Annals of Functional Analysis
Ann. Funct. Anal. 7 (2016), no. 1, 61–75
http://dx.doi.org/10.1215/20088752-3320340
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

STRONG CONVERGENCE THEOREMS BY HYBRID METHODS FOR SEMIGROUPS OF NOT NECESSARILY CONTINUOUS MAPPINGS IN HILBERT SPACES

WATARU TAKAHASHI[1*] and MAKOTO TSUKADA[2]

Dedicated to Professor Anthony To-Ming Lau

Communicated by B. Ricceri

Abstract. In this paper, we prove strong convergence theorems by two hybrid methods for semigroups of not necessarily continuous mappings in Hilbert spaces. Using these results, we prove strong convergence theorems for discrete semigroups generated by generalized hybrid mappings and semigroups of nonexpansive mappings in Hilbert spaces.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space, and let C be a nonempty subset of H. Let T be a mapping of C into itself. We denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* (see [15]) of T; that is,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \forall x \in C\}$.

We know from [15] that $A(T)$ is always closed and convex. Kocourek, Takahashi, and Yao [7] defined a broad class of nonlinear mappings in a Hilbert space. A mapping $T : C \to C$ is called *generalized hybrid* (see [7]) if there exist $\alpha, \beta \in \mathbb{R}$ such

Received Nov. 18, 2014; Accepted Mar. 25, 2015.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 47H20.
Keywords. generalized hybrid mapping, nonexpansive semigroup, invariant mean, fixed point, hybrid method.

that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. We call such a mapping (α, β)-*generalized hybrid*. A $(1, 0)$-generalized hybrid mapping is *nonexpansive* (see [4]). It is *nonspreading* (see [8]) for $\alpha = 2$ and $\beta = 1$. Furthermore, it is *hybrid* [14] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. In general, nonspreading and hybrid mappings are not continuous. We also know the concept of one-parameter nonexpansive semigroups in a Hilbert space. Let H be a Hilbert space, and let C be a nonempty subset of H. Let $S = \mathbb{R}^+ = \{t \in \mathbb{R} : 0 \le t < \infty\}$. A family $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself is called a *one-parameter nonexpansive semigroup* on C if $\mathcal{S}$ satisfies the following:

(1) $S(t+s)x = S(t)S(s)x$, $\forall x \in C, t, s \in \mathbb{R}^+$;
(2) $S(0)x = x$, $\forall x \in C$;
(3) for each $x \in C$, the mapping $t \mapsto S(t)x$ from $\mathbb{R}^+$ into C is continuous;
(4) for each $t \in \mathbb{R}^+$, $S(t)$ is nonexpansive.

Of course, $S(t)$ are continuous. Such one-parameter nonexpansive semigroups are used in the theory of nonlinear evolution equations (see [2]). Recently, using the concept of invariant means, Takahashi, Wong, and Yao [17] introduced the concept of semigroups of not necessarily continuous mappings in Hilbert spaces which contain discrete semigroups defined by generalized hybrid mappings and semigroups of nonexpansive mappings. They proved fixed-point, attractive-point, and mean convergence theorems for the semigroups in Hilbert spaces. Hussain and Takahashi [6] also proved weak and strong convergence theorems of Mann's type (see [9]) and of Halpern's type (see [5]) for semigroups of not necessarily continuous mappings. Such semigroups are defined by strongly asymptotically invariant nets instead of invariant means. Can we prove strong convergence theorems by hybrid methods (see [11], [16]) for the semigroups? This question is natural.

In this paper, we prove strong convergence theorems by two hybrid methods for semigroups of not necessarily continuous mappings in Hilbert spaces. Using these results, we prove strong convergence theorems for discrete semigroups defined by generalized hybrid mappings and semigroups of nonexpansive mappings in Hilbert spaces.

2. Preliminaries

Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [13], we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\big\|\lambda x + (1-\lambda)y\big\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.1}$$

We also know that, for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

From (2.2), we have that

$$2\langle x-y, z-y\rangle - \|z-y\|^2 = \|x-y\|^2 - \|x-z\|^2 \tag{2.3}$$

for all $x, y, z \in H$. Let C be a nonempty closed convex subset of H, and let $x \in H$. Then we know that there exists a unique nearest point $z \in C$ such that $\|x-z\| = \inf_{y\in C}\|x-y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive and that

$$\langle x - P_C x, P_C x - u\rangle \geq 0$$

for all $x \in H$ and $u \in C$ (see [13] for more details).

Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an (α, β)-generalized hybrid mapping. If x is a fixed point of T in (1.1), then, for any $y \in C$,

$$\alpha\|x-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 \leq \beta\|x-y\|^2 + (1-\beta)\|x-y\|^2$$

and hence $\|x-Ty\| \leq \|x-y\|$. This means that an (α, β)-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Furthermore, from [7] we have the following theorem for generalized hybrid mappings in a Hilbert space.

Theorem 2.1 (see [7])**.** *Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : C \to C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

As a direct consequence of Theorem 2.1, we have the following result.

Theorem 2.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H, and let T be a generalized hybrid mapping from C to itself. Then T has a fixed point.*

For a sequence $\{C_n\}$ of nonempty closed convex subsets of a Hilbert space H, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.4}$$

then it is said that $\{C_n\}$ *converges to C_0 in the sense of Mosco* [10] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [10]. We know the following theorem, which was proved by Tsukada [18].

Theorem 2.3 (see [18])**.** *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 = \text{M-lim}_{n\to\infty} C_n$ exists and is nonempty, then for each $x \in H$, $\{P_{C_n} x\}$ converges strongly to $P_{C_0} x$, where P_{C_n} and P_{C_0} are the metric projections of H onto C_n and C_0, respectively.*

Let ℓ^∞ be the Banach space of bounded sequences on S with supremum norm. Let μ be an element of $(\ell^\infty)^*$ (the dual space of ℓ^∞). Then, we denote by $\mu(f)$ the value of μ at $f = \{x_n\} \in \ell^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = \{1, 1, 1, \dots\}$. A mean μ is called a *Banach limit* on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$ for all $\{x_n\} \in \ell^\infty$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then, for $f = \{x_n\} \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = \{x_n\} \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of the existence of a Banach limit and its other elementary properties, see [12].

Generally, let S be a semitopological semigroup; that is, S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm, and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $\ell_s f$ and $r_s f$ as follows:

$$(\ell_s f)(t) = f(st) \qquad \text{and} \qquad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s\in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(\ell_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. If $S = \mathbb{N}$, then an invariant mean on $C(S) = B(S)$ is a Banach limit on ℓ^∞. The next theorem follows [12, Theorem 1.4.5].

Theorem 2.4 ([12, Theorem 1.4.5])**.** *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$; that is, there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Let H be a Hilbert space, and let C be a nonempty subset of H. Let S be a semitopological semigroup, and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \mapsto T_s x$ is continuous for each $x \in C$. We denote by $F(\mathcal{S})$ the set of common fixed points of $T_s, s \in S$; that is,

$$F(\mathcal{S}) = \bigcap \big\{ F(T_s) : s \in S \big\}.$$

A continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings on C is called a *nonexpansive semigroup* on C if each $T_s, s \in S$ is nonexpansive; that is,

$$\|T_s x - T_s y\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The following definition is crucial in the nonlinear ergodic theory of abstract semigroups. Let $u : S \to H$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded, and let μ be a mean on $C(S)$. Then there exists a unique point $z_0 \in \overline{\text{co}}\{u(s) : s \in S\}$ such that

$$\mu_s \big\langle u(s), y \big\rangle = \langle z_0, y \rangle, \quad \forall y \in H, \tag{2.5}$$

where $\overline{\text{co}}\, A$ is the closure of the convex hull of A. We call such z_0 the *mean vector* of u for μ. In particular, if $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as mappings on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$ and $u(s) = T_s x$ for all $s \in S$, then there exists $z_0 \in H$ such that

$$\mu_s \langle T_s x, y \rangle = \langle z_0, y \rangle, \quad \forall y \in H.$$

We denote such z_0 by $T_\mu x$.

Motivated by Takahashi and Takeuchi [15], Atsushiba and Takahashi [1] defined the set $A(\mathcal{S})$ of all common attractive points of a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself; that is,

$$A(\mathcal{S}) = \bigcap \big\{ A(T_s) : s \in S \big\}.$$

A net $\{\mu_\kappa\}$ of means on $C(S)$ is said to be *asymptotically invariant* if, for each $f \in C(S)$ and $s \in S$,

$$\mu_\kappa(f) - \mu_\kappa(\ell_s f) \to 0 \qquad \text{and} \qquad \mu_\kappa(f) - \mu_\kappa(r_s f) \to 0.$$

A net $\{\mu_\kappa\}$ of means on $C(S)$ is said to be *strongly asymptotically invariant* if, for each $s \in S$,

$$\|\ell_s^* \mu_\kappa - \mu_\kappa\| \to 0 \qquad \text{and} \qquad \|r_s^* \mu_\kappa - \mu_\kappa\| \to 0,$$

where ℓ_s^* and r_s^* are the adjoint operators of ℓ_s and r_s, respectively (see [3] and [12] for more details). Recently, Takahashi, Wong, and Yao [17] proved the following theorem.

Theorem 2.5 (see [17])**.** *Let H be a Hilbert space, and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Let $\{T_s x : s \in S\}$ be bounded for some $x \in C$, and let μ be a mean on $C(S)$. Suppose that*

$$\mu_s \|T_s x - T_t y\|^2 \leq \mu_s \|T_s x - y\|^2, \quad \forall y \in C, t \in S.$$

Then $A(\mathcal{S})$ is nonempty. In addition, if C is closed and convex, then $F(\mathcal{S})$ is nonempty.

3. Strong convergence theorems

In this section, using the hybrid method by Nakajo and Takahashi [11], we first prove a strong convergence theorem for semigroups of not necessarily continuous mappings in a Hilbert space. See Theorem 2.5 for the existence of common fixed points of the semigroups.

Theorem 3.1. *Let H be a real Hilbert space, and let C be a nonempty bounded convex closed subset of H. Let S be a commutative semitopological semigroup. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $F(\mathcal{S}) \neq \emptyset$. Suppose that*

$$\limsup_{\kappa} \sup_{x,y\in C} (\mu_\kappa)_s \big(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2\big) \leq 0, \quad \forall t \in S, \tag{3.1}$$

for every strongly asymptotically invariant net $\{\mu_\kappa\}$ of means on $C(S)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset [0,1]$ satisfies $0 \leq \alpha_n \leq a < 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{S})} x$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

Proof. Since S is commutative, we have from Theorem 2.4 that there exists an invariant mean on $C(S)$. Let μ be an invariant mean on $C(S)$, and put $\mu_\kappa = \mu$ in (3.1). Then we have that

$$\mu_s \|T_s x - T_t y\|^2 \leq \mu_s \|T_s x - y\|^2, \quad \forall x, y \in C, t \in S.$$

In particular, putting $x = u \in F(\mathcal{S})$, we have that

$$\|u - T_t y\|^2 \leq \|u - y\|^2, \quad \forall y \in C, t \in S. \tag{3.2}$$

Let us show that $F(\mathcal{S})$ is closed and convex. In fact, for proving that $F(\mathcal{S})$ is closed, take a sequence $\{z_n\} \subset F(\mathcal{S})$ with $z_n \to z$. Since C is closed, we have $z \in C$. Furthermore, we have from (3.2) that, for any $t \in S$,

$$\|z - T_t z\| \leq \|z - z_n\| + \|z_n - T_t z\| \leq 2\|z - z_n\| \to 0.$$

Thus z is a fixed point of T_t, and hence $F(\mathcal{S})$ is closed. Let us show that $F(\mathcal{S})$ is convex. For $x, y \in F(\mathcal{S})$ and $\alpha \in [0,1]$, put $z = \alpha x + (1-\alpha) y$. Then we have from (2.1) and (3.2) that, for any $t \in S$,

$$\begin{aligned} \|z - T_t z\|^2 &= \big\|\alpha x + (1-\alpha) y - T_t z\big\|^2 \\ &= \alpha\|x - T_t z\|^2 + (1-\alpha)\|y - T_t z\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\ &= \alpha(1-\alpha)^2\|x - y\|^2 + (1-\alpha)\alpha^2\|x - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \end{aligned}$$

$$= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|x-y\|^2$$
$$= 0,$$

and hence $T_t z = z$. This implies that $F(\mathcal{S})$ is convex. Thus $F(\mathcal{S})$ is closed and convex. Then there exists the metric projection of H onto $F(\mathcal{S})$. Since

$$\|y_n - z\|^2 \le \|x_n - z\|^2$$
$$\Longleftrightarrow \quad \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \le 0,$$

we also have that C_n, Q_n and $C_n \cap Q_n$ are closed and convex for all $n \in \mathbb{N}$. Let us show that $C_n \cap Q_n$ is nonempty. Let $z \in F(\mathcal{S})$. We have from $\|\mu_n\| = 1$ that, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\|T_{\mu_n} x_n - z\|^2 &= \langle T_{\mu_n} x_n - z, T_{\mu_n} x_n - z\rangle \\
&= (\mu_n)_t \langle T_t x_n - z, T_{\mu_n} x_n - z\rangle \\
&\le \|\mu_n\| \sup_t \big|\langle T_t x_n - z, T_{\mu_n} x_n - z\rangle\big| \\
&\le \sup_t \|T_t x_n - z\| \cdot \|T_{\mu_n} x_n - z\| \\
&\le \sup_t \|x_n - z\| \cdot \|T_{\mu_n} x_n - z\| \\
&= \|x_n - z\| \cdot \|T_{\mu_n} x_n - z\|,
\end{aligned}$$

and hence

$$\|T_{\mu_n} x_n - z\| \le \|x_n - z\|. \tag{3.3}$$

Using (3.3), we have that

$$\begin{aligned}
\|y_n - z\|^2 &= \big\|\alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n - z\big\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|T_{\mu_n} x_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\
&= \|x_n - z\|^2
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus we have $z \in C_n$, and hence $F(\mathcal{S}) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that $F(\mathcal{S}) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $F(\mathcal{S}) \subset Q_1$, it follows that $F(\mathcal{S}) \subset C_1 \cap Q_1$. Suppose that $F(\mathcal{S}) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. From $x_{k+1} = P_{C_k \cap Q_k} x$, we have that

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0, \quad \forall z \in C_k \cap Q_k.$$

Since $F(\mathcal{S}) \subset C_k \cap Q_k$, we also have that

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0, \quad \forall z \in F(\mathcal{S}).$$

This implies $F(\mathcal{S}) \subset Q_{k+1}$. Thus we have $F(\mathcal{S}) \subset C_{k+1} \cap Q_{k+1}$. By induction, we have that $F(\mathcal{S}) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \in Q_n$, we have from (2.2) that

$$\begin{aligned}
0 &\le 2\langle x - x_n, x_n - x_{n+1}\rangle \\
&= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\
&\le \|x - x_{n+1}\|^2 - \|x - x_n\|^2.
\end{aligned} \tag{3.4}$$

We get from (3.4) that

$$\|x - x_n\|^2 \le \|x - x_{n+1}\|^2. \tag{3.5}$$

Furthermore, since $x_n = P_{Q_n}x$ and $z \in F(\mathcal{S}) \subset Q_n$, we have

$$\|x - x_n\|^2 \le \|x - z\|^2. \tag{3.6}$$

We have from (3.5) and (3.6) that $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$ and $\{T_{\mu_n}x_n\}$ are also bounded. From (3.4), we have

$$\|x_n - x_{n+1}\|^2 \le \|x - x_{n+1}\|^2 - \|x - x_n\|^2,$$

and hence

$$\|x_n - x_{n+1}\| \to 0. \tag{3.7}$$

From $x_{n+1} \in C_n$, we have that $\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$. From (3.7), we have $\|y_n - x_{n+1}\| \to 0$. Since

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

we have that $\|y_n - x_n\| \to 0$. From

$$\|x_n - y_n\| = \big\|x_n - \alpha_n x_n - (1 - \alpha_n)T_{\mu_n}x_n\big\| = (1 - \alpha_n)\|x_n - T_{\mu_n}x_n\|$$

and $0 \le \alpha_n \le a < 1$, we have that

$$\|T_{\mu_n}x_n - x_n\| \to 0. \tag{3.8}$$

Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. We have from (3.8) that

$$T_{\mu_{n_i}}x_{n_i} \rightharpoonup v. \tag{3.9}$$

We have from (2.3) that, for $y \in C$ and $s, t \in S$,

$$2\langle T_s x_n - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 = \|T_s x_n - T_t y\|^2 - \|T_s x_n - y\|^2.$$

Applying μ_n to both sides of the equality, we have that

$$\begin{aligned} 2(\mu_n)_s\langle T_s x_n - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 \\ = (\mu_n)_s\big(\|T_s x_n - T_t y\|^2 - \|T_s x_n - y\|^2\big), \end{aligned}$$

and hence

$$\begin{aligned} 2\langle T_{\mu_n} x_n - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 \\ = (\mu_n)_s\big(\|T_s x_n - T_t y\|^2 - \|T_s x_n - y\|^2\big). \end{aligned}$$

Since $T_{\mu_{n_i}}x_{n_i} \rightharpoonup v$ and $\limsup_{i\to\infty}(\mu_{n_i})_s(\|T_s x_{n_i} - T_t y\|^2 - \|T_s x_{n_i} - y\|^2) \le 0$, we get that

$$2\langle v - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 \le 0.$$

Since $2\langle v - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 = \|v - T_t y\|^2 - \|v - y\|^2$, we have that

$$\|v - T_t y\|^2 \le \|v - y\|^2, \quad y \in C, t \in S. \tag{3.10}$$

Putting $y = v$, we have $v \in F(T_t)$. Therefore $v \in F(\mathcal{S})$.

Put $z_0 = P_{F(\mathcal{S})}x$. Since $z_0 = P_{F(\mathcal{S})}x \subset C_n \cap Q_n$ and $x_{n+1} = P_{C_n \cap Q_n}x$, we have that

$$\|x - x_{n+1}\|^2 \le \|x - z_0\|^2. \tag{3.11}$$

Since $\|\cdot\|^2$ is weakly lower semicontinuous, we also have from $x_{n_i} \rightharpoonup v$ that

$$\begin{aligned}\|x-v\|^2 &= \|x\|^2 - 2\langle x, v\rangle + \|v\|^2 \\ &\le \liminf_{i\to\infty}\big(\|x\|^2 - 2\langle x, x_{n_i}\rangle + \|x_{n_i}\|^2\big) \\ &= \liminf_{i\to\infty}\|x - x_{n_i}\|^2 \\ &\le \|x - z_0\|^2.\end{aligned}$$

From the definition of z_0 and $v \in F(\mathcal{S})$, we have $v = z_0$. This implies that $x_n \rightharpoonup z_0$. We finally show that $x_n \to z_0$. We have that

$$\|z_0 - x_n\|^2 = \|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle, \quad \forall n \in \mathbb{N}.$$

We have from (3.11) that

$$\begin{aligned}\limsup_{n\to\infty}\|z_0 - x_n\|^2 &= \limsup_{n\to\infty}\big(\|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle\big) \\ &\le \limsup_{n\to\infty}\big(\|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - x_n\rangle\big) \\ &= \|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - z_0\rangle \\ &= \|z_0 - x + x - z_0\|^2 \\ &= 0.\end{aligned}$$

Thus we obtain that $\lim_{n\to\infty}\|z_0 - x_n\| = 0$. Therefore, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

Next, we prove a strong convergence theorem by the shrinking projection method [16] for semigroups of not necessarily continuous mappings in a Hilbert space.

Theorem 3.2. *Let H be a real Hilbert space, and let C be a nonempty bounded convex closed subset of H. Let S be a commutative semitopological semigroup. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $F(\mathcal{S}) \neq \emptyset$. Suppose that*

$$\limsup_{\kappa}\sup_{x,y\in C}(\mu_\kappa)_s\big(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2\big) \le 0, \quad \forall t \in S \tag{3.12}$$

for every strongly asymptotically invariant net $\{\mu_\kappa\}$ of means on $C(S)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$. Let $C_1 = C$, and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, and $\{\alpha_n\} \subset [0,1]$ is a sequence such that $\liminf_{n\to\infty}\alpha_n < 1$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{S})}x$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

Proof. As in the proof of Theorem 3.1, $F(\mathcal{S})$ is closed and convex. Then there exists the metric projection of H onto $F(\mathcal{S})$. We shall show that C_n are closed and convex and that $F(\mathcal{S}) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from assumption that $C_1 = C$ is closed and convex and that $F(\mathcal{S}) \subset C_1$. Suppose that C_k is closed and convex and that$F(\mathcal{S}) \subset C_k$ for some $k \in \mathbb{N}$. We have that, for $z \in C_k$,

$$\begin{aligned}
&\|y_k - z\|^2 \leq \|x_k - z\|^2 \\
&\quad \iff \quad \|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z\rangle \leq 0.
\end{aligned}$$

Thus C_{k+1} is closed and convex. By induction, C_n are closed and convex for all $n \in \mathbb{N}$. Take $z \in F(\mathcal{S}) \subset C_k$. As in the proof of Theorem 3.1, we also have that

$$\|T_{\mu_k} x_k - z\|^2 \leq \|x_k - z\|^2.$$

Using this inequality, we have that

$$\begin{aligned}
\|y_k - z\|^2 &= \left\|\alpha_k x_k + (1-\alpha_k) T_{\mu_k} x_k - z\right\|^2 \\
&\leq \alpha_k \|x_k - z\|^2 + (1-\alpha_k)\|T_{\mu_k} x_k - z\|^2 \\
&\leq \alpha_k \|x_k - z\|^2 + (1-\alpha_k)\|x_k - z\|^2 \\
&= \|x_k - z\|^2.
\end{aligned}$$

This implies that $z \in C_{k+1}$. By induction, we have that $F(\mathcal{S}) \subset C_n$ for all $n \in \mathbb{N}$. Since C_n is nonempty, closed, and convex, there exists the metric projection P_{C_n} of H onto C_n. Thus $\{x_n\}$ is well defined.

Since $\{C_n\}$ is a nonincreasing sequence of nonempty closed convex subsets of H with respect to inclusion, it follows that

$$\emptyset \neq F(\mathcal{S}) \subset \operatorname*{M\text{-}lim}_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n. \tag{3.13}$$

Put $C_0 = \bigcap_{n=1}^{\infty} C_n$. By Theorem 2.3 we have that $\{P_{C_n} x\}$ converges strongly to $x_0 = P_{C_0} x$; that is,

$$x_n = P_{C_n} x \to x_0.$$

To complete the proof, it is sufficient to show that $x_0 = P_{F(\mathcal{S})} x$.

Since $x_n = P_{C_n} x$ and $x_{n+1} = P_{C_{n+1}} x \in C_{n+1} \subset C_n$, we have from (2.2) that

$$\begin{aligned}
0 &\leq 2\langle x - x_n, x_n - x_{n+1}\rangle \\
&= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\
&\leq \|x - x_{n+1}\|^2 - \|x - x_n\|^2.
\end{aligned} \tag{3.14}$$

Thus we get that

$$\|x - x_n\|^2 \leq \|x - x_{n+1}\|^2. \tag{3.15}$$

Furthermore, since $x_n = P_{C_n} x$ and $z \in F(\mathcal{S}) \subset C_n$, we have

$$\|x - x_n\|^2 \leq \|x - z\|^2. \tag{3.16}$$

Then $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$ and $\{T_{\mu_n} x_n\}$ are also bounded. From (3.14), we have

$$\|x_n - x_{n+1}\|^2 \leq \|x - x_{n+1}\|^2 - \|x - x_n\|^2.$$

Then we have that

$$\|x_n - x_{n+1}\|^2 \to 0. \tag{3.17}$$

From $x_{n+1} \in C_{n+1}$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Thus we get that $\|y_n - x_{n+1}\| \to 0$. Using this, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.18}$$

From $0 \leq \liminf_{n\to\infty} \alpha_n < 1$, we have a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that $\alpha_{n_i} \to \gamma$ and $0 \leq \gamma < 1$. From

$$\|x_n - y_n\| = \big\|x_n - \alpha_n x_n - (1 - \alpha_n) T_{\mu_n} x_n\big\| = (1 - \alpha_n)\|x_n - T_{\mu_n} x_n\|,$$

we have that

$$\|T_{\mu_{n_i}} x_{n_i} - x_{n_i}\| \to 0. \tag{3.19}$$

Since $x_{n_i} = P_{C_{n_i}} x \to x_0$, we have from (3.19) that $T_{\mu_{n_i}} x_{n_i} \to x_0$. As in the proof of Theorem 3.1, we have $x_0 \in F(\mathcal{S})$. Since $z_0 = P_{F(\mathcal{S})} x \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} x$, we have that

$$\|x - x_{n+1}\|^2 \leq \|x - z_0\|^2. \tag{3.20}$$

Then we have that

$$\|x - x_0\|^2 = \lim_{n\to\infty} \|x - x_n\|^2 \leq \|x - z_0\|^2.$$

From $z_0 = P_{F(\mathcal{S})} x$, we get $z_0 = x_0$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

4. Applications

In this section, we apply our results to get new results. Using Theorem 3.1, we can obtain a strong convergence theorem by the Nakajo and Takahashi hybrid method [11] for generalized hybrid mappings in a Hilbert space.

Theorem 4.1. *Let H be a Hilbert space, and let C be a closed convex subset of H. Let T be a generalized hybrid mapping of C into itself such that $F(T)$ is nonempty. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on ℓ^∞. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset [0, 1]$ satisfies $0 \leq \alpha_n \leq a < 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(T)} x$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Proof. Consider $S = \{0\} \cup \mathbb{N}$, and, from Theorem 3.1, let $\mathcal{S} = \{T^m : m \in \{0\} \cup \mathbb{N}\}$. Take $u \in F(T)$, and put $M = \{y \in C : \|y-u\| \leq \|x-u\|\}$. Then we have $x \in M$, $TM \subset M$ and M is bounded, closed, and convex. Since $T_{\mu_n} z \in M$ for all $z \in M$ and $n \in \mathbb{N}$, without loss of generality we may assume that C is bounded. Since $T : C \to C$ is generalized hybrid, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. We have that, for all $x, y \in C$ and $m \in \mathbb{N}$,

$$\begin{aligned}
0 &\leq \beta\|T^{m+1}x - y\|^2 + (1-\beta)\|T^m x - y\|^2 \\
&\quad - \alpha\|T^{m+1}x - Ty\|^2 - (1-\alpha)\|T^m x - Ty\|^2 \\
&= \beta\big\{\|T^{m+1}x - Ty\|^2 + 2\langle T^{m+1}x - Ty, Ty - y\rangle + \|Ty - y\|^2\big\} \\
&\quad + (1-\beta)\big\{\|T^m x - Ty\|^2 + 2\langle T^m x - Ty, Ty - y\rangle + \|Ty - y\|^2\big\} \\
&\quad - \alpha\|T^{m+1}x - Ty\|^2 - (1-\alpha)\|T^m x - Ty\|^2 \\
&= \|Ty - y\|^2 + 2\big\langle \beta T^{m+1}x + (1-\beta)T^m x - Ty, Ty - y\big\rangle \\
&\quad + (\beta - \alpha)\big\{\|T^{m+1}x - Ty\|^2 - \|T^m x - Ty\|^2\big\} \\
&= \|Ty - y\|^2 + 2\big\langle T^m x - Ty + \beta(T^{m+1}x - T^m x), Ty - y\big\rangle \\
&\quad + (\beta - \alpha)\big\{\|T^{m+1}x - Ty\|^2 - \|T^m x - Ty\|^2\big\},
\end{aligned}$$

and hence

$$\begin{aligned}
&2\langle T^m x - Ty, y - Ty\rangle - \|Ty - y\|^2 \\
&\quad \leq 2\beta\langle T^{m+1}x - T^m x, Ty - y\rangle + (\beta - \alpha)\big\{\|T^{m+1}x - Ty\|^2 - \|T^m x - Ty\|^2\big\}.
\end{aligned}$$

On the other hand, we have from (2.2) that

$$2\langle T^m x - Ty, y - Ty\rangle - \|Ty - y\|^2 = \|T^m x - Ty\|^2 - \|T^m x - y\|^2.$$

Then we have that

$$\begin{aligned}
&\|T^m x - Ty\|^2 - \|T^m x - y\|^2 \\
&\quad \leq 2\beta\langle T^{m+1}x - T^m x, Ty - y\rangle + (\beta - \alpha)\big\{\|T^{m+1}x - Ty\|^2 - \|T^m x - Ty\|^2\big\}.
\end{aligned}$$

If $\{\mu_\kappa\}$ is a strongly asymptotically invariant net of means on ℓ^∞, then we have that

$$\begin{aligned}
&(\mu_\kappa)_m\big(\|T^m x - Ty\|^2 - \|T^m x - y\|^2\big) \\
&\quad \leq 2\beta(\mu_\kappa)_m\langle T^{m+1}x - T^m x, Ty - y\rangle \\
&\qquad + (\beta - \alpha)(\mu_\kappa)_m\big(\big\{\|T^{m+1}x - Ty\|^2 - \|T^m x - Ty\|^2\big\}\big) \\
&\quad \leq 2|\beta|\|\ell_1^*\mu_\kappa - \mu_\kappa\| \sup_{m \in \mathbb{N}}\big|\langle T^m x, Ty - y\rangle\big| \\
&\qquad + |\beta - \alpha|\|\ell_1^*\mu_\kappa - \mu_\kappa\| \sup_{m \in \mathbb{N}}\|T^m x - Ty\|^2,
\end{aligned}$$

and hence

$$\limsup_\kappa \sup_{x,y \in C} (\mu_\kappa)_m\big(\|T^m x - Ty\|^2 - \|T^m x - y\|^2\big) \leq 0.$$

Therefore, we get the desired result from Theorem 3.1. $\square$

Using Theorem 3.1, we also get the following strong convergence theorem for semigroups of nonexpansive mappings in a Hilbert space.

Theorem 4.2. *Let H be a Hilbert space, and let C be a closed convex subset of H. Let S be a commutative semitopological semigroup with identity, and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset [0,1]$ satisfies $0 \le \alpha_n \le a < 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{S})} x$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

Proof. Take $u \in F(\mathcal{S})$, and put $M = \{y \in C : \|y - u\| \le \|x - u\|\}$. Then we have $x \in M$, $T_t M \subset M$ for all $t \in S$ and M is bounded, closed, and convex. Since $T_{\mu_n} z \in M$ for all $z \in M$ and $n \in \mathbb{N}$, without loss of generality we may assume that C is bounded. Since $\mathcal{S} = \{T_t : t \in S\}$ is a nonexpansive semigroup on C, we have that, for all $x, y \in C$ and $s, t \in S$,

$$\begin{aligned} &\|T_s x - T_t y\|^2 - \|T_s x - y\|^2 \\ &\quad = \|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2 + \|T_{s+t} x - T_t y\|^2 - \|T_s x - y\|^2 \\ &\quad \le \|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2 + \|T_s x - y\|^2 - \|T_s x - y\|^2 \\ &\quad = \|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2. \end{aligned}$$

If $\{\mu_\kappa\}$ is a strongly asymptotically invariant net of means on ℓ^∞, then we have that

$$\begin{aligned} &(\mu_\kappa)_s \big(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2 \big) \\ &\quad \le (\mu_\kappa)_s \big(\|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2 \big) \\ &\quad = (\mu_\kappa)_s \|T_s x - T_t y\|^2 - (\ell_t^* \mu_\kappa)_s \|T_s x - T_t y\|^2 \\ &\quad \le \|\mu_\kappa - \ell_t^* \mu_\kappa\| \sup_s \|T_s x - T_t y\|^2, \end{aligned}$$

and hence

$$\limsup_\kappa \sup_{x,y \in C} (\mu_\kappa)_s \big(\|T_s x - Ty\|^2 - \|T_s x - y\|^2 \big) \le 0.$$

Therefore, we get the desired result from Theorem 3.1. □

As in the proofs of Theorems 4.1 and 4.2, from Theorem 3.2 we also get the following strong convergence theorems by the shrinking projection method (see [16]).

Theorem 4.3. *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let T be a generalized hybrid mapping of C into itself such* *that $F(T)$ is nonempty. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on ℓ^∞. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $\{\alpha_n\} \subset [0,1]$ is a sequence such that $\liminf_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Theorem 4.4. *Let H be a Hilbert space, and let C be a closed convex subset of H. Let S be a commutative semitopological semigroup with identity, and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $\{\alpha_n\} \subset [0,1]$ is a sequence such that $\liminf_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{S})}x$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

Acknowledgment. The first author is partially supported by Grant-in-Aid for Scientific Research no. 15K04906 from the Japan Society for the Promotion of Science.

References

1. S. Atsushiba and W. Takahashi, *Nonlinear ergodic theorems without convexity for nonexpansive semigroups in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), no. 2, 209–219. Zbl 1279.47077. MR3086541. 65
2. F. E. Browder, "Nonlinear operators and nonlinear equations of evolution in Banach spaces" in *Nonlinear Functional Analysis (Chicago, 1968)*, Proc. Sympos. Pure Math. **XVIII**, Part 2, Amer. Math. Soc. Providence, 1976. MR0405188. 62
3. M. M. Day, *Amenable semigroup*, Illinois J. Math. **1** (1957), 509–544. MR0092128. 65
4. K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Stud. Adv. Math. **28**, Cambridge Univ. Press, Cambridge, 1990. Zbl 0708.47031. MR1074005. DOI 10.1017/CBO9780511526152. 62
5. B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961. Zbl 0177.19101. MR0218938. 62
6. N. Hussain and W. Takahashi, *Weak and strong convergence theorems for semigroups of mappings without continuity in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), no. 4, 769–783. Zbl pre06325417. MR3131145. 62
7. P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), no. 6, 2497–2511. Zbl 1226.47053. MR2761610. 61, 63
8. F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), no. 2, 166–177. Zbl 1149.47045. MR2430800. DOI 10.1007/s00013-008-2545-8. 62

9. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510. MR0054846. 62
10. U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. Math. **3** (1969), 510–585. MR0298508. 63
11. K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), no. 2, 372–379. Zbl 1035.47048. MR1974031. DOI 10.1016/S0022-247X(02)00458-4. 62, 66, 71
12. W. Takahashi, *Nonlinear Functional Analysis*, Yokohoma Publishers, Yokohoma, 2000. MR1864294. 64, 65
13. W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohoma Publishers, Yokohoma, 2009. Zbl 1183.46001. MR2548424. 62, 63
14. W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), no. 1, 79–88. MR2729999. 62
15. W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), no. 2, 399–406. Zbl 1230.47085. MR2858695. 61, 65
16. W. Takahashi, Y. Takeuchi, and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), no. 1, 276–286. Zbl 1134.47052. MR2394082. DOI 10.1016/j.jmaa.2007.09.062. 62, 69, 73
17. W. Takahashi, N.-C. Wong, and J.-C. Yao, *Attractive point and mean convergence theorems for semigroups of mappings without continuity in Hilbert spaces*, J. Nonlinear Convex Anal. **15** (2014), no. 6, 1087–1103. Zbl pre06400957. MR3293879. 62, 65
18. M. Tsukada, *Convergence of best approximations in a smooth Banach space*, J. Approx. Theory **40** (1984), no. 4, 301–309. MR0740641. DOI 10.1016/0021-9045(84)90003-0. 63

[1]Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 80702, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan.

E-mail address: wataru@is.titech.ac.jp; wataru@a00.itscom.net

[2]Department of Information Sciences, Toho University, Funabashi, Chiba 274-8510, Japan.

E-mail address: tsukada@is.sci.toho-u.ac.jp