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PROPERTY T OF REDUCED C^*-CROSSED PRODUCTS BY DISCRETE GROUPS

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Abstract. We generalize a result of Kamalov and we show that if G is an amenable discrete group with an action α on a finite nuclear unital C^*-algebra A such that the reduced crossed product $A \rtimes_{\alpha,r} G$ has property T, then G is finite and A is finite-dimensional. As an application, an infinite discrete group H is nonamenable if and only if the corresponding uniform Roe algebra $C^*_u(H)$ has property T.

1. Introduction

Property T for unital C^*-algebras was introduced by Bekka in [1] and has been studied by many different people. Recently, it was shown by Kamalov (in [6]) that

> "…if G is a discrete amenable group acting on a commutative unital C^*-algebra A such that the crossed product has property T, then G is finite and A is finite-dimensional."

The aims of this paper is to extend this result to the case of finite nuclear unital C^*-algebras, and to give an application concerning a characterization of the amenability of G. The main result of Brown in [2] is one of our essential tools.

2. The main results

Throughout this article, G is a discrete group acting on a unital C^*-algebra A through an action α (by automorphisms).

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Let $T(A)$ be the set of all traical states on A. For any $\tau \in T(A)$, we denote by $\pi_\tau : A \to \mathcal{B}(\mathcal{H}_\tau)$ the GNS representation corresponding to τ, and we denote by ξ_τ a norm 1 cyclic vector in $\mathcal{H}_\tau$ with

$$\tau(a) = \langle \pi_\tau(a)\xi_\tau, \xi_\tau \rangle, \quad a \in A.$$

Recall that A is said to be *finite* when $T(A)$ separates points of A_+ (see [4, Theorem 3.4]), and that if $T(A) = \emptyset$, then A has property T (see [1, Remark 2]). We use $T_\alpha(A)$ and $A \rtimes_{\alpha,r} G$ to denote the set of all α-invariant traical states on A and the reduced crossed product of α, respectively. We will regard $A \subseteq A \rtimes_{\alpha,r} G$ as well as $G \subseteq A \rtimes_{\alpha,r} G$ through their canonical embeddings.

Let us first give the following well-known facts. Since we cannot find the precise references for them, we present their simple arguments here.

Lemma 2.1. *We have the following:*

(a) $T_\alpha(A) \neq \emptyset$ *if and only if* $T(A \rtimes_{\alpha,r} G) \neq \emptyset$;
(b) *if* $T_\alpha(A)$ *separates points of* A_+, *then* $A \rtimes_{\alpha,r} G$ *is finite;*
(c) *if* G *is amenable and if* $T(A) \neq \emptyset$, *then* $T_\alpha(A) \neq \emptyset$.

Proof. Let us denote $B := A \rtimes_{\alpha,r} G$, and let us consider $\mathcal{E} : B \to A$ to be the canonical conditional expectation (see e.g., [3, Proposition 4.1.9]). Then the following hold.

(a) If $\sigma \in T(B)$, then $\sigma(\alpha_t(a)) = \sigma(tat^{-1}) = \sigma(a)$ ($a \in A; t \in G$), which means that $\sigma|_A \in T_\alpha(A)$. Conversely, consider any $\tau \in T_\alpha(A)$ and any $x = \sum_{s\in G} a_s s \in B$ such that $a_s = 0$ except for a finite number of s. Then

$$\tau\big(\mathcal{E}(x^*x)\big) = \tau\Big(\sum_{r\in G} \alpha_{r^{-1}}(a_r^* a_r)\Big) = \tau\Big(\sum_{r\in G} a_r a_r^*\Big) = \tau\big(\mathcal{E}(xx^*)\big).$$

Hence, $\tau \circ \mathcal{E}$ belongs to $T(B)$, because it is continuous.

(b) Since $\mathcal{E}$ is faithful, we know that B is a Hilbert A-module under the A-valued inner product

$$\langle x, y\rangle_A := \mathcal{E}(x^*y), \quad x, y \in B.$$

For any $\tau \in T_\alpha(A)$, we consider $B \otimes_{\pi_\tau} \mathcal{H}_\tau$ to be the Hilbert space as in [8, Proposition 4.5] (we regard a Hilbert $\mathbb{C}$-module as a Hilbert space by considering the conjugation of the inner product). If π_τ^B is the canoincal representation of B on $B \otimes_{\pi_\tau} \mathcal{H}_\tau$, then $(B \otimes_{\pi_\tau} \mathcal{H}_\tau, \pi_\tau^B)$ coincides with $(\mathcal{H}_{\tau\circ\mathcal{E}}, \pi_{\tau\circ\mathcal{E}})$, because $1 \otimes \xi_\tau$ is a cyclic vector for π_τ^B such that the state defined by $1 \otimes \xi_\tau$ is $\tau \circ \mathcal{E}$.

Let $(\mathcal{H}_0, \pi_0) := \bigoplus_{\tau\in T_\alpha(A)} (\mathcal{H}_\tau, \pi_\tau)$. As $T_\alpha(A)$ separates points of A_+, one knows that π_0 is faithful. It is easy to verify that the representation π_0^B of B on $B \otimes_{\pi_0} \mathcal{H}_0$ induced by π_0 is also faithful, and that π_0^B coincides with $\bigoplus_{\tau\in T_\alpha(A)} \pi_\tau^B$. Consequently, $\bigoplus_{\tau\in T_\alpha(A)} (\mathcal{H}_{\tau\circ\mathcal{E}}, \pi_{\tau\circ\mathcal{E}})$ is faithful. This means that the subset $\{\tau \circ \mathcal{E} : \tau \in T_\alpha(A)\}$ of $T(B)$ (see the argument of part (a)) separates points of B_+.

(c) Notice that $T(A)$ is a nonempty weak*-compact convex subset of A^* and α induces an action of G on $T(A)$ by continuous affine maps. Hence, Day's fixed point theorem (see [5, Theorem 1]) produces a fixed point $\tau_0 \in T(A)$ for this action. Obviously, $\tau_0 \in T_\alpha(A)$. □

We warn the readers that part (c) above does not hold for nonunital C^*-algebras.

Our main theorem concerns a relation between nuclearity and property T of $A \rtimes_{\alpha,r} G$. We recall from [2, Theorem 5.1] that if $A \rtimes_{\alpha,r} G$ is nuclear and has property T, then

$$A \rtimes_{\alpha,r} G = B_{\mathbf{nt}} \oplus B_{\mathbf{fd}}, \tag{2.1}$$

where $B_{\mathbf{nt}}$ is a nuclear C^*-algebra with no tracial state and $B_{\mathbf{fd}}$ is a finite-dimensional C^*-algebra (note that although all C^*-algebras in [2] are assumed to be separable, [2, Theorem 5.1] is true in the nonseparable case because one can use [3, Theorem 6.2.7] to replace [2, Theorem 4.2] in the proof of this result). The following theorem implies that if G is infinite, then we arrive at one of the extreme case of (2.1); namely, the algebra $B_{\mathbf{fd}}$ is zero. This proposition and its proof are essential ingredients in the argument of our main theorem.

Proposition 2.2. *Let G be an infinite discrete group acting on a unital C^*-algebra A through an action α. If $A \rtimes_{\alpha,r} G$ is nuclear and has property T, then $T(A \rtimes_{\alpha,r} G) = \emptyset$.*

Proof. Let $I_\alpha := \bigcap_{\tau \in T_\alpha(A)} \ker \pi_\tau$ and $A_\alpha := A/I_\alpha$. Suppose on contrary that $T(A \rtimes_{\alpha,r} G) \neq \emptyset$. Then $I_\alpha \neq A$ because of Lemma 2.1(a). Moreover, as

$$\ker \pi_\tau = \big\{x \in A : \tau(x^*x) = 0\big\}, \quad \tau \in T(A),$$

we know that I_α is α-invariant, and hence α produces an action β of G on A_α. Furthermore, every state in $T_\alpha(A)$ induces a state in $T_\beta(A_\alpha)$. This tells us that $T_\beta(A_\alpha)$ will separate points of $(A_\alpha)_+$.

Since $A_\alpha \rtimes_{\beta,r} G$ is a quotient C^*-algebra of $A \rtimes_{\alpha,r} G$ (see, e.g., [7]), the hypothesis implies that $A_\alpha \rtimes_{\beta,r} G$ is nuclear and has property T. Therefore, [2, Theorem 5.1] tells us that $A_\alpha \rtimes_{\beta,r} G = D_{\mathbf{nt}} \oplus D_{\mathbf{fd}}$, where $D_{\mathbf{fd}}$ is finite-dimensional and $T(D_{\mathbf{nt}}) = \emptyset$. However, the finiteness of $A_\alpha \rtimes_{\beta,r} G$ (which follows from Lemma 2.1(b)) tells us that $D_{\mathbf{nt}} = (0)$. Consequently, $A_\alpha \rtimes_{\beta,r} G$ is a nonzero finite-dimensional C^*-algebra, which contradicts the fact that G is infinite. □

Our main result concerns the other extreme case of (2.1)—that is, the algebra $B_{\mathbf{nt}}$ is zero. In order words, it gives a situation (that extends the one in [6]) ensuring that the reduced crossed product is finite-dimensional.

Notice that the finiteness assumption of A in this theorem is indispensable. In fact, if D is the direct sum of $\mathbb{C}$ with an infinite-dimensional nuclear unital C^*-algebra having no tracial state, then D is not finite (although it has a tracial state), the crossed product $D \rtimes H$ of the trivial action of a finite group H on D is nuclear and has property T, but $D \rtimes H$ is infinite-dimensional. We will see at the end of this article that the amenability of G is also indispensable (even if we assume that $A \rtimes_{\alpha,r} G$ is nuclear).

Theorem 2.3. *Let G be an amenable discrete group and let A be a finite nuclear unital C^*-algebra. If there is an action α of G on A such that $A \rtimes_{\alpha,r} G$ has property T, then G is finite and A is finite-dimensional.*

Proof. Let us set $I_\alpha := \bigcap_{\tau \in T_\alpha(A)} \ker \pi_\tau$ and $A_\alpha := A/I_\alpha$. Denote $B := A \rtimes_{\alpha,r} G$. The finiteness assumption of A as well as parts (a) and (c) of Lemma 2.1 imply that $T(B) \neq \emptyset$ and $I_\alpha \neq A$. By Proposition 2.2, we know that the group G has to be finite. Moreover, the argument of Proposition 2.2 tells us that I_α is α-invariant and $B_\alpha := A_\alpha \rtimes_{\beta,r} G$ is finite-dimensional. It suffices to show that $I_\alpha = \{0\}$.

Suppose on the contrary that $I_\alpha \neq \{0\}$. As in (2.1), we have

$$B = B_{\mathbf{nt}} \oplus B_{\mathbf{fd}}.$$

Thus, $I_\alpha \rtimes_{\alpha,r} G = J_{\mathbf{nt}} \oplus J_{\mathbf{fd}}$, with $J_{\mathbf{nt}}$ (respectively, $J_{\mathbf{fd}}$) being a closed ideal of $B_{\mathbf{nt}}$ (respectively, $B_{\mathbf{fd}}$). The short exact sequence

$$0 \to I_\alpha \to A \to A_\alpha \to 0,$$

induces a short exact sequence of the corresponding reduced crossed products, because G is amenable. From this, we obtain

$$B_\alpha = B/(I_\alpha \rtimes_{\alpha,r} G) = B_{\mathbf{nt}}/J_{\mathbf{nt}} \oplus B_{\mathbf{fd}}/J_{\mathbf{fd}}.$$

Hence, $B_{\mathbf{nt}}/J_{\mathbf{nt}}$ is a quotient C^*-algebra of the finite-dimensional C^*-algebra B_α. This forces $J_{\mathbf{nt}} = B_{\mathbf{nt}}$ (otherwise, $B_{\mathbf{nt}}$ will have a tracial state). Therefore, $B_\alpha \cong B_{\mathbf{fd}}/J_{\mathbf{fd}}$, or equivalently, $B_{\mathbf{fd}} \cong B_\alpha \oplus J_{\mathbf{fd}}$ (as $B_{\mathbf{fd}}$ is finite-dimensional). This gives

$$B \cong B_\alpha \oplus J_{\mathbf{fd}} \oplus B_{\mathbf{nt}} = B_\alpha \oplus J_{\mathbf{fd}} \oplus J_{\mathbf{nt}} = B_\alpha \oplus (I_\alpha \rtimes_{\alpha,r} G).$$

Consequently, $I_\alpha \rtimes_{\alpha,r} G$ is a unital C^*-algebra and so is I_α.

Now, by the finiteness assumption of A, one knows that $T(I_\alpha) \neq \emptyset$, and Lemma 2.1(c) will produce an element $\tau \in T_\alpha(I_\alpha)$. Let $\Phi : A \to M(I_\alpha) = I_\alpha$ be the canonical G-equivariant *-epimorphism. If we define

$$\tau'(a) := \big\langle \pi_\tau\big(\Phi(a)\big)\xi_\tau, \xi_\tau \big\rangle \quad (a \in A),$$

then $\tau' \in T_\alpha(A)$ and $\tau'|_{I_\alpha} = \tau$. However, the existence of such a τ' contradicts the definition of I_α. □

Corollary 2.4. *Let G be an infinite discrete group and α_G be the left translation action of G on $\ell^\infty(G)$. The following statements are equivalent:*

(1) *G is nonamenable,*
(2) *$\ell^\infty(G) \rtimes_{\alpha_G,r} G$ does not have a tracial state,*
(3) *$\ell^\infty(G) \rtimes_{\alpha_G,r} G$ has strong property T (see [9]),*
(4) *$\ell^\infty(G) \rtimes_{\alpha_G,r} G$ has property T,*
(5) *there is a finite nuclear unital C^*-algebra A and an action α of G on A such that $A \rtimes_{\alpha,r} G$ has property T.*

Proof. If G is nonamenable, then $T_{\alpha_G}(\ell^\infty(G)) = \emptyset$ and Lemma 2.1(a) tells us that statement (2) holds. On the other hand, if $\ell^\infty(G) \rtimes_{\alpha_G,r} G$ does not have a tracial state, then [9, Proposition 5.2] gives statement (3). Moreover, a strong property T C^*-algebra clearly has property T. Finally, suppose that statement (5) holds but G is amenable. Then Theorem 2.3 produces the contradiction that G is finite. □

The following comparison of Corollary 2.4 with the main result of Ozawa in [10] (see also Theorem 5.1.6 and Proposition 5.1.3 of [3]) may be worth mentioning:

> a discrete group G is exact if and only if $\ell^\infty(G) \rtimes_{\alpha_G,r} G$ is nuclear (or equivalently, the action α_G is amenable).

This result tells us that one cannot weaken the assumption of G being amenable in Theorem 2.3 to the assumptions that the action α is amenable and the crossed product $A \rtimes_{\alpha,r} G$ is nuclear. Indeed, if G is an exact nonamenable group, then α_G is amenable, the reduced crossed product $\ell^\infty(G) \rtimes_{\alpha_G,r} G$ has property T and is nuclear, while $\ell^\infty(G) \rtimes_{\alpha_G,r} G$ is infinite-dimensional.

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