Ann. Funct. Anal. 7 (2016), no. 1, 1–8
http://dx.doi.org/10.1215/20088752-3158073
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

GEOMETRIC MEAN AND NORM SCHWARZ INEQUALITY

TSUYOSHI ANDO

Dedicated to Professor Anthony To-Ming Lau

Communicated by M. S. Moslehian

Abstract. Positivity of a 2×2 operator matrix $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$ implies $\sqrt{\|A\| \cdot \|C\|} \geq \|B\|$ for operator norm $\| \cdot \|$. This can be considered as an operator version of the Schwarz inequality. In this situation, for $A, C \geq 0$, there is a natural notion of geometric mean $A \sharp C$, for which $\sqrt{\|A\| \cdot \|C\|} \geq \|A \sharp C\|$. In this paper, we study under what conditions on A, B, and C or on B alone the norm inequality $\sqrt{\|A\| \cdot \|C\|} \geq \|B\|$ can be improved as $\|A \sharp C\| \geq \|B\|$.

1. Introduction and preliminaries

Let $A, B, C, \ldots, W, X, Y$ be bounded linear operators on a Hilbert space $\mathcal{H}$ with inner product $\langle \ , \ \rangle$. Here $A \geq 0$ means that A is *positive* (*semidefinite*); that is, $\langle x | Ax \rangle \geq 0$ for all $x \in \mathcal{H}$. When $A \geq 0$ is invertible, we write $A > 0$. The order relation $X \geq Y$ means that both X and Y are self-adjoint and $X - Y \geq 0$.

A 2×2 operator matrix $\left[\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right]$ is considered as a bounded linear operator on the direct sum Hilbert space $\mathcal{H} \oplus \mathcal{H}$ in the natural way (see [5, Chapter 8]).

It is well known that positivity of a 2×2 operator matrix $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$ implies $\sqrt{\|A\| \cdot \|C\|} \geq \|B\|$ for operator norm $\| \cdot \|$. This can be considered as an operator version of the Schwarz inequality. In fact, since, for $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$,

$$\begin{bmatrix} \sum_{j=1}^n X_j X_j^* & \sum_{j=1}^n X_j Y_j^* \\ \sum_{j=1} Y_j X_j^* & \sum_{j=1}^n Y_j Y_j^* \end{bmatrix} = \sum_{j=1}^n \begin{bmatrix} X_j \\ Y_j \end{bmatrix} \cdot \begin{bmatrix} X_j \\ Y_j \end{bmatrix}^* \geq 0,$$

Received Dec. 5, 2014; Accepted Dec. 17, 2014.
2010 *Mathematics Subject Classification.* Primary 47A64; Secondary 47A30, 47B15, 47A63.
Keywords. geometric mean, norm Schwarz inequality, norm inequality, normal operator.

the following Schwarz inequality follows:

$$\sqrt{\Big\|\sum_{j=1}^{n} X_i X_j^*\Big\| \cdot \Big\|\sum_{j=1}^{n} Y_j Y_j^*\Big\|} \geq \Big\|\sum_{j=1}^{n} X_j Y_j^*\Big\|.$$

For other versions of the operator Schwarz-type inequalities, see [1] and [4] and references therein.

On the other hand, there is a well-established notion of geometric mean $A\sharp C$ for $A, C \geq 0$, for which $\sqrt{\|A\| \cdot \|C\|} \geq \|A\sharp C\|$ always.

In this paper, we study under what conditions on A, B, and C or on B alone $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$ implies $\|A\sharp C\| \geq \|B\|$. We say that the *norm Schwarz inequality* holds for $A, C \geq 0$ and B if $\|A\sharp C\| \geq \|B\|$.

In particular, we are interested in establishing conditions on B such that

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \quad \Longrightarrow \quad \|A\sharp C\| \geq \|B\|. \tag{†}$$

In the remaining part of this section we summarize known properties of a 2×2 operator matrix and also those of geometric mean.

Lemma 1.1 ([3, Chapter 1])**.** *The following statements are mutually equivalent:*

(1) $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$;
(2) $\left[\begin{smallmatrix} C & B^* \\ B & A \end{smallmatrix}\right] \geq 0$;
(3) $A, C \geq 0$ *and* $B = A^{1/2} W C^{1/2}$ *for some* W *with* $\|W\| \leq 1$;
(4) $A \geq 0$ *and* $C \geq B^*(A + \epsilon I)^{-1} B$ *for all* $\epsilon > 0$ *with identity operator* I; *when* $A > 0$, $B^*(A + \epsilon I)^{-1} B$ *can be replaced simply by* $B^* A^{-1} B$.

The *geometric mean* $A\sharp C$ for $A, C > 0$ is defined as

$$A\sharp C := A^{1/2} \cdot (A^{-1/2} C A^{-1/2})^{1/2} \cdot A^{1/2}. \tag{1.1}$$

Lemma 1.2 ([3, Chapter 4])**.** *Geometric mean for* $A, C > 0$ *has the following properties:*

(1) $A\sharp C = C\sharp A$;
(2) $A\sharp C = (AC)^{1/2}$ *when* $AC = CA$;
(3) $A^{-1}\sharp C^{-1} = (A\sharp C)^{-1}$;
(4) $(\alpha A)\sharp(\beta C) = \sqrt{\alpha\beta}(A\sharp C)$;
(5) $A \longmapsto A\sharp C$ *is monotone increasing;*
(6) $\left[\begin{smallmatrix} A & A\sharp C \\ A\sharp C & C \end{smallmatrix}\right] \geq 0$ *and* $A\sharp C = \max\{X \geq 0; \left[\begin{smallmatrix} A & X \\ X & C \end{smallmatrix}\right] \geq 0\}$;
(7) $(X^* A X)\sharp(X^* C X) \geq X^*(A\sharp C)X$ *for all* X.

In view of monotonicity, the notion of *geometric mean* is uniquely extended to all $A, C \geq 0$ as the limit in the strong operator topology:

$$A\sharp C := \lim_{\epsilon \downarrow 0} (A + \epsilon I)\sharp(C + \epsilon I).$$

Since

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \quad \Longleftrightarrow \quad \begin{bmatrix} A + \epsilon I & B \\ B^* & C + \epsilon I \end{bmatrix} \quad \forall \epsilon > 0$$

and

$$\|A\sharp C\| = \lim_{\epsilon\downarrow 0}\big\|(A+\epsilon I)\sharp(C+\epsilon I)\big\|,$$

throughout our discussions on the norm Schwarz inequality we may assume always that $A > 0$ and $C > 0$ in the inequality $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$.

2. A necessary condition for (†)

Let $r(X)$ denote the *spectral radius* of X; that is,

$$r(X) := \max\big\{|\lambda|; \lambda I - X \text{ is not invertible}\big\}.$$

Since it is known (see [5, p. 48]) that $r(X)$ is described by using norms of iterates of X as

$$r(X) = \lim_{n\to\infty} \|X^n\|^{1/n},$$

we can see

$$r(X) \leq \|X\| \qquad \text{and} \qquad r(XY) = r(YX) \quad \forall X, Y. \tag{2.1}$$

An operator B is called *normaloid* if $r(B) = \|B\|$. A *normal* operator B—that is, $B^*B = BB^*$—in particular, a self-adjoint operator, is normaloid (see [5, p. 110]).

Since, by Lemma 1.1,

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \quad \Longrightarrow \quad C \geq B^*A^{-1}B \qquad \text{and} \qquad \begin{bmatrix} A & B \\ B^* & B^*A^{-1}B \end{bmatrix} \geq 0 \tag{2.2}$$

and, by (1.1),

$$A\sharp(B^*A^{-1}B) = A^{1/2}\cdot|A^{-1/2}BA^{-1/2}|\cdot A^{1/2}, \tag{2.3}$$

where $|X| := (X^*X)^{1/2}$ is the *modulus* of X, property (†) for B is equivalent to the following:

$$\big\|A^{1/2}\cdot|A^{-1/2}BA^{-1/2}|\cdot A^{1/2}\big\| \geq \|B\| \quad \forall A > 0. \tag{‡}$$

Lemma 2.1. *We have* $\|A^{-1/2}BA^{1/2}\| \geq \|A^{1/2}\cdot|A^{-1/2}BA^{-1/2}|\cdot A^{1/2}\|$ $\forall A > 0, B$.

Proof. Since both sides are positive homogeneous of order 1 with respect to B, it suffices to prove that

$$1 = \|A^{-1/2}BA^{1/2}\| \quad \Longrightarrow \quad I \geq A^{1/2}\cdot|A^{-1/2}BA^{-1/2}|\cdot A^{1/2}.$$

Now $1 = \|A^{-1/2}BA^{1/2}\|$ implies $I \geq A^{1/2}B^*A^{-1}BA^{1/2}$, and hence

$$A^{-2} \geq |A^{-1/2}BA^{-1/2}|^2.$$

In view of the Löwner theorem (see [3, p. 22]), this implies

$$A^{-1} \geq |A^{-1/2}BA^{-1/2}|; \qquad \text{hence } I \geq A^{1/2}\cdot|A^{-1/2}BA^{-1/2}|\cdot A^{1/2}. \qquad \square$$

Theorem 2.2. *If* (†)*, equivalently* (‡)*, holds for* B*, then* B *is a normaloid.*

Proof. By Lemma 2.1 and (‡), we have

$$\|A^{-1/2}BA^{1/2}\| \geq \big\|A^{1/2} \cdot |A^{-1/2}BA^{-1/2}| \cdot A^{1/2}\big\| \geq \|B\| \quad \forall A > 0.$$

Now the assertion follows from the following characterization of the spectral radius (see [5, p. 82]):

$$r(B) = \inf\big\{\|A^{-1/2}BA^{1/2}\| : A > 0\big\}. \qquad \square$$

In the converse direction we have the following.

Theorem 2.3. *If B is normaloid, then $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$ implies $\|A^{1/2}C^{1/2}\| \geq \|B\|$.*

Proof. In view of Lemma 1.1, we have, by (2.1),

$$\begin{aligned}\|B\| &= r(B) = r(A^{1/2}WC^{1/2}) \\ &= r(WC^{1/2}A^{1/2}) \leq \|C^{1/2}A^{1/2}\| = \|A^{1/2}C^{1/2}\|. \end{aligned} \qquad \square$$

Finally, since by Lemma 1.2 $\left[\begin{smallmatrix} A & A\sharp C \\ A\sharp C & C \end{smallmatrix}\right] \geq 0$ and $A\sharp C$ is self-adjoint, and hence normaloid, we have, from Theorem 2.3,

$$\|A^{1/2}C^{1/2}\| \geq \|A\sharp C\| \quad \forall A, C \geq 0.$$

A little sharper inequality holds (see [2, Corollary 3.4]):

$$\|A^{1/2}C^{1/2}\| \geq \|A^{1/4}C^{1/2}A^{1/4}\| \geq \|A\sharp C\| \quad \forall A, C \geq 0.$$

3. Sufficient conditions

Lemma 3.1. *We have*

$$\begin{bmatrix} A_j & B \\ B^* & C_j \end{bmatrix} \geq 0 \quad (j = 1, 2) \quad \Longrightarrow \quad \begin{bmatrix} A_1\sharp A_2 & B \\ B^* & C_1\sharp C_2 \end{bmatrix} \geq 0.$$

Proof. In view of Lemma 1.1, the assumption means that $B^*A_j^{-1}B \leq C_j$ $(j = 1, 2)$, which implies, by Lemma 1.2,

$$B^*(A_1\sharp A_2)^{-1}B = B^*(A_1^{-1}\sharp A_2^{-1})B \leq (B^*A_1^{-1}B)\sharp(B^*A_2^{-1}B) \leq C_1\sharp C_2.$$

Again, by Lemma 1.1, this implies $\left[\begin{smallmatrix} A_1\sharp A_2 & B \\ B^* & C_1\sharp C_2 \end{smallmatrix}\right] \geq 0$. $\square$

Theorem 3.2. *We have*

$$\sqrt{\big\|A\sharp(B^*A^{-1}B)\big\| \cdot \big\|A\sharp(BA^{-1}B^*)\big\|} \geq \|B\| \quad \forall A > 0, B.$$

Proof. This follows from Lemma 3.1 via

$$\begin{bmatrix} A & B \\ B^* & B^*A^{-1}B \end{bmatrix} \geq 0 \qquad \text{and} \qquad \begin{bmatrix} BA^{-1}B^* & B \\ B^* & A \end{bmatrix} \geq 0.$$

$\square$

Recall that the *partial transpose* of a 2×2 operator matrix $\left[\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right]$ is defined as its block-wise transpose, that is, as $\left[\begin{smallmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \end{smallmatrix}\right]$.

Theorem 3.3. *If a 2×2 positive operator matrix $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right]$ has positive partial transpose, then $\|A\sharp C\| \geq \|B\|$; that is,*

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \qquad \text{and} \qquad \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0 \quad \Longrightarrow \quad \|A\sharp C\| \geq \|B\|.$$

Proof. Since by Lemma 1.1 the second assumption is equivalent to $\left[\begin{smallmatrix} C & B \\ B^* & A \end{smallmatrix}\right] \geq 0$, it follows from Lemma 3.1 that $\left[\begin{smallmatrix} A\sharp C & B \\ B^* & C\sharp A \end{smallmatrix}\right] \geq 0$, which implies

$$\|A\sharp C\| = \sqrt{\|A\sharp C\| \cdot \|C\sharp A\|} \geq \|B\|.$$ □

Theorem 3.3 can be automatically applied to the case of self-adjoint B. But a little more can be said as an extension of Lemma 1.2.

Theorem 3.4. *If B is self-adjoint and $\left[\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right] \geq 0$, then $A\sharp C \geq \pm B$.*

Proof. In the proof of Theorem 3.3 it is shown that $\left[\begin{smallmatrix} A\sharp C & B \\ B & A\sharp C \end{smallmatrix}\right] \geq 0$, which implies immediately that $A\sharp C \geq \pm B$. □

Theorem 3.5. *If B is a scalar multiple of a unitary operator, then*

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \quad \Longrightarrow \quad \|A\sharp C\| \geq \|B\|.$$

Proof. We may assume that B is unitary, and we will prove via (2.2) that

$$\left\|A\sharp(B^*A^{-1}B)\right\| \geq 1 = \|B\|.$$

Suppose, by contradiction, that

$$\left\|A\sharp(B^*A^{-1}B)\right\| \leq \frac{1}{1+\epsilon} \quad \exists \epsilon > 0$$

or, equivalently, by Lemma 1.2,

$$A^{-1}\sharp(B^*AB) \geq (1+\epsilon)I \quad \exists \epsilon > 0.$$

Since $A^{-1}\sharp(B^*AB) = A^{-1/2}|A^{1/2}BA^{1/2}|A^{-1/2}$ by (2.3), this leads to

$$|A^{1/2}BA^{1/2}| \geq (1+\epsilon)A, \qquad \text{and hence} \qquad \|A\| \geq (1+\epsilon)\|A\|,$$

which is a contradiction. □

Theorem 3.6. *Positivity $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$ implies $\|A\sharp C\| \geq \|B\|$ if one of the following conditions is satisfied:*

(1) $AB = BA$;
(2) $B^*A^{-1}B = BA^{-1}B^*$;
(3) $C = \alpha A$ $\exists \alpha > 0$.

Proof. (1) Commutativity implies $B^*A^{-1}B = |B|A^{-1}|B|$, so that $\left[\begin{smallmatrix} A & |B| \\ |B| & C \end{smallmatrix}\right] \geq 0$ by Lemma 1.1. Now appeal to Lemma 1.2.

(2) Appeal to Theorem 3.2.

(3) Appeal to Lemma 1.2. □

4. The case of normal B

Theorems 3.4 and 3.5 suggest that (†), equivalently (‡), will hold for all normal B. At present we can settle (‡) for all normal B only when $\dim(\mathcal{H}) = n = 2$. The proof, as seen below, is quite specialized to the case $n = 2$. We are rather pessimistic even for the case $n = 3$, but we cannot find a counterexample.

Denote by $\mathbb{M}_2$ the space of 2×2 complex matrices, identified with the space of all (bounded) linear operators on the 2-dimensional Hilbert space.

Theorem 4.1. *For $0 < A \in \mathbb{M}_2$ and normal $B \in \mathbb{M}_2$,*

$$\big\|A^{1/2} \cdot |A^{-1/2}BA^{-1/2}| \cdot A^{1/2}\big\| \geq \|B\|. \tag{‡}$$

Proof. We may assume that $B \neq 0$. Since both sides of the inequality in (‡) are positive homogeneous of order 1 with respect to B, it suffices to show that

$$I \geq D := A^{1/2} \cdot |A^{-1/2}BA^{-1/2}| \cdot A^{1/2} \quad \Longrightarrow \quad 1 \geq \|B\|. \tag{4.1}$$

Now we have, by definition of modulus,

$$A^{-1/2}DA^{-1}DA^{-1/2} = A^{-1/2}B^*A^{-1}BA^{-1/2},$$

and hence, with $S := A^{-1} > 0$,

$$B^*SB = DSD. \tag{4.2}$$

Since $B \in \mathbb{M}_2$ is normal, we may assume that it is a diagonal matrix (see [5, p. 92]):

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{4.3}$$

Write

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}. \tag{4.4}$$

Then it follows from $D \geq 0$ and $S > 0$ that

$$d_{11}, d_{22} \geq 0, \qquad \overline{d_{12}} = d_{21} \qquad \text{and} \qquad s_{11}, s_{22} > 0, \qquad \overline{s_{12}} = s_{21}. \tag{4.5}$$

If $\min(d_{11}, d_{22}) = 0$, then $D \geq 0$ implies $d_{12} = 0 = d_{21}$. Then it follows from (4.2) and (4.3) that $|\lambda_j| = d_{jj}$ $(j = 1, 2)$, so that

$$\|B\| = \max\big(|\lambda_1|, |\lambda_2|\big) = \max(d_{11}, d_{22}) \leq \|D\| \leq 1.$$

Therefore we may assume that $d_{jj} > 0$ $(j = 1, 2)$.

Taking determinants of both sides of (4.2) we have by (4.3)

$$|\lambda_1\lambda_2| = \det(D) \geq 0. \tag{4.6}$$

Computing the $(1,1)$-entry and the $(2,2)$-entry of each side of (4.2), we have by (4.3), (4.4), and (4.5),

$$s_{11}|\lambda_1|^2 = s_{11}d_{11}^2 + d_{11}(s_{12}d_{21} + s_{21}d_{12}) + s_{22}|d_{12}|^2$$

and

$$s_{22}|\lambda_2|^2 = s_{11}|d_{12}|^2 + d_{22}(s_{12}d_{21} + s_{21}d_{12}) + s_{22}d_{22}^2.$$

Then it follows from (4.6) that

$$\begin{aligned} &d_{22}s_{11}|\lambda_1|^2 - d_{11}s_{22}|\lambda_2|^2 \\ &\quad = d_{22}s_{11}d_{11}^2 - d_{11}s_{11}|d_{12}|^2 + d_{22}s_{22}|d_{12}|^2 - d_{11}s_{22}d_{22}^2 \\ &\quad = d_{11}s_{11}\big(d_{11}d_{22} - |d_{12}|^2\big) + d_{22}s_{22}\big(|d_{12}|^2 - d_{11}d_{22}\big) \\ &\quad = (d_{11}s_{11} - d_{22}s_{22})\det(D) \\ &\quad = (d_{11}s_{11} - d_{22}s_{22})|\lambda_1|\cdot|\lambda_2|. \end{aligned}$$

Therefore, we have

$$\big(s_{11}|\lambda_1| + s_{22}|\lambda_2|\big)\big(d_{22}|\lambda_1| - d_{11}|\lambda_2|\big) = 0. \tag{4.7}$$

Since $s_{11}, s_{22} > 0$ and $\max(|\lambda_1|, |\lambda_2|) > 0$, (4.7) implies

$$d_{22}|\lambda_1| = d_{11}|\lambda_2|. \tag{4.8}$$

Then it follows from (4.6) and (4.8) that

$$\begin{aligned} |d_{12}|^2 &= d_{11}d_{22} - |\lambda_1|\cdot|\lambda_2| \\ &= \big(d_{11}^2 - |\lambda_1|^2\big)\frac{d_{22}}{d_{11}} \\ &= \big(d_{22}^2 - |\lambda_2|^2\big)\frac{d_{11}}{d_{22}}, \end{aligned}$$

and hence, by (4.6),

$$|\lambda_1|^2 = \frac{d_{11}}{d_{22}}\big(d_{11}d_{22} - |d_{12}|^2\big) = \frac{d_{11}}{d_{22}}\det(D)$$

and, correspondingly,

$$|\lambda_2|^2 = \frac{d_{22}}{d_{11}}\det(D).$$

Since $\det(D) \le d_{11}d_{22}$, we can conclude that

$$|\lambda_1| \le d_{11} \qquad \text{and} \qquad |\lambda_2| \le d_{22}$$

and, finally,

$$\|B\| = \max\big(|\lambda_1|, |\lambda_2|\big) \le \max(d_{11}, d_{22}) \le \|D\| \le 1.$$

This completes the proof of (4.1). □

Acknowledgment. The author thanks Prof. A. Uchiyama of Yamagata University for his suggestions on simplifying the proof of Theorem 4.1.

References

1. L. Abramović, D. Bakić, and M. S. Moslehian, *A treatment of the Cauchy–Schwarz inequality in C^*-modules*, J. Math. Anal. Appl. **381** (2011), no. 2, 546–556. MR2802092. DOI 10.1016/j.jmaa.2011.02.062. 2
2. T. Ando and F. Hiai, *Log majorization and complementary Golden–Thompson inequalities*, Linear Algeba Appl. **197/198** (1994), 113–131. MR1275611. DOI 10.1016/0024-3795(94)90484-7. 4
3. R. Bhatia, *Positive Definite Matrices*, Princeton Ser. Appl. Math., Princeton Univ. Press, Princeton, 2007. MR2284176. 2, 3

4. J. I. Fujii, *Operator-valued inner product and operator inequalities*, Banach J. Math. Anal. **2** (2008), no. 2, 59–67. Zbl 1151.47024. MR2404103. 2
5. P. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Grad. Texts in Math. **19**, Springer, New York, 1989. MR0675952. 1, 3, 4, 6

Hokkaido University (emeritus), Sapporo, Japan.
E-mail address: ando@es.hokudai.ac.jp