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SOME CONSEQUENCES OF SPECTRAL SYNTHESIS IN HYPERGROUP ALGEBRAS

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Dedicated to Professor Anthony To-Ming Lau, in recognition of his extraordinary contributions to our discipline and in great appreciation for all that he has done for the author and for so many others

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Abstract. Properties of spectral synthesis are exploited to show that, for a large class of commutative hypergroups and for every compact hypergroup, every closed, reflexive, left-translation-invariant subspace of $L^\infty(K)$ is finite-dimensional. Also, we show that, for a class of hypergroups which includes many commutative hypergroups and all $\mathcal{Z}$-hypergroups, every derivation of $L^1(K)$ into an arbitrary Banach L^1-bimodule is continuous.

1. Introduction

Spectral synthesis has been studied for hypergroups by various authors, including Chilana and Ross [2], Lasser [8], and Vogel [15], [16]. In this paper, we will consider two problems related to spectral synthesis of finite sets.

In Section 3, we show that if K is either compact or if K belongs to a rich class of commutative hypergroups, then every closed, reflexive, translation-invariant subspace of $L^\infty(K)$ must be finite-dimensional. This extends a beautiful theorem of Glicksberg [4] for compact groups and locally compact commutative groups.

In Section 4, we consider derivations of the hypergroup algebra $L^1(K)$. We prove that if K is a $\mathcal{Z}$-hypergroup or if K is in a subclass of those commutative hypergroups for which finite sets admit synthesis, then every derivation of $L^1(K)$

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into an arbitrary Banach $L^1(K)$-bimodule is automatically continuous. For related results on group algebras, we refer the reader to [18].

For the standard properties of hypergroups and their measure algebras, and for the notation which we will use throughout this paper, the reader is referred to [6].

2. Preliminaries

Throughout this paper, K will denote a hypergroup which is either commutative or a $\mathcal{Z}$-hypergroup. In either case, K admits a Haar measure λ (see [14]), and $L^1(K)$ will denote the closed ideal of $M(K)$ consisting of those measures which are absolutely continuous with respect to λ.

If K is commutative, then $\Omega_b(K)$ will denote the space of all bounded continuous multiplicative functions on K; $\hat{K}$ will denote the functions φ in $\Omega_b(K)$ for which $\varphi(x^{\sim}) = \overline{\varphi(x)}$. In general, $\hat{K}$ may be a proper subset of $\Omega_b(K)$, and $\hat{K}$ need not be a hypergroup (see [6]). We will follow Chilana and Ross [2] and assume that K also satisfies

(H1) $\hat{K}$ is a hypergroup under pointwise multiplication, and

(H2) $\hat{K} = \Omega_b(K)$.

Under these assumptions, $L^1(K)$ can be identified via the Fourier transform with the pointwise algebra $A(\hat{K})$; $A(\hat{K})$ is a regular Banach algebra of continuous functions which vanish at infinity on its structure space $\hat{K}$. Moreover, $A(\hat{K}) \cap C_{00}(\hat{K})$ is dense in $A(\hat{K})$. For these and many other properties of $L^1(K)$ and $A(\hat{K})$, we refer the reader to [2].

It follows from [2, Theorem 2.6] that if $\alpha \in \hat{K}$ and U_α is a neighborhood of α in $\hat{K}$, then there exists $g \in L^1(K)$ such that $\hat{g}(\alpha) = 1$ and $\hat{g}(\beta) = 0$ if $\beta \notin U_\alpha$. In case K is a group, we may choose g such that $\|g\|_1 = 1$. In general, for hypergroups, the norm of g depends on U_α. In Section 4, we will need to assume that K satisfies the following:

(H3) For every $\alpha \in \hat{K}$ there exists $M_\alpha > 0$ such that if U_α is a neighborhood of α in $\hat{K}$, then there exists $g \in L^1(K)$ such that $\hat{g}(\alpha) = 1$ and $\hat{g}(\beta) = 0$ if $\beta \notin U_\alpha$ and
$$\|g\|_1 \le M_\alpha \quad \text{(see [8])}.$$

Let $\mathcal{A}$ be a commutative semisimple Banach algebra with maximal ideal space $\Delta(\mathcal{A})$. We will consider $\mathcal{A}$ to be an algebra of continuous functions on $\Delta\mathcal{A}$ via the Gelfand transform.

Given a closed subset of A of $\Delta\mathcal{A}$, define

$$I(A) = \{u \in \mathcal{A} \mid u(\alpha) = 0 \text{ for every } \alpha \in A\},$$
$$j(A) = \{u \in \mathcal{A} \mid u \in I(A) \text{ and } \operatorname{supp} u \text{ is compact and disjoint from } A\},$$
$$J(A) \quad \text{is the closure of } j(A) \text{ in } \mathcal{A}.$$

Then $I(A)$, $J(A)$, and $j(A)$ are ideals in $\mathcal{A}$ with $I(A)$ and $J(A)$ being closed.

Given an ideal I in $\mathcal{A}$, define
$$\mathcal{Z}(I) = \{\alpha \in \Delta(\mathcal{A}) \mid u(\alpha) = 0 \text{ for every } u \in I\}.$$

Then $\mathcal{Z}(I)$ is a closed subset of $\Delta(\mathcal{A})$. A set $A \subseteq \Delta(\mathcal{A})$ is called a set of *spectral synthesis* or simply a *spectral set* if $I(A)$ is the only closed ideal I in $\mathcal{A}$ with $\mathcal{Z}(I) = A$. If $\mathcal{A}$ is regular, then A is a spectral set if and only if $J(A) = I(A)$ (see [5, Theorem 39.18]).

Let K be a commutative hypergroup satisfying (H1) and (H2). Let X be a weak-* closed translation invariant subspace of $L^\infty(K)$. The spectrum $\Sigma(X)$ of X is defined to be the set $X \cap \hat{K}$. By [6, Theorem 5.1D] and [2, Theorem 2.11], $X^\perp$ is a closed ideal in $L^1(K)$. Then $\Sigma(X) = \mathcal{Z}(X^\perp)$ is closed in $\hat{K}$. If $X \neq \{0\}$, then $\Sigma(X) \neq \emptyset$ [5, Theorem 39.27] (see also [5, Theorem 40.7]). For $f \in L^\infty(K)$, let $[f]_*$ be the weak-* closure of the linear span of $\{{}_xf \mid x \in K\}$. Then $[f]_*$ is translation invariant and the spectrum $\Sigma(f)$ of f is by definition $\Sigma([f]_*)$.

Lemma 2.1. *Let A be a closed set in $\hat{K}$ and $f \in L^\infty(K)$. Then the following are equivalent:*

(i) *$\Sigma(f) \subseteq A$;*
(ii) *$f \in J(A)^\perp$;*
(iii) *$I(A) * f^\sim = \{0\}$.*

Proof. (i) $\leftrightarrow$ (ii) This follows as in [5, Theorem 40.8].

(ii) $\leftrightarrow$ (iii) If (ii) holds, then for $g \in I(A)$, $g * f^\sim(x) = \int_K {}_xg(y)\overline{f(y)}\,dy = 0$, by [1, Theorem 2.11]. Thus (iii) holds.

If (iii) holds, then $\int_K g\overline{f}\,d\lambda = g * f^\sim(e) = 0$. Hence (ii) holds. □

Proposition 2.2. *Let $A \subseteq \hat{K}$ be closed. Then the following are equivalent:*

(i) *A is a spectral set of $L^1(K)$.*
(ii) *If $f \in L^\infty(K)$ and $\Sigma(f) \subseteq A$, then $f \in I(A)^\perp$.*

Proof. (i) $\to$ (ii) follows from Lemma 2.1.

Conversely, if (i) fails, then there is an $h \in I(A)\backslash J(A)$. Hence there exists $f \in J(A)^\perp$ such that $\int_K h\overline{f}\,d\lambda \neq 0$. By the previous lemma, $\Sigma(F) \subseteq A$, which is impossible if (ii) holds. □

Remark 2.3. It can be shown that [5, Lemma 40.9] and hence [5, Theorem 40.10] are valid for hypergroups. We will omit the details.

Corollary 2.4. *Let X be a weak-* closed invariant subspace of $L^\infty(K)$ such that $\Sigma(X)$ is a spectral set. Then X is the smallest weak-* closed subspace of $L^\infty(K)$ containing $\Sigma(X)$.*

3. On a theorem of Glicksberg

Glicksberg proved in [4] that if G is a locally compact abelian group or a compact group, then any closed, reflexive invariant subspace of $L^\infty(G)$ is finite-dimensional. In this section, we extend this result to a large class of commutative hypergroups and to all compact hypergroups.

Let $AP(K)$ denote the space of almost periodic functions in $C(K)$. Then $AP(K)$ is a norm-closed, conjugate-closed, translation-invariant subspace of $C(K)$. Furthermore, $\Omega_b(K) \subseteq AP(K)$. Let K_a denote the almost periodic compactification of K. Then there is a linear isometry $f \to f^\sim$ of $AP(K)$ onto a closed subspace $\mathcal{A}(K_a)$ of $C(K_a)$ which separates points of K_a (see [8, Proposition 3]). $\mathcal{A}(K_a)$ becomes a commutative Banach *-algebra with a unit in a natural way. If K is a locally compact abelian group, then $\mathcal{A}(K_a)^* = M(K_a)$, the measure algebra of the Bohr compactification K_a of K. Following Lasser [8], we will assume throughout this section that K also satisfies the following:

(A1) For each $\alpha \in \Omega_b(K)$ there exists $\nu_\alpha \in \mathcal{A}(K_a)^*$ such that $\nu_\alpha(\tilde{\alpha}) = 1$ and $\nu_\alpha(L_S F) = \tilde{\alpha}(S)\nu_\alpha(F)$ for each $S \in K_a$, $F \in \mathcal{A}(K_a)$.

Let K be any hypergroup with a left Haar measure λ. Let

$$UC_r(K) = \big\{f \in C(K) \mid x \mapsto {}_x f \text{ is continuous}\big\}.$$

Then it is easy to see that $UC_r(K)$ is a left Banach $L^1(K)$-module. In fact, by Cohen's factorization theorem, we have $UC_r(K) = L^1(K) * UC_r(K)$. Similarly, $L^1(K) * L^p(K) = L^p(K)$ for $1 \le p < \infty$. The next result is due to Mitchell [11, Theorem 7] for locally compact groups. The proof uses a method of Johnson [7, pp. 26–27].

Proposition 3.1. *Let $f \in L^\infty(K)$, and suppose that $x \mapsto {}_x f$ is weakly continuous. Then $f \in UC_r(K)$.*

Proof. Let $\mathcal{U}$ be the set of all neighborhoods of e directed by containment. For each $U \in \mathcal{U}$, choose a function $\varphi_U \in C_{00}^+(K)$ with $\int_K \varphi_U \, d\lambda = 1$ and $\operatorname{supp} \varphi \subseteq U$. Then $\{\varphi_U\}_{U \in \mathcal{U}}$ is a bounded approximate identity for $L^1(K)$. Since $x \mapsto {}_x f$ is weakly continuous, the weak vector-valued integral

$$\int_K \delta_x * f(Y)\varphi_U(Y)\, dY$$

exists (see [13, Theorem 3.27]) and is equal to $\varphi_U * f$ because $L^1(K)$ is weak-* dense in $L^\infty(K)^*$. Thus, $\varphi_U * f \xrightarrow{\text{weakly}} f$, so $f \in UC_r(K)$. □

Theorem 3.2. *Let K be a commutative hypergroup such that (H1), (H2), and (A1) hold. Assume also that every finite subset of $\hat{K}$ is a spectral set. Then, any closed reflexive invariant subspace X of $L^\infty(K)$ is finite-dimensional.*

Proof. As in [4], X is weak-* closed, and if $f \in X$, then the mapping $x \mapsto {}_x f$ is weakly continuous. Hence f is (uniformly) continuous by Proposition 3.1. Assume that X contains a sequence $\{\gamma_n\}$ of distinct elements of $\hat{K}$. Since $\|\gamma_n\|_\infty = 1$ and the unit ball of X is weakly compact, we can assume that γ_n converges weakly to γ and $\gamma \neq \gamma_n$ for any $n = 1, 2, \ldots$. Since $\gamma_n \longrightarrow \gamma$ pointwise, the Lebesgue dominated convergence theorem shows that $\gamma \in \hat{K}$.

Choose $\nu_\gamma \in \mathcal{A}(K_a)^*$ such that $\nu_\gamma(\tilde{\gamma}) = 1$ and $\nu_\gamma(L_S F) = \tilde{\gamma}(S)\nu_\gamma(F)$ for every $S \in K_a$ and $F \in \mathcal{A}(K_a)$. By the proof of [8, Theorem 1], $\nu_\gamma(\tilde{\gamma}_n) = 0$ for each n. So $\tilde{\gamma}_n \not\longrightarrow \tilde{\gamma}$ weakly. However, this is a contradiction since $f \longrightarrow \tilde{f}$ is weak–weak continuous. Hence, $\Sigma(X)$ is finite. By Corollary 2.4, X is the finite-dimensional subspace spanned by $\Sigma(X)$. □

Corollary 3.3. *Let I be a closed ideal in $L^1(K)$. If I is infinite-codimensional, then $L^1(K)/I$ is not reflexive.*

Proof. If $L^1(K)/I$ is reflexive, then so is $(L^1(K)/I)^* = I^\perp$. It follows that $(L^1(K)/I)^*$ is finite-dimensional by Theorem 3.2 and hence I has finite codimension. □

When K is a locally compact abelian group, the next result is due to Rosenthal [12, Corollary 2.6].

Corollary 3.4. *Let $A \subseteq K$ be closed. Then $L^1(K)/I(A)$ is reflexive if and only if A is finite.*

Proof. $(L^1(K)/I(A))^* = I(A)^\perp$ and $\Sigma(I(A)^\perp) = \mathcal{Z}(I(A)) = A$. If $L^1(K)/I(A)$ is reflexive, then so is $I(A)^\perp$. Hence, $I(A)^\perp$ is finite-dimensional. By (A1), the characters are linearly independent. Hence A is finite. The converse follows from Corollary 2.4. □

Examples. We give some examples of hypergroups which satisfy (H1), (H2), and (A1) and for which finite sets are spectral.

(1) All locally compact abelian groups have these properties.
(2) Let G be a locally compact group, and let I_G denote the (topological) group of inner automorphisms of G. Let $I_G \subseteq B$ and $G \in [FD]^- \cap [SIN]_B \subseteq [FIA]_B^-$. Let G_B be the commutative hypergroup of conjugacy classes of G. It is shown in [6, Section 3] that G_B satisfies (H1), (H2), and (A1). It is also proved in [8, Proposition 3.1] that every finite subset of $(G_B)\hat{}$ is a spectral set in $L^1(G_B)$.
(3) All compact commutative hypergroups for which $\hat{K}$ is a hypergroup have the desired properties.

Compact hypergroups always satisfy (H2), but (H1) may fail even for a three-element hypergroup (see [6, Example 9.1C]). As $\hat{K}$ is discrete, all subsets are spectral sets for $L^1(K)$. To see that K satisfies (A1), for each $\alpha \in \hat{K}$, let $\nu_\alpha \in \mathcal{A}(K_a)^* = M(K)$ be given by

$$\nu_\alpha(f) = \frac{1}{\int_K |\alpha(x)|^2 \, d\alpha(x)} \hat{f}(\alpha).$$

The remainder of this section will be devoted to showing that Glicksberg's result holds for all compact hypergroups, including those compact commutative hypergroups for which (H1) fails.

Lemma 3.5. *Let X be a closed subspace of either $UC_r(K)$ or $L^p(K)$ for $1 \le p < \infty$. Then X is left translation invariant if and only if $L^1(K) * X \subseteq X$. In this case, $L^1(K) * X = X$. Moreover, if K is compact, then the trigonometric polynomials in X are dense in X.*

Proof. Assume that $X \subseteq L^p(K)$ such that $L^1(K) * X \subseteq X$. Let $x \in K$ and $f \in X$. Let $\{\varphi_\alpha\}_{\alpha\in I} \in L^1(K)^+$ be such that $\varphi_\alpha\lambda \longrightarrow \delta_x$ in the cone topology (see [9, Section 3]). Then $\|\varphi_\alpha * f - \delta_x * f\|_p \longrightarrow 0$ (see [6, Lemma 5.4H]).

The converse is obtained similarly.

If $X \subseteq UC_r(K)$, for $f \in X$ write $f = g * h$, where $g \in L^1(K)$ and $h \in UC_r(K)$. Then

$$\|\mu * f - f\|_\infty \le \|\mu * g - g\|_1 \|h\|_\infty$$

for every $\mu \in M(K)$. Hence, if $\mu_\alpha \longrightarrow \mu$ in $M^+(K)$, then $\|\mu_\alpha * f - f\|_\infty \longrightarrow 0$. Therefore, if $X \subseteq UC_r(K)$, then the result follows just as above.

Finally, if K is compact, then we can choose a bounded approximate identity $\{h_\alpha\}$ in $L^1(K)$ which consists of trigonometric polynomials (see [17, Lemma 2.12]). If $f \in X$, then $h_\alpha * f \longrightarrow f$ is norm and each $h_\alpha * f$ is a trigonometric polynomial. □

Theorem 3.6. *Let K be a compact hypergroup. Let X be a closed, reflexive, invariant subspace of $L^\infty(K)$. Then X is finite-dimensional.*

Proof. By Lemma 3.5 the trigonometric polynomials in X are dense in X. It follows that X is the closed linear span of minimal finite-dimensional left-invariant subspaces (i.e., minimal left ideals in the Hilbert algebra $L^2(K)$). Each minimal left ideal is contained in a minimal closed two-sided ideal. Minimal closed ideals are pairwise orthogonal (see [10, Section 27]). Since minimal closed ideals in $L^2(K)$ are finite-dimensional (see [6, Theorem 7.2C]), each minimal closed two-sided ideal contains at most finitely many closed minimal left ideals.

If X is not the linear span of finitely many minimal left ideals, then it contains a sequence $\{I_n\}$ of pairwise orthogonal minimal left ideals. Each I_n contains a positive definite function ψ_n with $\psi_n(e) = 1$. As in the proof of Theorem 3.2, $\psi_n \longrightarrow \psi$ weakly. Therefore, $\psi(e) = 1$. This is a contradiction since

$$0 = \lim_n \lim_n \int \psi_n \overline{\psi}_m \, d\lambda = \int \psi \overline{\psi} \, d\lambda > 0.$$ □

4. Automatic continuity and spectral synthesis

We begin this section with a series of results which hold for a large class of commutative Banach algebras. We will assume that $\mathcal{A}$ is a commutative, semisimple, regular Banach algebra. As such, $\mathcal{A}$ may be viewed as an algebra of functions in $C_0(\Delta(\mathcal{A}))$. We will also assume that $\mathcal{A}$ has a bounded approximate identity $\{u_i\}_{i \in \mathcal{I}}$ with $\|u_i\|_{\mathcal{A}} \leq M$ and $u_i \in C_{00}(\Delta(\mathcal{A}))$ for each $i \in \mathcal{I}$.

Given a closed subset A of $\Delta(\mathcal{A})$, let

$$\mathcal{F}(A) = \bigl\{F \subseteq \Delta(\mathcal{A}) \bigm| F \cap A = \emptyset \text{ and } F \text{ is compact}\bigr\}.$$

We say that A can be *uniformly separated* if there is an L such that for each $F \in \mathcal{F}(A)$ there exists a $u_F \in \mathcal{A}$ with $u_F(\alpha) = 1$ if $\alpha \in A$ and $u_F(\alpha) = 0$ if $\alpha \in F$, and $\|u_F\|_{\mathcal{A}} \leq L$. We say that $\mathcal{A}$ has the uniform separation property or (USP) if $\{\alpha\}$ is uniformly separated for every $\alpha \in \Delta(\mathcal{A})$.

Lemma 4.1. *Assume that A is a uniformly separated spectral set. Then $I(A)$ has a bounded approximate identity $\{v_j\}_{j \in \mathcal{J}}$ with $v_j \in C_{00}(\Delta(\mathcal{A}))$ for every $j \in \mathcal{J}$.*

Proof. Let $F \in \mathcal{F}(A)$, and let u_F be as above. Let $v \in \mathcal{A}$ with $\operatorname{supp} v \subseteq F$. Then $(u_i - u_i u_F)v = u_i v$. Hence, $\lim_i (u_i - u_i u_F)v = v$. Moreover, $\|u_i - u_i u_F\|_{\mathcal{A}} \leq M(L+1)$. Order $\mathcal{F}(A)$ by inclusion. Let $v_{i,F} = u_i - u_i u_F$ for each $(i,F) \in \mathcal{I} \times \mathcal{F}(A)$. If $u \in \mathcal{A}$ is such that $\operatorname{supp} u \cap A = \emptyset$ and $\operatorname{supp} u$ is compact, then $\lim_{(i,F)} v_{i,F} u = u$. However, since A is a spectral set, such functions are dense in $I(A)$. Therefore, $\{v_{i,F}\}_{(i,F) \in \mathcal{I} \times \mathcal{F}(A)}$ is the desired bounded approximate identity. □

Theorem 4.2. *Assume that $\mathcal{A}$ has the USP. Assume also that $\{\alpha\}$ is a spectral set for every $\alpha \in \Delta(\mathcal{A})$. Then the following hold:*

1. *$I(A)$ has a bounded approximate identity for every finite set $A \subseteq \Delta(\mathcal{A})$.*
2. *Every finite subset of $\Delta(\mathcal{A})$ is a spectral set.*
3. *An ideal I in $\mathcal{A}$ is cofinite if and only if $I = I(A)$ for some finite set $A \subseteq \Delta(\mathcal{A})$.*

Proof. (1) Let $A = \{\alpha_1, \ldots, \alpha_n\}$. By Lemma 4.1, the ideal $I(\{\alpha_i\})$ has a bounded approximate identity $\{v_{j_i}\}_{j_i \in \mathcal{J}_i}$. Standard arguments show that $\{v_{j_i} \cdots v_{j_n}\}_{(j_1,\ldots,j_n)\in\mathcal{J}_1\times\cdots\times\mathcal{J}_n}$ is a bounded approximate identity for $I(A)$.

(2) If A is finite, then by (1), $I(A)$ has a bounded approximate identity which can be constructed so as to be in $C_{00}(\Delta(\mathcal{A}))$. Since $\mathcal{A}$ is regular, A is a spectral set (see [5, Theorem 39.18]).

(3) If A is finite, then $I(A)$ is clearly cofinite.
Conversely, assume that I is closed and cofinite. Let $A = \mathcal{Z}(I)$. Then A must be finite. Since A is a spectral set, $I = I(A)$. Therefore, by (2), every closed cofinite ideal of $\mathcal{A}$ has a bounded approximate identity and hence is idempotent. By [3, Theorem 2.3], every cofinite ideal is closed. □

Corollary 4.3. *Assume that $\mathcal{A}$ has the USP. If $\{\alpha\}$ is a spectral set for every $\alpha \in \Delta(\mathcal{A})$, then every homomorphism from $\mathcal{A}$ with finite-dimensional range is continuous.*

Proof. This follows immediately from Theorem 4.2 and from [3, Theorem 2.3]. □

Lemma 4.4. *Let I be a closed ideal in $\mathcal{A}$ for which $\mathcal{Z}(I)$ is infinite. Then there exists a sequence $\{u_n\} \subset \mathcal{A}$ such that $u_n u_m = 0$ if $n \neq m$ and $u_n^2 \notin I$ for every n.*

Proof. Since $\mathcal{Z}(I)$ is infinite, we can find a sequence $\{\alpha_n\} \subseteq \mathcal{Z}(I)$ and a sequence $\{v_n\}$ of compact neighborhoods of the α_n's such that $v_n \cap (\bigcup_{i=1}^{n-1} v_i) = \emptyset$. We can also find $u_n \in \mathcal{A}$ such that $u_n(\alpha_n) = 1$ and $\operatorname{supp} u_n \subseteq v_n$. Clearly, $u_n u_m = 0$ if $n \neq m$. Since $u_n^2(\alpha_n) = 1$, $u_n \notin I$. □

Theorem 4.5. *Assume that $\mathcal{A}$ has the USP. If $\{\alpha\}$ is a spectral set for every $\alpha \in \Delta(\mathcal{A})$, then every derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is continuous.*

Proof. It follows from Theorem 4.2 that every cofinite ideal in $\mathcal{A}$ has a bounded approximate identity and that if I is a closed ideal with infinite codimension, then $\mathcal{Z}(I)$ is infinite. The theorem now follows immediately from Lemma 4.4 and from [1, Corollary 2.6]. □

Theorem 4.6. *Let K be a commutative hypergroup which satisfies (H1), (H2), and (H3). If $\{\alpha\}$ is a spectral set for every $\alpha \in \hat{K}$, then every derivation from $L^1(K)$ into a Banach $L^1(K)$-bimodule is continuous.*

Proof. $L^1(K)$ is commutative, regular, and semisimple (see [2]). Condition (H3) is simply that $L^1(K)$ has the USP. The theorem follows from Theorem 4.5. □

Corollary 4.7. *Let $K = G_B$, where $G \in [FD]^- \cap [SIN]_B$ and $I_G \subseteq B$. Then every derivation from $L^1(K)$ into a Banach $L^1(K)$-bimodule is continuous.*

Proof. K satisfies all of the conditions of Theorem 4.6 (see [8, Proposition 7]). □

Remark 4.8. If K is a locally compact abelian group, then conditions (H1), (H2), and (H3) always hold. In this case, Theorem 4.6 is due to Willis (see [18]).

The class of Banach algebras considered here also includes the Fourier algebra of any locally compact amenable group. In fact, for any $1 < p < \infty$, the Figà–Talamanca–Herz algebras, $A_p(G)$, satisfy the conditions of Theorem 4.5, provided that G is amenable. Hence, these algebras all have automatically continuous derivations. Moreover, this can be shown to characterize the class of amenable groups among all locally compact groups.

In contrast, spectral synthesis for finite sets fails for many commutative hypergroups. Vogel considers spectral synthesis for hypergroups which arise from algebras of orthogonal polynomial series in [16]. He gives many examples of hypergroups for which even singletons need not be spectral sets (see also [2, Example 4.5]). In lieu of [16, Corollary 3.13], the automatic continuity of derivations on $L^1(K)$ for an arbitrary commutative hypergroup K seems to be in doubt.

For noncommutative algebras, the procedures used to establish automatic continuity of derivations are essentially the same as for commutative algebras. However, in general, the technical difficulties become much greater. We are nonetheless able to establish the analogue of Theorem 4.6 for a large class of hypergroups which includes all compact hypergroups.

A hypergroup K is a $\mathcal{Z}$-hypergroup if $K/\mathcal{Z}(K)$ is compact, where $\mathcal{Z}(K)$ denotes the intersection of the center of K with the maximal subgroup of K.

If K is a $\mathcal{Z}$-hypergroup, let $C^*(K)$ denote the enveloping C^*-algebra of $L^1(K)$. Let $\hat{K} = C^*(K)\hat{}$ denote the set of equivalence classes of irreducible representations of K. Let $\operatorname{Prim}_* L^1(K) = \{\ker \rho \mid \rho \in \hat{K}\}$ be endowed with the hull-kernel topology. Let $\operatorname{Prim} C^*(K)$ and $\operatorname{Max} L^1(K)$ denote the set of primitive ideals of $C^*(K)$ and the set of maximal modular ideals of $L^1(K)$, respectively. Then, for a $\mathcal{Z}$-hypergroup, $\hat{K} \cong \operatorname{Prim} C^*(K) \cong \operatorname{Prim}_* L^1(K) = \operatorname{Max} L^1(K)$ (see [15, Lemma 2.7]).

Given $E \subseteq \operatorname{Prim}_* L^1(K)$, define $I(E) = \bigcap\{P \mid P \in E\}$. Given an ideal I in $L^1(K)$, define

$$\mathcal{Z}(I) = \big\{P \in \operatorname{Prim}_* L^1(K) \bigm| I \subseteq P\big\}.$$

A set $E \subseteq \operatorname{Prim}_* L^1(K)$ is called a *spectral set* if $I(E)$ is the only closed ideal I in $L^1(K)$ such that $\mathcal{Z}(I) = E$.

Theorem 4.9. *Let K be a $\mathcal{Z}$-hypergroup. Then every homomorphism from $L^1(K)$ with finite-dimensional range is continuous.*

Proof. If I is a closed cofinite ideal in $L^1(K)$, then I has a bounded approximate identity (see [15, Remark 3.10(b)]). Therefore, every closed cofinite ideal of $L^1(K)$ is idempotent. It follows from [3, Theorem 2.3] that every homomorphism with finite-dimensional range is continuous. □

Theorem 4.10. *Let K be a $\mathcal{Z}$-hypergroup. Then every derivation from $L^1(K)$ into a Banach $L^1(K)$-bimodule is continuous.*

Proof. As before, closed cofinite ideals have bounded approximate identities.

Let I be a closed ideal with infinite codimension. Since every finite subset of $\hat{K}$ is a spectral set (see [15, Corollary 3.6]), $\mathcal{Z}(I)$ is infinite. Since $\hat{K}$ is Hausdorff, we can find a sequence $\{\rho_n\} \subseteq \mathcal{Z}(I)$ and a sequence $\{U_n\}$ of neighborhoods of the ρ_n's such that $U_n \cap (\bigcap_{k=1}^{n-1} U_k) = \emptyset$ (see [15, Corollary 2.8]). We can also find $g_n \in L^1(K)$ such that $\rho_n(g_n) = Id_{\mathcal{H}_{\rho_n}}$ and $\tau(g_n) = 0$ for every $\tau \notin U_n$. If $n \neq m$, then $\tau(g_n * g_m) = 0$ for every $\tau \in \hat{K}$. Hence, $g_n * g_m = 0$. Finally, $\rho_n(g_n * g_n) = Id_{\mathcal{H}_{\rho_n}}$, so $g_n * g_n \notin I$. By [1, Corollary 2.6], every derivation from $L^1(K)$ into a Banach $L^1(K)$-bimodule is continuous. $\square$

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