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A CHARACTERIZATION OF THE RIESZ FAMILY OF SHIFTS OF FUNCTIONS ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract. For a locally compact Abelian group G, we give a necessary and sufficient condition for shifts of a function $\phi \in L_2(G)$ to be a Riesz family. Also, for a finite family Φ of compactly supported functions in $L_2(G)$, we show that the shifts of Φ constitute a Riesz family if and only if the nets $(\phi(\xi\eta))_{\eta \in L^\perp}$, $\phi \in \Phi$, are linearly independent for all $\xi \in \hat{G}$.

1. Introduction and preliminaries

Shift-invariant spaces and Riesz families play an increasingly important role in various areas of mathematical analysis and their applications. They appear in the study of spline wavelets, approximation, regular sampling, Gabor systems, and several others (see [1], [4], [8], [14]). A sequence $(x_n)_{n\in\mathbb{N}}$ for a Hilbert space H is called a *Riesz basis* if it is a frame and is also a basis for H. Frames provide a useful tool to obtain signal decomposition in cases where redundancy, oversampling, and irregular sampling play a role (see [2], [6]). A frame for a vector space equipped with an inner product also allows each vector in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required. Intuitively, one can think about a frame as a basis to which one has added more elements (see [5]). One of the main purposes of this paper is to give a necessary and sufficient condition for a finite family of functions to constitute a Riesz family via linear independence of the

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Fourier transform of basis functions. The problem of linear independence of integer translates of basis functions stemmed from some questions about multivariate splines. In [3], de Boor and Höllig considered the linear independence problem for integer translates of a box spline. They gave a necessary condition for the integer translates of a box spline to be linearly independent and conjectured that their condition would be also sufficient. Their conjecture was confirmed independently by Dahmen and Micchelli [7] and by Jia [11]. The general problem of linear independence of integer translates of a function was studied in [7]. A characterization for a Riesz family of shifts of a finite number of compactly supported functions in $L_p(\mathbb{R})$ for $1 < p \leq \infty$ was established in [13], based on the linear independence of the Fourier transform of functions. Then Jia extended the result for a finite number of compactly supported distributions in $L_p(\mathbb{R})$ for $0 < p \leq \infty$ (see [12]).

In this paper we give conditions under which shifts of a finite number of compactly supported functions constitute a Riesz family on a locally compact Abelian (LCA) group.

Let G be an LCA group with identity 1, and let $\hat{G}$ be its dual group with Haar measure μ. For a closed subgroup H of G, the annihilator of H in $\hat{G}$ is denoted by $H^\perp$ and is defined by $\{\xi \in \hat{G}; \xi(H) = \{1\}\}$, which is a closed subgroup of $\hat{G}$. A subgroup L of G is called a *uniform lattice* if it is discrete and cocompact (i.e., G/L is compact). Let L be a uniform lattice in G; then the subgroup $L^\perp$ is a uniform lattice in $\hat{G}$ (see [9, Theorem 4.39]). For a uniform lattice L in G, a fundamental domain is a measurable set S_L in G, such that every $x \in G$ can be uniquely written as $x = ks$, for $k \in L$ and $s \in S_L$. The existence of a relatively compact fundamental domain for a uniform lattice in an LCA group G is guaranteed by [15, Lemma 2], and it has been shown that S_L has positive measure (see [14], [15]). For a uniform lattice L, a closed subspace $V \subseteq L_2(G)$ is called L-*invariant* if it is invariant under translations by elements of L. In other words, V is called *shift-invariant* if $f \in V$ implies $T_k f \in V$, where T_k is the translation operator on $L_2(G)$ defined by $T_k f(x) = f(k^{-1}x)$ for all $x \in G$, $k \in L$. Also in [15], it was seen that $L_2(G/L) \cong L_2(S_L)$, when G is a second countable and LCA group and L is a uniform lattice in G.

We denote by $l(L)$ the linear space of all functions on L, and by $c_c(L)$ the linear space of all finitely supported functions on L. Given $a \in l(L)$, the formal Laurent series $\sum_{k\in L} a(k)\overline{\xi(k)}$ for $\xi \in \hat{G}$ is called the *symbol* of a and is denoted by $\tilde{a}(\xi)$. If $a \in l_1(L)$, then the symbol $\tilde{a}$ is a continuous function on $S_{L^\perp}$. If f and g are measurable functions on G, then the convolution of f and g is the function defined by

$$f * g(x) = \int f(y)g(y^{-1}x)\,dy,$$

for all $x \in G$ whenever the integral exists. Suppose that $1 \leq p \leq \infty$, $f \in L_1(G)$, and that $g \in L_p(G)$. By [9, Proposition 2.39], the above integral converges absolutely for almost every x, and we have

$$f * g \in L_p(G) \qquad \text{and} \qquad \|f * g\|_p \leq \|f\|_1 \|g\|_p. \tag{1.1}$$

In the case of discrete groups, letting $a, b \in l(L)$, the convolution of a and b is given by

$$a * b(l) = \sum_{k \in L} a(k^{-1}l)b(k), \quad l \in L,$$

whenever the above series exists. If $\delta(k) = 1$ for $k = 1$ and $\delta(k) = 0$ for $k \neq 1$, then $\delta \in l_1(L)$ and $a * \delta = a$. Note that if $a, c \in l_1(L)$ and $\tilde{a}(\xi)\tilde{c}(\xi) = 1$ for all $\xi \in S_{L^\perp}$, then $a * c = \delta$, which is followed by [9, Proposition 4.36].

Let $\Phi = \{\phi_1, \ldots, \phi_n\}$ be a finite family of functions in $L_2(G)$. The family of all shifts of Φ is called a *Riesz family* for the span$\{T_k\phi, k \in L, \phi \in \Phi\}$ if there exist two positive constants C_1 and C_2 such that

$$\begin{aligned} C_1 \sum_{i=1}^{n} \|a_i\|_{l_2(L)} &\leq \Big\| \sum_{i=1}^{n} \sum_{k \in L} a_i(k) T_k \phi_i \Big\|_{L^2(G)} \\ &\leq C_2 \sum_{i=1}^{n} \|a_i\|_{l_2(L)}, \end{aligned} \tag{1.2}$$

for all elements $a_1, a_2, \ldots, a_n \in l_2(L)$.

In this paper we show that the set of shifts of a function $\phi \in L_2(G)$ is a Riesz family if and only if $0 < \operatorname{ess\,inf}_{\xi \in \hat{G}} \sum_{\eta \in L^\perp} |\hat{\phi}(\xi\eta)|^2$ and $\operatorname{ess\,sup}_{\xi \in \hat{G}} \sum_{\eta \in L^\perp} |\hat{\phi}(\xi\eta)|^2 < \infty$. We define a set of nets, $\{(\hat{\phi}_1(\xi\eta))_{\eta \in L^\perp}, \ldots, (\hat{\phi}_n(\xi\eta))_{\eta \in L^\perp}\}$, to be linearly independent on $\hat{G}$ if $\sum_{i=1}^{n} c_i(\hat{\phi}_i(\xi\eta))_{\eta \in L^\perp} = 0$ implies that $c_1 = \cdots = c_n = 0$. We show that the shifts of a finite number of compactly supported functions $\{\phi_1, \ldots, \phi_n\}$ in $L_2(G)$ constitute a Riesz family if and only if the nets $(\hat{\phi}_i(\xi\eta))_{\eta \in L^\perp}, i = 1, \ldots, n$, are linearly independent for all $\xi \in \hat{G}$. This paper is organized as follows. In the second section, we investigate conditions under which shifts of a function $\phi \in L_2(G)$ constitute a Riesz family. In the third section, we give a necessary and sufficient condition for shifts of a finite number of compactly supported functions in $L_2(G)$ to be a Riesz family.

2. Riesz family of a function

In this section, we establish a necessary and sufficient condition for shifts of a single function in $L_2(G)$ to form a Riesz family.

Theorem 2.1. *Let G be a second countable LCA group, and let $\phi \in L_2(G)$. The shifts of ϕ constitute a Riesz family in the sense of (1.2), if and only if $0 < \operatorname{ess\,inf}_{\xi \in \hat{G}} \sum_{\eta \in L^\perp} |\hat{\phi}(\xi\eta)|^2$ and $\operatorname{ess\,sup}_{\xi \in \hat{G}} \sum_{\eta \in L^\perp} |\hat{\phi}(\xi\eta)|^2 < \infty$.*

Proof. Let $V_0 := \overline{\operatorname{span}}\{T_k\phi, k \in L\}$, and let $f \in V_0$. Then

$$f(x) = \sum_{k \in L} a(k)\phi(k^{-1}x), \quad \big(a(k)\big)_{k \in L} \in l_2(L).$$

Also, $\mathcal{T}$ defined by $\mathcal{T}(\xi) = \sum_{k\in L} a(k)\bar{\xi}(k)$ is in $L_2(\hat{L})$. Indeed, using the fact that $\{\xi(k), k \in L\}$ is an orthonormal basis for $L_2(\hat{L})$, we have

$$\begin{aligned}\|\mathcal{T}\|^2_{L_2(\hat{L})} &= \int_{\hat{L}} \big|\mathcal{T}(\xi)\big|^2\, d\xi \\ &= \int_{\hat{L}} \sum_{k\in L} a(k)\bar{\xi}(k) \sum_{l\in L} \bar{a}(l)\xi(l)\, d\xi \\ &= \int_{\hat{L}} \sum_{k\in L} \big|a(k)\big|^2\, d\xi < \infty.\end{aligned}$$

Taking the Fourier transform of f, we obtain

$$\hat{f}(\xi) = \sum_{k\in L} a(k)\widehat{(T_k\phi)}(\xi) = \sum_{k\in L} a(k)\bar{\xi}(k)\hat{\phi}(\xi) = \mathcal{T}(\xi)\hat{\phi}(\xi).$$

Therefore,

$$\begin{aligned}\|f\|_2 = \|\hat{f}\|_2 &= \int_{\hat{G}} \big|\mathcal{T}(\xi)\big|^2 \big|\hat{\phi}(\xi)\big|^2\, d\xi \\ &= \int_{\hat{L}} \sum_{\eta\in L^\perp} \big|\mathcal{T}(\xi\eta)\big|^2 \big|\hat{\phi}(\xi\eta)\big|^2\, d\xi \\ &= \int_{\hat{L}} \big|\mathcal{T}(\xi)\big|^2 \sum_{\eta\in L^\perp} \big|\hat{\phi}(\xi\eta)\big|^2\, d\xi.\end{aligned}$$

Let $F(\xi) = \sum_{\eta\in L^\perp} |\hat{\phi}(\xi\eta)|^2$. Then $F(\xi)$ is measurable, and $\sum_{\eta\in L^\perp} |\hat{\phi}(\xi\eta)|^2 \le \infty$. Set

$$M_r = \operatorname*{ess\,sup}_{\xi\in\hat{\mathrm{G}}} F(\xi), \qquad M_l = \operatorname*{ess\,inf}_{\xi\in\hat{\mathrm{G}}} F(\xi).$$

If $M_r < \infty$ and $M_l > 0$, then (1.2) follows. On the other hand, if $M_l = 0$, then, for some $\varepsilon > 0$, the measure of the set E_ε defined as $\{\xi \in \hat{G}, F(\xi) \le \varepsilon\}$ is positive. Set $\mu(E_\varepsilon) = \delta$, where μ is the Haar measure of $\hat{G}$. Let $\mathcal{T}(\xi)$ be defined by

$$\mathcal{T}(\xi) = \begin{cases} \frac{1}{\sqrt{\delta}} & \xi \in E_\varepsilon, \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathcal{T} \in L_2(\hat{L})$ and $\|\mathcal{T}\|_{L_2(\hat{L})} = 1$. Suppose that $\mathcal{T}(\xi) = \sum_{k\in L} a(k)\bar{\xi}(k)$ and $f(x) = \sum_{k\in L} a(k)T_k\phi(x)$. Then $\sum_{k\in L} |a(k)|^2 = 1$. On the other hand,

$$\begin{aligned}\|f\|^2 &= \|\hat{f}\|^2 \\ &= \int_{\hat{L}} \big|\mathcal{T}(\xi)\big|^2 \sum_{\eta\in L^\perp} \big|\hat{\phi}(\xi\eta)\big|^2\, d\xi \\ &= \int_{E_\varepsilon} \big|\mathcal{T}(\xi)\big|^2 F(\xi)\, d\xi \le \varepsilon,\end{aligned}$$

which implies that there is no constant $A > 0$ such that the left-hand side of (1.2) holds for all $\{a(k)\}_{k\in L} \in l_2(L)$. Similarly, if $M_r = \infty$, then there exists $M \in \mathbb{R}$, such that the measure of E_M defined as $\{\xi \in \hat{G}, |F(\xi)| > M\}$ is positive. Then, as before, one can show that there is no constant $B > 0$ such that the right-hand side of (1.2) holds for all $\{a(k)\}_{k\in L} \in l_2(L)$. □

We show that, by Theorem 2.1, the following example does not satisfy the Riesz family condition (1.2).

Example 2.2. Let $\phi(x) = \chi_{[0,2)}(x)$. Then $\hat{\phi}(\xi) = \frac{1-e^{-2i\xi}}{i\xi}$, and

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\xi + 2k\pi)\right|^2 &= \sum_{k\in\mathbb{Z}} \frac{\sin^2 \xi}{(\frac{\xi}{2} + k\pi)^2} \\ &= \sin^2 \xi \sum_{k\in\mathbb{Z}} \frac{1}{(\frac{\xi}{2} + k\pi)^2} \\ &= \frac{\sin^2 \xi}{\sin^2(\frac{\xi}{2})} = 4\cos^2\Big(\frac{\xi}{2}\Big).\end{aligned}$$

Hence $\sum_{k\in\mathbb{Z}} |\hat{\phi}(\cdot + 2k\pi)|^2 = 4\cos^2(\frac{\cdot}{2})$ is a continuous function vanishing at $\xi = \pi$. Therefore, $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is not a Riesz basis for its span.

3. Riesz family of a finite number of functions

In this section, we give a necessary and sufficient condition for shifts of a finite number of compactly supported functions in $L_2(G)$ to be a Riesz family. To express and prove our results, we require the following lemmas.

Lemma 3.1. *Suppose that $a_{ij} \in c_c(L)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, where L is a uniform lattice in G. Let the matrix $A(\xi) = (\tilde{a}_{ij}(\xi))_{m\times n}$ have rank n for every $\xi \in S_{L^\perp}$; then there exist two positive constants C_1 and C_2 such that*

$$C_1 \sum_{j=1}^{n} \|u_j\|_2 \le \sum_{i=1}^{m} \Big\| \sum_{j=1}^{n} a_{ij} * u_j \Big\|_2 \le C_2 \sum_{j=1}^{n} \|u_j\|_2, \tag{3.1}$$

for all $u_1, u_2, \ldots, u_n \in l_2(L)$.

Proof. The right-hand inequality in (3.1) is followed by (1.1). To prove the left-hand inequality in (3.1), set

$$\nu_i = \sum_{j=1}^{n} a_{ij} * u_j, \quad i = 1, \ldots, m. \tag{3.2}$$

For $i = 1, \ldots, m$ and $j = 1, \ldots, n$, let $b_{ji}(k)$ be the complex conjugate of $a_{ij}(k^{-1})$; that is, $b_{ji}(k) = \overline{a_{ij}(k^{-1})}$ for all $k \in L$. Then $\tilde{b}_{ji}(\xi) = \overline{\tilde{a}_{ij}(\xi)}$ for all $\xi \in S_{L^\perp}$. Set

$$B(\xi) = \big(\tilde{b}_{ji}(\xi)\big)_{n\times m} \qquad \text{and} \qquad G(\xi) = B(\xi)A(\xi), \xi \in S_{L^\perp}.$$

The $n\times n$ matrix $G(\xi)$ has n-rank, so it is nonsingular. Hence there exist functions $h_{ij}, i, j = 1, 2, \ldots, n$, such that $H(\xi)B(\xi)A(\xi) = I$ for all $\xi \in S_{L^\perp}$, where $H(\xi) =$

$(\tilde{h}_{ij}(\xi))_{n\times n}$ and I is the $n\times n$ identity matrix. As a result, if we set

$$c_{ri}=\sum_{t=1}^{n}h_{rt}*b_{ti},\quad r=1,\dots,n,i=1,\dots,m,$$

then

$$\sum_{i=1}^{m}c_{ri}*a_{ij}=\begin{cases}\delta & r=j,\\ 0 & r\neq j.\end{cases}$$

By (3.2) we have

$$\sum_{i=1}^{m}c_{ri}*\nu_i=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ri}*a_{ij}*u_j=u_r,\quad r=1,\dots,n.$$

Hence by (1.1), there exists $C>0$ such that

$$\begin{aligned}\sum_{r=1}^{n}\|u_r\|_2&\leq\sum_{r=1}^{n}\sum_{i=1}^{m}\|c_{ri}\|_1\|\nu_i\|_2\\&\leq C\sum_{i=1}^{m}\|\nu_i\|_2.\end{aligned}$$

This completes the proof. □

Lemma 3.2. *Let U be a compact subset of G, and let L be a uniform lattice in G. Then $U\cap kS_L\neq\emptyset$, only for a finite number $k\in L$.*

Proof. Without loss of generality, we can suppose that $1\in S_L$. As G is locally compact, and S_L has compact closure, there exists a symmetric neighborhood V of 1 with compact closure such that $S_L\subseteq V$, and so $\bigcup_{k\in L}kV$ is an open covering for U. Therefore, there are $k_1,\dots,k_n\in L$ such that $U\subseteq\bigcup_{j=1}^{n}k_jV$. Set $w=\bigcup_{j=1}^{n}k_jV$ and $W=w\cup w^{-1}$; then W is a symmetric neighborhood of 1 with compact closure. Note that W contains k_iS_L for all $i=1,\dots,n$. Fix $1\leq i_0\leq n$; therefore $\bar{W}\cap kk_{i_0}S_L\neq\emptyset$ for finitely many $k\in L$. Indeed, $\bar{W}\cap kk_{i_0}S_L\neq\emptyset$ if and only if $k\in\bar{W}\bar{W}$. Also $\bar{W}\bar{W}\cap L$ is compact and discrete. So it is finite in L. Thus $U\cap kk_{i_0}S_L\neq\emptyset$ for finitely many $k\in L$. □

Now we state and prove the main result of this section. We assume that G is a second countable and LCA group.

Theorem 3.3. *Let $\Phi=\{\phi_1,\dots,\phi_n\}$ be a finite number of compactly supported functions in $L_2(G)$. Then shifts of $\Phi=\{\phi_1,\dots,\phi_n\}$ constitute a Riesz family if and only if, for any $\xi\in\hat{G}$, the nets $(\hat{\phi}(\xi\eta))_{\eta\in L^{\perp}}$, $\phi\in\Phi$ are linearly independent.*

Proof. Let $S(\Phi)$ be the shift-invariant space generated by Φ, defined as

$$S(\Phi)=\operatorname{span}\{T_k\phi,\phi\in\Phi,k\in L\}.$$

Set $U_i=\operatorname{supp}(\phi_i),i=1,\dots,n$. Then $kU_i\cap S_L\neq\emptyset$ (or $U_i\cap kS_L\neq\emptyset$) for only finitely many $k\in L$ by Lemma 3.2. Therefore, $S(\Phi)|_{S_L}$ is finite-dimensional. Hence there exist functions $\psi_1,\dots,\psi_m\in L_2(G)$ with support in S_L such that

$\{\psi_j|_{S_L}\}_{j=1,\dots,m}$ forms a basis for $S(\Phi)|_{S_L}$. Every $\phi_i, i = 1,\dots,n$, can be represented as

$$\phi_i(x) = \sum_{j=1}^{m} \sum_{k\in L} a_{ji}(k)\psi_j(k^{-1}x), \tag{3.3}$$

where $a_{ji} \in c_c(L), j = 1,\dots,m, i = 1,\dots,n$. Consider $u_1,\dots,u_n \in l_2(L)$ and

$$f(x) = \sum_{i=1}^{n} \sum_{k\in L} \phi_i(k^{-1}x)u_i(k).$$

Using (3.3) we obtain

$$\begin{aligned} f(x) &= \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k\in L} \sum_{l\in L} a_{ji}(k)u_i(l)\psi_j(k^{-1}l^{-1}x) \\ &= \sum_{j=1}^{m} \sum_{\gamma\in L} \nu_j(\gamma)\psi_j(\gamma^{-1}x), \end{aligned}$$

where

$$\nu_j = \sum_{i=1}^{n} a_{ji} * u_i, \quad j = 1,\dots,m. \tag{3.4}$$

Thus $f(kx) = \sum_{j=1}^{m} \nu_j(k)\psi_j(x)$ for $x \in S_L$ and $k \in L$ (note that $G = \bigcup_{k\in L} kS_L$). Accordingly, there exist two positive constants C_1 and C_2 such that, for all $k \in L$,

$$C_1\Big(\sum_{j=1}^{m} \big|\nu_j(k)\big|^2\Big)^{\frac{1}{2}} \le \|f\|_{L_2(kS_L)} \le C_2\Big(\sum_{j=1}^{m} \big|\nu_j(k)\big|^2\Big)^{\frac{1}{2}}.$$

As $\|f\|_2^2 = \sum_{k\in L} \|f\|_{L_2(kS_L)}$, it follows that

$$C_1\Big(\sum_{j=1}^{m} \|\nu_j\|_2^2\Big)^{\frac{1}{2}} \le \|f\|_2 \le C_2\Big(\sum_{j=1}^{m} \|\nu_j\|_2^2\Big)^{\frac{1}{2}}. \tag{3.5}$$

This, together with (3.4), yields

$$\|f\|_2 \le C_2\Big(\sum_{j=1}^{m} \Big\|\sum_{i=1}^{n} a_{ji} * u_i\Big\|_2^2\Big)^{1/2} \le C_3 \sum_{i=1}^{n} \|u_i\|_2 \tag{3.6}$$

for all $u_1,\dots,u_n \in l_2(L)$, where $C_3 > 0$ is a constant independent of $u_1,\dots,u_n$. Suppose that the Fourier transforms of $\phi_1,\dots,\phi_n$ exist. Taking the Fourier transforms of both sides of (3.3), we get

$$\begin{aligned} \hat{\phi}_i(\xi) &= \sum_{j=1}^{m} \sum_{k\in L} a_{ji}(k)\bar{\xi}(k)\hat{\psi}_j(\xi) \\ &= \sum_{j=1}^{m} \tilde{a}_{ji}(\xi)\hat{\psi}_j(\xi), \quad \xi \in \hat{G}, i = 1,\dots,n. \end{aligned}$$

Thus, if the nets $(\hat{\phi}_i(\xi\eta))_{\eta\in L^\perp}, i = 1, \ldots, n$, are linearly independent for every $\xi \in \hat{G}$, then the matrix $A(\xi) := (\tilde{a}_{ji}(\xi))_{m\times n}$ has rank n for every $\xi \in S_{L^\perp}$. By Lemma 3.1, there exists a constant $C_4 > o$ such that

$$\sum_{i=1}^{n} \|u_i\|_2 \le C_4 \sum_{j=1}^{m} \|\nu_j\|_2.$$

This, along with (3.5), yields

$$\sum_{i=1}^{n} \|u_i\|_2 \le C_5 \|f\|_2 \tag{3.7}$$

for all $u_1, \ldots, u_n \in l_2(L)$, where $C_5 > 0$ is a constant independent of $u_1, \ldots, u_n$. By combining (3.6) and (3.7), the sufficient part of the theorem is proved. To prove the necessity part of the theorem, suppose that the nets $(\hat{\phi}_i(\xi\eta))_{\eta\in L^\perp}, i = 1, \ldots, n$, are linearly dependent for some $\xi \in \hat{G}$. Therefore, $\sum_{i=1}^n c_i\hat{\phi}_i(\xi\eta) = 0$ for all $\eta \in L^\perp$ for some complex numbers $c_1, \ldots, c_n$, which are not all zero. It follows that

$$\sum_{i=1}^{n} \sum_{k\in L} c_i \xi(k)\phi_i(k^{-1}x) = 0. \tag{3.8}$$

Indeed, if

$$f(x) := \sum_{i=1}^{n} \sum_{k\in L} c_i \bar{\xi}(k^{-1}x)\phi_i(k^{-1}x),$$

then f is L-periodic and well defined. Consider

$$\nu(\eta) = \int_{S_L} f(x)\bar{\eta}(x)\, dx, \quad \eta \in L^\perp.$$

We have

$$\begin{aligned}\nu(\eta) &= \int_{S_L} \sum_{i=1}^{n} \sum_{k\in L} c_i \bar{\xi}(k^{-1}x)\phi_i(k^{-1}x)\bar{\eta}(x)\, dx \\ &= \sum_{i=1}^{n} \int_G c_i \bar{\xi}(x)\phi_i(x)\bar{\eta}(x)\, dx \\ &= \sum_{i=1}^{n} c_i \hat{\phi}_i(\xi\eta).\end{aligned}$$

Thus (3.8) is true if and only if $\sum_{i=1}^n c_i\hat{\phi}_i(\xi\eta) = 0$ for all $\eta \in L^\perp$. Suppose that V is a symmetric compact neighborhood of 1 in G such that $\phi_1, \ldots, \phi_n$ are supported in V. Now, let U be a symmetric compact neighborhood of the identity 1 in G such that $V \subseteq U$ and U contains at least one nontrivial element of L. Set

$U_m = UU\cdots U$ (m factors). Then every U_m is a compact set that contains finitely many $k \in L$. For $m \in \mathbb{N}$, $i = 1, \dots, n$, set

$$a_{i,m}(k) = \begin{cases} c_i\xi(k) & \text{if } k \in U_m, \\ 0 & \text{if } k \in U_m^c, \end{cases}$$

where $a_{i,m} \in c_c(L)$. Let $f_m(x) = \sum_{i=1}^{n} \sum_{k\in L} a_{i,m}(k)\phi_i(k^{-1}x)$. Then (3.8) implies that

$$\begin{aligned} f_m(x) &= \sum_{i=1}^{n} \sum_{k\in U_m} c_i\xi(k)\phi_i(k^{-1}x) \\ &= -\sum_{i=1}^{n} \sum_{k\in U_m^c} c_i\xi(k)\phi_i(k^{-1}x). \end{aligned}$$

Therefore $f_m(x) = 0$ for $x \in (U_m V^c) \cup (U_m^c V^c)$. In other words, the function f_m is supported in $E := (U_m V^c)^c \cap (U_m^c V^c)^c$. Since U_m contains the identity element of G and $V \subseteq U$, we have $E \subseteq U$. Clearly, $\phi_i(k^{-1}x) \neq 0$ only if $k \in xV$. Now, if $x \in E$ and $\phi_i(k^{-1}x) \neq 0$, then $k \in F := L \cap EV \subseteq U_2$. In other words, $x \in E$ and $k \notin F$ imply that $f_m(x) = 0$. As $\mathrm{supp}(f_m) \subseteq E$, we have

$$\|f_m\|_{L_2(G)} \le \sum_{i=1}^{n} |c_i| \|\phi_i\|_{L_2(G)} \sum_{k\in U_2} 1$$

and

$$\sum_{i=1}^{n} \|a_{i,m}\|_{l_2(L)} = \sum_{i=1}^{n} \Big(\sum_{k\in U_m} \big|c_i\xi(k)\big|^2 \Big)^{\frac{1}{2}} \ge \sum_{i=1}^{n} |c_i| . m^{\frac{1}{2}}.$$

Consequently,

$$\lim_{m\to\infty} \frac{\sum_{i=1}^{n} \|a_{i,m}\|_{l_2(L)}}{\|f_m\|_{L_2(G)}} \ge \frac{\sum_{i=1}^{n} |c_i| \cdot m^{\frac{1}{2}}}{\sum_{i=1}^{n} |c_i| \|\phi_i\|_{L_2(G)} \sum_{k\in U_2} 1} = \infty.$$

This shows that the set of shifts of $\phi_1, \dots, \phi_n$ is not a Riesz family. The proof of the theorem is complete. □

As an example of Theorem 3.3, we give the following example.

Example 3.4. Let $\{V_i\}_{i\in I}$ be a multiresolution of $L_2(\mathbb{R})$ that is a nested sequence as follows:

$$\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$$

such that

(i) $\bigcap_{i\in\mathbb{Z}} V_i = \{0\}$;
(ii) $\overline{\bigcup_{i\in\mathbb{Z}} V_i} = L_2(\mathbb{R})$;
(iii) $f(\cdot) \in V_i$ if and only if $f(2\cdot) \in V_{i+1}$; and
(iv) there exists a function $\phi \in V_0$ such that $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of V_0.

As $\phi \in V_0$ is also in V_1, we can expand ϕ into a linear combination of the basis of V_1; that is,

$$\phi(x/2) = \sum_{k\in\mathbb{Z}} a_k \phi(x-k), \quad (a_k)_{k\in\mathbb{Z}} \in l_2(\mathbb{Z}). \tag{3.9}$$

Taking the Fourier transform of (3.9), we obtain

$$\hat{\phi}(\xi) = m_\phi(\xi/2)\hat{\phi}(\xi/2), \tag{3.10}$$

where $m_\phi(\xi) = \sum_{k\in\mathbb{Z}} \frac{a_k}{2} e^{-2\pi i k\xi}$. Denoting by W_i the orthogonal complement of V_i in V_{i+1}, we have the orthogonal decomposition $W_i \oplus V_i = V_{i+1}$. Let $\psi \in W_0$. Then there exists a sequence $(b_k)_{k\in\mathbb{Z}} \in l_2(\mathbb{Z})$ such that

$$\psi = \sum_{k\in\mathbb{Z}} b_k (D_{2^{-1}} T_k \phi), \tag{3.11}$$

which induces a Riesz basis $\{\psi(2^i \cdot -k), k \in \mathbb{Z}\}$ of W_i (see [6]). Moreover, $\{\phi(2^i \cdot -k), \psi(2^i \cdot -k), k \in \mathbb{Z}\}$ forms a Riesz basis of V_{i+1}. By taking the Fourier transform of (3.11), we have

$$\hat{\psi}(\xi) = m_\psi(\xi/2)\hat{\phi}(\xi/2), \tag{3.12}$$

where $m_\psi(\xi) = \sum_{k\in\mathbb{Z}} \frac{b_k}{2} e^{-2\pi i k\xi}$. Therefore, by (3.10) we have $\hat{\psi}(\xi) = \frac{1}{m_\phi(\xi/2)} m_\psi(\xi/2)\hat{\phi}(\xi)$. For more details on multiresolution, we refer to [6].

It is clear that $\frac{1}{m_\phi((\xi+2k\pi)/2)} m_\psi((\xi+2k\pi)/2)$ is not constant for every $k \in \mathbb{Z}$. So, for every $\xi \in \mathbb{R}$, $\{(\hat{\phi}(\xi + 2\pi k))_{k\in\mathbb{Z}}, (\hat{\psi}(\xi + 2\pi k))_{k\in\mathbb{Z}}\}$ is linearly independent.

Using Theorem 3.3 we obtain the following example, which confirms [10, Example 4.1].

Example 3.5. Let the scaling function $\phi := (\phi_1, \phi_2)$ be given as follows (see Figure 1):

$$\begin{cases} \phi_1 := (1 - 3x^2 - 2x^3)\chi_{[-1,0]} + (1 - 3x^2 + 2x^3)\chi_{[0,1]}, \\ \phi_2 := (x + 2x^2 + x^3)\chi_{[-1,0]} + (x - 2x^2 + x^3)\chi_{[0,1]}. \end{cases}$$

Then $\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$, where the mask a is given by

$$a(-1) = \begin{bmatrix} 1/4 & 3/8 \\ -1/16 & -1/16 \end{bmatrix}, \qquad a(0) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}, \qquad a(1) = \begin{bmatrix} 1/4 & -3/8 \\ 1/16 & -1/16 \end{bmatrix},$$

with $a(k) = 0$ for all $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$. Consider a sequence b supported on $\{-1, 0, 1\}$ as

$$b(-1) = \frac{1}{4}\begin{bmatrix} -2 & -15 \\ \frac{125}{512} & \frac{1875}{512} - 2 \end{bmatrix}, \qquad b(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad b(1) = \frac{1}{4}\begin{bmatrix} -2 & 15 \\ \frac{-125}{512} & \frac{1875}{512} - 2 \end{bmatrix},$$

and $b(k) = 0$ for all $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$. Define

$$\begin{aligned} \psi(x) &= 2 \sum_{k\in\mathbb{Z}} b(k)\phi(2x - k) \\ &= 2\big(b(-1)\phi(2x + 1) + b(0)\phi(2x) + b(1)\phi(2x - 1)\big). \end{aligned}$$

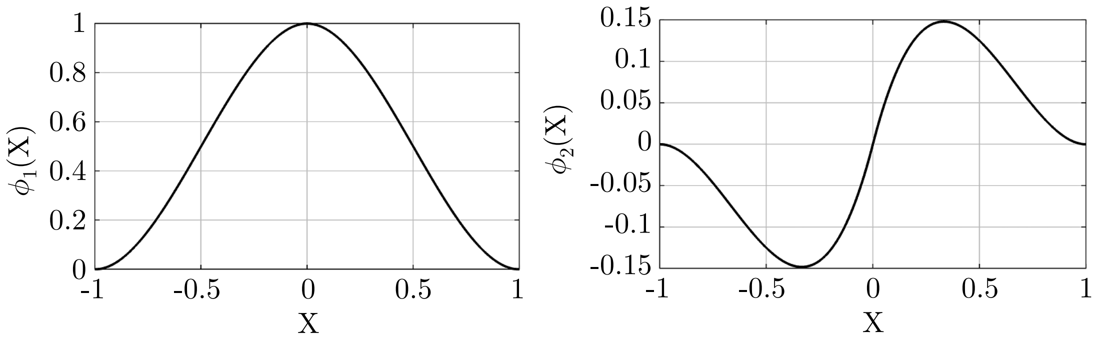

FIGURE 1. $\phi_1(x), \phi_2(x)$.

Then, by Theorem 3.3, $\psi = (\psi_1, \psi_2)$ generates a Riesz wavelet basis for $L_2(\mathbb{R})$, since the set $\{(\hat{\psi}_1(\xi + 2k\pi))_{k\in\mathbb{Z}}, (\hat{\psi}_2(\xi + 2k\pi))_{k\in\mathbb{Z}}\}$ for all $\xi \in \mathbb{R}$ is linearly independent.

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