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PRECONDITIONERS IN SPECTRAL APPROXIMATION

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ABSTRACT. Let $\mathcal{H}$ be a complex separable Hilbert space, and let A be a bounded self-adjoint operator on $\mathcal{H}$. Consider the orthonormal basis $\mathcal{B} = \{e_1, e_2, \ldots\}$ and the projection P_n of $\mathcal{H}$ onto the finite-dimensional subspace spanned by the first n elements of $\mathcal{B}$. The finite-dimensional truncations $A_n = P_nAP_n$ shall be regarded as a sequence of finite matrices by restricting their domains to $P_n(\mathcal{H})$ for each n. Many researchers used the sequence of eigenvalues of A_n to obtain information about the spectrum of A. But in many situations, these A_n's need not be simple enough to make the computations easier. The natural question *Can we use some simpler sequence of matrices B_n instead of A_n?* is addressed in this article. The notion of preconditioners and their convergence in the sense of eigenvalue clustering are used to study the problem. The connection between preconditioners and compact perturbations of operators is identified here. The usage of preconditioners in the spectral gap prediction problems is also discussed. The examples of Toeplitz and block Toeplitz operators are considered as an application of these results. Finally, some future possibilities are discussed.

1. INTRODUCTION

Given a complex separable Hilbert space $\mathcal{H}$ and a bounded self-adjoint operator A on $\mathcal{H}$, how to approximate the spectrum of A is a fundamental question in operator theory. The usage of finite-dimensional truncations led to some positive results in the literature (see [1], [7]–[9], [12]). Let $\mathcal{B} = \{e_1, e_2, \ldots\}$ be an orthonormal basis for $\mathcal{H}$, and for each n let P_n be the projection of $\mathcal{H}$ onto the finite-dimensional subspace spanned by the first n elements of $\mathcal{B}$. We will regard

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the finite-dimensional truncation $A_n = P_nAP_n$ as a finite matrix by restricting the domain to $P_n(\mathcal{H})$ for each n. The usage of the eigenvalue sequence of the truncations A_n to study the spectrum of A is usually referred to as the *truncation method*. This method involves computation of the sequence of eigenvalues of $A_n = P_nAP_n$ and their limits as n tends to infinity. The bounds of the essential spectrum $\sigma_e(A)$, and the discrete eigenvalues that lie above and below these bounds, were approximated by this method (see [7]). Also, there were attempts to determine the spectral gaps that may exist between the bounds of $\sigma_e(A)$ using this method (see [12]). The location of eigenvalues inside spectral gaps was also studied in some articles (see [8], [9]).

In this article, we try to modify the truncation method with the help of the notions of preconditioners and the convergence of matrix sequences in the sense of eigenvalue clustering. Recall that, in the numerical analysis literature, the preconditioner associated with a matrix is used to make the iteration process more efficient for solving, for example, linear systems of large dimensions. The Frobenius optimal approximation of matrices was used in the special case of Toeplitz matrices for the design of efficient solvers of complicated linear systems of large size (see [15], [16]). Here we use different notions of convergence of matrix sequences in the sense of eigenvalue clustering to study the spectral approximation by preconditioners. To be more precise, we will replace the matrix sequence A_n by its preconditioner sequence B_n, and we will consider the eigenvalue sequence of B_n to approximate the spectrum of A. This also makes sense numerically, because the preconditioners usually are chosen in a class of matrices for which the eigenvalues are known in closed form or can be computed very efficiently using fast Fourier transform-type algorithms (see [4], [5]).

We start with defining different notions of convergence of matrix sequences in the sense of eigenvalue clustering. Such notions were used in the special case of Toeplitz matrices in [15] and were generalized into the case of arbitrary symmetric matrices in [11]. The definition given below is valid for arbitrary matrix sequences.

Definition 1.1. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ matrices. We say that $A_n - B_n$ converges to the zero matrix in a *strong cluster* if for every $\epsilon > 0$ there exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that

$$A_n - B_n = R_n + N_n, \quad \text{for all } n \geq N_{2,\epsilon},$$

where the rank of R_n is bounded above by $N_{1,\epsilon}$ and $\|N_n\| \leq \epsilon$.

If the number $N_{1,\epsilon}$ does not depend on ϵ, then we say that $A_n - B_n$ converges to 0 in a *uniform cluster*. If $N_{1,\epsilon}$ depends on ϵ, n and is of $o(n)$, we say that $A_n - B_n$ converges to 0 in a *weak cluster*.

This article aims to modify the truncation method by replacing A_n by some other *simpler* sequence of matrices B_n, where $\{A_n\} - \{B_n\}$ converges to the zero matrix in a strong cluster (resp., weak or uniform cluster). We study the effect of this replacement in the well-known results obtained by the truncation method. A brief outline of the truncation method is given below.

1.1. **Truncation method.** Let A be a bounded self-adjoint operator defined on a complex separable Hilbert space $\mathbb{H}$. The spectrum of A is denoted by $\sigma(A)$ with

m, M as its lower and upper bounds. Consider the finite-dimensional truncations of A; that is, $A_n = P_n A P_n$, where P_n is the projection of $\mathbb{H}$ onto the span of first n elements $\{e_1, e_2, \dots, e_n\}$ of the basis. Let ν, μ be the lower and upper bounds of the essential spectrum $\sigma_e(A)$, respectively, with A being self-adjoint. Let $\lambda_R^+(A) \leq \cdots \leq \lambda_2^+(A) \leq \lambda_1^+(A)$ be the discrete eigenvalues of A lying above μ, and let $\lambda_1^-(A) \leq \lambda_2^-(A) \leq \cdots \leq \lambda_S^-(A)$ be the eigenvalues of A lying below ν. Here R and S can be infinity. Denote by $\lambda_1(A_n) \geq \lambda_2(A_n) \geq \cdots \geq \lambda_n(A_n)$ the eigenvalues of A_n.

Now we recall the notion of essential points and transient points introduced in [1].

Definition 1.2. A real number λ is an *essential point* of A if for every open set U containing λ, $\lim_{n\to\infty} N_n(U) = \infty$, where $N_n(U)$ is the number of eigenvalues of A_n in U.

Definition 1.3. A real number λ is a *transient point* of A if there is an open set U containing λ, such that $\sup N_n(U)$ with n varying on the set of all natural number is finite.

Remark 1.4. Note that a number can be neither transient nor essential.

Denote $\Lambda = \{\lambda \in R; \lambda = \lim \lambda_n, \lambda_n \in \sigma(A_n)\}$ and Λ_e as the set of all essential points. The following spectral inclusion result for a bounded self-adjoint operator A is of high importance.

Theorem 1.5 (see [1, Theorem 2.3]). *The spectrum of a bounded self-adjoint operator is contained in the set of all limit points of the eigenvalue sequences of its truncations. Also, the essential spectrum is contained in the set of all essential points; that is,*

$$\sigma(A) \subseteq \Lambda \subseteq [m, M] \qquad \text{and} \qquad \sigma_e(A) \subseteq \Lambda_e.$$

The following result from [7, Theorem 3.1] enables us to approximate discrete eigenvalues that lie above and below the upper and lower bounds of the essential spectrum.

Theorem 1.6. *For every fixed integer k,*

$$\lim_{n\to\infty} \lambda_k(A_n) = \begin{cases} \lambda_k^+(A) & \text{if } R = \infty \text{ or } 1 \leq k \leq R, \\ \mu & \text{if } R < \infty \text{ and } k \geq R+1, \end{cases}$$

$$\lim_{n\to\infty} \lambda_{n+1-k}(A_n) = \begin{cases} \lambda_k^-(A) & \text{if } S = \infty \text{ or } 1 \leq k \leq S, \\ \nu & \text{if } S < \infty \text{ and } k \geq S+1. \end{cases}$$

In particular, $\lim_{k\to\infty} \lim_{n\to\infty} \lambda_k(A_n) = \mu$ *and* $\lim_{k\to\infty} \lim_{n\to\infty} \lambda_{n+1-k}(A_n) = \nu$.

The subsequent theorem taken from [7, Theorem 4.1] denies the existence of spurious eigenvalues (points in Λ which are not spectral values) under the assumption that the essential spectrum is connected.

Theorem 1.7. *If A is a bounded self-adjoint operator and if $\sigma_e(A)$ is connected, then $\sigma(A) = \Lambda$.*

Hence, the remaining task is to predict the existence of spectral gaps that may arise in between the bounds of the essential spectrum. There were attempts in this direction using the truncation method (see [12]). The following theorem is taken from [12, Theorem 3.1], which is an attempt to predict the spectral gaps between the bounds of the essential spectrum.

Theorem 1.8. *Let A be a bounded self-adjoint operator, and let $\lambda_{n1}(A_n) \geq \lambda_{n2}(A_n) \geq \cdots \geq \lambda_{nn}(A_n)$ be the eigenvalues of A_n arranged in decreasing order. For each positive integer n, let $a_n = \sum_{k=1}^{n} w_{nk}\lambda_{nk}$ be the convex combination of eigenvalues of A_n. If there exists a $\delta > 0$ and $K > 0$ such that*

$$\#\big\{\lambda_{nj}; |a_n - \lambda_{nj}| < \delta\big\} < K$$

and, in addition, if $\sigma_e(A)$ and $\sigma(A)$ have the same upper and lower bounds, then $\sigma_e(A)$ has a gap.

The article is organized as follows. In the next section, we give a characterization for convergence in the strong, uniform, and weak cluster in the case of Hermitian matrices. We also prove that the strong and uniform convergence amounts to a compact and finite-rank perturbation in the operator. The third section deals with the spectral gap prediction problems and the usage of Frobenius optimal preconditioners. The concrete example of the Toeplitz case and its preconditioners is considered in this section. A concluding section ends the paper.

2. Main results

The following theorem gives a characterization of the convergence in a strong, uniform, or weak cluster in the case of Hermitian matrices.

Theorem 2.1. *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ Hermitian matrices. Then $\{A_n\} - \{B_n\}$ converges to 0 in a strong cluster (resp., weak or uniform cluster) if and only if, for every given $\epsilon > 0$, there exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that the spectrum $\sigma(A_n - B_n)$ lies in the interval $(-\epsilon, \epsilon)$, except for at most $N_{1,\epsilon}$ (independent of the size n) eigenvalues for all $n > N_{2,\epsilon}$.*

Proof. First we suppose that $\{A_n\} - \{B_n\}$ converges to 0 in a strong cluster (resp., weak or uniform cluster). Therefore, by definition, for $\epsilon > 0$, there exist natural numbers $N_{1,\epsilon}, N_{2,\epsilon}$ with the following decomposition:

$$A_n - B_n = R_n + N_n, \quad \text{for all } n \geq N_{2,\epsilon},$$

where the rank of R_n is bounded above by $N_{1,\epsilon}$ and $\|N_n\| \leq \epsilon$.

Since the rank of R_n is bounded by $N_{1,\epsilon}$, by the rank-nullity theorem, R_n has at most $N_{1,\epsilon}$ nonzero eigenvalues. Also, since $\|N_n\| \leq \epsilon$, we have, except for at most $N_{1,\epsilon}$ eigenvalues, all the eigenvalues of $A_n - B_n = R_n + N_n$ that lie in the interval $(-\epsilon, \epsilon)$ whenever $n \geq N_{2,\epsilon}$.

Conversely, suppose that, for any given $\epsilon > 0$, there exist integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that all eigenvalues of $A_n - B_n$ lie in the interval $(-\epsilon, \epsilon)$ except for at most $N_{1,\epsilon}$ (resp., $N_{1,\epsilon} = o(n)$) eigenvalues whenever $n \geq N_{2,\epsilon}$. Hence, by the spectral theorem, there exists a unitary matrix sequence U_n such that

$$U_n(A_n - B_n){U_n}^{-1} = D_n, \quad {U_n}^{-1} = {U_n}^{*},$$

where D_n is a diagonal matrix sequence with diagonal entries that lie in the interval $(-\epsilon, \epsilon)$ except for at most $N_{1,\epsilon}$ (resp., $N_{1,\epsilon} = o(n)$) entries whenever $n \geq N_{2,\epsilon}$. Therefore, we can write $D_n = R_n' + N_n'$, where R_n' and N_n' are diagonal matrices with all the entries in N_n' in the interval $(-\epsilon, \epsilon)$ whenever $n \geq N_{2,\epsilon}$ and the entries in R_n' are 0 except for at most $N_{1,\epsilon}$ (resp., $N_{1,\epsilon} = o(n)$) entries. Therefore, we have

$$(A_n - B_n) = {U_n}^{-1} D_n U_n = {U_n}^{-1}(R_n' + N_n')U_n = R_n + N_n.$$

Also, the rank of $R_n = {U_n}^{-1} R_n' U_n$ is bounded above by $N_{1,\epsilon}$ and $\|N_n\| = \|N_n'\| \leq \epsilon$. Hence, for $\epsilon > 0$, there exist natural numbers $N_{1,\epsilon}, N_{2,\epsilon}$ with the following decomposition:

$$A_n - B_n = R_n + N_n, \quad \text{for all } n \geq N_{2,\epsilon},$$

where the rank of R_n is bounded above by $N_{1,\epsilon}$ and $\|N_n\| \leq \epsilon$. □

Remark 2.2. Theorem 2.1 is not true for non-Hermitian matrices. However, we shall obtain the necessary part for normal matrices in terms of disks around the origin. This follows easily by noticing that we used only the spectral theorem, which is also true for normal matrices.

Remark 2.3. Theorem 2.1 shall be generalized if the singular values are considered in place of the eigenvalues.

2.1. **Perturbation of operators and eigenvalue clustering.** Here we establish the connection between compact perturbations of operators and convergence in the eigenvalue cluster.

Theorem 2.4. *Let $A, B \in B(\mathcal{H})$ be self-adjoint operators. Then the operator $R = A - B$ is compact if and only if the truncation $A_n - B_n$ converges to the zero matrix in the strong cluster.*

Proof. First assume that $R = A - B$ is compact and its spectrum $\sigma(R) = \{\lambda_k(R) : k = 1, 2, 3, \ldots\} \cup \{0\}$. Here 0 is the only accumulation point of the spectrum. Hence, $\lambda_k(R) \to 0$ as $k \to \infty$. Hence, for any given $\epsilon > 0$, there exists a positive integer $N_{1,\epsilon}$ such that

$$\lambda_k(R) \in \left(\frac{-\epsilon}{2}, \frac{\epsilon}{2}\right), \quad \text{for every } k > N_{1,\epsilon}.$$

Also, since R is compact, the truncation $R_n = A_n - B_n$ converges to R in the operator norm topology. Therefore, the eigenvalues of truncations converge to the eigenvalues of R.

In addition to this, if we arrange the eigenvalues of R and R_n in such a way that nonnegative eigenvalues occur at the odd places in nonincreasing order and negative eigenvalues at even places in a nondecreasing order (i.e., $\lambda_k \geq 0$ if k is odd and $\lambda_k < 0$ if k is even), then we have the following inequality (see [6, pp. 176–178]):

$$\left|\lambda_k(R_n) - \lambda_k(R)\right| \leq \|R_n - R\|. \tag{2.1}$$

Therefore, we have

$$\lambda_k(R_n) \to \lambda_k(R) \quad \text{as } n \to \infty, \text{for each } k.$$

In particular, for every $k > N_{1,\epsilon}$, there exists a positive integer $N_{2,\epsilon}$ such that

$$\lambda_k(R_n) - \lambda_k(R) \in \left(\frac{-\epsilon}{2}, \frac{\epsilon}{2}\right), \quad \text{for every } n > N_{2,\epsilon}.$$

Also, notice that this $N_{2,\epsilon}$ can be chosen independently of k by inequality (2.1). Therefore, when $n > N_{2,\epsilon}$, all the eigenvalues $\lambda_k(R_n)$ of $R_n = A_n - B_n$, except for the first $N_{1,\epsilon}$ eigenvalues, are in the interval $(-\epsilon, \epsilon)$; that is, $A_n - B_n$ converges to 0 in the strong cluster.

For the converse part, assume that $A_n - B_n$ converges to the zero matrix in the strong cluster. Then for any $\lambda \neq 0$, choose an $\epsilon > 0$ such that λ is outside the interval $(-\epsilon, \epsilon)$. Corresponding to this ϵ, there exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that $\sigma(A_n - B_n)$ is contained in $(-\epsilon, \epsilon)$, for every $n > N_{2,\epsilon}$, except for possibly $N_{1,\epsilon}$ eigenvalues. Now consider the counting function $N_n(U)$ of eigenvalues of $A_n - B_n$ in $U \subseteq \mathbb{R}$. For any neighborhood U of λ that does not intersect with $(-\epsilon, \epsilon)$, $N_n(U)$ is bounded by the number $N_{1,\epsilon}$. Hence, λ is not an essential point of $A - B$. Therefore, by Theorem 1.5, λ is not in the essential spectrum. Since $\lambda \neq 0$ was arbitrary, this shows that the essential spectrum of $A - B$ is the singleton set $\{0\}$. Hence, it is a compact operator and the proof is completed. □

Theorem 2.5. *Let $A, B \in B(\mathcal{H})$ be self-adjoint operators. Then the operator $R = A - B$ is of finite rank if and only if the truncation $A_n - B_n$ converges to the zero matrix in the uniform cluster.*

Proof. The proof is an imitation of the proof of Theorem 2.4 and differs only in the choice of $N_{1,\epsilon}$ to be independent of ϵ; however, the details are given below. First assume that $R = A - B$ is a finite-rank operator with rank N_1, and its spectrum $\sigma(R) = \{\lambda_k(R) : k = 1, 2, 3, \ldots, N_1\} \cup \{0\}$. Since the truncation $R_n = A_n - B_n$ converges to R in the operator norm topology, the eigenvalues of truncations converge to the eigenvalues of R; that is,

$$\lambda_k(R_n) \to \lambda_k(R) \quad \text{as } n \to \infty, \text{for each } k = 1, 2, 3, \ldots, N_1.$$

For every $k > N_1$, $\lambda_k(R_n)$ converges to 0 by [7, Theorem 3.1]. Hence, for a given $\epsilon > 0$, there exists a positive integer $N_{2,\epsilon}$ such that

$$\lambda_k(R_n) \in (-\epsilon, \epsilon), \quad \text{for every } n > N_{2,\epsilon} \text{ and for each } k > N_1.$$

Therefore, when $n > N_{2,\epsilon}$, all the eigenvalues $\lambda_k(R_n)$ of $R_n = A_n - B_n$, except for the first N_1 eigenvalues, are in the interval $(-\epsilon, \epsilon)$; that is, $A_n - B_n$ converges to 0 in the uniform cluster.

For the converse part, assume that $A_n - B_n$ converges to the zero matrix in the uniform cluster. Then, for any $\epsilon > 0$, there exist positive integers $N_1, N_{2,\epsilon}$ such that $\sigma(A_n - B_n)$ is contained in $(-\epsilon, \epsilon)$, for every $n > N_{2,\epsilon}$, except for possibly N_1 eigenvalues. As in the proof of Theorem 2.4, we obtain that 0 is the only element in the essential spectrum. Hence, $R = A - B$ is a compact operator. In addition to this, R can have at most N_1 eigenvalues. To see this, note that all the eigenvalues of a compact operator are obtained as the limits of a sequence of eigenvalues of its truncations. In this case at most N_1 such sequences can go to a nonzero limit. Hence, R is a finite-rank operator and the proof is completed. □

Remark 2.6. The above results have the following implications. Since a compact perturbation may change the discrete eigenvalues, the above results show that the convergence of preconditioners in the sense of eigenvalue clustering is not sufficient to use them in the spectral approximation problems. Nevertheless, one can use it in the spectral gap prediction problems, since the compact perturbation preserves the essential spectrum. In particular, it can be used to compute the upper and lower bound of the essential spectrum.

Remark 2.7. The effect of convergence of truncations in a weak cluster has to be investigated in detail. This could be one possibility to modify the approximation techniques for discrete spectral values.

Remark 2.8. Since the eigenvalue clustering results in compact perturbation only, in Theorem 1.8 one can replace the role of the eigenvalue sequence of A_n by any sequence of matrices B_n (truncation) such that $A_n - B_n$ converges to the zero matrix in a strong cluster.

3. Frobenius optimal preconditioners and other examples

In this section we introduce the Frobenius optimal preconditioners (see [11]). These preconditioners were used in the special case of Toeplitz operators in [15] and [16], and in the general case in [11].

Let $\{U_n\}$ be a sequence of unitary matrices over $\mathbb{C}$, where U_n is of order n for each n. For each n, we define the commutative algebra M_{U_n} of matrices as follows:

$$M_{U_n} = \left\{A \in M_n(\mathbb{C}); {U_n}^* A U_n \text{ complex diagonal}\right\}.$$

Recall that $M_n(\mathbb{C})$ is a Hilbert space with the Frobenius norm

$$\|A\|_2^2 = \sum_{j,k=1}^{n} |A_{j,k}|^2,$$

induced by the classical Frobenius scalar product

$$\langle A, B\rangle = \operatorname{trace}(B^* A).$$

Observe that M_{U_n} is a closed convex set in $M_n(\mathbb{C})$, and, hence, corresponding to each $A \in M_n(\mathbb{C})$, there exists a unique matrix $P_{U_n}(A)$ in M_{U_n} such that

$$\|A - X\|_2^2 \geq \left\|A - P_{U_n}(A)\right\|_2^2 \quad \text{for every } X \in M_{U_n}.$$

Definition 3.1 (see [11, Definition 3.3]). For each $A \in \mathbb{B}(\mathbb{H})$, $\Phi_n : \mathbb{B}(\mathbb{H}) \to \mathbb{M}_n(\mathbb{C})$ is defined as

$$\Phi_n(A) = P_{U_n}(A_n),$$

where $P_{U_n}(A_n)$ is the matrix which minimizes the Frobenius distance of A_n to M_{U_n}, for each positive integer n. $\Phi_n(A)$ is called the *preconditioner* of A.

Lemma 3.2 (see [15, Lemma 2.1]). *With $A,B \in M_n(\mathbb{C})$ and α, β complex numbers, we have*

$$P_{U_n}(A) = U_n \sigma({U_n}^* A U_n){U_n}^*,$$

where $\sigma(X)$ is the diagonal matrix having X_{ii} as the diagonal elements

$$\begin{aligned} P_{U_n}(\alpha A + \beta B) &= \alpha P_{U_n}(A) + \beta P_{U_n}(B), \\ P_{U_n}(A^*) &= P_{U_n}(A)^*, \\ \text{Trace}\, P_{U_n}(A) &= \text{Trace}(A), \\ \left\| P_{U_n}(A) \right\| &= 1 \quad \text{(operator norm)}, \\ \left\| P_{U_n}(A) \right\|_F &= 1 \quad \text{(Frobenius norm)}, \\ \left\| A - P_{U_n}(A) \right\|_F^2 &= \|A\|_F^2 - \left\| P_{U_n}(A) \right\|_F^2. \end{aligned}$$

Lemma 3.3 (see [4], [5])**.** *If A is a Hermitian matrix, then the eigenvalues of $P_{U_n}(A)$ are contained in the closed interval $[\lambda_1(A), \lambda_n(A)]$, where $\lambda_j(A)$ are the eigenvalues of A arranged in a nondecreasing way. Hence, if A is positive definite, then $P_{U_n}(A)$ is positive definite as well.*

Since by Lemma 3.2 the trace of A_n and $B_n = P_{U_n}(A_n)$ is equal, one can use it for spectral gap prediction results. The following theorem is an application of Theorem 1.8.

Theorem 3.4. *Let A be a bounded self-adjoint operator, and let $\lambda_{n1} \geq \lambda_{n2} \geq \cdots \geq \lambda_{nn}$ be the eigenvalues of $B_n = P_{U_n}(A_n)$ arranged in decreasing order. For each positive integer n, let $a_n = \frac{\text{Trace}(B_n)}{n}$. If there exists a $\delta > 0$ and $K > 0$ such that*

$$\#\left\{\lambda_{nj}; |a_n - \lambda_{nj}| < \delta\right\} < K$$

and, in addition, if $\sigma_e(A)$ and $\sigma(A)$ have the same upper and lower bounds, then $\sigma_e(A)$ has a gap.

Proof. This follows easily by taking $w_{nk} = \frac{1}{n}$, for all k, in Theorem 1.8, and also by using Lemma 3.2. □

Now we give the concrete examples for which the preconditioners are useful in the truncation method of spectral approximation. We make use of the results from [11] and [15] to construct useful examples. We discuss the case of the well-known Toeplitz operator and its preconditioners.

3.1. **Toeplitz operator with continuous periodic symbols.** Consider the Toeplitz operator $A = A(f)$, where the symbol function $f \in C[-\pi, \pi]$ and $\mathbb{H} = L^2[-\pi, \pi]$. It can be easily observed that the truncation of such operators comprises the finite Toeplitz matrices with symbol f. The notation $A_n(f)$ is used to denote the finite Toeplitz matrix with symbol f.

Let $v = \{v_n\}_{n\in N}$ with $v_n = (v_{nj})_{j\leq n-1}$ be a sequence of trigonometric functions on an interval I. Let $S = \{S_n\}_{n\in N}$ be a sequence of grids of n points on I, namely, $S_n = \{x_i^n, i = 0, 1, \ldots, n-1\}$. Let us suppose that the generalized Vandermonde matrix

$$V_n = \left(v_{nj}(x_i^n)\right)_{i;j=0}^{n-1}$$

is a unitary matrix. Then the algebra of the form M_{U_n} is a trigonometric algebra if $U_n = {V_n}^*$ with V_n a generalized trigonometric Vandermonde matrix.

We get examples of trigonometric algebras with the following choice of the sequence of matrices U_n and grid S_n:

$$U_n = F_n = \Big(\frac{1}{\sqrt{n}} e^{ijx_i^n}\Big), \quad i,j = 0,1,\dots,n-1,$$
$$S_n = \Big\{x_i^n = \frac{2i\pi}{n}, i = 0,1,\dots,n-1\Big\} \subset I = [-\pi,\pi] \quad \text{(circulant algebra)},$$
$$U_n = G_n = \Big(\sqrt{\frac{2}{n+1}}\sin(j+1)x_i^n\Big), \quad i,j = 0,1,\dots,n-1,$$
$$S_n = \Big\{x_i^n = \frac{(i+1)\pi}{n+1}, i = 0,1,\dots,n-1\Big\} \subset I = [0,\pi] \quad \text{(algebra } \tau),$$
$$U_n = H_n = \Big(\frac{1}{\sqrt{n}}\big[\sin(jx_i^n) + \cos(jx_i^n)\big]\Big), \quad i,j = 0,1,\dots,n-1,$$
$$S_n = \Big\{x_i^n = \frac{2i\pi}{n}, i = 0,1,\dots,n-1\Big\} \subset I = [-\pi,\pi] \quad \text{(Hartly algebra)}.$$

Now we consider the preconditioners $B_n = P_{U_n}(A_n(f))$ of $A_n(f)$ corresponding to the matrix algebras M_{U_n} with the above choices of U_n's. We see that the eigenvalues of the preconditioners are much easier to handle since they are obtained as the evaluation of some trigonometric functions at some grid points. The following results taken from [15, Section 4.1] illustrate this fact.

(1) Consider the preconditioner $B_n = P_{U_n}(A_n(f))$ of $A_n(f)$ associated with the circulant algebra. Observe that from Lemma 3.2 the eigenvalues of B_n are obtained as the evaluation of a trigonometric function at certain grid points; that is, the jth eigenvalue of B_n, $\lambda_j(B_n) = \sigma(U_n A_n(f) U_n^*)_{jj}$ is the evaluation of certain trigonometric functions at the grid points ${x_j}^n$. Now let $L_n[U_n](f)(x)$ denote the function obtained by replacing the grids ${x_j}^n$ in the expression of $\lambda_j(B_n)$ by $x \in I$. In the case of the circulant algebra, the eigenvalue function $L_n[U_n](\cdot)$ is the Cesàro sum $[C_n(\cdot)](x)$. Also, $L_n[U_n](f)$ converges to f uniformly on $[-\pi,\pi]$ and has a rate of convergence of order n^{-1} on a class of functions (see [17, pp. 122–123]) which contains the polynomials.

(2) Consider the matrix algebra of all the matrices simultaneously diagonalized by the transform $U_n = G_n$ given above and generated by the symmetric Toeplitz matrix $A_n(\cos(x))$.

The explicit expression of the eigenvalues of the Frobenius-optimal approximation $P_{U_n}(A_n(f))$ is given by the following results taken from [3, Theorems 4.3 and 4.4].

Theorem 3.5. *The eigenvalues of $B_n = P_{U_n}(A_n(f))$ are given by the values taken on the grid $\frac{i\pi}{n+1}$ by the function $L_n[U_n](f)(x)$ defined by*

$$L_n[U_n](f)(x) = \big[K_n(f)\big](x) - \frac{2}{n+1}h(x), \qquad h(x) = s'(x) - s(x)\cot(x),$$

where $s(x) = \sum_{j=1}^{n-1} a_j \sin(jx)$ and $K_n(f)$ denotes the nth Fourier sum of f.

Theorem 3.6. *$L_n[U_n](f)(x)$ can be written as*

$$L_n[U_n](f)(x) = \big[Cn(f)\big](x) + \frac{\cos(x)}{n+1}\sum_{j=0}^{n-2}\big(a_{j+1}U_j\big(\cos(x)\big)\big),$$

where U_j denotes the jth Chebyshev polynomial of the second kind.

Remark 3.7. These examples justify the usage of preconditioners in the spectral approximation problems since the computation of the eigenvalue sequence is given explicitly when using these preconditioners; that is, while we apply the truncation method to compute the spectrum of operators discussed above, we shall make use of the eigenvalues of the preconditioners B_n, which are *simpler* in the sense that the explicit form is available. This is useful in the computations.

3.2. **Ten martini conjecture.** We conclude this section by mentioning one important future possibility of the above results. Consider the Schrödinger operator defined by

$$\tilde{A}(u) = -\ddot{u} + V \cdot u$$

on some suitable subspace of $L^2(\mathbb{R})$, where V is an essentially bounded function called the *potential*. The classical Borg theorem states that the Schrödinger operator with periodic potential has a connected essential spectrum if and only if the periodic potential reduces to a constant (see [14] and references therein).

If the operator $\tilde{A}$ is discretized by replacing differentiation by finite differences, then we get a bounded operator A on $l^2(\mathbb{Z})$, and after some scaling by a constant and translating by a constant multiple of the identity operator, A is defined by

$$A\big(\{x_n\}_{n\in\mathbb{Z}}\big) = \{x_{n-1} + x_{n+1} + v_n x_n\}_{n\in\mathbb{Z}},$$

where v_n is a periodic bounded sequence called the *potential*. The discretized version of Borg's theorem (see [10]) states that the essential spectrum of A is connected if and only if the p-periodic potential $(v_j)_{j\in\mathbb{Z}}$ is constant.

The well-known ten martini conjecture asserted by Barry Simon states the following: If we consider the almost-periodic Mathieu potential, say, $v_j = \cos(2\pi j\alpha)$, where α is an irrational number, then the spectrum of the associated operator is a Cantor-like set. This conjecture was settled, and many modified proofs are also available in the literature (see [2] and references therein for more details). Here we propose one possible approach to prove this conjecture.

Consider a sequence of rational numbers α_n that converges to α. The operators A_n with potential $\cos(2\pi j\alpha_n)$ will converge to the operator A with potential $\cos(2\pi j\alpha)$ in norm. Even though A is not periodic, each A_n is periodic and will have spectral gaps for each n. The major task is to estimate the size and number of spectral gaps for each n. If the size decreases and the number increases as n increases, then we have a large number of smaller spectral gaps for large n; that is, the spectrum of A_n will be obtained after removing these spectral gaps from the interval. As in the construction of the Cantor set, after each stage, we are removing more numbers of open intervals. Using A_n as preconditioners for A, and proceeding as above, we expect a much simpler proof for the ten martini conjecture. However, estimating the size of the spectral gaps and counting their number remain difficult.

4. Concluding remarks

Finally, we list a few related problems in this regard. We hope that these problems will lead to future research in this area.

- The search for optimal preconditioners, which might be useful in spectral approximation problems, is a future possibility for research. The qualities one is likely to have for the optimal preconditioners are the following. First, it must give better information regarding the spectrum of the operator under concern. Second, it must be of practical use; that is, it must have a better rate of convergence and be useful in computations.
- The random versions and perturbed versions of the spectral approximation results are the current area of research (see [12]). One further possibility is to address the preconditioner problems when the operator is subjected to an analytic perturbation and for random operators.
- The preconditioners for non-self-adjoint operators can be considered with modifying the notion of convergence in the language of disks instead of intervals. Also, one can look at the preconditioner as a finite rank and small norm perturbation. Another important problem is to develop the theory of preconditioners for unbounded operators.

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