Omega)$ under the Luxemburg-type norm

$$\|u\| := \inf\Big\{\mu > 0; \int_\Omega \Phi_p\Big(\frac{|\nabla u(x)|}{\mu}\Big)\, dx \le 1\Big\}$$

(see Section 2 for more details regarding the definition and properties of Φ_p and $\mathcal{D}_0^{1,\Phi_p}(\Omega)$).

Definition 1.1. We say that u is a *weak solution* of equation (1.3) if there exists $u \in \mathcal{D}_0^{1,\Phi_p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla w\, dx + \int_\Omega |\nabla u|^{p-2} \nabla u \nabla w\, dx = \lambda \int_\Omega V(x) u w\, dx, \quad \forall w \in \mathcal{D}_0^{1,\Phi_p}(\Omega). \tag{1.4}$$

The main result of our paper is formulated in the following theorem.

Theorem 1.2. *Assume condition (1.1) is fulfilled. Then the set of parameters λ for which problem (1.3) possesses nontrivial solutions is exactly the open interval $(\lambda_1, +\infty)$, where λ_1 is given by*

$$\lambda_1 := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega V(x) u^2(x)\, dx}. \tag{1.5}$$

Note that by [6, Theorem 2.2] it is obvious that λ_1 defined in (1.5) is achieved in $\mathcal{D}_0^{1,2}(\Omega)$, which is larger than $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ (see Section 2 for details).

On the other hand, arguments similar to those used in the proof of Theorem 1.2 can be applied here to establish the following result.

Theorem 1.3. *Assume that $\Omega \subset \mathbb{R}^N$ ($N \ge 3$) is an open set and that $V : \Omega \to [0, \infty)$ is a function satisfying the following hypotheses:*

$$\begin{cases} V \in L^1_{\mathrm{loc}}(\Omega), V = V_1 + V_2, V_1 \in L^{N/p}(\Omega), \\ \lim_{|x|\to\infty} |x|^p V_2(x) = 0, \lim_{x\to y} |x-y|^p V_2(x) = 0 \text{ for any } y \in \overline{\Omega}. \end{cases} \tag{1.6}$$

Then the set of parameters λ for which the problem

$$-\Delta u - \Delta_p u = \lambda V(x) |u|^{p-2} u, \quad u \in \mathcal{D}_0^{1,\Phi_p}(\Omega), \tag{1.7}$$

possesses nontrivial weak solutions is exactly the open interval $(\mu_1, +\infty)$*, where* μ_1 *is given by*

$$\mu_1 := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega V(x)|u|^p\,dx}. \tag{1.8}$$

The structure of the paper is as follows. In Section 2, we introduce the function spaces that will be used throughout the paper and state a couple of properties of these spaces. In Section 3 we present the variational setup for the main problem (1.3) and provide the proof of Theorem 1.2. The basic idea of the proof will be to associate to problem (1.3) the so-called *energy functional* whose nontrivial critical points offer solutions to our problem. The specificity of the proof will be given by the fact that the energy functional will be defined on a corresponding *Nehari manifold* on which it achieves its minimum in an element which proves to be a critical point of the functional. This method is described, for instance, in [2, Section 2.3.3] (see also [5] for more applications of the method). However, here we cannot apply the method exactly as in [2, Section 2.3.3] since our problem possesses some particularities which ask for a more careful analysis.

2. Function spaces

Assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is an open set. In this section we will point out the definitions and some elementary properties of the function spaces which will be used in the analysis of problem (1.3).

For each $q \in (1, N)$ denote by $L^q(\Omega)$ the Lebesgue space endowed with the norm

$$\|u\|_{L^q(\Omega)} := \Big(\int_\Omega |u|^q\,dx\Big)^{1/q},$$

and denote by $\mathcal{D}_0^{1,q}(\Omega)$ the Sobolev space defined as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_q := \|\nabla u\|_{L^q(\Omega)}.$$

Further, our aim is that of introducing the Orlicz–Sobolev-type space where problem (1.3) will be analyzed. We provide its definition and a brief review of the basic properties of that space. (For more details, see the books by Adams [1] and Rao and Ren [4] and the paper by Fukagai et al. [3].)

For $p \in (1, N) \setminus \{2\}$, define $\phi_p : \mathbb{R} \to \mathbb{R}$ by

$$\phi_p(t) = t + |t|^{p-2}t.$$

It is easy to check that ϕ_p is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Next, define

$$\Phi_p(t) := \int_0^t \phi_p(s)\,ds = \frac{t^2}{2} + \frac{|t|^p}{p}.$$

Note that $\Phi_p(0) = 0$, Φ_p is convex, and $\lim_{t\to\infty} \Phi_p(t) = +\infty$, which makes Φ_p a *Young function*. Moreover, since $\Phi_p(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \Phi_p(t)/t = 0$, and $\lim_{t\to\infty} \Phi_p(t)/t = +\infty$, Φ_p is an *N-function* (see [1] or [3] for more details). Let $\Phi_p^\star$ be the *complementary* function of Φ_p given by

$$\Phi_p^\star(t) = \sup\{st - \Phi_p(s) : s \geq 0\} \quad \text{for all } t \geq 0.$$

$\Phi_p^\star$ is also an N-function.

Define

$$\phi_p^- := \inf_{t>0} \frac{t\phi_p(t)}{\Phi_p(t)} \qquad \text{and} \qquad \phi_p^+ := \sup_{t>0} \frac{t\phi_p(t)}{\Phi_p(t)}.$$

It is elementary to check that

$$\phi_p^- = p \qquad \text{and} \qquad \phi_p^+ = 2 \quad \text{if } p \in (1,2),$$

and that

$$\phi_p^- = 2 \qquad \text{and} \qquad \phi_p^+ = p \quad \text{if } p \in (2,\infty).$$

Thus we always have

$$1 < \phi_p^- \leq \frac{t\phi_p(t)}{\Phi_p(t)} \leq \phi_p^+ < \infty \quad \text{for all } t > 0. \tag{2.1}$$

Moreover, relation (2.1) and [3, Lemma 2.5, (2.7)] imply that

$$1 < \frac{\phi_p^+}{\phi_p^+ - 1} \leq \frac{t\phi_p^{-1}(t)}{\Phi_p^\star(t)} \leq \frac{\phi_p^-}{\phi_p^- - 1} < \infty \quad \text{for all } t > 0. \tag{2.2}$$

Further, define the Orlicz space $L^{\Phi_p}(\Omega)$ as the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^{\Phi_p}} := \sup\Big\{\int_\Omega uv\,dx : \int_\Omega \Phi_p^\star(|v|)\,dx \leq 1\Big\} < \infty. \tag{2.3}$$

Endowed with the *Orlicz norm* (2.3), $L^{\Phi_p}(\Omega)$ is a Banach space. An equivalent norm on $L^{\Phi_p}(\Omega)$ is the *Luxemburg norm* defined by

$$\|u\|_{\Phi_p} := \inf\Big\{\mu > 0 : \int_\Omega \Phi_p\Big(\frac{u(x)}{\mu}\Big)\,dx \leq 1\Big\}. \tag{2.4}$$

The Orlicz–Sobolev space $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm $\|u\| := \||\nabla u|\|_{\Phi_p}$. We note that (2.1) and (2.2) imply that Φ_p and $\Phi_p^\star$ satisfy the Δ_2-*condition*

$$\Phi_p(2t) \leq K\Phi_p(t) \quad \forall t \geq 0 \tag{2.5}$$

for some constant $K > 0$ (see [1, p. 232]). Thus $L^{\Phi_p}(\Omega)$ and $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ are reflexive Banach spaces (see [1, Theorem 8.19] and [1, p. 232]). On the other hand (see, e.g., [3, Lemma 2.1]), we have

$$\|u\|_{\Phi_p}^{\phi_p^+} \leq \int_\Omega \Phi_p\big(|u(x)|\big)\,dx \leq \|u\|_{\Phi_p}^{\phi_p^-} \quad \forall u \in L^{\Phi_p}(\Omega), \|u\|_{\Phi_p} < 1, \tag{2.6}$$

and

$$\|u\|_{\Phi_p}^{\phi_p^-} \leq \int_\Omega \Phi_p\big(|u(x)|\big)\,dx \leq \|u\|_{\Phi_p}^{\phi_p^+} \quad \forall u \in L^{\Phi_p}(\Omega), \|u\|_{\Phi_p} > 1. \tag{2.7}$$

Finally, note that, by the definition of the Luxemburg norm, we have

$$1 \geq \int_\Omega \Phi_p\Big(\frac{|\nabla u(x)|}{\|\nabla u\|_{\Phi_p}}\Big)\,dx = \frac{1}{p}\int_\Omega \frac{|\nabla u(x)|^p}{\|\nabla u\|_{\Phi_p}^p}\,dx + \frac{1}{2}\int_\Omega \frac{|\nabla u(x)|^2}{\|\nabla u\|_{\Phi_p}^2}$$

for any $u \in \mathcal{D}_0^{1,\Phi_p}(\Omega)$. The above inequality implies that

$$\begin{aligned}&\|\nabla u\|_{L^p(\Omega)} \leq \sqrt[p]{p}\|\nabla u\|_{\Phi_p} \qquad \text{and}\\ &\|\nabla u\|_{L^2(\Omega)} \leq \sqrt{2}\|\nabla u\|_{\Phi_p} \quad \text{for any } u \in \mathcal{D}_0^{1,\Phi_p}(\Omega),\end{aligned}$$

or

$$\|u\|_p \leq \sqrt[p]{p}\|u\| \qquad \text{and} \qquad \|u\|_2 \leq \sqrt{2}\|u\| \quad \text{for any } u \in \mathcal{D}_0^{1,\Phi_p}(\Omega).$$

It follows that $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ is continuously embedded in $\mathcal{D}_0^{1,p}(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)$.

On the other hand, by [6, Lemma 2.1] we know that the application $T : \mathcal{D}_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$T(u) = \int_\Omega V(x)u^2\,dx$$

is weakly continuous; that is, if $\{u_n\} \subset \mathcal{D}_0^{1,2}(\Omega)$ weakly converges to $u \in \mathcal{D}_0^{1,2}(\Omega)$ in $\mathcal{D}_0^{1,2}(\Omega)$, then $T(u_n) \to T(u)$. In particular, we deduce that application T is weakly continuous on $\mathcal{D}_0^{1,\Phi_p}(\Omega)$.

3. Proof of the main result

We start by pointing out a result related with λ_1 introduced by relation (1.5) from Theorem 1.2. This quantity will play a key role in our analysis of problem (1.3).

Lemma 3.1. *Let λ_1 be defined by relation (1.5) from Theorem 1.2. Define*

$$\nu_1 := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx}{\frac{1}{2}\int_\Omega V(x)u^2\,dx}. \tag{3.1}$$

Then $\nu_1 = \lambda_1$.

Proof. First, note that, for each $u \in C_0^\infty(\Omega)\setminus\{0\}$, we have

$$\inf_{w \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega V(x)w^2\,dx} \leq \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega V(x)u^2\,dx} \leq \frac{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx}{\frac{1}{2}\int_\Omega V(x)u^2\,dx}.$$

Thus we get

$$\inf_{w \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega V(x)w^2\,dx} \leq \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx}{\frac{1}{2}\int_\Omega V(x)u^2\,dx},$$

or

$$\lambda_1 \leq \nu_1.$$

On the other hand, for each $u \in C_0^\infty(\Omega)\setminus\{0\}$ and each $t > 0$, we have

$$\inf_{w \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\frac{1}{p}\int_\Omega |\nabla w|^p\,dx + \frac{1}{2}\int_\Omega |\nabla w|^2\,dx}{\frac{1}{2}\int_\Omega V(x)w^2\,dx} \leq \frac{\frac{t^{p-2}}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx}{\frac{1}{2}\int_\Omega V(x)u^2\,dx}.$$

Letting $t \to +\infty$ if $p \in (1,2)$ or $t \to 0$ if $p \in (2,\infty)$, we obtain

$$\inf_{w \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\frac{1}{p}\int_\Omega |\nabla w|^p\,dx + \frac{1}{2}\int_\Omega |\nabla w|^2\,dx}{\frac{1}{2}\int_\Omega V(x)w^2\,dx} \le \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega V(x)u^2\,dx}$$

for any $u \in C_0^\infty(\Omega)\setminus\{0\}$. Taking the inf as $u \in C_0^\infty(\Omega)\setminus\{0\}$ in the right-hand side of the above inequality, we conclude that $\nu_1 \le \lambda_1$, and, consequently, the conclusion of our lemma holds true. □

The conclusion of Theorem 1.2 will be a simple consequence of the next two propositions.

Proposition 3.2. *For each $\lambda \in (-\infty, \lambda_1]$, problem (1.3) has no nontrivial solution.*

Proof. First, note that, assuming that for some $\lambda \in (-\infty, 0]$ problem (1.3) has a nontrivial solution, we arrive at a contradiction by taking $w = u$ in relation (1.4). Thus, for any $\lambda \in (-\infty, 0]$, we cannot find nontrivial solutions of problem (1.3).

Next, let $\lambda \in (0, \lambda_1)$. We assume that there exists a solution of (1.3) with $u \neq 0$. By the definition of λ_1 given by (1.5) and taking $w = u$ in (1.4), we get

$$\begin{aligned} 0 &< \frac{\lambda_1 - \lambda}{2}\int_\Omega V(x)u^2\,dx \\ &\le \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega V(x)u^2\,dx \\ &\le \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{2}\int_\Omega V(x)u^2\,dx = 0, \end{aligned}$$

which represents a contradiction.

Finally, we show that for $\lambda = \lambda_1$ problem (1.3) has no nontrivial solution. We assume by contradiction that there exists $u \neq 0$ a solution of problem (1.3) with $\lambda = \lambda_1$. Then, by relations (1.4) with $w = u$ and (1.5), we deduce

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla u|^p\,dx = \lambda_1\int_\Omega V(x)u^2\,dx \le \int_\Omega |\nabla u|^2\,dx,$$

which implies that $\int_\Omega |\nabla u|^p\,dx = 0$, or $\|u\|_p = 0$. It follows that $u = 0$. This is a contradiction. □

Proposition 3.3. *For each $\lambda > \lambda_1$ problem (1.3) has nontrivial weak solutions.*

In order to prove Proposition 3.3, we start by defining for each $\lambda > \lambda_1$ the so-called *energy functional* associated to problem (1.3) as $J_\lambda : \mathcal{D}_0^{1,\Phi_p}(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega V(x)u^2\,dx.$$

It is standard to check that $J_\lambda \in C^1(\mathcal{D}_0^{1,\Phi_p}(\Omega)\setminus\{0\}, \mathbb{R})$ with its derivative given by

$$\langle J_\lambda'(u), w\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w\,dx + \int_\Omega \nabla u \nabla w\,dx - \lambda \int_\Omega V(x) u w\,dx$$

for all $u \in \mathcal{D}_0^{1,\Phi_p}(\Omega) \setminus \{0\}$ and $w \in \mathcal{D}_0^{1,\Phi_p}(\Omega)$.

Note that we cannot establish the coercivity of J_λ on $\mathcal{D}_0^{1,\Phi_p}(\Omega)$, and consequently we cannot apply the direct method in the calculus of variations in order to find critical points for this functional. In this context, our idea will be to analyze the energy functional on a subset of $\mathcal{D}_0^{1,\Phi_p}(\Omega)$, namely, the so-called *Nehari manifold* defined by

$$\begin{aligned}\mathcal{N}_\lambda &:= \left\{u \in \mathcal{D}_0^{1,\Phi_p}(\Omega) \setminus \{0\}; \langle J_\lambda'(u), u\rangle = 0\right\} \\ &= \left\{u \in \mathcal{D}_0^{1,\Phi_p}(\Omega) \setminus \{0\}; \int_\Omega |\nabla u|^p\,dx + \int_\Omega |\nabla u|^2\,dx = \lambda \int_\Omega V(x) u^2\,dx\right\}.\end{aligned}$$

We point out that, for each $u \in \mathcal{N}_\lambda$, we have

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{2}\right) \int_\Omega |\nabla u|^p\,dx, \tag{3.2}$$

and

$$\lambda \int_\Omega V(x) u^2\,dx > \int_\Omega |\nabla u|^2\,dx. \tag{3.3}$$

Next, we will establish a few properties of $\mathcal{N}_\lambda$ and of the restriction of J_λ to $\mathcal{N}_\lambda$ which will prove to be useful in establishing the conclusion of Proposition 3.3.

Lemma 3.4. *We have that* $\mathcal{N}_\lambda \neq \emptyset$.

Proof. Indeed, since $\lambda > \lambda_1$, it follows that there exists $w \in \mathcal{D}_0^{1,\Phi_p}(\Omega) \setminus \{0\}$ for which

$$\int_\Omega |\nabla w|^2\,dx < \lambda \int_\Omega V(x) w^2\,dx.$$

Then there exists $t > 0$ such that $tw \in \mathcal{N}_\lambda$; that is,

$$t^2 \int_\Omega |\nabla w|^2\,dx + t^p \int_\Omega |\nabla w|^p\,dx = \lambda t^2 \int_\Omega V(x) w^2\,dx,$$

which is obvious with

$$t = \left(\frac{\lambda \int_\Omega V(x) w^2\,dx - \int_\Omega |\nabla w|^2\,dx}{\int_\Omega |\nabla w|^p\,dx}\right)^{1/(p-2)} > 0. \qquad \square$$

Let

$$m := \inf_{\tau \in \mathcal{N}_\lambda} J_\lambda(\tau).$$

By (3.2) we deduce that $J_\lambda(u) \geq 0$ for all $u \in \mathcal{N}_\lambda$ if $p \in (1,2)$ and $J_\lambda(u) < 0$ for all $u \in \mathcal{N}_\lambda$ if $p \in (2,\infty)$. Thus $m \geq 0$ if $p \in (1,2)$ and $m < 0$ if $p \in (2,\infty)$.

Next, our idea will be to prove that m can be achieved on $\mathcal{N}_\lambda$. In order to show that, we will analyze two separate cases: $p \in (1,2)$ and $p \in (2,N)$.

The case $p \in (1,2)$.

Lemma 3.5. *Every minimizing sequence of functional* J_λ *on* $\mathcal{N}_\lambda$ *is bounded in* $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ *provided that* $p \in (1,2)$.

Proof. Let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence of J_λ on $\mathcal{N}_\lambda$; that is,

$$\int_\Omega |\nabla u_n|^p \, dx \longrightarrow \Big(\frac{1}{p} - \frac{1}{2}\Big)^{-1} m \quad \text{as } n \to \infty.$$

We assume by contradiction that $\int_\Omega |\nabla u_n|^2 \, dx \to \infty$ as $n \to \infty$. Then, since for each n we have $u_n \in \mathcal{N}_\lambda$, we deduce that $\int_\Omega V(x) u_n^2 \, dx \to \infty$ as $n \to \infty$.

Let

$$w_n := \frac{u_n}{(\int_\Omega V(x) u_n^2 \, dx)^{1/2}}.$$

Since inequality (3.3) holds true for any n, we deduce that $\int_\Omega |\nabla w_n|^2 \, dx < \lambda$ for any n. Thus the sequence $\{w_n\}_n$ is bounded in $\mathcal{D}_0^{1,2}(\Omega)$.

Since here it holds that the sequence $\{\int_\Omega V u_n^2 \, dx\}_n$ is unbounded while the sequence $\{\int_\Omega |\nabla u_n|^p \, dx\}_n$ is bounded, we have

$$\int_\Omega |\nabla w_n|^p \, dx = \frac{\int_\Omega |\nabla u_n|^p \, dx}{(\int_\Omega V(x) u_n^2 \, dx)^{p/2}} \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.4}$$

The above relation shows that $\{w_n\}_n$ is bounded in $\mathcal{D}_0^{1,p}(\Omega)$. Consequently, $\{w_n\}_n$ is bounded in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$, too. It follows that there exists $w_0 \in \mathcal{D}_0^{1,\Phi_p}(\Omega)$ such that w_n converges weakly to w_0 in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ (and consequently in $\mathcal{D}_0^{1,p}(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)$) and $\lim_{n\to\infty} \int_\Omega V(x) w_n^2 \, dx = \int_\Omega V(x) w_0^2 \, dx$.

By (3.4), we have

$$\int_\Omega |\nabla w_0|^p \, dx \le \liminf_{n\to\infty} \int_\Omega |\nabla w_n|^p \, dx = 0,$$

which implies that $w_0 = 0$.

On the other hand, since $\int_\Omega V(x) w_n^2 \, dx = 1$ for any n, we get that $\int_\Omega V(x) w_0^2 \, dx = 1$, and that is a contradiction with $w_0 = 0$. Thus the sequence $\{\int_\Omega |\nabla u_n|^2 \, dx\}_n$ is bounded, and since $\lambda_1 \int_\Omega V(x) u_n^2 \, dx \le \int_\Omega |\nabla u_n|^2 \, dx$ for each n, the sequence $\{\int_\Omega V(x) u_n^2 \, dx\}_n$ is bounded, too. Taking into account that $u_n \in \mathcal{N}_\lambda$, it follows that the sequence $\{\int_\Omega |\nabla u_n(x)|^p \, dx\}_n$ is also bounded, and the proof of this lemma is clear. □

Lemma 3.6. *If* $p \in (1,2)$, *then* $m > 0$.

Proof. As we already noted, $m \ge 0$ when $p \in (1,2)$. We assume by contradiction that $m = 0$. Let $\{u_n\}_n \subset \mathcal{N}_\lambda$ be a minimizing sequence for $m = 0$; that is,

$$\int_\Omega \big|\nabla u_n(x)\big|^p \, dx \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Using the same arguments as in the proof of Lemma 3.5, we infer that $\{u_n\}_n$ is bounded in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ and, consequently, that it is also bounded in $\mathcal{D}_0^{1,p}(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)$. It follows that there exists $u_0 \in \mathcal{D}_0^{1,\Phi_p}(\Omega)$ such that $\{u_n\}_n$ converges weakly to u_0 in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$, $\mathcal{D}_0^{1,p}(\Omega)$, and $\mathcal{D}_0^{1,2}(\Omega)$, and $\lim_{n\to\infty}\int_\Omega V(x)u_n^2\,dx = \int_\Omega V(x)u_0^2\,dx$. Clearly,

$$\int_\Omega |\nabla u_0|^p\,dx \le \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^p\,dx = 0.$$

Thus $u_0 = 0$. Consequently, $\{u_n\}_n$ converges weakly to 0 in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$, $\mathcal{D}_0^{1,p}(\Omega)$, and $\mathcal{D}_0^{1,2}(\Omega)$, and

$$\lim_{n\to\infty}\int_\Omega V(x)u_n^2\,dx = 0.$$

Let

$$w_n := \frac{u_n}{(\int_\Omega V(x)u_n^2\,dx)^{1/2}}.$$

Taking into account that inequality (3.3) holds true for each n, we obtain that

$$\int_\Omega |\nabla w_n|^2\,dx = \frac{\int_\Omega |\nabla u_n|^2\,dx}{\int_\Omega V(x)u_n^2\,dx} < \lambda \tag{3.6}$$

for any n. Consequently, the sequence $\{w_n\}_n$ is bounded in $\mathcal{D}_0^{1,2}(\Omega)$.

On the other hand, since for each n we have $u_n \in \mathcal{N}_\lambda$ and $p \in (1,2)$, we infer that

$$\int_\Omega |\nabla w_n|^p\,dx = \Big(\int_\Omega V(x)u_n^2\,dx\Big)^{(2-p)/2}\Big(\lambda - \int_\Omega |\nabla w_n|^2\,dx\Big) \to 0 \quad \text{as } n\to\infty. \tag{3.7}$$

Relations (3.7) and (3.6) imply that $\{w_n\}_n$ is bounded in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$. It follows that there exists $w_0 \in \mathcal{D}_0^{1,\Phi_p}(\Omega)$ such that w_n converges weakly to w_0 in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$, $\mathcal{D}_0^{1,p}(\Omega)$, and $\mathcal{D}_0^{1,2}(\Omega)$, and

$$\lim_{n\to\infty}\int_\Omega V(x)w_n^2\,dx = \int_\Omega V(x)w_0^2\,dx.$$

Taking into account (3.7), we find that

$$\int_\Omega |\nabla w_0|^p\,dx \le \liminf_{n\to\infty}\int_\Omega |\nabla w_n|^p\,dx = 0,$$

or $w_0 = 0$. But $\int_\Omega V(x)w_n^2\,dx = 1$ for each n, and thus $\int_\Omega V(x)w_0^2\,dx = 1$, a contradiction with $w_0 = 0$. In conclusion, $m > 0$, and consequently the conclusion of Lemma 3.6 holds true. □

Lemma 3.7. *If $p \in (1,2)$, then there exists $u \in \mathcal{N}_\lambda$ such that $J_\lambda(u) = m$.*

Proof. Let $\{u_k\}_k \subset \mathcal{N}_\lambda$ be a minimizing sequence for m; that is, $J_\lambda(u_k) \longrightarrow m$ as $k\to\infty$. By Lemma 3.5, we have that $\{u_k\}_k$ is bounded in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$. Thus there exists $u \in \mathcal{D}_0^{1,\Phi_p}(\Omega)$ such that u_k converges weakly to u in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$, $\mathcal{D}_0^{1,p}(\Omega)$, and $\mathcal{D}_0^{1,2}(\Omega)$, and $\lim_{k\to\infty}\int_\Omega V(x)u_k^2\,dx = \int_\Omega V(x)u^2\,dx$ and $u_k(x) \to u(x)$ for a.e. $x \in \Omega$.

By the above pieces of information, we deduce that

$$J_\lambda(u) \le \liminf_{k\to\infty} J_\lambda(u_k) = m.$$

Since $u_k \in \mathcal{N}_\lambda$ for each k, we have

$$\int_\Omega |\nabla u_k|^2\,dx + \int_\Omega |\nabla u_k|^p\,dx = \lambda \int_\Omega V(x)u_k^2\,dx \quad \text{for all } k. \tag{3.8}$$

If $u \equiv 0$ in Ω, then $\int_\Omega V(x)u_k^2\,dx \to 0$ as $k \to \infty$, and by (3.8) we obtain $\int_\Omega |\nabla u_k|^2\,dx + \int_\Omega |\nabla u_k|^p\,dx \to 0$ or $\int_\Omega \Phi_p(|\nabla u_k|)\,dx \to 0$ as $k \to \infty$. Combining that fact with relation (2.6), we infer that u_k converges strongly to 0 in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$, and consequently in $\mathcal{D}_0^{1,p}(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)$.

Thus we deduce that

$$0 < \lambda \int_\Omega V(x)u_k^2\,dx - \int_\Omega |\nabla u_k|^2\,dx = \int_\Omega |\nabla u_k|^p\,dx \longrightarrow 0$$

as $k \to \infty$. Next, we can apply a similar argument as the one used in the proof of Lemma 3.6 in order to arrive at a contradiction. Consequently, $u \neq 0$.

Letting $k \to \infty$ in (3.8), we deduce that

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla u|^p\,dx \le \lambda \int_\Omega V(x)u^2\,dx.$$

If we have the equality above, then $u \in \mathcal{N}_\lambda$ and everything is clear. Otherwise, if we have

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla u|^p\,dx < \lambda \int_\Omega V(x)u^2\,dx,$$

then we will show that we will arrive at a contradiction. We assume that

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla u|^p\,dx < \lambda \int_\Omega V(x)u^2\,dx. \tag{3.9}$$

Let $t > 0$ be such that $tu \in \mathcal{N}_\lambda$; that is, $t = (\frac{\lambda \int_\Omega V(x)u^2\,dx - \int_\Omega |\nabla u|^2\,dx}{\int_\Omega |\nabla u|^p\,dx})^{1/(p-2)}$. Note that, by (3.9) and $\frac{1}{p-2} < 0$, we get $t \in (0,1)$. Finally, since $tu \in \mathcal{N}_\lambda$ with $t \in (0,1)$, we have

$$\begin{aligned} 0 < m \le J_\lambda(tu) &= \Big(\frac{1}{p} - \frac{1}{2}\Big) \int_\Omega \big|\nabla(tu)\big|^p\,dx = t^p \Big(\frac{1}{p} - \frac{1}{2}\Big) \int_\Omega |\nabla u|^p\,dx \\ &\le t^p \liminf_{k\to\infty} J_\lambda(u_k) = t^p m < m, \end{aligned}$$

a contradiction. Thus (3.9) cannot hold true. Therefore, $u \in \mathcal{N}_\lambda$, and by

$$J_\lambda(u) \le \liminf_{k\to\infty} J_\lambda(u_k) = m,$$

we conclude that $J_\lambda(u) = m$. The proof of Lemma 3.7 is complete. □

The case $p \in (2, N)$.

Lemma 3.8. *The Nehari manifold $\mathcal{N}_\lambda$ is bounded in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$ provided $p \in (2, N)$.*

Proof. First, we prove that if $\{u_n\}_n \subset \mathcal{N}_\lambda$, then $\{\int_\Omega |\nabla u_n|^2\,dx\}_n$ is bounded. Assume the contrary; that is, $\int_\Omega |\nabla u_n|^2\,dx \to +\infty$ as $n \to \infty$.

Let $w_n := u_n/(\int_\Omega |\nabla u_n|^2\,dx)^{1/2}$. Then $\int_\Omega |\nabla w_n|^2\,dx = 1$ for any n, which means that $\{w_n\}_n$ is bounded in $\mathcal{D}_0^{1,2}(\Omega)$. Thus there exists $w \in \mathcal{D}_0^{1,2}(\Omega)$ such that w_n converges weakly to w in $\mathcal{D}_0^{1,2}(\Omega)$, $\int_\Omega V(x)w_n^2\,dx \to \int_\Omega V(x)w^2\,dx$, and $w_n(x) \to w(x)$ for a.e. $x \in \Omega$ as $n \to \infty$.

Since $u_n \in \mathcal{N}_\lambda$, for each n, it follows that (3.3) holds true for u_n, or $\lambda \int_\Omega V w_n^2\,dx > 1$ for each n. Passing to the limit as $n \to \infty$, we obtain that

$$\lambda \int_\Omega V(x)w^2\,dx \geq 1. \tag{3.10}$$

On the other hand, since $u_n \in \mathcal{N}_\lambda$, for each n, and $p \in (2, N)$, we have

$$\int_\Omega |\nabla w_n|^p\,dx = \Big(\int_\Omega |\nabla u_n|^2\,dx\Big)^{(2-p)/2}\Big(\lambda \int_\Omega V(x)w_n^2\,dx - 1\Big) \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Relation (3.11) implies that w_n converges strongly to 0 in $\mathcal{D}_0^{1,p}(\Omega)$. In particular, this means that $w_n(x) \to 0$ for a.e. $x \in \Omega$, and consequently $w = 0$, which contradicts (3.10). It follows that $\{\int_\Omega |\nabla u_n|^2\,dx\}_n$ is bounded provided that $\{u_n\}_n \subset \mathcal{N}_\lambda$.

We recall that

$$\int_\Omega |\nabla u|^2\,dx \geq \lambda_1 \int_\Omega V(x)u^2\,dx \quad \text{for any } u \in \mathcal{D}_0^{1,2}(\Omega)$$

and that $\mathcal{N}_\lambda \subset \mathcal{D}_0^{1,\Phi_p}(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega)$, and consequently

$$\int_\Omega |\nabla u_n|^2\,dx \geq \lambda_1 \int_\Omega V(x)u_n^2\,dx \quad \text{for any } n.$$

Thus the sequence $\{\int_\Omega V(x)u_n^2\,dx\}_n$ is bounded. Since $u_n \in \mathcal{N}_\lambda$ for each n, we deduce that the sequence $\{\int_\Omega |\nabla u_n|^p\,dx\}_n$ is bounded, too. It follows that actually $\{\int_\Omega \Phi_p(|\nabla u_n|)\,dx\}_n$ is bounded, which, in view of relations (2.6) and (2.7), shows that $\{u_n\}_n$ is bounded in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$. This completes the proof of Lemma 3.8. □

Lemma 3.9. *If $p \in (2, N)$, then $m \in (-\infty, 0)$.*

Proof. We already pointed out that $m < 0$. We will show that $m \neq -\infty$. By Lemma 3.8 there exists a positive constant M such that $\|u\|_{\Phi_p} \leq M$ for every $u \in \mathcal{N}_\lambda$. Using (2.6) and (2.7), we deduce that there exists a positive constant M_1 such that $\int_\Omega |\nabla u|^p\,dx \leq M_1$ for all $u \in \mathcal{N}_\lambda$. Since $p \in (2, N)$, the above inequality yields $J_\lambda(u) = (\frac{1}{p} - \frac{1}{2}) \int_\Omega |\nabla u|^p\,dx \geq (\frac{1}{p} - \frac{1}{2})M_1$.

We obtain in this way that J_λ is bounded from below on $\mathcal{N}_\lambda$, which implies that $m \neq -\infty$. Thus $m \in (-\infty, 0)$, and the proof of Lemma 3.9 is complete. □

Lemma 3.10. *There exists $u \in \mathcal{N}_\lambda$ such that $J_\lambda(u) = m$ provided that $p \in (2, N)$.*

Proof. Let $\{u_n\}_n \subset \mathcal{N}_\lambda$ be a minimizing sequence for J_λ on $\mathcal{N}_\lambda$; that is,

$$J_\lambda(u_n) = \Big(\frac{1}{p} - \frac{1}{2}\Big) \int_\Omega |\nabla u_n|^p\,dx \to m \quad \text{as } n \to \infty.$$

Since by Lemma 3.8 we have that $\mathcal{N}_\lambda$ is bounded, we deduce that there exists $u_0 \in \mathcal{D}_0^{1,\Phi_p}(\Omega)$ such that u_n converges weakly to u_0 in $\mathcal{D}_0^{1,\Phi_p}(\Omega)$, and consequently in $\mathcal{D}_0^{1,2}(\Omega)$ and $\mathcal{D}_0^{1,p}(\Omega)$, and $\int_\Omega V(x)u_n^2\,dx \to \int_\Omega V(x)u_0^2\,dx$ as $n\to\infty$. Taking into account the above pieces of information, we deduce that

$$\int_\Omega |\nabla u_0|^2\,dx \le \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^2\,dx$$

and that

$$\begin{aligned}&\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(\int_\Omega |\nabla u_0|^2\,dx - \lambda\int_\Omega V(x)u_0^2\,dx\Big)\\ &\quad\le \liminf_{n\to\infty}\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(\int_\Omega |\nabla u_n|^2\,dx - \lambda\int_\Omega V(x)u_n^2\,dx\Big).\end{aligned}$$

The above facts yield

$$\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(\int_\Omega |\nabla u_0|^2\,dx - \lambda\int_\Omega V(x)u_0^2\,dx\Big) \le \liminf_{n\to\infty} J_\lambda(u_n) = m < 0. \tag{3.12}$$

Thus

$$\int_\Omega |\nabla u_0|^2\,dx < \lambda\int_\Omega V(x)u_0^2\,dx,$$

and then $u_0 \neq 0$.

By the fact that $u_n \in \mathcal{N}_\lambda$ for every n, we have

$$\int_\Omega |\nabla u_n|^p\,dx + \int_\Omega |\nabla u_n|^2\,dx = \lambda\int_\Omega V(x)u_n^2\,dx, \quad \forall n.$$

Letting $n\to\infty$ in the above relation and taking into account that u_n converges weakly to u_0 in $\mathcal{D}_0^{1,p}(\Omega)$ and $\mathcal{D}_0^{1,2}(\Omega)$ and $\int_\Omega V(x)u_n^2\,dx \to \int_\Omega V(x)u_0^2\,dx$ as $n\to\infty$, we get

$$\int_\Omega |\nabla u_0|^p\,dx + \int_\Omega |\nabla u_0|^2\,dx \le \lambda\int_\Omega V(x)u_0^2\,dx. \tag{3.13}$$

Assume by contradiction that in (3.13) the strict inequality holds; that is,

$$\int_\Omega |\nabla u_0|^p\,dx + \int_\Omega |\nabla u_0|^2\,dx < \lambda\int_\Omega V(x)u_0^2\,dx. \tag{3.14}$$

Taking

$$t_0 = \Big(\frac{\lambda\int_\Omega V(x)u_0^2\,dx - \int_\Omega |\nabla u_0|^2\,dx}{\int_\Omega |\nabla u_0|^p\,dx}\Big)^{1/(p-2)},$$

we have $t_0u_0 \in \mathcal{N}_\lambda$, and it follows by (3.14) that $t_0 > 1$. Then some simple computations yield

$$\begin{aligned}J_\lambda(t_0u_0) &= \Big(\frac{1}{2}-\frac{1}{p}\Big)t_0^2\Big(\int_\Omega |\nabla u_0|^2\,dx - \lambda\int_\Omega V(x)u_0^2\,dx\Big)\\ &< \Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(\int_\Omega |\nabla u_0|^2\,dx - \lambda\int_\Omega V(x)u_0^2\,dx\Big)\\ &\le \liminf_{n\to\infty} J_\lambda(u_n) = m,\end{aligned}$$

which represents a contradiction. Thus inequality (3.14) cannot hold true.

Therefore, in (3.13) only the equality holds true, which means $u_0 \in \mathcal{N}_\lambda$. By (3.12) it follows that $J_\lambda(u_0) \le m$, and thus $J_\lambda(u_0) = m$. In other words, J_λ attains its infimum on $\mathcal{N}_\lambda$ in u_0. The proof of Lemma 3.10 is now complete. □

Proof of Proposition 3.3. Let $u \in \mathcal{N}_\lambda$ be such that $J_\lambda(u) = m$ found in Lemma 3.7 for $p \in (1,2)$ and in Lemma 3.10 for $p \in (2,N)$. Since $u \in \mathcal{N}_\lambda$, we have $\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla u|^p\,dx = \lambda \int_\Omega V(x)u^2\,dx$, and the fact that $u \ne 0$ implies $\int_\Omega |\nabla u|^2\,dx < \lambda \int_\Omega V(x)u^2\,dx$. Let $w \in \mathcal{D}_0^{1,\Phi_p}(\Omega)$ be arbitrary but fixed. Then there exists $\delta > 0$ small enough such that, for each $s \in (-\delta,\delta)$, function $u + sw$ does not vanish everywhere in Ω and $\lambda \int_\Omega V(x)(u+sw)^2\,dx > \int_\Omega |\nabla(u+sw)|^2\,dx$.

For each $s \in (-\delta,\delta)$, let $t(s) > 0$ be given by

$$t(s) := \Big(\frac{\lambda \int_\Omega V(x)(u+sw)^2\,dx - \int_\Omega |\nabla(u+sw)|^2\,dx}{\int_\Omega |\nabla(u+sw)|^p\,dx}\Big)^{1/(p-2)},$$

and note that $t(s)\cdot(u+sw) \in \mathcal{N}_\lambda$. Function $t(s)$ is the composition of some differentiable functions, and consequently it is differentiable. On the other hand, since $u \in \mathcal{N}_\lambda$, we infer that $t(0) = 1$.

Define $\gamma : (-\delta,\delta) \to \mathbb{R}$ by $\gamma(s) := J_\lambda(t(s)(u+sw))$. Obviously, we have $\gamma \in C^1(-\delta,\delta)$ and $\gamma(0) = \min_{s\in(-\delta,\delta)} \gamma(s)$. Thus we deduce

$$\begin{aligned} 0 = \gamma'(0) &= \big\langle J_\lambda'\big(t(0)u\big), t'(0)u + t(0)w\big\rangle \\ &= t'(0)\big\langle J_\lambda'(u), u\big\rangle + \big\langle J_\lambda'(u), w\big\rangle = \big\langle J_\lambda'(u), w\big\rangle, \end{aligned}$$

where the latter equality holds because $u \in \mathcal{N}_\lambda$. The proof of Proposition 3.3 is now complete. □

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