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COMMUTATOR IDEALS IN C^*-CROSSED PRODUCTS BY HEREDITARY SUBSEMIGROUPS

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Abstract. Let (G, G_+) be a lattice-ordered abelian group with positive cone G_+, and let H_+ be a hereditary subsemigroup of G_+. In previous work, the author and Pryde introduced a closed ideal I_{H_+} of the C^*-subalgebra B_{G_+} of $\ell^\infty(G_+)$ spanned by the functions $\{1_x : x \in G_+\}$. Then we showed that the crossed product C^*-algebra $B_{(G/H)_+} \times_\beta G_+$ is realized as an induced C^*-algebra $\mathrm{Ind}_{H^\perp}^{\widehat{G}}(B_{(G/H)_+} \times_\tau (G/H)_+)$. In this paper, we prove the existence of the following short exact sequence of C^*-algebras:

$$0 \to I_{H_+} \times_\alpha G_+ \to B_{G_+} \times_\alpha G_+ \to \mathrm{Ind}_{H^\perp}^{\widehat{G}}\big(B_{(G/H)_+} \times_\tau (G/H)_+\big) \to 0.$$

This relates $B_{G_+} \times_\alpha G_+$ to the structure of $I_{H_+} \times_\alpha G_+$ and $B_{(G/H)_+} \times_\beta G_+$. We then show that there is an isomorphism ι of $B_{H_+} \times_\alpha H_+$ into $B_{G_+} \times_\alpha G_+$. This leads to nontrivial results on commutator ideals in C^*-crossed products by hereditary subsemigroups involving an extension of previous results by Adji, Raeburn, and Rosjanuardi.

1. Introduction

Suppose that (G, G_+) is a lattice-ordered abelian group. Denote by $\{\varepsilon_x : x \in G_+\}$ the usual basis for the Hilbert space $\ell^2(G_+)$. For each $x \in G_+$, there is an isometry T_x on $\ell^2(G_+)$ satisfying $T_x(\varepsilon_y) = \varepsilon_{x+y}$ for all $y \in G_+$. The *Toeplitz algebra* of G is the C^*-subalgebra $\mathcal{T}(G)$ of $B(\ell^2(G_+))$ generated by the isometries $\{T_x : x \in G_+\}$. Recall that the C^*-algebra $C^*(G, G_+)$ is the crossed product $B_{G_+} \times_\alpha G_+$ of the dynamical system (B_{G_+}, G_+, α).

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In [4], we showed that, for a hereditary subsemigroup H_+ of the positive cone G_+,

$$I_{H_+} = \overline{\operatorname{span}}\{1_x - 1_{x+h} : h \in H_+, x \in G_+\}$$

is an extendibly α_z-invariant ideal of B_{G_+} for all $z \in G_+$, where α is the action given by

$$\alpha_x(y) = 1_{xy} \quad \text{for all } x, y \in G_+. \tag{1.1}$$

Then we showed that there is an isomorphism Ω of the crossed product $(B_{G_+}/I_{H_+}) \times_{\widetilde{\alpha}} G_+$ onto the crossed product $B_{(G/H)_+} \times_\beta G_+$, where $\widetilde{\alpha}_x(1_y + I_{H_+}) = \alpha_x(1_y) + I_{H_+}$, and β is an action of G_+ on $B_{(G/H)_+}$ by extendible endomorphisms. Indeed $\beta := \tau \circ q$, where $\tau : (G/H)_+ \to \operatorname{End}(B_{(G/H)_+})$ satisfies $\tau_{x+H}(1_{y+H}) = 1_{x+y+H}$ and every τ_{x+H} is extendible because $B_{(G/H)_+}$ is unital. Moreover, $q : G \to G/H$ is the quotient map of G onto G/H. We then showed (see [4, Theorem 6.7]) that $B_{(G/H)_+} \times_\beta G_+$ is realized as the induced C^*-algebra $\operatorname{Ind}_{H^\perp}^{\widehat{G}}(B_{(G/H)_+} \times_\tau (G/H)_+)$. Adji in [1] (see [1, Lemma 3.2] and [1, Remark 3.3]) proved a result about the commutator ideal in the case of totally ordered groups (see also [2] and [3]). Here, we are extending her results to more general cases (lattice-ordered groups) so extra work needs to be done and the proofs are more involved.

We begin with a preliminaries section in which we discuss lattice-ordered groups (G, G_+) and hereditary subsemigroups. We then review semigroup dynamical systems, recall the basic properties, and set up our notation. In Section 3, we show the existence of a surjective homomorphism

$$\theta_H : B_{G_+} \times_\alpha G_+ \to B_{(G/H)_+} \times_\beta G_+.$$

We then describe our structure theorem, which is the existence of the following short exact sequence of C^*-algebras:

$$0 \to I_{H_+} \times_\alpha G_+ \to B_{G_+} \times_\alpha G_+ \to \operatorname{Ind}_{H^\perp}^{\widehat{G}}\big(B_{(G/H)_+} \times_\tau (G/H)_+\big) \to 0.$$

This enables us to show that the ideal $I_{H_+} \times_\alpha G_+$ is generated by $\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\}$. In Section 4, we present an interesting result that allows us to view the crossed product $B_{H_+} \times_\alpha H_+$ as a C^*-subalgebra of the crossed product $B_{G_+} \times_\alpha G_+$. Then we show the existence of the exact sequence of C^*-algebras

$$0 \to B_{H_+,\infty} \times_\alpha H_+ \xrightarrow{\phi} B_{H_+} \times_\alpha H_+ \to C(\widehat{H}) \to 0,$$

which leads us to identify the commutator ideal of $B_{H_+} \times_\alpha H_+$.

2. Preliminaries

Let G be a discrete group. A binary relation "$\leq$" defined on G is a *partial order* if for $x, y, z \in G$, we have

(1) $x \leq x$ (reflexivity),
(2) $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry),
(3) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity),
(4) $x \leq y \Rightarrow zx \leq zy$ and $xz \leq yz$.

A nonempty group G together with a partial order $\leq$ is called a *partially ordered group*. The *positive cone* of a partially ordered group G is the set of all positive elements of G ($x \in G$ is positive if $x \geq e$, where e is the identity element of G), which is a semigroup.

Let G_+ be a subsemigroup of a group G with identity e such that $G_+ \cap G_+^{-1} = \{e\}$. There is a relation $\leq$ on G with respect to G_+ where $x \leq y$ if $x^{-1}y \in G_+$. This relation is a partial order on G which is left invariant in the sense that $x \leq y$ implies $zx \leq zy$ for any $x, y, z \in G$. It is the natural partial order determined by G_+.

Convention. We now use (G, G_+) to refer to the group G with the natural partial order $\leq$ on G determined by G_+.

Definition 2.1. The partially ordered group (G, G_+) is said to be a *lattice-ordered* group if every two elements of G have a least upper bound in G.

Notation. The least upper bound or *sup* of the elements x and y will be denoted by $x \vee y$.

One can easily verify that for a lattice-ordered group (G, G_+), every two elements of G_+ have a least upper bound in G_+.

Definition 2.2. Let (G, G_+) be a lattice-ordered group, and let $H \subset G_+$. Then H is said to be *hereditary* if for any $x, y \in G_+$, $e \leq x \leq y$ and $y \in H$ imply that $x \in H$ (see [8, Definition 2.3]).

Let (G, G_+) be a lattice-ordered group. We now consider a particular C^*-subalgebra of $\ell^\infty(G_+)$. Denote by 1_x the function on G_+ defined by

$$1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The lattice condition gives

$$1_x 1_y = \begin{cases} 1_{x \vee y} & \text{if } x, y \text{ have a common upper bound,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The algebra $B_{G_+} := \overline{\operatorname{span}}\{1_x : x \in G_+\}$ is an abelian C^*-algebra with multiplication satisfying (2.2) (see [5, Section 2]).

Definition 2.3. Let (G, G_+) be a lattice-ordered group, let B be a unital C^*-algebra, and let V be a map from G_+ to B. Then V is said to be an *isometric representation* of G_+ if it satisfies the following three conditions:

(i) $V_e = 1_B$;
(ii) $V_x^* V_x = 1_B$ for all $x \in G_+$;
(iii) $V_x V_y = V_{xy}$ for all $x, y \in G_+$.

If in addition V satisfies $V_x V_x^* V_y V_y^* = V_{x \vee y} V_{x \vee y}^*$ for all $x, y \in G_+$, then V is a *covariant isometric representation*.

We now give our definition of semigroup dynamical systems.

Definition 2.4. A *semigroup dynamical system* is a triple (A, G_+, α) where A is a C^*-algebra and α is an action of the semigroup G_+ on A by endomorphisms (i.e., $\alpha : G_+ \to \mathrm{End}(A)$ is a homomorphism such that α_x is an endomorphism of A for each $x \in G_+$). Two dynamical systems (A, G_+, α) and (B, G_+, β) are equivalent (isomorphic) if there is an isomorphism $\phi : A \to B$ such that $\phi \circ \alpha_x = \beta_x \circ \phi$ for all $x \in G_+$. A covariant representation of a dynamical system (A, G_+, α) is a pair (π, V), where π is a nondegenerate representation of A on a Hilbert space $\mathcal{H}$, and V is an isometric representation of G_+ on $\mathcal{H}$ satisfying

$$\pi\big(\alpha_x(a)\big) = V_x \pi(a) V_x^* \quad \text{for all } x \in G_+, a \in A.$$

Definition 2.5. A *crossed product* for a dynamical system (A, G_+, α) is a C^*-algebra B together with a nondegenerate homomorphism $i_A : A \to B$ and a homomorphism i_{G_+} of G_+ into the semigroup of isometries in $M(B)$ (the multiplier algebra of B) such that:

(1) $i_A(\alpha_x(a)) = i_{G_+}(x) i_A(a) i_{G_+}(x)^*$ for $x \in G_+$ and $a \in A$;
(2) for every covariant representation (π, V) of (A, G_+, α) there is a nondegenerate representation $\pi \times V$ of B such that

$$(\pi \times V) \circ i_A = \pi \qquad \text{and} \qquad \overline{\pi \times V} \circ i_{G_+} = V;$$

(3) B is generated by $\{i_A(a) i_{G_+}(x) : a \in A, x \in G_+\}$.

The extension of a faithful nondegenerate representation ϕ of a C^*-algebra B to its multiplier algebra $M(B)$ is denoted $\overline{\phi}$.

Notation. We write $A \times_\alpha G_+$ to denote the crossed product for the dynamical system (A, G_+, α). The homomorphisms (i_A, i_{G_+}) are the universal covariant representation.

Remark 2.6.

(1) If A is unital and (A, G_+, α) has a nontrivial covariant representation, then it is shown in [5, Proposition 2.1] that there is a crossed product and it is unique up to isomorphism.
(2) Let G_+ be an Ore semigroup (a cancellative semigroup which is right-reversible, in the sense that $G_+x \cap G_+y \neq \emptyset$ for all $x, y \in G_+$), and let (A, G_+, α) be a dynamical system with extendible endomorphisms that has a nonzero covariant representation. Then there exists a crossed product for the system which is unique up to isomorphism (see [6, Proposition 1.4]).
(3) If A has a unit (see [7, p. 11]), then the representation π of Definition 2.4 and the homomorphism i_A of Definition 2.5 must be unital, and condition (2) of Definition 2.5 reduces to the existence of a unital representation $\pi \times V$ of B such that

$$(\pi \times V) \circ i_A = \pi \qquad \text{and} \qquad (\pi \times V) \circ i_{G_+} = V.$$

Definition 2.7. An endomorphism ϕ of a C^*-algebra A is called *extendible* if it extends to a strictly continuous endomorphism $\overline{\phi}$ of the multiplier algebra $M(A)$. This happens precisely when there is an approximate identity (i_λ) and a projection $p \in M(A)$ such that $\phi(i_\lambda)$ converges strictly to p in $M(A)$ (see [1, Section 2]).

Definition 2.8. Suppose that α is an extendible endomorphism of a C^*-algebra A and that I is an ideal of A. Let $\psi : A \to M(I)$ denote the canonical nondegenerate homomorphism defined by $\psi(a)b = ab$, $a \in A$, $b \in I$. Let $\overline{\psi}$ be the strictly continuous extension of $M(A)$ into $M(I)$. Then I is called *extendibly α-invariant* if it is α-invariant, in the sense that $\alpha(I) \subset I$, and there exists an approximate identity (i_λ) for I such that $\alpha(i_\lambda)$ converges strictly to $\overline{\psi}(\overline{\alpha}(1_{M(A)}))$ in $M(I)$ (see [1, Section 3]).

3. Structure theorem

If H is a subgroup of G, then $(G/H)^\wedge$ is isomorphic to $H^\perp = \{\xi \in \widehat{G} : \xi(x) = 1 \text{ for all } x \in H\}$ and $\widehat{G}/H^\perp$ is isomorphic to $\widehat{H}$ (see [4, Remark 6.4]). Recall that the *induced algebra* $\mathrm{Ind}_{H^\perp}^{\widehat{G}}(B_{(G/H)_+} \times_\tau (G/H)_+)$ consists of the continuous functions $f : \widehat{G} \to B_{(G/H)_+} \times_\tau (G/H)_+$ satisfying $f(\gamma\mu) = \widehat{\tau}_\mu^{-1}(f(\gamma))$ for $\mu \in H^\perp$.

Proposition 3.1. *Let $(i_{B_{G_+}}, i_{G_+})$ and $(j_{B_{(G/H)_+}}, j_{G_+})$ denote the universal representations of the dynamical systems (B_{G_+}, G_+, α) and $(B_{(G/H)_+}, G_+, \beta)$, respectively, and let q be the quotient map of G onto G/H. Then there exists a surjective homomorphism*

$$\theta_H : B_{G_+} \times_\alpha G_+ \to B_{(G/H)_+} \times_\beta G_+$$

such that $\theta_H \circ i_{B_{G_+}}(1_x) = j_{B_{(G/H)_+}}(1_{q(x)})$ and $\theta_H \circ i_{G_+}(y) = j_{G_+}(y)$ for all $x, y \in G_+$.

Proof. Lemma 5.5 in [4] says that there is a surjective homomorphism $\phi : B_{G_+} \to B_{(G/H)_+}$ satisfying $\phi(1_x) = 1_{q(x)}$ for $x \in G_+$, so the map $j_{B_{(G/H)_+}} \circ \phi : B_{G_+} \to B_{(G/H)+} \times_\beta G_+$ is a unital homomorphism. The map j_{G_+} is a covariant isometric representation of G_+ into the semigroup of isometries of $B_{(G/H)+} \times_\beta G_+$. For $x, y \in G_+$, we have

$$\begin{aligned} j_{B_{(G/H)_+}} \circ \phi\big(\alpha_x(1_y)\big) &= j_{B_{(G/H)_+}}(1_{q(x+y)}) \\ &= j_{B_{(G/H)_+}}\big(\beta_x(1_{q(y)})\big) \\ &= j_{G_+}(x) j_{B_{(G/H)_+}}(1_{q(y)}) j_{G_+}(x)^* \\ &= j_{G_+}(x) j_{B_{(G/H)_+}}\big(\phi(1_y)\big) j_{G_+}(x)^*. \end{aligned} \tag{3.1}$$

Hence by linearity and continuity of $j_{B_{(G/H)_+}}$, ϕ, and α_x, the pair $(j_{B_{(G/H)_+}} \circ \phi, j_{G_+})$ is a covariant representation of the dynamical system (B_{G_+}, G_+, α) in the C^*-algebra $B_{(G/H)_+} \times_\beta G_+$. Thus, there exists a unital homomorphism

$$\theta_H : B_{G_+} \times_\alpha G_+ \to B_{(G/H)_+} \times_\beta G_+$$

such that $\theta_H \circ i_{G_+}(y) = j_{G_+}(y)$ and $\theta_H \circ i_{B_{G_+}}(1_x) = j_{B_{(G/H)_+}}(\phi(1_x)) = j_{B_{(G/H)_+}}(1_{q(x)})$ for all $x, y \in G_+$. Moreover, since the range of θ_H is a C^*-subalgebra of $B_{(G/H)_+} \times_\beta G_+$ containing all the generators, θ_H is surjective. □

Recall the following facts from [4]. For a lattice-ordered group (G, G_+) and a hereditary subsemigroup H_+ of the positive cone G_+,

$$I_{H_+} = \overline{\mathrm{span}}\{1_x - 1_{x+h} : h \in H_+, x \in G_+\}$$

is an extendibly α_z-invariant ideal of B_{G_+} for all $z \in G_+$. Moreover, in [4, Theorem 6.7] we showed that there is an isomorphism Ψ of the crossed product $B_{(G/H)_+} \times_\beta G_+$ onto the induced C^*-algebra $\mathrm{Ind}_{H^\perp}^{\widehat{G}}(B_{(G/H)_+} \times_\tau (G/H)_+)$ such that $\Psi(a)(\gamma) = Q(\widehat{\beta}_\gamma^{-1}(a))$ for $a \in B_{(G/H)_+} \times_\beta G_+$ and $\gamma \in \widehat{G}$. We now give our structure theorem.

Theorem 3.2. *Let I_{H_+} be the extendibly α_x-invariant ideal of B_{G_+} in [4, Corollary 4.8], let Ψ be the isomorphism of [4, Theorem 6.7], let $(i_{B_{G_+}}, i_{G_+})$ and $(j_{B_{(G/H)_+}}, j_{G_+})$ denote the universal homomorphisms of the crossed products $B_{G_+} \times_\alpha G_+$ and $B_{(G/H)_+} \times_\beta G_+$, respectively, and let θ_H be the homomorphism of Proposition 3.1. Define $\Upsilon = \Psi \circ \theta_H$. Then the following is a short exact sequence of C^*-algebras*

$$0 \to I_{H_+} \times_\alpha G_+ \xrightarrow{\phi} B_{G_+} \times_\alpha G_+ \xrightarrow{\Upsilon} \mathrm{Ind}_{H^\perp}^{\widehat{G}}\big(B_{(G/H)_+} \times_\tau (G/H)_+\big) \to 0 \tag{3.2}$$

in which ϕ is an isomorphism of $I_{H_+} \times_\alpha G_+$ onto the ideal

$$D := \overline{\mathrm{span}}\big\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in I_{H_+}, x, y \in G_+\big\}.$$

Proof. We will apply Theorem 1.7 of [6]. To do so, we first need to check that G_+ is an Ore semigroup of G. Since G_+ is a subset of G, it is cancellative. We still need G_+ to be right-reversible, so for $y, z \in G_+$, we have $y + G_+ \cap z + G_+ \neq \emptyset$ since $y+z \in y+G_+$ and $z+y \in z+G_+$; therefore, $z+y \in y+G_+ \cap z+G_+$. Hence G_+ is an Ore semigroup of G. Therefore, [6, Theorem 1.7] implies that there is a short exact sequence

$$0 \to I_{H_+} \times_\alpha G_+ \xrightarrow{\phi} B_{G_+} \times_\alpha G_+ \xrightarrow{\varphi} B_{G_+}/I_{H_+} \times_{\tilde{\alpha}} G_+ \to 0$$

in which

$$\varphi \circ i_{B_{G_+}}(1_x) = j_{B_{G_+}/I_{H_+}}(1_x + I_{H_+}) \qquad \text{and} \qquad \varphi \circ i_{G_+}(y) = j_{G_+}(y),$$

and $I_{H_+} \times_\alpha G_+$ is isomorphic to the ideal $D := \overline{\mathrm{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in I_{H_+}, x, y \in G_+\}$ in $B_{G_+} \times_\alpha G_+$. But Lemma 6.2 of [4] says that $B_{(G/H)_+} \times_\beta G_+$ is isomorphic to $B_{G_+}/I_{H_+} \times_{\tilde{\alpha}} G_+$. Therefore, there is a short exact sequence

$$0 \to I_{H_+} \times_\alpha G_+ \xrightarrow{\phi} B_{G_+} \times_\alpha G_+ \xrightarrow{\theta_H} B_{(G/H)_+} \times_\beta G_+ \to 0 \tag{3.3}$$

in which

$$\theta_H \circ i_{B_{G_+}}(1_x) = j_{B_{(G/H)_+}}(1_{q(x)}) \qquad \text{and} \qquad \theta_H \circ i_{G_+}(y) = j_{G_+}(y).$$

Now as $B_{(G/H)_+} \times_\beta G_+$ is isomorphic to $\mathrm{Ind}_{H^\perp}^{\widehat{G}}(B_{(G/H)_+} \times_\tau (G/H)_+)$, then $\Upsilon = \Psi \circ \theta_H$ is a map from $B_{G_+} \times_\alpha G_+$ onto $\mathrm{Ind}_{H^\perp}^{\widehat{G}}(B_{(G/H)_+} \times_\tau (G/H)_+)$ with kernel $I_{H_+} \times_\alpha G_+$ (this is true by exactness of (3.3) and because Ψ is an isomorphism of $B_{(G/H)_+} \times_\beta G_+$ onto $\mathrm{Ind}_{H^\perp}^{\widehat{G}}(B_{(G/H)_+} \times_\tau (G/H)_+)$). Thus, we have the following short exact sequence

$$0 \to I_{H_+} \times_\alpha G_+ \xrightarrow{\phi} B_{G_+} \times_\alpha G_+ \xrightarrow{\Upsilon} \mathrm{Ind}_{H^\perp}^{\widehat{G}}\big(B_{(G/H)_+} \times_\tau (G/H)_+\big) \to 0. \qquad \square$$

Corollary 3.3. *Let* $(i_{B_{G_+}}, i_{G_+})$ *be the universal homomorphisms of the crossed product* $B_{G_+} \times_\alpha G_+$. *Then the ideal* $D = \overline{\operatorname{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in I_{H_+}, x, y \in G_+\}$ *of* $B_{G_+} \times_\alpha G_+$ *in Theorem 3.2 is generated by* $\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\}$.

Proof. Since $i_{G_+}(x)^*, i_{G_+}(y) \in B_{G_+} \times_\alpha G_+$, D is generated by $\{i_{B_{G_+}}(a) : a \in I_{H_+}\}$. So to prove this corollary, it suffices to show that, for $a \in I_{H_+}$, $i_{B_{G_+}}(a)$ is in the ideal generated by $\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\}$. To see this, fix $x \in G_+$ and $h \in H_+$. Then

$$\begin{aligned} i_{B_{G_+}}(1_x - 1_{x+h}) &= i_{B_{G_+}}(1_x) - i_{B_{G_+}}(1_{x+h}) \\ &= i_{G_+}(x) i_{G_+}(x)^* - i_{G_+}(x+h) i_{G_+}(x+h)^* \\ &= i_{G_+}(x)\big(1 - i_{G_+}(h) i_{G_+}(h)^*\big) i_{G_+}(x)^* \\ &= i_{G_+}(x) i_{B_{G_+}}(1 - 1_h) i_{G_+}(x)^*. \end{aligned}$$

Hence $i_{B_{G_+}}(1_x - 1_{x+h})$ is in the ideal generated by $\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\}$. Therefore, by continuity of $i_{B_{G_+}}$ we have that $i_{B_{G_+}}(a)$ is in the ideal generated by $\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\}$ for all $a \in I_{H_+}$. □

Remark 3.4. Let $(i_{B_{G_+}}, i_{G_+})$ be the universal covariant representation of the dynamical system (B_{G_+}, G_+, α). Then $i_{B_{G_+}}(1_x) = i_{G_+}(x) i_{G_+}(x)^*$ and from [5, Corollary 2.4] we know that the map $i_{B_{G_+}}$ is injective, so for simplicity we write 1_x for $i_{G_+}(x) i_{G_+}(x)^*$. Hence one can say that the crossed product $I_{H_+} \times_\alpha G_+$ in (3.3) is generated by the set $\{1 - 1_u : u \in H_+\}$.

4. The crossed product $B_{H_+} \times_\alpha H_+$ and its commutator ideal

The following proposition is interesting as it allows us to view the crossed product $B_{H_+} \times_\alpha H_+$ as a C^*-subalgebra of the crossed product $B_{G_+} \times_\alpha G_+$.

Proposition 4.1. *Let* (G, G_+) *be a lattice-ordered group with* G *abelian, let* H_+ *be a hereditary subsemigroup of* G_+, *and let* $(i_{B_{G_+}}, i_{G_+})$ *denote the universal representation of the dynamical system* (B_{G_+}, G_+, α). *Then there is an isomorphism* ι *of* $B_{H_+} \times_\alpha H_+$ *into* $B_{G_+} \times_\alpha G_+$.

Proof. The existence of the crossed product $B_{H_+} \times_\alpha H_+$ follows directly from Remark 2.6. Let $V := i_{G_+}|_{H_+}$. Then V is a covariant isometric representation of H_+. Since $B_{H_+} \times_\alpha H_+$ is universal for covariant isometric representations, there is a unital representation $\pi_V : B_{H_+} \to B_{G_+} \times_\alpha G_+$ such that $\pi_V(1_x) = V_x V_x^*$ for all $x \in H_+$. Hence, there is a unital representation $\pi_V \times V : B_{H_+} \times_\alpha H_+ \to B_{G_+} \times_\alpha G_+$ such that $(\pi_V \times V) \circ i_{B_{H_+}} = \pi_V$ and $(\pi_V \times V) \circ i_{H_+} = V$.

Note that

$$\begin{aligned} \pi_V(1_x) = V_x V_x^* &= i_{G_+}(x) i_{G_+}(x)^* \\ &= i_{B_{G_+}}(1_x). \end{aligned}$$

This is true since $(i_{B_{G_+}}, i_{G_+})$ is the universal representation.

Then π_V and $i_{B_{G_+}}$ agree on the generators of B_{H_+}. Therefore, $\pi_V = i_{B_{G_+}}|_{B_{H_+}}$ and so π_V is faithful. By Proposition 3.1 and Theorem 3.7 of [5], $\pi_V \times_\alpha V$ is faithful. Taking $\iota := \pi_V \times_\alpha V$, we obtain the desired result. □

Definition 4.2. Let A be a C^*-algebra. The *commutator* ideal $\mathcal{C}$ of A is the closed ideal generated by $\{ab - ba : a, b \in A\}$.

Remark 4.3. The commutator ideal of a C^*-algebra A is the smallest closed ideal $\mathcal{C}$ in A such that $A/\mathcal{C}$ is commutative (see [9, Section 3.5]).

The following results will allow us to identify the commutator ideal of the C^*-algebra $B_{H_+} \times_\alpha H_+$. We first introduce the algebra

$$B_{H_+,\infty} := \big\{ f \in B_{H_+} : \lim_{h\to\infty} f(h) = 0 \big\}. \tag{4.1}$$

Proposition 4.4. *Suppose that* (G, G_+) *is a lattice-ordered group with* G *abelian and that* H_+ *is a hereditary subsemigroup of* G_+*. Then the algebra* $B_{H_+,\infty}$ *is the closed span of* $\{1 - 1_h : h \in H_+\}$.

Proof. Let A be the closed span of $\{1 - 1_h : h \in H_+\}$. Fix $h \in H_+$. For $u \geq h$, we have

$$(1 - 1_h)(u) = 1(u) - 1_h(u) = 0.$$

Therefore, $\lim_{u\to\infty}(1 - 1_h)(u) = 0$ and so $1 - 1_h \in B_{H_+,\infty}$.

For any $f \in A$, $f = \lim_{n\to\infty} f_n$ where $f_n = \sum_{h_i \in F_n} \lambda_i (1 - 1_{h_i})$ and F_n is a finite subset of H_+. Fix $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $\|f - f_n\| < \varepsilon$. Let $h_n = \vee F_n$. Then for $u \geq h_n$, we have

$$\begin{aligned} \big|f(u)\big| &= \big|f(u) - f_n(u) + f_n(u)\big| \\ &\leq \big|f(u) - f_n(u)\big| + \big|f_n(u)\big| \\ &< \varepsilon + 0 = \varepsilon, \quad \text{since } \big|f(u) - f_n(u)\big| \leq \|f - f_n\|. \end{aligned}$$

Hence $f \in B_{H_+,\infty}$ and so $A \subset B_{H_+,\infty}$.

To show that $B_{H_+,\infty} \subset A$, we first need to show that for any $f \in B_{H_+}$, $\lim_{u\to\infty} f(u)$ exists. To see this, suppose that $f \in B_{H_+}$. Then $f = \lim_{n\to\infty} f_n$, where $f_n = \sum_{h_i \in F_n} \lambda_i 1_{h_i}$ and F_n is a finite subset of H_+.

Claim. *Suppose that* $x_n := \lim_{u\to\infty} f_n(u)$*. Then* $\{x_n\}$ *converges.*

Proof. Note that every $x_n \in \mathbb{C}$ so it is enough to show that $\{x_n\}$ is a Cauchy sequence (this is true since $\mathbb{C}$ is a Hilbert space). But $\{f_n\}$ is a Cauchy sequence in B_{H_+}; therefore, $\{x_n\}$ is a Cauchy sequence. To see this, fix $\varepsilon > 0$. Then there exists N such that

$$\|f_n - f_m\| < \varepsilon \quad \text{for all } n, m > N,$$

where $\|f_n - f_m\| = \sup_{x \in H_+} |f_n(x) - f_m(x)|$. Now

$$\begin{aligned} |x_n - x_m| &= \big|\lim_{u\to\infty} f_n(u) - \lim_{u\to\infty} f_m(u)\big| \\ &= \big|\lim_{u\to\infty}\big(f_n(u) - f_m(u)\big)\big| \\ &= \lim_{u\to\infty}\big|f_n(u) - f_m(u)\big| \end{aligned}$$

$$\leq \|f_n - f_m\|$$
$$< \varepsilon. \qquad \square$$

Fix $\varepsilon > 0$, and choose $m \in \mathbb{N}$ such that $\|f - f_m\| < \varepsilon/2$ and $|\lim_{n\to\infty} x_n - x_m| < \varepsilon/2$. Let $h_n = \vee F_n$. Then for $u \geq h_n$, we have

$$\begin{aligned} \big|f(u) - \lim_{n\to\infty} x_n\big| &= \big|f(u) - f_m(u) + f_m(u) - \lim_{n\to\infty} x_n\big| \\ &\leq \big|f(u) - f_m(u)\big| + \big|f_m(u) - \lim_{n\to\infty} x_n\big| \\ &< \varepsilon/2 + \big|x_m - \lim_{n\to\infty} x_n\big|, \quad \text{as } u \geq h_n \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $\lim_{u\to\infty} f(u)$ exists.

To complete the proof, take $f \in B_{H_+}$ such that $\lim_{u\to\infty} f(u) = 0$. Then there exists $\{f_n\}$ such that $f_n \to f$, where $f_n = \sum_{h_i \in F_n} \lambda_i 1_{h_i}$ and F_n is a finite subset of H_+. Let $x_n = \lim_{u\to\infty} f_n(u)$. Then $\lim_{n\to\infty} x_n = 0$ (by the previous part of this proof). Define $g_n := f_n - x_n 1$. Then $g_n = \sum_{h_i \in F_n} -\lambda_i(1 - 1_{h_i}) \in A$ and

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} (f_n - x_n 1) = f.$$

Therefore, $B_{H_+,\infty} \subset A$. Consequently, $A = B_{H_+,\infty}$. $\square$

Lemma 4.5. *Suppose that (G, G_+) is a lattice-ordered group with G abelian, that H_+ is a hereditary subsemigroup of G_+, and that α is the action in (1.1). Then the algebra $B_{H_+,\infty}$ is an extendibly α-invariant ideal of B_{H_+}.*

Proof. To see that $B_{H_+,\infty}$ is a closed ideal, fix $t, u \in H_+$. Then

$$1_t(1 - 1_u) = 1_t - 1_{t\vee u} = (1 - 1_{t\vee u}) - (1 - 1_t) \in B_{H_+,\infty},$$

and by continuity of multiplication in B_{H_+} we conclude that $B_{H_+,\infty}$ is a closed ideal of B_{H_+}. Calculations show that the set $S = \{1 - 1_u : u \in H_+\}$ is an approximate identity for $B_{H_+,\infty}$.

For $z \in H_+$, α_z is linear and continuous so routine calculations show that $B_{H_+,\infty}$ is α-invariant. Another routine calculation shows that for $(1 - 1_t) \in B_{H_+,\infty}$ the approximate identity S satisfies

$$\alpha_z(1 - 1_u)(1 - 1_t) \to \psi\big(\alpha_z(1)\big)(1 - 1_u), \tag{4.2}$$

where ψ is the canonical map in Definition 2.8. For any $b \in B_{H_+,\infty}$, a standard $\varepsilon/3$ argument shows that it satisfies (4.2) with $(1 - 1_t)$ replaced by b. Thus this completes the proof that $B_{H_+,\infty}$ is an extendibly α-invariant ideal of B_{H_+}. $\square$

Remark 4.6. In [1, Section 3], Adji shows that for a totally ordered group Γ with positive cone Γ^+, there is a short exact sequence

$$0 \to B_{\Gamma^+,\infty} \xrightarrow{\iota} B_{\Gamma^+} \xrightarrow{\delta} \mathbb{C} \to 0,$$

where $\delta : B_{\Gamma^+} \to \mathbb{C}$ is defined by $\delta(f) = \lim_{x\to\infty} f(x)$. This result still holds for a lattice-ordered group (G, G_+).

Corollary 4.7. *Suppose that (G, G_+) is a lattice-ordered group with G abelian, that H_+ is a hereditary subsemigroup of G_+, that α is the action in (1.1), that $(i_{B_{H_+}}, i_{H_+})$ is the universal covariant representation of (B_{H_+}, H_+, α), and that $B_{H_+,\infty}$ is the extendibly α-invariant ideal in Lemma 4.5. Then there is a short exact sequence of C^*-algebras*

$$0 \to B_{H_+,\infty} \times_\alpha H_+ \xrightarrow{\phi} B_{H_+} \times_\alpha H_+ \to C(\widehat{H}) \to 0$$

in which ϕ is an isomorphism of $B_{H_+,\infty} \times_\alpha H_+$ onto the ideal $D = \overline{\operatorname{span}}\{i_{G_+}(x)^* \times i_{B_{G_+}}(a) i_{G_+}(y) : a \in B_{H_+,\infty}, x, y \in H_+\}$ of $B_{H_+} \times_\alpha H_+$. Moreover, $B_{H_+,\infty} \times_\alpha H_+$ is the commutator ideal of $B_{H_+} \times_\alpha H_+$.

Proof. Since H_+ is an Ore semigroup of H (this is true, because in the proof of Theorem 3.2 we showed that G_+ is an Ore semigroup of G and as H_+ is a subset of H), then [6, Theorem 1.7] implies that there exists the following short exact sequence

$$0 \to B_{H_+,\infty} \times_\alpha H_+ \to B_{H_+} \times_\alpha H_+ \to (B_{H_+}/B_{H_+,\infty}) \times_{\tilde{\alpha}} H_+ \to 0, \tag{4.3}$$

with $B_{H_+,\infty} \times_\alpha H_+$ isomorphic to the ideal $D = \overline{\operatorname{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in B_{H_+,\infty}, x, y \in H_+\}$ of $B_{H_+} \times_\alpha H_+$.

We know from Remark 4.6 that $B_{H_+}/B_{H_+,\infty}$ is isomorphic to $\mathbb{C}$. Moreover, note that $\mathbb{C}$ has only the trivial action, that is, id, so the crossed product $B_{H_+}/B_{H_+,\infty} \times_{\tilde{\alpha}} H_+$ will be isomorphic to $\mathbb{C} \times_{\mathrm{id}} H_+$. Since $\mathbb{C}$ has only the unital representation $z \mapsto z1$, then the covariance condition gives that the system $(\mathbb{C}, H_+, \mathrm{id})$ consists of unitaries. Moreover, since $H = H_+ - H_+$, [10] gives that $\mathbb{C} \times_{\mathrm{id}} H_+$ is isomorphic to $C^*(H)$, and as H is abelian, $C^*(H)$ is isomorphic to $C(\widehat{H})$. Thus we have the desired short exact sequence.

We know from Corollary 3.3 that the ideal $D = \overline{\operatorname{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in B_{H_+,\infty}, x, y \in H_+\}$ of $B_{H_+} \times_\alpha H_+$ is generated by $\{1 - 1_u : u \in H_+\}$. For $u \in H_+$, $1 - 1_u = i_{H_+}(u)^* i_{H_+}(u) - i_{H_+}(u) i_{H_+}(u)^* \in \mathcal{C}_H$ (the commutator ideal) of $B_{H_+} \times_\alpha H_{H_+}$, which means that $B_{H_+,\infty} \times_\alpha H_+ \subset \mathcal{C}_H$. Moreover, since $(B_{H_+} \times_\alpha H_+ / B_{H_+,\infty} \times_\alpha H_+) \simeq C(\widehat{H})$ is commutative, $\mathcal{C}_H \subset B_{H_+,\infty} \times_\alpha H_+$. Thus $B_{H_+,\infty} \times_\alpha H_+$ is the commutator ideal of $B_{H_+} \times_\alpha H_+$. □

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