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ON A NOTION OF CLOSENESS OF GROUPS

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Dedicated to Professor Anthony To-Ming Lau

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Abstract. Enlightened by the notion of perturbation of C^*-algebras, we introduce, and study briefly in this article, a notion of closeness of groups. We show that if two groups are "close enough" to each other, and one of them has the property that the orders of its elements have a uniform finite upper bound, then these two groups are isomorphic (but in general they are not). We also study groups that are close to abelian groups, as well as an equivalence relation induced by closeness.

1. Introduction

In 1972, Kadison and Kastler introduced in [2] the notion of "perturbation of C^*-algebras," and they asked what kinds of properties will be preserved if A is close enough to B in this sense. Many people followed this question and produced interesting results in C^*-algebras (see [1] and the references therein).

Using this idea, one can define a notion of closeness among (discrete) groups. We will give a brief study of this notion in this article. More precisely, we say that a group G is δ*-contained in* another group H if there are injective unitary representations of them on the same Hilbert space such that the image of G is contained in the image of H up to a distance of δ.

We obtained several results in this direction. In particular, let H be a "k-bounded" group for some $k \geq 2$ (i.e., the orders of all elements in H are

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bounded by k) and $\delta \in (0, \frac{1}{2}\sin\frac{\pi}{k})$; then any group G that is δ-contained in H is isomorphic to a subgroup of H (see Theorem 3.4). We will also study those groups that are arbitrarily close to abelian groups. Furthermore, using this notion of closeness, one can define an equivalence relation on different classes of groups, and we will consider certain interesting classes in this article.

2. Definitions and basic facts

Notation 2.1. Throughout this article, G and H are groups, and $\mathrm{R}_{\mathrm{inj}}(G)$ and $\mathrm{R}_{\mathrm{inj}}(H)$ are the collections of all injective unitary representations of G and H, respectively. Moreover, if $\mathfrak{H}$ is a Hilbert space, then $\mathcal{L}(\mathfrak{H})$ is the C^*-algebra of all bounded linear operators on $\mathfrak{H}$, while $U(\mathfrak{H})$ is the unitary group of $\mathcal{L}(\mathfrak{H})$. For any normed space E, we denote by $\mathbf{B}_E$ the closed unit ball of E.

Let us begin with the precise definition of closeness.

Definition 2.2.

(a) G is said to be *δ-contained in* (resp., *δ-close to*) H if there exist $(\mathfrak{H}, \phi) \in \mathrm{R}_{\mathrm{inj}}(G)$ and $(\mathfrak{H}, \psi) \in \mathrm{R}_{\mathrm{inj}}(H)$ such that
$$\phi(G) \subseteq \psi(H) + \delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})}$$
(resp., $\phi(G) \subseteq \psi(H) + \delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})}$ and $\psi(H) \subseteq \phi(G) + \delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})}$).

(b) G is said to be *asymptotically contained in* (resp., *asymptotically close to*) H if it is δ-contained in (resp., δ-close to) H for each $\delta > 0$.

Parts (b)–(d) of the following example tell us that groups which are asymptotically close to each other can have very different properties.

Example 2.3.

(a) $\mathbb{Z}_2$ is not δ-contained in $\mathbb{Z}_3$ if $0 < \delta < 2 - \sqrt{3}$. In fact, if $\mathfrak{H}$ is a Hilbert space and $u, v \in U(\mathfrak{H}) \setminus \{1\}$ with $u^2 = 1$ and $v^3 = 1$, then $\|1 - u\| = 2$ and $\|1 - v\| = \sqrt{3}$. Thus, $\|u - v\| \geq \|1 - u\| - \|1 - v\| \geq 2 - \sqrt{3}$.

(b) Let $t \in (0, 1)$ be an irrational number. Then $\theta : n \mapsto e^{2nt\pi \mathrm{i}}$ is an injective group homomorphism from $\mathbb{Z}$ into $\mathbb{T} := U(\mathbb{C})$ $(= \{z \in \mathbb{C} : |z| = 1\})$ with dense range. Therefore, $\mathbb{T}$ and $\mathbb{Z}$ are asymptotically close to each other. Note that $\mathbb{Z}$ is countable but $\mathbb{T}$ is not.

(c) Let $n_1, n_2, \ldots$ be a sequence of relatively prime numbers. By considering the canonical injective representation from $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ into $\mathbb{T}$ (sending $(\bar{k}_j)_{j\in\mathbb{N}}$ to $\prod_{j=1}^{\infty} e^{2k_j\pi \mathrm{i}/n_j}$), we see that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ is asymptotically contained in $\mathbb{Z}$. In the case when $\{p_1, p_2, \ldots\}$ is the set of all prime numbers, the image of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is dense in $\mathbb{T}$. This means that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is asymptotically close to $\mathbb{Z}$. Note that $\mathbb{Z}$ has a single generator, but the minimal number of generators of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is infinite.

(d) Let $D := \varinjlim \mathbb{Z}_{2^k}$ and $T := \varinjlim \mathbb{Z}_{3^k}$. Then both of them can be considered as dense subgroups of $\mathbb{T}$, and hence they are asymptotically close to each other. Note that all elements in both D and T are of finite order, but the order of any element in D is a power of 2, while the order of any element in T is a power of 3.

The following result gives a connection between closeness of groups and the notion of perturbation of Banach algebras in the sense of Kadison and Kastler [2].

Proposition 2.4. *If $(\mathfrak{H},\phi) \in \mathrm{R}_{\mathrm{inj}}(G)$ and $(\mathfrak{H},\psi) \in \mathrm{R}_{\mathrm{inj}}(H)$ satisfying $\phi(G) \subseteq \psi(H) + \delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})}$, then $\tilde\phi(\mathbf{B}_{\ell^1(G)}) \subseteq \tilde\psi(\mathbf{B}_{\ell^1(H)}) + 2\delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})}$ (where $\tilde\phi$ and $\tilde\psi$ are the induced representations).*

Proof. For every $r \in G$, we take $\theta(r) \in H$ with $\|\phi(r)-\psi(\theta(r))\| < \delta$. If $f \in \mathbf{B}_{\ell^1(G)}$, for each $u \in H$, then we set $g(u) := 0$ if $\theta^{-1}(u) = \emptyset$ and $g(u) := \sum_{r\in\theta^{-1}(u)} f(r)$ otherwise. Then $\sum_{u\in H} |g(u)| \le \sum_{r\in G} |f(r)| \le 1$. As $g \in \mathbf{B}_{\ell^1(H)}$ and

$$\big\|\tilde\phi(f) - \tilde\psi(g)\big\| = \Big\|\sum_{t\in G} f(t)\big(\phi(t) - \psi\big(\theta(t)\big)\big)\Big\| \le \delta,$$

we obtain the required inclusion. □

3. The main results

We say that a group H is *k-bounded* for some integer $k \ge 2$, if $o(t) \le k$ for each $t \in H$, where $o(t)$ is the order of t. The following observations are useful in our study.

Lemma 3.1.

(a) Let u be a unitary in a C^*-algebra, and let G_u be the subgroup of $\mathbb{T}$ generated by its spectrum $\sigma(u)$. Then the order, $o(G_u)$, of G_u equals $o(u)$.

(b) For each pair of positive integers (N, m), there is a strictly positive constant $\kappa_{N,m}$ such that for any Hilbert space $\mathfrak{H}$ and $u \in U(\mathfrak{H})$ with $o(u) = m$, if $v \in U(\mathfrak{H})$ satisfying $o(v) \le N$ and $\|u - v\| < \kappa_{N,m}$, then $o(v)$ divides m.

(c) Suppose that H is a k-bounded group. For any $(\psi,\mathfrak{H}) \in \mathrm{R}_{\mathrm{inj}}(H)$ and any two distinct elements $r, s \in H$, one has $\|\psi(r) - \psi(s)\| \ge 2\sin\frac{\pi}{k}$.

Proof. (a) This statement can be obtained easily by considering the C^*-subalgebra, $C^*(u)$, generated by u as well as the fact that $C^*(u) \cong C(\sigma(u))$.

(b) Let $\mathcal{G}_k$ be the collection of all generating subsets of $\mathbb{T}_k := \{z \in \mathbb{T} : z^k = 1\}$ ($k \in \mathbb{N}$), and let $Q_{N,m} := \{n \in \mathbb{N} : n \le N; n \nmid m\}$. If $Q_{N,m} = \emptyset$, then we can set $\kappa_{N,m} := 1$. Otherwise, for any $n \in Q_{N,m}$ and $S \in \mathcal{G}_n$, we set

$$\kappa^S := \max_{s\in S} \min\big\{|e^{\frac{2l\pi\mathrm{i}}{m}} - s| : l = 1,\dots,m\big\}.$$

Note that $\kappa^S > 0$ as $S \nsubseteq \mathbb{T}_m$. Define

$$\kappa_{N,m} := \min\{\kappa^S : S \in \mathcal{G}_n; n \in Q_{N,m}\} > 0.$$

Suppose on the contrary that $\|u-v\| < \delta < \kappa_{N,m}$ but $o(v) \in Q_{N,m}$. Then $\delta < \kappa^{\sigma(v)}$ because $\sigma(v) \in \mathcal{G}_{o(v)}$ by part (a). This implies that

$$\sigma(v) \nsubseteq \mathbb{T}_m + \delta\mathbf{B}_{\mathbb{C}}. \tag{3.1}$$

On the other hand, observe that if $\lambda \notin \sigma(u) + \delta\mathbf{B}_{\mathbb{C}}$, then

$$\big\|(\lambda - u)^{-1}\big\| = \sup\big\{|\lambda - \mu|^{-1} : \mu \in \sigma(u)\big\} \le \delta^{-1}.$$

This implies that $\|(\lambda - v) - (\lambda - u)\| < \delta < \|(\lambda - u)^{-1}\|^{-1}$ and $\lambda \notin \sigma(v)$. Therefore, we have $\sigma(v) \subseteq \sigma(u) + \delta \mathbf{B}_{\mathbb{C}}$. This contradicts (3.1) because $\sigma(u) \subseteq \mathbb{T}_m$ (by part (a)).

(c) The assertion follows from the following inequalities:

$$\big\|\psi(r) - \psi(s)\big\| = r_\sigma\big(1 - \psi(r^{-1}s)\big) \geq |1 - e^{2\pi \mathrm{i}/o(r^{-1}s)}| \geq 2\sin(\pi/k), \tag{3.2}$$

where $r_\sigma(\cdot)$ stands for the spectral radius. □

Remark 3.2.

(a) Let $n \neq m$ and $u, v \in U(\mathfrak{H})$ with $o(u) = m$ and $o(v) = n$. It follows from Lemma 3.1(b) that $\|u - v\| \geq \min\{\kappa_{n,m}, \kappa_{m,n}\} > 0$. This generalizes the situation in Example 2.3(a).

(b) By Example 2.3(d), one has $\inf\limits_{N\in\mathbb{N}} \kappa_{N,3} = 0$. Similarly, for any $m \in \mathbb{N}$, by considering $\varinjlim \mathbb{Z}_{m^k}$ and $\varinjlim \mathbb{Z}_{p^k}$ with p being a prime number not dividing m, one can show that $\inf\limits_{N\in\mathbb{N}} \kappa_{N,m} = 0$.

Example 3.3. For each integer $k \geq 2$, pick any $\delta \in \mathbb{R}_+ \setminus \{0\}$ with

$$\delta < \min\{\kappa_{N,m} : N, m = 1, 2, \ldots, k\} \qquad \text{and} \qquad \delta < \sin(\pi/k).$$

If G and H are two abelian groups with order less than k, which are δ-close to each other, then G and H are isomorphic.

In fact, let $(\phi, \mathfrak{H}) \in \mathrm{R}_{\mathrm{inj}}(G)$ and $(\psi, \mathfrak{H}) \in \mathrm{R}_{\mathrm{inj}}(H)$ be as in Definition 2.2. Then Lemma 3.1(c) gives a bijection θ from G onto H such that $\|\phi(r) - \psi(\theta(r))\| < \delta$ $(r \in G)$. Furthermore, Lemma 3.1(b) shows that θ preserves orders of elements. Now suppose that the common order of G and H is $n = p_1^{r_1} \cdots p_k^{r_k}$, where $p_1, \ldots, p_k$ are distinct prime numbers and $r_1, \ldots, r_k \in \mathbb{N}$. Then we can write G and H as direct sums of their Sylow subgroups

$$G = G(p_1) \oplus \cdots \oplus G(p_k) \qquad \text{and} \qquad H = H(p_1) \oplus \cdots \oplus H(p_k).$$

Since θ preserves the order of every element, we see that θ maps bijectively from the Sylow p_i-subgroup $G(p_i)$ onto the Sylow p_i-subgroup $H(p_i)$ $(i = 1, \ldots, k)$. Thus, one can assume that $n = p^r$.

In this case, $G = \bigoplus_{k=1}^{i} \mathbb{Z}_{p^{m_k}}$ and $H = \bigoplus_{l=1}^{j} \mathbb{Z}_{p^{n_l}}$, where $m_1 \leq \cdots \leq m_i$ and $n_1 \leq \cdots \leq n_j$. It is clear that $m_i = n_j$. Let $G_0 := \bigoplus_{k=1}^{i-1} \mathbb{Z}_{p^{m_k}}$ and $H_0 := \bigoplus_{l=1}^{j-1} \mathbb{Z}_{p^{n_l}}$. Suppose that $O_\Gamma(m)$ is the number of order m elements in a group Γ $(m \in \mathbb{N})$. If $(x, y) \in G_0 \oplus \mathbb{Z}_{p^{m_i}}$ and $l \in \mathbb{N}$, then $o(x, y) = p^l$ if and only if either $o(x) = p^l$ and $o(y) \leq p^l$, or $o(x) < p^l$ and $o(y) = p^l$. This shows that

$$O_G(p) = O_{G_0}(p)p + (p - 1)$$

as well as

$$O_G(p^l) = O_{G_0}(p^l)p^l + \big(1 + \cdots + O_{G_0}(p^{l-1})\big)(p^l - p^{l-1}) \quad (l = 2, 3, \ldots, m_i),$$

and similar equations hold for O_H and O_{H_0}. Since $O_G = O_H$, we know that $O_{G_0} = O_{H_0}$, which gives $m_{i-1} = n_{j-1}$. Inductively, one has $i = j$ and $m_k = n_k$ $(k = 1, \ldots, i)$.

Recall that the *spectrum* of a finite group G is the set $\omega(G) := \{o(g) : g \in G\}$, where $o(g)$ is the order of g. The problem of determining finite simple groups by their spectra was initiated in [3] and studied by some people. One may regard the function O_G in the above example as the *extended spectrum* of G (notice that $n \in \omega(G)$ if and only if $O_G(n) \neq 0$). The argument of Example 3.3 tells us that two asymptotically close finite groups will have the same extended spectrums. Moreover, the argument in this example also shows that the extended spectrum of a finite abelian group determines the group completely (which is likely a known fact). We do not know if the extended spectrum is a complete invariant for all finite groups. However, we will see in Theorem 3.4 below that if two finite groups are asymptotically close to each other, then they are indeed isomorphic.

3.1. **Asymptotic containment in groups of bounded order.** Motivated by Example 3.3, our first task is to consider in what situation closeness of groups will give rise to isomorphisms. The best situation we can think of is the following result (note that Example 2.3(d) tells us that the conclusion of this result can be false if one only assumes that every element in H has a finite order).

Theorem 3.4. *Suppose that $k \in \{2, 3, 4, \dots\}$ and that G and H are two groups such that H is k-bounded. Take any $\delta \in (0, \frac{1}{2}\sin\frac{\pi}{k})$.*

(a) If G is δ-contained in H, then G is isomorphic to a subgroup of H.
(b) If G is δ-close to H, then they are isomorphic.

Proof. (a) Let $(\mathfrak{H}, \phi, \psi)$ be as in Definition 2.2. If $u, v \in \psi(H)$ are distinct elements, then we have $\|u - v\| \geq 2\sin\frac{\pi}{k}$ (by Lemma 3.1(c)). Thus, if $r \in G$, then there exists exactly one $\theta(r) \in H$ with $\|\phi(r) - \psi(\theta(r))\| < \delta$. Since

$$\big\|\phi(r^{-1}) - \psi\big(\theta(r)^{-1}\big)\big\| = \big\|\psi\big(\theta(r)\big) - \phi(r)\big\| < \delta < \sin(\pi/k),$$

we see that $\theta(r^{-1}) = \theta(r)^{-1}$. Moreover, if $r_1, r_2 \in G$, we have

$$\begin{aligned}
&\big\|\phi(r_1 r_2) - \psi\big(\theta(r_1)\theta(r_2)\big)\big\| \\
&\quad\leq \big\|\phi(r_1)\phi(r_2) - \phi(r_1)\psi\big(\theta(r_2)\big)\big\| + \big\|\phi(r_1)\psi\big(\theta(r_2)\big) - \psi\big(\theta(r_1)\big)\psi\big(\theta(r_2)\big)\big\| \\
&\quad< \sin(\pi/k).
\end{aligned}$$

This implies that $\theta(r_1 r_2) = \theta(r_1)\theta(r_2)$. Therefore, θ is a group homomorphism. Finally, suppose on the contrary that θ is not injective. By the construction of θ, we have

$$\phi(\ker\theta) \subseteq 1 + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})}.$$

Therefore, $\sigma(\phi(s)) \subseteq 1 + 1/2\mathbf{B}_{\mathbb{C}}$ for every $s \in \ker\theta$. Now, if we take any $s_0 \in \ker\theta \setminus \{e\}$, then there always exists $n \in \mathbb{N}$ with $\sigma(\phi(s_0)^n) \nsubseteq 1 + 1/2\mathbf{B}_{\mathbb{C}}$ (by considering the C^*-subalgebra generated by the nontrivial unitary $\phi(s_0)$), which is a contradiction (because $s_0^n \in \ker\theta$).

(b) Let $(\mathfrak{H}, \phi, \psi)$ be as in Definition 2.2. As in the above, one can construct an injective homomorphism $\theta : G \to H$. To see that θ is also surjective, let $v \in H$. By assumption, there exists $t \in G$ with $\|\psi(v) - \phi(t)\| < \delta$. Thus, $\|\psi(v) - \psi(\theta(t))\| < \sin\frac{\pi}{k}$, and the inequality (3.2) ensures that $v = \theta(t)$. □

One may also reformulate Theorem 3.4 as follows.

Corollary 3.5. *For any integer $k \geq 2$, there exists $r > 0$ such that if $\mathfrak{H}$ is a Hilbert space and H is a k-bounded subgroup of $U(\mathfrak{H})$, then any subgroup $G \subseteq U(\mathfrak{H})$ satisfying $G \subseteq H + r\mathbf{B}_{\mathcal{L}(\mathfrak{H})}$ is a subgroup of H.*

3.2. **Asymptotic containment in abelian groups.** Next, we would like to consider when a group is asymptotically contained in an abelian group. A natural question is whether such a group should be abelian. For the moment, we only know that it is the case when every element in G is of finite order (or when H is k-bounded for some $k \in \mathbb{N}$). Nevertheless, we find some properties of such groups which are interesting by themselves. Notice that the trivial group $\{e\}$ is 2-bounded.

Definition 3.6.

(a) A subset $E \subseteq G$ is said to be *asymptotically included* in another subset $F \subseteq G$ if, for any $\delta > 0$, there is $(\mathfrak{H}, \phi) \in \mathrm{R}_{\mathrm{inj}}(G)$ with $\phi(E) \subseteq \phi(F) + \delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})}$.

(b) G is said to be *asymptotically abelian* if the set $C_G := \{s^{-1}r^{-1}sr : r, s \in G\}$ of commutators is asymptotically included in $\{e\}$.

(c) G is said to be *pairwise asymptotically abelian* if the subset $\{s^{-1}r^{-1}sr\}$ is asymptotically included in the subset $\{e\}$ for every $r, s \in G$.

Clearly, G is asymptotically abelian if and only if, for each $\epsilon > 0$, there is $(\mathfrak{H}, \phi) \in \mathrm{R}_{\mathrm{inj}}(G)$ such that $\|\phi(rs) - \phi(sr)\| < \epsilon$ $(r, s \in G)$. Moreover, if we set

$$N_G := \big\{s \in G : \{s\} \text{ is asymptotically included in } \{e\}\big\},$$

then G being pairwise asymptotically abelian is equivalent to $C_G \subseteq N_G$.

Lemma 3.7. *If $r, s \in G$ both have finite orders and $\{r\}$ is asymptotically included in $\{s\}$, then $o(r) = o(s)$. Consequently, any nontrivial element in N_G is of infinite order.*

Proof. This follows from Remark 3.2(a). □

It follows from Proposition 3.8(b) below that the above can fail if only one of the elements is assumed to have finite order. This proposition also tells us that N_G can be very big even when G is abelian.

Proposition 3.8.

(a) If every element in G has a finite order and G is pairwise asymptotically abelian, then G is abelian.

(b) If G is a finitely generated infinite abelian group, then N_G contains all the elements of G with infinite orders. Consequently, $G = \{s + t : s, t \in N_G\}$.

Proof. (a) By Lemma 3.7 and the hypothesis, we have $N_G = \{e\}$, and hence $C_G = \{e\}$ (as G is pairwise asymptotically abelian).

(b) We may write $G = \mathbb{Z}^i \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_j}$ (notice that this includes the case of $G = \mathbb{Z}^i$ if we take $j = 1$ and $m_1 = 1$). Let $(n_1^0, \ldots, n_i^0, \bar{l}_1^0, \ldots, \bar{l}_j^0) \in G$ be an element of infinite order. Without loss of generality, we can assume $n_1^0 \neq 0$. For any $k \in \{1, \ldots, j\}$ and $\epsilon > 0$, there exists an irrational number $t_k \in (0, 1)$ such that $|1 - e^{2\pi\mathrm{i}(n_1^0 t_k + l_k^0/m_k)}| < \epsilon$. On the other hand, we choose any irrational numbers $s_2, \dots, s_i \in (0,1)$ with $|1 - e^{2\pi \mathrm{i} n_k^0 s_k}| < \epsilon$ $(k = 2, \dots, i)$. Then $\phi : G \to \mathbb{T}^{i+j-1} \subseteq U(\mathbb{C}^{i+j-1})$ given by

$$\phi(n_1, \dots, n_i, \bar{l}_1, \dots, \bar{l}_j) := (e^{2\pi \mathrm{i} n_2 s_2}, \dots, e^{2\pi \mathrm{i} n_i s_i}, e^{2\pi \mathrm{i}(n_1 t_1 + l_1/m_1)}, \dots, e^{2\pi \mathrm{i}(n_1 t_j + l_j/m_j)})$$

is an injective representation of G satisfying $\|1 - \phi(n_1^0, \dots, n_i^0, \bar{l}_1^0, \dots, \bar{l}_j^0)\| < \epsilon$. This shows that $(n_1^0, \dots, n_i^0, \bar{l}_1^0, \dots, \bar{l}_j^0) \in N_G$. Finally, for each $(\bar{l}_1^0, \dots, \bar{l}_j^0) \in \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_j}$, we have $(-1, 0, \dots, 0, \bar{0}, \dots, \bar{0}), (1, 0, \dots, 0, \bar{l}_1^0, \dots, \bar{l}_j^0) \in N_G$. Thus, any element in G is a sum of two elements in N_G. □

In the following, let $C_G^n := \{t_1 \cdots t_n : t_1, \dots, t_n \in C_G\}$ $(n \in \mathbb{N})$.

Proposition 3.9.

(a) If G is asymptotically contained in an abelian group H, then G is asymptotically abelian.

(b) If G is asymptotically abelian, then C_G^n contains no nontrivial subgroup of G for all $n \in \mathbb{N}$.

(c) If G contains a finite nonabelian subgroup H, then G is not asymptotically contained in an abelian group.

Proof. (a) For any $\delta > 0$, let $(\mathfrak{H}, \phi, \psi)$ be as in Definition 2.2. For any $r, r' \in G$, there are $u, u' \in H$ with $\|\phi(r) - \psi(u)\| < \delta$ and $\|\phi(r') - \psi(u')\| < \delta$, which implies that $\|\phi(r)\phi(r') - \phi(r')\phi(r)\| < 4\delta$.

(b) Let $H \subseteq C_G^n$ be a subgroup of G. As G is asymptotically abelian, it is easy to see that C_G^n is asymptotically included in $\{e\}$. Hence, H is asymptotically contained in $\{e\}$ (which is a 2-bounded group). Now, Theorem 3.4(a) ensures that $H = \{e\}$.

(c) By Proposition 3.8(a), G is not pairwise asymptotically abelian (otherwise, H will be abelian). Thus, G is not asymptotically contained in any abelian group, because of part (a). □

We end this subsection with the following example, which tells us that a pairwise asymptotically abelian group need not be asymptotically abelian.

Example 3.10. (a) Let γ be the action of $\mathbb{Z}_2 := \{1, -1\}$ (with the usual multiplication) on $\mathbb{Z}$ given by $\gamma_\alpha(k) = \alpha k$ ($\alpha \in \mathbb{Z}_2$ and $k \in \mathbb{N}$), and let G be the semidirect product $\mathbb{Z} \rtimes_\gamma \mathbb{Z}_2$. For any $(k, \alpha), (l, \beta) \in G$, we have

$$(k, \alpha)^{-1}(l, \beta)^{-1}(k, \alpha)(l, \beta) = (\alpha\beta k + \beta l - \alpha\beta l - \alpha k, 1),$$

which equals either $(0, 1)$, $(-2k, 1)$, $(2l, 1)$, or $(2k - 2l, 1)$. This shows that $C_G = 2\mathbb{Z} \times \{1\}$. Thus, Proposition 3.9(b) tells us that G is not asymptotically abelian and hence is not asymptotically contained in any abelian group. Next, we show that G is pairwise asymptotically abelian. Let

$$V := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad W_\theta := \begin{pmatrix} e^{-\pi \mathrm{i}\theta} & 0 \\ 0 & e^{\pi \mathrm{i}\theta} \end{pmatrix} \quad \big(\theta \in (0, 1/2) \setminus \mathbb{Q}\big).$$

As $VW_\theta V = W_\theta^*$, the map $\phi_\theta : G \to U(\mathbb{C}^2)$ given by $\phi_\theta(k, 1) := W_\theta^k$ and $\phi_\theta(k, -1) := W_\theta^k V$ $(k \in \mathbb{Z})$ is an injective group homomorphism. Moreover, for any $k \in \mathbb{Z}$,

$$\|1 - W_\theta^{2k}\| = \|1 - W_{2k\theta}\| = 2 \sin k\theta$$

can be made arbitrarily small if θ is small enough. This shows that any singleton subset of C_G is asymptotically included in $\{e\}$ as required.

(b) Let γ be the action of $\mathbb{Z}$ on $\mathbb{Z}\oplus\mathbb{Z}$ given by $\gamma_k(m,n)=(m+kn,n)$, and $G:=(\mathbb{Z}\oplus\mathbb{Z})\rtimes_\gamma\mathbb{Z}$. It is not hard to see that $C_G=(\mathbb{Z}\oplus\{0\})\times\{0\}$, and G is not asymptotically abelian by Proposition 3.9(b). Suppose that $\theta\in(0,2)$ is irrational and $u,v\in U(\mathfrak{H})$ are such that $uv=e^{\theta\pi\mathrm{i}}vu$. Then $\phi(((m,n),k)):=e^{m\theta\pi\mathrm{i}}v^nu^k$ is an injective representation of G. In fact, if $e^{m\theta\pi\mathrm{i}}v^nu^k=1_{\mathfrak{H}}$, then $e^{(m-nk)\theta\pi\mathrm{i}}u^kv^n=1_{\mathfrak{H}}$, which implies that $k=0=n$ and hence $m=0$ as well. Therefore, any singleton subset of C_G is asymptotically included in $\{e\}$, and G is pairwise asymptotically abelian.

In Example 3.10(b), the C^*-subalgebra generated by u and v is the irrational rotation algebra (yet this C^*-algebra is far from being the group C^*-algebra of $(\mathbb{Z}\oplus\mathbb{Z})\rtimes_\gamma\mathbb{Z}$).

3.3. **Asymptotic equivalence.** One can define an equivalence relation in a collection of groups using asymptotic closeness.

Definition 3.11. If $G_1,\dots,G_n$ are groups in a class $\mathcal{G}$ such that G_i is asymptotically close to G_{i+1} for every $i=1,\dots,n-1$, then we say that G_1 and G_n are *asymptotically equivalent inside* $\mathcal{G}$. When $\mathcal{G}$ is the class of all groups, we simply say that G_1 and G_n are *asymptotically equivalent*.

It is natural to ask how this equivalence relation behaves in some well-known classes of groups. Note, first of all, that by Propositions 3.8(a) and 3.9(a), we have the following result.

Corollary 3.12. *Let $\mathcal{G}_f$ be the class of groups whose elements are all of finite orders. If G and H are asymptotically equivalent inside $\mathcal{G}_f$, then G is abelian if and only if H is abelian.*

Note that the groups in Corollary 3.12 need not be isomorphic (see Example 2.3(d)). Yet, they will be isomorphic if we impose a stronger assumption on them, as stated in the following corollary, which is a direct consequence of Theorem 3.4(b).

Corollary 3.13.

(a) *If G is k-bounded for some integer $k\geq 2$ and is asymptotically equivalent to H, then G is isomorphic to H.*
(b) *If two groups are asymptotically equivalent, then either they are both finite and isomorphic or they are both infinite.*

Finally, we have the following result concerning finitely generated abelian groups.

Corollary 3.14. *Suppose that $\mathcal{H}$ is the class of all finitely generated infinite abelian groups. Any two elements in $\mathcal{H}$ are asymptotically equivalent inside $\mathcal{H}$.*

Proof. Suppose that $G=\mathbb{Z}^k\oplus\mathbb{Z}_{m_1}\oplus\cdots\oplus\mathbb{Z}_{m_j}$ (this includes the case of $G=\mathbb{Z}^k$ when $j=1=m_1$). There exist irrational numbers $t_1,\dots,t_k\in(0,1)$ such that

$\{1, t_1, \ldots, t_k\}$ are $\mathbb{Q}$-linearly independent (notice that as $\mathbb{Q} + \mathbb{Q}t_1 + \cdots + \mathbb{Q}t_k$ is countable, one can obtain this claim by induction). Then

$$(n_1, \ldots, n_k) \mapsto e^{(n_1 t_1 + \cdots + n_k t_k)2\pi \mathrm{i}}$$

is an injective representation of $\mathbb{Z}^k$ in $U(\mathbb{C})$ with dense range. Thus, $\mathbb{Z}$ is asymptotically close to $\mathbb{Z}^k$ (see Example 2.3(b)). Consequently, G is asymptotically close to $G_1 := \mathbb{Z} \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_j}$. If $t \in (0,1)$ is an irrational number, then $(n, k_1) \mapsto e^{(nt + \frac{k_1}{m_1})2\pi \mathrm{i}}$ is an injective representation of $\mathbb{Z} \oplus \mathbb{Z}_{m_1}$ in $U(\mathbb{C})$ with dense range. Therefore, $\mathbb{Z}$ is asymptotically close to $\mathbb{Z} \oplus \mathbb{Z}_{m_1}$, and so are G_1 and $G_2 := \mathbb{Z} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_j}$. Inductively, we see that G is asymptotically equivalent to $\mathbb{Z}$ inside $\mathcal{H}$, and so is H. $\square$

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References

1. E. Christensen, A. M. Sinclair, R. R. Smith, S. A. White, and W. Winter, *Perturbation of nuclear C^*-algebras*, Acta Math. **208** (2012), no. 1, 93–150. MR2910797. DOI 10.1007/s11511-012-0075-5. 24
2. R. V. Kadison and D. Kastler, *Perturbations of von Neumann algebras, I: Stability of type*, Amer. J. Math. **94** (1972), 38–54. Zbl 0239.46070. MR0296713. 24, 26
3. W. J. Shi, *A characteristic property of* $\mathrm{PSL}_2(7)$, J. Austral. Math. Soc. Ser. A **36** (1984), no. 3, 354–356. MR0733907. 28

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