Ann. Funct. Anal. 7 (2016), no. 3, 434–441
http://dx.doi.org/10.1215/20088752-3605510
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

A NOTE CONCERNING THE NUMERICAL RANGE OF A BASIC ELEMENTARY OPERATOR

MOHAMED BOUMAZGOUR[1*] and HOSSAM A. NABWEY[1,2]

Communicated by T. Yamazaki

Abstract. Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H, and let $\mathcal{S}$ be a norm ideal in $\mathcal{B}(H)$. For $A, B \in \mathcal{B}(H)$, define the elementary operator $M_{\mathcal{S},A,B}$ on $\mathcal{S}$ by $M_{\mathcal{S},A,B}(X) = AXB$ $(X \in \mathcal{S})$. The aim of this paper is to give necessary and sufficient conditions under which the equality $V(M_{\mathcal{S},A,B}) = \overline{\text{co}}(W(A)W(B))$ holds. Here $V(T)$ and $W(T)$ denote the algebraic numerical range and spatial numerical range of an operator T, respectively, and $\overline{\text{co}}(\Omega)$ denotes the closed convex hull of a subset $\Omega \subseteq \mathbb{C}$.

1. Introduction

Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators acting on a complex Hilbert space H. For A and B in $\mathcal{B}(H)$, define the operators L_A and R_B on $\mathcal{B}(H)$ by $L_A(X) = AX$ and $R_B(X) = XB$ $(X \in \mathcal{B}(H))$, respectively. The basic elementary operator $M_{A,B}$, induced by the operators A and B, is the multiplication operator on $\mathcal{B}(H)$ defined by $M_{A,B} = L_A R_B$. An elementary operator on $\mathcal{B}(H)$ is a finite sum $R = \sum_{i=1}^{n} M_{A_i,B_i}$ of basic ones. A familiar example of elementary operators is the generalized derivation $\delta_{A,B}$ defined by $\delta_{A,B} = L_A - R_B$.

Let $\mathcal{S}$ be a nonzero two-sided ideal of the algebra $\mathcal{B}(H)$. We say that $\mathcal{S}$ is a *norm ideal* if $\mathcal{S}$ is equipped with a norm $\|\cdot\|_{\mathcal{S}}$ satisfying the following conditions:

(1) $\mathcal{S}$ is a Banach space with respect to the norm $\|\cdot\|_{S}$;
(2) $\|X\|_{\mathcal{S}} = \|X\|$ for all $X \in \mathcal{S}$ with 1-dimensional range;
(3) $\|AXB\|_{\mathcal{S}} \leq \|A\|\|X\|_{\mathcal{S}}\|B\|$ for all $A, B \in \mathcal{B}(H)$ and $X \in \mathcal{S}$.

Received Nov. 6, 2015; Accepted Jan. 14, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47A12; Secondary 47A30, 47B47.
Keywords. numerical range, spectrum, elementary operators, norm ideals.

Familiar examples of norm ideals are the Schatten p-ideals $(C_p(H), \|\cdot\|_p)$ $(1 \leq p \leq \infty)$ of operators on a given Hilbert space H. Here we denote by $C_\infty(H)$ the ideal of all compact operators on H. Recall that the Hilbert–Schmidt ideal $\mathcal{C}_2(H)$ is a Hilbert space when equipped with the inner product defined by

$$\langle X, Y\rangle = \operatorname{tr}(XY^*),$$

where tr denotes the usual functional trace. (We refer the reader to [10] for more details about norm ideals.)

Let $\mathcal{S}$ be a norm ideal in $\mathcal{B}(H)$, and let $A, B \in \mathcal{B}(H)$. Then $M_{A,B}(\mathcal{S}) \subset \mathcal{S}$, and we denote the restriction of $M_{A,B}$ to $\mathcal{S}$ by $M_{\mathcal{S},A,B}$. Since $\|AXB\|_{\mathcal{S}} \leq \|A\|\|X\|_{\mathcal{S}}\|B\|$ for all $X \in \mathcal{S}$, clearly $M_{\mathcal{S},A,B} \in \mathcal{B}(\mathcal{S})$. Moreover, one can easily show that $\|M_{\mathcal{S},A,B}\| = \|A\|\|B\|$. In a case where $\mathcal{S} = \mathcal{C}_p(H)$ $(1 \leq p \leq \infty)$, we denote $M_{\mathcal{S},A,B}$ by $M_{p,A,B}$.

Let E be a complex Banach space, and let E' be its dual space. For $T \in \mathcal{B}(H)$, the spatial numerical range of T, denoted by $W(T)$, is defined to be the set

$$W(T) = \big\{f(Tx) : x \in E, \|x\| = 1 \text{ and } f \in D(x)\big\},$$

where

$$D(x) = \big\{f \in E' : f(x) = \|f\| = \|x\|\big\}.$$

If H is a Hilbert space and $T \in \mathcal{B}(H)$, then the numerical range of T is given by

$$W(T) = \big\{\langle Tx, x\rangle : x \in H \text{ and } \|x\| = 1\big\}.$$

Let $T \in \mathcal{B}(E)$. The algebraic numerical range of T is defined by

$$V(T) = \big\{F(T) : F \in \big(\mathcal{B}(E)\big)' \text{ and } \|F\| = F(I) = 1\big\};$$

here I denotes the identity operator on E.

It is well known that $V(T)$ $(T \in \mathcal{B}(E))$ is a compact convex subset of the plane and that $V(T)$ contains the spectrum of T (see [2]). Furthermore, $V(T)$ coincides with the closed convex hull of $W(T)$ whenever T is a bounded operator on a Banach space. (For basic facts about numerical ranges, see [2].)

Elementary operators appear and are used in branches of mathematics such as Banach algebras, the theory of complete positive linear maps, quantum information theory, and many others. Their spectral and structural properties have been intensively studied over the past decades (see [4], [5], and the references therein). However, the problem of computing the numerical range of an elementary operator when restricted to a norm ideal is still open (see [5, Problem 4.5]). Besides the case of generalized derivations on norm ideals for which Shaw [12] found an explicit formula for the numerical range, no formula is known even for computing the numerical range of the operator $M_{\mathcal{S},A,B}$; here $A, B \in \mathcal{B}(H)$ and $\mathcal{S}$ is a norm ideal in $\mathcal{B}(H)$. In [11], Seddik established the inclusion $W(A)W(B) \subseteq V(M_{\mathcal{S},A,B})$ for every $A, B \in \mathcal{B}(H)$ and every norm ideal $\mathcal{S}$. Since $V(M_{\mathcal{S},A,B})$ is a compact convex subset, it follows that $\overline{\operatorname{co}}(W(A)W(B)) \subseteq V(M_{\mathcal{S},A,B})$. Further, this inclusion may be strict. It is then natural and interesting to understand the class of operators A and B which satisfy the equation

$$V(M_{\mathcal{S},A,B}) = \overline{\mathrm{co}}\big(W(A)W(B)\big), \tag{1.1}$$

where $\mathcal{S}$ is a given norm ideal in $\mathcal{B}(H)$.

In [3], Chraibi proved that if $\mathcal{S}$ is the ideal of Hilbert–Schmidt operators, then the equality in (1.1) holds whenever either A or B is a subnormal operator.

Motivated by results of Chraibi [3], we aim in this article to characterize the class of operators A and B which satisfy equation (1.1). In Section 2, we consider the equality in (1.1) in the case when A and B are convexoid operators and $\mathcal{S}$ is an arbitrary norm ideal. Section 3 is devoted to equation (1.1) when one of the operators A and B has a normal dilation; as a consequence of the obtained result, we get a generalization of the main result of Chraibi [3, Théorème 10]. Section 4 contains some remarks and a discussion about the numerical range of an elementary operator when restricted to a Schatten p-ideal $(1 \le p \le \infty)$.

For $T \in \mathcal{B}(E)$, let T^*, $\sigma(T)$, $r(T)$, and $w(T)$ denote the adjoint, the spectrum, the spectral radius, and the numerical radius of T, respectively. If Ω is a subset of $\mathbb{C}$, then we denote by $\overline{\Omega}$ its closure. If H and K are Hilbert spaces, then we denote by $\mathcal{C}_2(H, K)$ the class of Hilbert–Schmidt operators from H to K.

2. Convexoid operators

Recall that a bounded linear operator T on a Banach space is said to be *convexoid* if $\mathrm{co}(\sigma(T)) = V(T)$. Note that the class of convexoid operators on a Hilbert space includes hyponormal operators.

The main result of this section is the following.

Theorem 2.1. *Let $A, B \in \mathcal{B}(H)$, and let $\mathcal{S}$ be a norm ideal in $\mathcal{B}(H)$. Then*

(1) *if $M_{\mathcal{S},A,B}$ is convexoid, then the equality in (1.1) holds;*
(2) *if A and B are convexoid operators, then $M_{\mathcal{S},A,B}$ is convexoid if and only if the equality in (1.1) holds.*

For the proof, we need the following auxiliary lemmas.

Lemma 2.2. *If Ω_1 and Ω_2 are two subsets of $\mathbb{C}$, then*

$$\mathrm{co}(\Omega_1\Omega_2) = \mathrm{co}\big(\mathrm{co}(\Omega_1)\,\mathrm{co}(\Omega_2)\big).$$

Proof. For the proof, see [7, p. 683]. □

Lemma 2.3. *Let $A, B \in \mathcal{B}(H)$, and let $\mathcal{S}$ be a norm ideal in $\mathcal{B}(H)$. Then*

$$\sigma(M_{\mathcal{S},A,B}) \subseteq \overline{W}(A)\overline{W}(B) \subseteq V(M_{\mathcal{S},A,B}).$$

Proof. The first inclusion follows from the fact that $\sigma(M_{\mathcal{S},A,B}) = \sigma(A)\sigma(B)$ (see [4]), and the second follows from [11]. □

Proof of Theorem 2.1. The property (1) follows from Lemma 2.3.

To prove (2), assume that A and B are convexoid operators. Then the sufficiency follows from Part (1) so that we only need to prove the necessity. In fact, if $V(M_{\mathcal{S},A,B}) = \overline{\text{co}}(W(A)W(B))$, then, by virtue of Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\overline{\text{co}}\big(W(A)W(B)\big) &= \overline{\text{co}}\big(\text{co}\big(\sigma(A)\big)\,\text{co}\big(\sigma(B)\big)\big)\\ &= \text{co}\big(\sigma(A)\sigma(B)\big)\\ &= \text{co}\big(\sigma(M_{\mathcal{S},A,B})\big)\\ &\subseteq \overline{\text{co}}\big(W(A)W(B)\big),\end{aligned}$$

and so

$$\text{co}\big(\sigma(M_{\mathcal{S},A,B})\big) = \overline{\text{co}}\big(W(A)W(B)\big) = V(M_{\mathcal{S},A,B});$$

that is, $M_{\mathcal{S},A,B}$ is convexoid, as desired. □

Corollary 2.4. *Let $A, B \in \mathcal{B}(H)$, and let $\mathcal{S}$ be a norm ideal in $\mathcal{B}(H)$. If $M_{\mathcal{S},A,B}$ is convexoid, then $w(A) = r(A)$ and $w(B) = r(B)$.*

Proof. If $M_{\mathcal{S},A,B}$ is convexoid, then it follows from Theorem 2.1 that

$$w(M_{\mathcal{S},A,B}) = r(M_{\mathcal{S},A,B}) = r(A)r(B),$$

where the second equality follows from [4]. Since we always have $w(A)w(B) \leq w(M_{\mathcal{S},A,B})$ (see [11]), and $r(A) \leq w(A)$, $r(B) \leq w(B)$, we deduce that $w(A) = r(A)$ and $w(B) = r(B)$. □

The next example shows that the converse of Corollary 2.4 does not hold in general.

Example 2.5. Consider the operator matrices $A = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M \end{bmatrix}$ and $B = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. One can easily check that

$$W(A) = W(B) = \text{co}\Big(\{1\} \cup \Big\{z : |z| \leq \frac{1}{2}\Big\}\Big), \qquad \text{co}\big(\sigma(A)\sigma(B)\big) = [0,1],$$

and

$$w(A) = r(A) = w(B) = r(B) = 1.$$

Since, by Lemma 2.3, $W(A)W(B) \subseteq V(M_{\mathcal{S},A,B})$ for every norm ideal $\mathcal{S}$, clearly $\text{co}(\sigma(A)\sigma(B))$ is strictly contained in $V(M_{\mathcal{S},A,B})$, and so $M_{\mathcal{S},A,B}$ is not convexoid.

Example 2.6. If one of the operators A and B is not convexoid, then the equivalence in Theorem 2.1, part (2), is no longer true. Indeed, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $W(A) = [0,1]$ and $W(B) = \{z : |z| \leq 1/2\}$, and thus

$$V(M_{2,A,B}) = \text{co}\big(W(A)W(B)\big) = \Big\{z : |z| \leq \frac{1}{2}\Big\}.$$

However, $\sigma(M_{2,A,B}) = \{0\}$, and therefore $M_{2,A,B}$ is not convexoid.

3. Operators having normal dilations

Let A and B be bounded linear operators on the complex Hilbert spaces H and K, respectively. A is said to be *dilated* to B (or B is a *dilation* of A) if B is unitarily equivalent to a 2×2 operator matrix of the form $\left[\begin{smallmatrix} A & * \\ * & * \end{smallmatrix}\right]$. This is equivalent to requiring the existence of an isometry V from H to K such that $A = V^*BV$.

In this section, we consider the equality in (1.1) when one of the operators A or B has a normal dilation. The main result here is the following.

Theorem 3.1. *Let $A, B \in \mathcal{B}(H)$. If A has a normal dilation N on K such that $\sigma(N) \subseteq \sigma(A)$, then*

$$V(M_{2,A,B}) = \overline{\mathrm{co}}\big(W(A)W(B)\big). \tag{3.1}$$

Proof. Since A has a normal dilation N on K, then there is an operator V from H to K such that $V^*V = I$ and $A = V^*NV$; hence

$$M_{2,A,B} = M_{2,V^*NV,B} = L_{2,V^*}M_{2,N,B}L_{2,V}.$$

Since $L_{2,V^*} = L_{2,V}^*$, we have $L_{2,V}^*L_{2,V} = I$, and

$$M_{2,A,B} = L_{2,V}^*M_{2,N,B}L_{2,V};$$

that is, $M_{2,A,B}$ is dilated to $M_{2,N,B}$ on $\mathcal{C}_2(H,K)$. From this we derive that

$$V(M_{2,A,B}) \subseteq V(M_{2,N,B}). \tag{3.2}$$

Next, since N is normal, by virtue of [3, Théorème 10], we have

$$V(M_{2,N,B}) = \overline{\mathrm{co}}\big(W(N)W(B)\big).$$

Since N is convexoid, this last equality implies that

$$\begin{aligned} V(M_{2,N,B}) &= \overline{\mathrm{co}}\big(\mathrm{co}\big(\sigma(N)\big)W(B)\big) \\ &\subseteq \overline{\mathrm{co}}\big(\mathrm{co}\big(\sigma(A)\big)W(B)\big) \\ &\subseteq \overline{\mathrm{co}}\big(W(A)W(B)\big). \end{aligned} \tag{3.3}$$

From Lemma 2.3, we have $\overline{\mathrm{co}}(W(A)W(B)) \subseteq V(M_{2,A,B})$. Hence, by combining (3.2) and (3.3), we obtain

$$V(M_{2,A,B}) = V(M_{2,N,B}) = \overline{\mathrm{co}}\big(W(A)W(B)\big).$$

This completes the proof. □

Note that the class of operators having normal dilations with the same condition as in the above theorem includes Toeplitz operators and hyponormal operators (see [8]).

Remark 3.2. Let $A, B \in \mathcal{B}(H)$, and let $X \in \mathcal{C}_2(H)$ be such that $\|X\|_2 = 1$. Then $\|X^*\|_2 = 1$ and $\mathrm{tr}(AXBX^*) = \mathrm{tr}(BX^*A(X^*)^*)$. From this we derive that

$$V(M_{2,A,B}) = V(M_{2,B,A}) \quad \text{for all } A, B \in \mathcal{B}(H).$$

Thus, if B has a normal dilation whose spectrum contains $\sigma(B)$, then the equality in (3.1) still holds.

The main result in [3] states that, if either A or B is a subnormal operator, then the equality in (3.1) holds. As a consequence of Theorem 3.1, we get the next generalization of this result. Here we recall that every subnormal operator is hyponormal.

Corollary 3.3. *Let $A, B \in \mathcal{B}(H)$ be such that either A or B is hyponormal. Then the equality in (3.1) holds.*

Proof. This follows from Theorem 3.1, Remark 3.2, and the fact that every hyponormal operator A on H may be dilated to a normal operator N with $\sigma(N) \subseteq \sigma(A)$ (see [8, Theorem 3.2]). $\square$

Remark 3.4. If $A \in \mathcal{B}(H)$ is hyponormal, then it follows from the above corollary that $w(M_{2,A,B}) = w(A)w(B) = \|A\|w(B)$ for any operator B in $\mathcal{B}(H)$.

Recall that a bounded linear operator on a Banach space is called *spectraloid* if its spectral radius coincides with its numerical radius. As an application of Corollary 3.3, we get the next proposition.

Proposition 3.5. *Let $A, B \in \mathcal{B}(H)$ be given such that A is hyponormal. Then $M_{2,A,B}$ is spectraloid if and only if B is spectraloid.*

Proof. If $M_{2,A,B}$ is spectraloid, then

$$w(M_{2,A,B}) = r(M_{2,A,B}) = r(A)r(B).$$

Since, by Remark 3.4, $w(M_{2,A,B}) = w(A)w(B)$, then it follows that

$$r(A)r(B) = w(A)w(B).$$

Thus $r(B) = w(B)$.

The converse is obvious. $\square$

Let us give an example showing that the condition $\sigma(N) \subseteq \sigma(A)$ in Theorem 3.1 may not be dropped even in finite dimensions. Here we recall that every operator A may dilated to a normal operator N, but the operator N may not satisfy $\sigma(N) \subseteq \sigma(A)$.

Example 3.6. Let $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then clearly A has no normal dilation N such that $\sigma(N) \subseteq \sigma(A) = \{0\}$. If $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, then $w(A) = w(B) = 1/2$.

On the other hand, a straightforward computation shows that $w(M_{2,B,A}) = 1/2$, and so $w(A)w(B) < w(M_{2,B,A})$. This shows that $\overline{\text{co}}(W(A)W(B))$ is strictly contained in $V(M_{2,A,B})$.

4. Concluding remarks

Let $A, B \in \mathcal{B}(H)$. Since the operators $L_{2,A}$ and $R_{2,B}$ satisfy $L_{2,A}R_{2,B} = R_{2,B}L_{2,A}$ and $L_{2,A}^* R_{2,B} = R_{2,B}L_{2,A}^*$, it follows from [9] that

$$w(M_{2,A,B}) = w(L_{2,A}R_{2,B}) \leq w(L_{2,A})\|R_{2,B}\| = w(A)\|B\|.$$

Suppose that $W(A)$ is a disk centered at the origin and B is a normaloid operator; that is, $w(B) = \|B\|$. Since, by Lemma 2.3, $W(A)W(B) \subseteq V(M_{2,A,B})$, one easily checks that

$$V(M_{2,A,B}) = \overline{\mathrm{co}}\big(W(A)W(B)\big).$$

Thus we get another class of operators satisfying equation (3.1).

In the remainder of this section, we consider the numerical range of the operator $M_{p,A,B}$ $(1 \le p \le \infty)$. Recall from [10] that $(\mathcal{C}_\infty(H))' = \mathcal{C}_1(H)$ and $(\mathcal{C}_1(H))' = \mathcal{B}(H)$. Further, $(\mathcal{C}_p(H))' = (\mathcal{C}_q(H))'$ for every $1 < p, q < \infty$ with $1/p + 1/q = 1$. Under these identifications, we have

$$M^*_{\infty,A,B} = M_{1,B,A}, \qquad M_{A,B} = M^{**}_{\infty,A,B}, \qquad \text{and} \qquad M_{p,A,B} = M^*_{q,B,A} \tag{4.1}$$

(see [6]).

Denote by $\mathcal{U}(H)$ the set of all unitary operators acting on H.

Proposition 4.1. *Let $A, B \in \mathcal{B}(H)$. Then*

(1) $V(M_{1,B,A}) = V(M_{\infty,A,B}) = V(M_{A,B}) = [\bigcup_{U\in\mathcal{U}(H)} \overline{W}(AUBU^*)]^-$;
(2) $V(M_{p,A,B}) = V(M_{q,B,A})$ *for any* $p, q \in (1, \infty)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Note from [2, Corollary 6] that if E is a Banach space and if $T \in \mathcal{B}(E)$, then $V(T) = V(T^*)$. Thus the equalities $V(M_{1,B,A}) = V(M_{\infty,A,B}) = V(M_{A,B})$ and $V(M_{p,A,B}) = V(M_{q,B,A})$ follow directly from (4.1). The equality $V(M_{A,B}) = [\bigcup_{U\in\mathcal{U}(H)} \overline{W}(AUBU^*)]^-$ follows from [1]. □

The following corollary is an immediate consequence of Proposition 4.1.

Corollary 4.2. *Let $A, B \in \mathcal{B}(H)$. Then*

(1) $w(M_{1,B,A}) = w(M_{\infty,A,B}) = w(M_{A,B}) = \sup\{w(AUBU^*) : U \in \mathcal{U}(H)\}$;
(2) $w(M_{p,A,B}) = w(M_{q,B,A})$.

Example 4.3. One might expect that the equalities in Corollary 4.2, part (1), still hold in case of an arbitrary norm ideal of $\mathcal{B}(H)$. But this is not true. To see this, consider the matrices $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. Since $AB = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, we have $1 \in W(AB) \subseteq V(M_{A,B})$ (see [1]). Thus

$$w(M_{A,B}) = 1.$$

On the other hand, we have $w(M_{2,B,A}) = 1/2$. This shows that

$$V(M_{A,B}) \neq V(M_{2,B,A}).$$

Remark 4.4. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be elements of $\mathcal{B}(H)$. From (4.1) we have

$$\Big(\sum_{i=1}^n M_{\infty,A_i,B_i}\Big)^* = \sum_{i=1}^n M_{1,B_i,A_i} \qquad \text{and} \qquad \Big(\sum_{i=1}^n M_{\infty,A_i,B_i}\Big)^{**} = \sum_{i=1}^n M_{A_i,B_i}.$$

Thus the equalities established in Proposition 4.1 and Corollary 4.2 still hold for the elementary operator $\sum_{i=1}^n M_{A_i,B_i}$.

Acknowledgments. We thank the anonymous referees for valuable comments and suggestions. The authors' work was partially supported by the Deanship of Scientific Research at Sattam bin Abdulaziz University under the Research Project 3991/01/2015.

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[1]Department of Mathematics, Faculty of Science and Humanitarian Studies, Sattam bin Abdulaziz University, P.O. Box 83, Alkharj 11942, Saudi Arabia.

E-mail address: boumazgour@hotmail.com

[2]Department of Engineering Basic Sciences, Faculty of Engineering, Menofia University, Menofia, Egypt.

E-mail address: eng_hossam21@yahoo.com