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RECENT DEVELOPMENTS OF MATRIX VERSIONS OF THE ARITHMETIC–GEOMETRIC MEAN INEQUALITY

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Dedicated to Professor Anthony To-Ming Lau

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Abstract. The main aim of this survey article is to present recent developments of matrix versions of the arithmetic–geometric mean inequality. Among others, we show improvements and generalizations of the arithmetic–geometric mean inequality for unitarily invariant norms via the Hadamard product, and for singular values via the operator monotone functions.

1. Introduction

A capital letter means an $n \times n$ matrix in the matrix algebra $\mathbb{M}_n$. For two Hermitian matrices A, B, the order relation $A \geq B$ means, by definition, that $A - B$ is positive semidefinite. Incidentally, $A \geq 0$ means that A is positive semidefinite. Let us denote $A > 0$ if A is positive definite, that is, A is positive semidefinite and invertible.

Matrix inequalities in this paper are of two kinds. One is based on the singular values. For a matrix $A \in \mathbb{M}_n$, the eigenvalues of $|A| := (A^*A)^{1/2}$ are called the *singular values of* A and are denoted by $s(A) := (s_1(A), s_2(A), \ldots, s_n(A))$, which is arranged in decreasing order. For two matrices $A, B \in \mathbb{M}_n$, the order relation $s(A) \geq s(B)$ means that $s_j(A) \geq s_j(B)$ for $j = 1, \ldots, n$. Notice the known fact that

$$|A| \geq |B| \quad \Longrightarrow \quad s(A) \geq s(B)$$

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and that the converse is true up to a unitarily equivalence; that is,

$$s(A) \geq s(B) \quad \Longrightarrow \quad U^*|A|U \geq |B| \quad \text{for some unitary } U.$$

Similarly, we arrange the eigenvalues of a Hermitian matrix A in decreasing order and denote it by $\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$.

The other kind are unitarily invariant norm inequalities; that is, a norm $|||\cdot|||$ is unitarily invariant if $|||UAV||| = |||A|||$ for all unitary U, V and all $A \in \mathbb{M}_n$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be given vectors that are arranged in decreasing order. We say that x *weakly majorizes* y, denoted by $x \succ_w y$, if $\sum_{j=1}^k x_j \geq \sum_{j=1}^k y_j$ for all $1 \leq k \leq n$. Then, by Fan's dominance theorem, we have

$$s(A) \succ_w s(B) \qquad \Longrightarrow \qquad |||A||| \geq |||B|||$$

for all unitarily invariant norms. Hence it follows that the implication

$$|A| \geq |B| \Longrightarrow s(A) \geq s(B) \Longrightarrow |||A||| \geq |||B|||$$

holds for all $A, B \in \mathbb{M}_n$.

The main aim of this survey article is to present recent developments of the arithmetic–geometric mean inequality based on the singular values and unitarily invariant norms.

For positive real numbers a and b, the arithmetic–geometric mean inequality says that

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

It is also extended to complex numbers:

$$|\overline{a}b| \leq \frac{|a|^2 + |b|^2}{2}. \tag{1.1}$$

We would expect the following matrix inequality of (1.1):

$$|A^*B| \leq \frac{AA^* + BB^*}{2} \quad \text{for } A, B \in \mathbb{M}_n. \tag{1.2}$$

Unfortunately, we have a counterexample of (1.2). For example, put

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In [5], Bhatia and Kittaneh showed the arithmetic–geometric mean inequality for the singular values corresponding to (1.1):

$$s(A^*B) \leq s\Big(\frac{AA^* + BB^*}{2}\Big)$$

holds for all matrices $A, B \in \mathbb{M}_n$. Moreover, Bhatia and Davis [4] showed the arithmetic–geometric mean inequality for matrices: for arbitrary matrices $A, B, X \in \mathbb{M}_n$,

$$2|||A^*XB||| \leq |||AA^*X + XBB^*||| \tag{1.3}$$

for every unitarily invariant norm. Mathias [13] gave a simple proof of (1.3) using the Hadamard product.

On the other hand, for two positive real numbers a and b, the Heinz mean in the parameter $r \in [0,1]$ is defined as

$$H_r(a,b) := \frac{a^r b^{1-r} + a^{1-r} b^r}{2}.$$

The path of a Heinz mean interpolates between the geometric mean and the arithmetic mean:

$$\sqrt{ab} \le H_r(a,b) \le \frac{a+b}{2} \quad \text{for } r \in [0,1].$$

Bhatia and Davis [4, Theorem 2] extended this to the Heinz inequality for matrices.

Theorem BD. *For positive semidefinite A, B and arbitrary $X \in \mathbb{M}_n$,*

$$2|||A^{1/2}XB^{1/2}||| \le |||A^r XB^{1-r} + A^{1-r}XB^r||| \le |||AX + XB||| \tag{1.4}$$

for all $r \in [0,1]$.

The following is known as the Hermite–Hadamard integral inequality for convex functions: for a real-valued convex function f on $[a,b]$,

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{1.5}$$

In [11, Theorem 1], Kittaneh showed a refinement of the first inequality in (1.4) by applying (1.5) to the convex function $f(t) = |||A^t XB^{1-t} + A^{1-t}XB^t|||$: for positive semidefinite A, B, arbitrary $X \in \mathbb{M}_n$, and $r \in [0,1]$,

$$\begin{aligned} 2|||A^{1/2}XB^{1/2}||| &\le \frac{1}{|1-2r|}\Big|\int_r^{1-r} |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||\,d\nu\Big| \\ &\le |||A^r XB^{1-r} + A^{1-r}XB^r||| \end{aligned} \tag{1.6}$$

for every unitarily invariant norm. By virtue of the Hadamard product, Kaur, Moslehian, Singh, and Conde [10, Theorem 4.1] showed a further refinement of (1.6): let $A, B, X \in \mathbb{M}_n$ with A, B positive semidefinite. Then, for any real numbers α, β,

$$\begin{aligned} &|||A^{\frac{\alpha+\beta}{2}}XB^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}}XB^{\frac{\alpha+\beta}{2}}||| \\ &\quad \le \frac{1}{|\alpha-\beta|}\Big|\Big|\Big|\int_\alpha^\beta (A^v XB^{1-v} + A^{1-v}XB^v)\,dv\Big|\Big|\Big| \\ &\quad \le \frac{1}{2}|||A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha + A^\beta XB^{1-\beta} + A^{1-\beta}XB^\beta|||. \end{aligned} \tag{1.7}$$

In fact, if $\alpha+\beta = 1$ in (1.7), then we have a refinement of (1.6).

Next, in [11, Theorem 4], Kittaneh showed another refinement of the second inequality in (1.4): for positive semidefinite A, B, arbitrary X, and $r \in [0,1]$,

$$|||A^r XB^{1-r} + A^{1-r}XB^r||| \le 4r_0|||A^{1/2}XB^{1/2}||| + (1-2r_0)|||AX+XB||| \tag{1.8}$$

for every unitarily invariant norm, where $r_0 = \min\{r, 1-r\}$. Also, Kaur et al. [10, Theorem 4.4] showed a further refinement of (1.8): let $A, B, X \in \mathbb{M}_n$ with A, B positive semidefinite and $X \in \mathbb{M}_n$. Then, for $r \in [0,1]$,

$$|||A^r X B^{1-r} + A^{1-r} X B^r||| \le \left|\left|\left|4r_0 A^{\frac{1}{2}} X B^{\frac{1}{2}} + (1-2r_0)(AX + XB)\right|\right|\right|$$

holds for every unitarily invariant norm, where $r_0 = \min\{r, |\frac{1}{2} - r|, 1-r\}$.

Now, the arithmetic–geometric mean inequality (1.4) for matrices due to Bhatia and Davis leads to

$$2|||AXB||| \le |||A^r X B^{2-r} + A^{2-r} X B^r||| \le |||A^2 X + XB^2||| \quad \text{for } 0 \le r \le 2. \tag{1.9}$$

The right-hand side of (1.9) was generalized by Zhan [18, Theorem 6] as follows.

Theorem ZH. *Let $A, B, X \in \mathbb{M}_n$ with A, B positive semidefinite. Then*

$$|||A^r X B^{2-r} + A^{2-r} X B^r||| \le \frac{2}{t+2} |||A^2 X + tAXB + XB^2||| \tag{1.10}$$

for any real numbers r, t satisfying $1 \le 2r \le 3$ and $-2 < t \le 2$.

In [15], Singh and Vasudeva showed an operator monotone function version of Theorem ZH: if $A, B, X \in M_n$ with A, B positive definite, and f is any operator monotone function on $(0, \infty)$, then, for $-2 < t \le 2$,

$$\begin{aligned} &\left|\left|\left|A^{\frac{1}{2}} f(A) X f(B)^{-1} B^{\frac{3}{2}} + A^{\frac{3}{2}} f(A)^{-1} X f(B) B^{\frac{1}{2}}\right|\right|\right| \\ &\quad \le \frac{2}{t+2} |||A^2 X + tAXB + XB^2|||. \end{aligned} \tag{1.11}$$

In fact, if $f(x) = x^{r-1/2}$, then f is operator monotone for $1 \le 2r \le 3$, and (1.11) implies Theorem ZH. Afterward, Wang, Zou, and Jiang [17] improved Theorem ZH as follows:

$$\begin{aligned} &|||A^r X B^{2-r} + A^{2-r} X B^r||| \\ &\quad \le 2(2r_0 - 1)|||AXB||| + \frac{4 - 4r_0}{t+2} |||A^2 X + tAXB + XB^2||| \end{aligned} \tag{1.12}$$

for $1 \le 2r \le 3$ and $-2 < t \le 2$, where $r_0 = \min\{r, 2-r\}$.

Following [12], a continuous real-valued function f defined on an interval (a, b) with $a \ge 0$ is called a *Kwong function* if the matrix

$$K_f = \left(\frac{f(\lambda_i) + f(\lambda_j)}{\lambda_i + \lambda_j}\right)_{i,j=1,2,\dots,n}$$

is positive semidefinite for any (distinct) $\lambda_1, \dots, \lambda_n$ in (a, b). By the use of the Kwong function, Najafi [14] proposed a more general norm inequality of the Heinz inequality (1.4) as follows:

$$\left|\left|\left|f(A) X g(B) + g(A) X f(B)\right|\right|\right| \le |||AX + XB||| \tag{1.13}$$

for any continuous functions f and g with $\frac{f(x)}{g(x)}$ Kwong and $f(x)g(x) \le x$.

For a comprehensive inspection of the results concerning the above norm inequalities, we refer the reader to [9], [11], [12], and [15].

We return to the arithmetic–geometric mean inequality for the singular values: for positive semidefinite $A, B \in \mathbb{M}_n$,

$$2s(A^{1/2}B^{1/2}) \le s(A+B). \tag{1.14}$$

In [16, Theorem 3], Tao showed that

$$s(A^{\frac{1}{4}}B^{\frac{3}{4}} + A^{\frac{3}{4}}B^{\frac{1}{4}}) \le s(A+B)$$

for any positive semidefinite $A, B \in \mathbb{M}_n$.

In [3, Theorem 2], Audenaert showed a singular value inequality for Heinz means (1.4), which is the affirmative answer to Zhan's conjecture (see [19, Conjecture 4]).

Theorem AZ. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite. Then, for $0 \le r \le 1$,*

$$s(A^rB^{1-r} + A^{1-r}B^r) \le s(A+B). \tag{1.15}$$

However, the singular value version of the left-hand sides of (1.4) is false. For example, put

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{pmatrix}$$

and $2s_2(A^{1/2}B^{1/2}) > s_2(A^{0.1}B^{0.9} + A^{0.9}B^{0.1})$ for $r = 0.1$ (see [3, Remark]).

Afterward, Bhatia and Kittaneh [6] proved that, for positive semidefinite $A, B \in \mathbb{M}_n$,

$$s(A^{\frac{1}{2}}B^{\frac{3}{2}} + A^{\frac{3}{2}}B^{\frac{1}{2}}) \le \frac{1}{2}s\big((A+B)^2\big). \tag{1.16}$$

Moreover, Zhan posed the following conjecture (see [19, Conjecture 3]), that if $A, B \in \mathbb{M}_n$ are positive semidefinite, then, for each $1 \le 2r \le 3$ and $-2 < t \le 2$,

$$s(A^rB^{2-r} + A^{2-r}B^r) \le \frac{2}{t+2}s(A^2 + tAB + B^2). \tag{1.17}$$

The inequality (1.17) was proved to hold for $r = \frac{1}{2}, 1, \frac{3}{2}$, and all $-2 < t \le 2$ by Dumitru, Levanger, and Visinescu [7]. Furthermore, it was shown that the function $f(t) = \frac{2}{t+2}\lambda_j(A^2 + B^2 + \frac{t}{2}AB + \frac{t}{2}BA)$ is nonincreasing on $(-2, \infty)$.

The purpose of this survey article is to show some improvements and generalizations of the unitarily invariant norm inequalities (1.9)–(1.13) via Hadamard products, as well as the general singular value inequality for Theorem AZ, and refine the Heinz mean inequality for the singular values.

2. Refinement of the Heinz mean inequality

First of all, we recall the norm of the Hadamard product multiplication. We define the Hadamard product of $A = (a_{ij})$ and $B = (b_{ij}) \in \mathbb{M}_n$ to be $A \circ B = (a_{ij}b_{ij}) \in \mathbb{M}_n$. Given $A \in \mathbb{M}_n$, we define the linear map $S_A : \mathbb{M}_n \mapsto \mathbb{M}_n$ by $S_A(B) = A \circ B$. Let us denote the spectral norm on $\mathbb{M}_n$ by $\|\cdot\|$, and define $\|S_A\|$

the induced norm of S_A to be $\|S_A\| := \max\{\|A \circ B\| : \|B\| \leq 1\}$. Then, for any unitarily invariant norm $|||\cdot|||$,

$$|||A \circ B||| \leq \|S_A\| |||B||| \quad \text{for all } A, B \in \mathbb{M}_n.$$

It follows from [13, Lemma 1.2] that

$$\|S_A\| = \max\Big\{\sum_{i=1}^{n} s_i(A \circ B) : \sum_{i=1}^{n} s_i(B) \leq 1\Big\}$$

for all $A \in \mathbb{M}_n$. It is not easy to compute $\|S_A\|$ for a general matrix A, but, in the special case that A is positive semidefinite, it is known that

$$\|S_A\| = \max\{a_{ii} : i = 1, \ldots, n\}.$$

This fact has been observed in [1, p. 363]; it also follows very easily from the second part of Theorem 5.5.19 in [9]. So we pick it as a key lemma.

Lemma 2.1. *If $X = (x_{ij})$ is positive semidefinite, then, for any matrix Y,*

$$|||X \circ Y||| \leq \max_{1 \leq i \leq n} x_{ii} |||Y|||.$$

We next cite the following lemma due to Zhan [18].

Lemma 2.2 ([18, Lemma 5])**.** *Let $\sigma_1, \sigma_2, \ldots, \sigma_n > 0$, and let $r \in [-1, 1]$, $-2 < t \leq 2$. Then the $n \times n$ matrix*

$$\Big(\frac{\sigma_i^r + \sigma_j^r}{t\sigma_i\sigma_j + \sigma_i^2 + \sigma_j^2}\Big)$$

is positive semidefinite.

By using Lemmas 2.1 and 2.2, we have the following result related to (1.12) due to Wang, Zou, and Jiang [17], which removes the restrictions of t in Zhan's inequality (1.10).

Theorem 2.3 ([8, Theorem 2.1])**.** *Let $A, B, X \in M_n$ with A, B positive semidefinite, and let $1 \leq 2r \leq 3$. Then, for $\beta > 0$,*

$$\begin{aligned} &|||A^r X B^{2-r} + A^{2-r} X B^r||| \\ &\quad \leq \Big|\Big|\Big| 2(1 - 2\beta + 2\beta r_0) AXB + \frac{4\beta(1 - r_0)}{t + 2}(A^2 X + tAXB + XB^2) \Big|\Big|\Big| \end{aligned}$$

holds for $-2 < t \leq 2\beta - 2$, where $r_0 = \min\{\frac{1}{2} + |1 - r|, 1 - |1 - r|\}$.

By using (1.7), we have the following refinement of the Heinz inequality (1.9).

Theorem 2.4. *Let $A, B, X \in M_n$ such that A and B are positive semidefinite, and let $r \in [\frac{1}{2}, \frac{3}{2}]$. Then*

$$\begin{aligned} 2|||AXB||| &\leq \frac{1}{|2 - 2r|} \Big|\Big|\Big| \int_r^{2-r} (A^\xi X B^{2-\xi} + A^{2-\xi} X B^\xi)\, d\xi \Big|\Big|\Big| \\ &\leq \frac{1}{2} \Big|\Big|\Big| 2AXB + (A^{2-r} X B^r + A^r X B^{2-r}) \Big|\Big|\Big| \leq |||A^{2-r} X B^r + A^r X B^{2-r}|||. \end{aligned}$$

3. Generalized Zhan's inequality

In this section, we show some generalized versions of Singh and Vasudeva's results (1.11). The following lemma is a Kwong function version of Lemma 2.2.

Lemma 3.1. *Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be any positive real numbers, and let $-2 < t \leq 2$. If f and g are two continuous functions on $(0, \infty)$ such that $\frac{f(x)}{g(x)}$ is Kwong, then the $n \times n$ matrix*

$$W = \Big(\frac{f(\sigma_i)g^{-1}(\sigma_i) + f(\sigma_j)g^{-1}(\sigma_j)}{\sigma_i^2 + t\sigma_i\sigma_j + \sigma_j^2} \Big)_{i,j=1,2,\ldots,n}$$

is positive semidefinite.

Theorem 3.2. *Let $A, B, X \in M_n$ such that A, B are positive definite matrices. If f and g are two continuous functions on $(0, \infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is Kwong, then*

$$(t+2)\Big|\Big|\Big| A^{\frac{1}{2}} \big[f(A)Xg(B) + g(A)Xf(B) \big] B^{\frac{1}{2}} \Big|\Big|\Big| \leq 2k |||A^2X + tAXB + XB^2|||$$

holds for any $-2 < t \leq 2$, where

$$k = k(f, g; A, B) = \max \Big\{ \frac{f(\lambda)g(\lambda)}{\lambda} \;\Big|\; \lambda \in \sigma(A) \cup \sigma(B) \Big\}.$$

Remark 3.3. Theorem 3.2 is a stronger version of Singh and Vasudeva's inequality (1.11), which will be refined more in the next section.

We have shown the generalized Zhan's inequality as Theorem 3.2 and the improved Wang's inequality as Theorem 2.3 when $t \in (-2, 2\beta - 2]$ for $\beta > 0$. Combining Theorems 2.3 and 3.2, we have the following theorem.

Theorem 3.4. *Let $A, B, X \in M_n$ such that A, B are positive definite, and let f, g be two continuous functions on $(0, \infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is Kwong. Then, for $\beta > 0$,*

$$\begin{aligned} &\Big|\Big|\Big| A^{\frac{1}{2}} \big(f(A)Xg(B) + g(A)Xf(B) \big) B^{\frac{1}{2}} \Big|\Big|\Big| \\ &\quad \leq k \Big|\Big|\Big| 2(1 - 2\beta + 2\beta r_0)AXB + \frac{4\beta(1 - r_0)}{t+2}(A^2X + tAXB + XB^2) \Big|\Big|\Big| \end{aligned}$$

holds for $1 \leq 2r \leq 3$, $-2 < t \leq 2\beta - 2$, where $r_0 = \min\{\frac{1}{2} + |1 - r|, 1 - |1 - r|\}$, $k = k(f, g; A, B)$.

Take $\beta = 1$ in Theorem 3.4; then the following corollary is immediately obtained.

Corollary 3.5. *Let $A, B, X \in M_n$ with A, B positive definite, and let f, g be two continuous functions on $(0, \infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is Kwong. Then*

$$\begin{aligned} &\Big|\Big|\Big| A^{\frac{1}{2}} \big(f(A)Xg(B) + g(A)Xf(B) \big) B^{\frac{1}{2}} \Big|\Big|\Big| \\ &\quad \leq k \Big|\Big|\Big| 2(2r_0 - 1)AXB + \frac{4 - 4r_0}{t+2}(A^2X + tAXB + XB^2) \Big|\Big|\Big| \end{aligned}$$

holds for $1 \leq 2r \leq 3$, $-2 < t \leq 0$, where $r_0 = \min\{\frac{1}{2} + |1 - r|, 1 - |1 - r|\}$, $k = k(f, g; A, B)$.

Let f be a positive operator monotone function on $(0,\infty)$, and let $g(x) := x/f(x)$. Then $f^2(x)/x$ is a Kwong function, so that f and g satisfy the conditions of Theorem 3.4, and hence we have the following corollary.

Corollary 3.6. *Let $A, B, X \in M_n$ such that A, B are positive definite. If f is an operator monotone function on $(0,\infty)$ such that $f(t) > 0$, then, for $\beta > 0$,*

$$\begin{aligned}&\left|\left|\left| A^{\frac{1}{2}} f(A) X f(B)^{-1} B^{\frac{3}{2}} + A^{\frac{3}{2}} f(A)^{-1} X f(B) B^{\frac{1}{2}} \right|\right|\right| \\ &\quad \le \left|\left|\left| 2(1-2\beta+2\beta r_0) AXB + \frac{4\beta(1-r_0)}{t+2}(A^2X + tAXB + XB^2) \right|\right|\right|\end{aligned}$$

holds for each $1 \le 2r \le 3$ and $-2 < t \le 2\beta - 2$, where $r_0 = \min\{\frac{1}{2} + |1-r|, 1 - |1-r|\}$.

Take $f(x) = \log(1+x)$ on $(0,\infty)$ and $g(x) = 1$ in Theorem 3.4. Then $f(x)g(x)^{-1} = f(x)$ is operator monotone, and hence, by Kwong [12], we have the following corollary.

Corollary 3.7. *Let $A, B, X \in M_n$ with A, B positive semidefinite. Then, for $\beta > 0$,*

$$\begin{aligned}&\left|\left|\left| A^{\frac{1}{2}} \big(\log(I+A)X + X\log(I+B)\big) B^{\frac{1}{2}} \right|\right|\right| \\ &\quad \le k \left|\left|\left| 2(1-2\beta+2\beta r_0) AXB + \frac{4\beta(1-r_0)}{t+2}(A^2X + tAXB + XB^2) \right|\right|\right|\end{aligned}$$

holds for $1 \le 2r \le 3$, $-2 < t \le 2\beta - 2$, where $r_0 = \min\{\frac{1}{2} + |1-r|, 1-|1-r|\}$, $k = \max\{\frac{\log(1+\lambda)}{\lambda} \mid \lambda \in \sigma(A) \cup \sigma(B)\}$.

Remark 3.8. In the infinite-dimensional case, the unitarily invariant norm $|||\cdot|||$ is characterized by the invariance property $|||UTV||| = |||T|||$ for all compact operators T in the norm ideal associated with $|||\cdot|||$ and unitary operators U, V on a complex Hilbert space. The Heinz mean inequality for Hilbert space operators has been shown in [10].

4. Refined Zhan's inequality

In this section, we show a refined version of Zhan's inequality (1.10). First of all, we study the function at the right-hand side of the inequality (1.10).

Theorem 4.1. *Let $A, B, X \in \mathbb{M}_n$ with A, B positive semidefinite. Suppose that*

$$\Psi(t) = \frac{2}{t+2} |||A^2X + tAXB + XB^2|||, \quad t \in (-2, 2].$$

Then $\Psi(t)$ is monotone decreasing on $(-2, 2]$. In particular,

$$\begin{aligned}|||A^rXB^{2-r} + A^{2-r}XB^r||| &\le \frac{1}{2}|||A^2X + 2AXB + XB^2||| \\ &\le \frac{2}{t+2}|||A^2X + tAXB + XB^2|||\end{aligned} \tag{4.1}$$

holds for $1 \le 2r \le 3$ and $t \in (-2, 2]$.

Proof. It suffices to prove the monotonicity of $\Psi(t)$. For any $-2 < s < t \le 2$, there exists $\alpha \in (0,1)$ such that $\frac{2}{t+2} = \frac{2\alpha}{s+2}$. Then we have

$$\begin{aligned}
\Psi(t) &= \frac{2}{t+2}|||A^2X + tAXB + XB^2||| \\
&= \left|\left|\left|\frac{2}{t+2}(A^2X + XB^2) + \Big(1 - \frac{2}{t+2}\Big)2AXB\right|\right|\right| \\
&= \left|\left|\left|\frac{2\alpha}{s+2}(A^2X + XB^2) + \Big(1 - \frac{2\alpha}{s+2}\Big)2AXB\right|\right|\right| \\
&= \left|\left|\left|\frac{2\alpha}{s+2}(A^2X + sAXB + XB^2) + \Big(1 - \frac{2\alpha}{s+2} - \frac{\alpha s}{s+2}\Big)2AXB\right|\right|\right| \\
&= \left|\left|\left|\frac{2\alpha}{s+2}(A^2X + sAXB + XB^2) + (1-\alpha)2AXB\right|\right|\right| \\
&\le \frac{2\alpha}{s+2}|||A^2X + sAXB + XB^2||| + (1-\alpha)2|||AXB||| \\
&\le \frac{2}{s+2}|||A^2X + sAXB + XB^2||| = \Psi(s),
\end{aligned}$$

where the last inequality follows from (1.9) and Theorem ZH. Therefore, $\Psi(t)$ is monotone decreasing on $(-2, 2]$.

Besides, Zhan's inequality and the monotonicity of $\Psi(t)$ yield the inequality (4.1). □

By Theorem 4.1, we have a refinement of Theorem 3.2.

Theorem 4.2. *Suppose that $A, B, X \in \mathbb{M}_n$ such that A, B are positive definite, and f, g are two continuous functions on $(0,\infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is Kwong. Then*

$$\left|\left|\left|A^{\frac{1}{2}}\big(f(A)Xg(B) + g(A)Xf(B)\big)B^{\frac{1}{2}}\right|\right|\right| \le \frac{k(f,g;A,B)}{2}|||A^2X + 2AXB + XB^2|||.$$

Remark 4.3. Theorem 4.2 can be seen as a refinement of (1.11). Let f be an operator monotone function on $(0,\infty)$ and $g(x) = x/f(x)$. Since f is operator monotone, we have $\sqrt{x}(f/g)(\sqrt{x}) = f^2(\sqrt{x})$ is operator monotone. Hence it follows from Audenaert [3, Theorem 2.1] that $f(x)/g(x) = f^2(x)/x$ is Kwong and $k(f,g;A,B) = 1$, and this implies that, for $-2 < t \le 2$,

$$\begin{aligned}
\left|\left|\left|A^{\frac{1}{2}}f(A)Xf(B)^{-1}B^{\frac{3}{2}} + A^{\frac{3}{2}}f(A)^{-1}Xf(B)B^{\frac{1}{2}}\right|\right|\right| &\le \frac{1}{2}|||A^2X + 2AXB + XB^2||| \\
&\le \frac{2}{t+2}|||A^2X + tAXB + XB^2|||
\end{aligned}$$

holds for $A, B, X \in \mathbb{M}_n$ with A, B positive definite.

Using the method in the proof of Theorem 4.2 we again get the following improvement of Najafi's result (1.13).

Theorem 4.4. *Suppose that $A, B, X \in \mathbb{M}_n$ such that A, B are positive definite. If f and g are two continuous functions on $(0,\infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is Kwong,* *then*

$$|||f(A)Xg(B)+g(A)Xf(B)||| \le \frac{k_1}{2}|||A^2X+2AXB+XB^2|||$$

holds for $k_1 = \max\{\frac{f(\lambda)g(\lambda)}{\lambda^2} \mid \lambda \in \sigma(A)\cup\sigma(B)\}$.

Example 4.5. Take $f(x)=\log(1+x)$ and $g(x)=x$ defined on $(0,\infty)$. Then $f(x)/g(x)$ is not operator monotone but Kwong. Theorem 4.2 leads to the following inequality:

$$\begin{aligned}&|||A^{\frac{1}{2}}\big(\log(I+A)XB+AX\log(I+B)\big)B^{\frac{1}{2}}|||\\ &\le \frac{\log(1+\lambda_0)}{2}|||A^2X+2AXB+XB^2|||\end{aligned}$$

for all matrices $A,B,X\in M_n$ with A, B positive semidefinite, in which $\lambda_0 = \max\{\lambda \mid \lambda\in\sigma(A)\cup\sigma(B)\}$.

Example 4.6. Let $s,r\in\mathbb{R}$. Since $F(x)=x^{s-r}$ defined on $(0,\infty)$ is Kwong if and only if $-1\le s-r\le 1$, it follows from Theorem 4.2 that, if $A,B,X\in M_n$ with A, B positive semidefinite and $|s-r|\le 1$, then

$$|||A^{\frac{1}{2}}(A^sXB^r+A^rXB^s)B^{\frac{1}{2}}||| \le \frac{k}{2}|||A^2X+2AXB+XB^2|||$$

for $k=\max\{\lambda^{s+r-1} \mid \lambda\in\sigma(A)\cup\sigma(B)\}$. In particular, if we put $r\mapsto r-\frac{1}{2}$ and $s\mapsto\frac{3}{2}-r$ in the above inequality for $1\le 2r\le 3$, then we have $|r-s|\le 1$ and $k=1$. Hence we have Zhan's inequality (1.10) by Theorem 4.1.

5. Refined integral Heinz mean inequality

In this section, we improve the integral Heinz mean inequality (1.7) by Kaur et al. [10]. By Lemmas 2.1 and 3.1, we have a stronger matrix version of (1.7).

Theorem 5.1. *Let $A,B,X\in\mathbb{M}_n$ with A, B positive semidefinite. Then, for any real positive numbers α and β,*

$$\begin{aligned}&\frac{1}{|\alpha-\beta|}\left|\left|\left|\int_\alpha^\beta (A^vXB^{1-v}+A^{1-v}XB^v)\,dv\right|\right|\right|\\ &\le \frac{1}{4}|||2A^{\frac{\alpha+\beta}{2}}XB^{1-\frac{\alpha+\beta}{2}}+2A^{1-\frac{\alpha+\beta}{2}}XB^{\frac{\alpha+\beta}{2}}+A^\alpha XB^{1-\alpha}+A^{1-\alpha}XB^\alpha\\ &\quad +A^\beta XB^{1-\beta}+A^{1-\beta}XB^\beta|||.\end{aligned}$$

As a corollary of Theorem 5.1, we have a refinement of the left-hand sides of the Heinz mean inequality (1.9).

Corollary 5.2. *Let $A,B,X\in\mathbb{M}_n$ with A, B positive semidefinite and $r\in[\frac{1}{2},\frac{3}{2}]$. Then*

$$\begin{aligned}2|||AXB||| &\le \frac{1}{|2-2r|}\left|\left|\left|\int_r^{2-r}(A^vXB^{2-v}+A^{2-v}XB^v)\,dv\right|\right|\right|\\ &\le \frac{1}{2}|||2AXB+A^rXB^{2-r}+A^{2-r}XB^r||| \le |||A^rXB^{2-r}+A^{2-r}XB^r|||.\end{aligned}$$

Further,

$$\lim_{r\to 1}\frac{1}{|2-2r|}\left|\!\left|\!\left|\int_r^{2-r}\left(A^vXB^{2-v}+A^{2-v}XB^v\right)dv\right|\!\right|\!\right| = 2|||AXB|||.$$

By Theorem 5.1, we have the following integral inequalities as an improvement of (1.4).

Corollary 5.3. *Let $A, B, X \in \mathbb{M}_n$ with A, B positive semidefinite and $r \in [0,1]$. Then*

$$\begin{aligned}
2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le \frac{1}{|1-2r|}\left|\!\left|\!\left|\int_r^{1-r}\left(A^vXB^{1-v}+A^{1-v}XB^v\right)dv\right|\!\right|\!\right| \\
&\le \frac{1}{2}|||2A^{\frac{1}{2}}XB^{\frac{1}{2}}+A^rXB^{1-r}+A^{1-r}XB^r||| \\
&\le \left|\!\left|\!\left|\alpha(A^rXB^{1-r}+A^{1-r}XB^r)+2(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\!\right|\!\right| \quad \left(\frac{1}{2}\le\alpha\le 1\right) \\
&\le |||A^rXB^{1-r}+A^{1-r}XB^r|||.
\end{aligned}$$

By the symmetry of the integral function and Theorem 5.1, we have the following corollary.

Corollary 5.4. *Let $A, B, X \in \mathbb{M}_n$ with A, B positive semidefinite and $r \in [0,1]$. Then*

$$\begin{aligned}
&2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \\
&\quad\le \frac{1}{|1-2r|}\left|\!\left|\!\left|\int_r^{1-r}\left(A^vXB^{1-v}+A^{1-v}XB^v\right)dv\right|\!\right|\!\right| \\
&\quad\le \frac{1}{4}|||4A^{\frac{1}{2}}XB^{\frac{1}{2}}+A^{1-r}XB^r+A^rXB^{1-r}+A^{\frac{1+2r}{4}}XB^{\frac{3-2r}{4}}+A^{\frac{3-2r}{4}}XB^{\frac{1+2r}{4}}||| \\
&\quad\le |||A^{1-r}XB^r+A^rXB^{1-r}|||.
\end{aligned}$$

6. The singular values inequality

In this section, we show a unified form of Heinz mean inequalities for singular values. The following results due to Tao [16, Theorem 1] and Audenaert [2, Corollary 1] play an important role in what follows.

Theorem B (Tao)**.** *Given any positive semidefinite block matrix $\left(\begin{smallmatrix} M & K \\ K^* & N\end{smallmatrix}\right)$, where $M, N \in \mathbb{M}_n$, we have*

$$2s_j(K) \le s_j\begin{pmatrix} M & K \\ K^* & N\end{pmatrix} \quad \text{for } j = 1, 2, \ldots, n.$$

Theorem C (Audenaert)**.** *If $A, B \in \mathbb{M}_n$ are positive semidefinite, then*

$$\frac{1}{2}\lambda_j\big((A+B)\big(f(A)+f(B)\big)\big) \le \lambda_j\big(Af(A)+Bf(B)\big) \quad \text{for } j = 1, \ldots, n,$$

for any operator monotone function f.

We need the following known fact.

Lemma 6.1. *For any matrices $X, Y \in \mathbb{M}_n$, $\lambda_j(XY) = \lambda_j(YX)$ for $j = 1, \dots, n$.*

By Theorem B, Theorem C, and Lemma 6.1, we have the following extension of Theorem AZ.

Theorem 6.2. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite, and let f, g be real valued continuous functions on $[0, \infty)$. Further suppose that f and g satisfy either of the following conditions:*

(i) *g is monotone on $[0, \infty)$ and $h_1(t) = f(g^{-1}(\sqrt{t}))^2$ is operator monotone;*
(ii) *f is monotone on $[0, \infty)$ and $h_2(t) = g(f^{-1}(\sqrt{t}))^2$ is operator monotone.*

Then

$$s\big(f(A)\big(g(A)^2 + g(B)^2\big)f(B)\big) \le s\big(f(A)^2 g(A)^2 + f(B)^2 g(B)^2\big).$$

If we put $f(t) = t$ or $g(t) = t$ in Theorem 6.2, then we have the following corollary.

Corollary 6.3. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite, and let f be a semi operator monotone function on $[0, \infty)$; that is, $f(\sqrt{t})^2$ is operator monotone. Then*

$$s\big(f(A)[A^2 + B^2]f(B)\big) \le s\big(A^2 f(A)^2 + B^2 f(B)^2\big), \tag{i}$$

$$s\big(A\big[f(A)^2 + f(B)^2\big]B\big) \le s\big(A^2 f(A)^2 + B^2 f(B)^2\big). \tag{ii}$$

By Theorem 6.2 we have the generalized Heinz mean inequality for singular values, which is a generalization of the Audenaert–Zhan inequality (1.15).

Theorem 6.4. *Let $A, B \in \mathbb{M}_n$ be positive definite, and let $r, s \in \mathbb{R}$ such that $rs \ge 0$. Then*

$$s\big(A^{\frac{r}{2}}(A^s + B^s)B^{\frac{r}{2}}\big) \le \frac{1}{2}\lambda\big((A^r + B^r)(A^s + B^s)\big) \le s(A^{r+s} + B^{r+s}).$$

If we put $s = \frac{1}{2} - r$ in Theorem 6.4, then we have the Audenaert–Zhan inequality (1.15) for singular values.

Corollary 6.5. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite. Then, for $0 \le r \le 1$,*

$$s(A^r B^{1-r} + A^{1-r} B^r) \le \frac{1}{2}\lambda\big((A^{2r_0} + B^{2r_0})(A^{1-2r_0} + B^{1-2r_0})\big) \le s(A + B),$$

where $r_0 = \min\{r, 1 - r\}$.

Remark 6.6. If we put $r = \frac{1}{4}$ in Corollary 6.5 and replace A and B by A^2 and B^2, respectively, then we have the result (1.16) due to Bhatia and Kittaneh.

Remark 6.7. In the case where $r = \frac{1}{2}$, we can obtain the following equality for singular values. Note that

$$\begin{pmatrix} A^{\frac12} & A^{\frac12} \\ B^{\frac12} & B^{\frac12} \end{pmatrix}^* \begin{pmatrix} A^{\frac12} & A^{\frac12} \\ B^{\frac12} & B^{\frac12} \end{pmatrix} = \begin{pmatrix} A+B & A+B \\ A+B & A+B \end{pmatrix}, \qquad \text{and}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}\begin{pmatrix} A+B & A+B \\ A+B & A+B \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \begin{pmatrix} 2(A+B) & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, we say that $\begin{pmatrix} A+B & A+B \\ A+B & A+B \end{pmatrix}$ and $\begin{pmatrix} 2(A+B) & 0 \\ 0 & 0 \end{pmatrix}$ are unitarily similar; that is,

$$s_j(A+B) = \frac{1}{2} s_j^2 \begin{pmatrix} A^{\frac{1}{2}} & A^{\frac{1}{2}} \\ B^{\frac{1}{2}} & B^{\frac{1}{2}} \end{pmatrix} \quad \text{for } j = 1, 2, \ldots, n.$$

If we put $s = 1 - r$ in Theorem 6.4, then we have the following complementary inequality related to Zhan's conjecture (1.17).

Corollary 6.8. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite. Then, for $0 \le r \le 2$ and $-2 < t \le 0$,*

$$\begin{aligned} s(A^r B^{2-r} + A^{2-r} B^r) &\le \frac{1}{2} \lambda\big((A^{2r_1} + B^{2r_1})(A^{2-2r_1} + B^{2-2r_1})\big) \\ &\le \frac{2}{2+t} s(A^2 + tAB + B^2), \end{aligned}$$

where $r_1 = \min\{r, 2-r\}$.

In [16], Tao showed the following generalization of the Bhatia–Kittaneh inequality (1.16): if A and B are positive semidefinite and m is a positive integer, then

$$2s\big(A^{\frac{1}{2}}(A+B)^{m-1}B^{\frac{1}{2}}\big) \le s\big((A+B)^m\big). \tag{6.1}$$

Based on Tao's technique [16], we show a variant of Tao's inequality (6.1).

Theorem 6.9. *Let $A, B \in \mathbb{M}_n$ be positive definite, and let $r, s \in \mathbb{R}$. Then, for $j = 1, 2, \ldots, n$,*

$$\begin{aligned} &2s\big(A^r(A^{r+s} + B^{r+s})^{m-1}(A^{2s} + B^{2s})(A^{r+s} + B^{r+s})^{m-1}B^r\big) \\ &\quad \le \lambda\big((A^{r+s} + B^{r+s})^{m-1}(A^{2s} + B^{2s})(A^{r+s} + B^{r+s})^{m-1}(A^{2r} + B^{2r})\big). \end{aligned}$$

Remark 6.10.

(i) If we put $m = 1$ in Theorem 6.9 and replace A and B by $A^{1/2}$ and $B^{1/2}$, respectively, then we get the first inequality in Theorem 6.4 for all $r, s \in \mathbb{R}$.

(ii) If we put $r = s = \frac{1}{2}$ in Theorem 6.9, then we have $T^* = S$, and this implies Tao's inequality (6.1) because $(TT^*)^m$ is positive semidefinite.

(iii) Moreover, if we put $r = s = \frac{1}{2}$ and $m = 1$ in Theorem 6.9, then the Bhatia–Kittaneh inequality (1.16) holds.

7. The Heinz mean inequality for eigenvalues

In this final section, we consider the Heinz mean inequality for the eigenvalues. We recall the following basic majorization inequalities.

Lemma 7.1 (H. Weyl)**.** *Let $\lambda_1(A), \ldots, \lambda_n(A)$ be the eigenvalues of a matrix A ordered by $|\lambda_1(A)| \ge \cdots \ge |\lambda_n(A)|$. Then $\prod_{j=1}^k |\lambda_j(A)| \le \prod_{j=1}^k s_j(A)$ for all $1 \le k \le n$.*

Lemma 7.2 (A. Horn)**.** *For any matrices A, B, $\prod_{j=1}^k s_j(AB) \le \prod_{j=1}^k \{s_j(A)s_j(B)\}$ for all $1 \le k \le n$.*

Note that the eigenvalues of the product of two positive semidefinite matrices are nonnegative, since $\lambda(AB) = \lambda(A^{1/2}BA^{1/2})$. If A, B are positive semidefinite, then, by Lemmas 7.1 and 7.2,

$$\prod_{j=1}^{k} \lambda_j(AB) \le \prod_{j=1}^{k} s_j(AB) \le \prod_{j=1}^{k} \{\lambda_j(A)\lambda_j(B)\} \quad \text{for all } 1 \le k \le n.$$

Theorem 7.3. *Let $A, B \in M_n$ be positive semidefinite. Then, for $\frac{1}{2} \le r \le \frac{3}{2}$,*

$$\prod_{j=1}^{k} \lambda_j(A^r B^{2-r} + A^{2-r}B^r) \le \prod_{i=1}^{k} \frac{1}{2}\lambda_j(A+B)^2 \quad \textit{for all } 1 \le k \le n.$$

Proof. By Lemma 7.1, Lemma 7.2, and Theorem AZ, we have

$$\begin{aligned}
&\prod_{j=1}^{k} \lambda_j(A^r B^{2-r} + A^{2-r}B^r)\\
&= \prod_{j=1}^{k} \lambda_j\big(A^{\frac{1}{2}}(A^{r-\frac{1}{2}}B^{\frac{3}{2}-r} + A^{\frac{3}{2}-r}B^{r-\frac{1}{2}})B^{\frac{1}{2}}\big)\\
&:= \prod_{j=1}^{k} \lambda_j\big(A^{\frac{1}{2}}(A^{r'}B^{1-r'} + A^{1-r'}B^{r'})B^{\frac{1}{2}}\big)\\
&\quad (\text{put } r' = r - \tfrac{1}{2} \text{ and } 0 \le r' \le 1)\\
&= \prod_{j=1}^{k} \lambda_j\big((A^{r'}B^{1-r'} + A^{1-r'}B^{r'})B^{\frac{1}{2}}A^{\frac{1}{2}}\big) \quad (\text{by Lemma 6.1})\\
&\le \prod_{j=1}^{k} s_j\big((A^{r'}B^{1-r'} + A^{1-r'}B^{r'})B^{\frac{1}{2}}A^{\frac{1}{2}}\big) \quad (\text{by Lemma 7.1})\\
&\le \prod_{j=1}^{k} s_j(A^{r'}B^{1-r'} + A^{1-r'}B^{r'}) \prod_{j=1}^{k} s_j(B^{\frac{1}{2}}A^{\frac{1}{2}}) \quad (\text{by Lemma 7.2})\\
&\le \prod_{j=1}^{k} \lambda_j(A+B)\lambda_j\Big(\frac{A+B}{2}\Big) \quad (\text{by Theorem AZ and (1.14)})\\
&= \prod_{j=1}^{k} \frac{1}{2}\lambda_j(A+B)^2
\end{aligned}$$

for all $1 \le k \le n$. ∎

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