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DENSITY PROPERTIES FOR FRACTIONAL SOBOLEV SPACES WITH VARIABLE EXPONENTS

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Abstract. In this article we show some density properties of smooth and compactly supported functions in fractional Sobolev spaces with variable exponents. The additional difficulty in this nonlocal setting is caused by the fact that the variable exponent Lebesgue spaces are not translation-invariant.

1. Introduction

Function spaces with variable exponents have been the subject of intensive investigation in recent years. Examples of such spaces include Lebesgue and Sobolev spaces with variable exponents. Introduced by Orlicz [20] in 1931, their properties were further developed by Nakano [19] as special cases of the theory of modular spaces. In the ensuing decades they were primarily considered as important examples of modular spaces or the class of Musielak–Orlicz spaces. Initially of theoretical interest, by the end of the twentieth century these function spaces moved beyond theory and into the area of variational problems and studies of the $p(x)$-Laplacian operator, which in turn further fueled the development of this theory. Also stimulating the wide-ranging investigation of these spaces were their application to various problems in applied mathematics, for example, in the areas of nonlinear elasticity theory, fluid mechanics, and mathematical modeling of physical phenomena. (For more details on these spaces, see the monographs [6], [17], [18]; see also [22].)

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Let Ω be a bounded domain of $\mathbb{R}^n$. We fix $s \in (0,1)$, and we consider two variable exponents—that is, $q : \Omega \to [1,+\infty)$ and $p : \Omega \times \Omega \to [1,+\infty)$—to be two continuous functions. In the following, we use the symbol $:=$ to make the left-hand side equal by definition to the right-hand side.

The variable exponent Lebesgue space $L^{q(\cdot)}(\Omega)$ is defined by

$$L^{q(\cdot)}(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \text{ measurable}, \int_\Omega |u(x)|^{q(x)}\, dx < \infty \Big\}.$$

We define a norm, the so-called *Luxembourg norm*, in this space by

$$\|u\|_{L^{q(\cdot)}(\Omega)} := \inf\Big\{ \lambda > 0 : \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{q(x)} dx \le 1 \Big\}.$$

We define the fractional Sobolev space with variable exponents via the Gagliardo approach as follows:

$$\begin{aligned} &W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega) \\ &\quad := \Big\{ u \in L^{q(\cdot)}(\Omega), \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{n+sp(x,y)}}\, dx\, dy < \infty, \text{for some } \lambda > 0 \Big\}. \end{aligned}$$

Let

$$[u]^{s,p(\cdot,\cdot)}(\Omega) := \inf\Big\{ \lambda > 0 : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{n+sp(x,y)}}\, dx\, dy \le 1 \Big\}$$

be the corresponding variable exponent Gagliardo seminorm. It is easy to see that $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)} := \|u\|_{L^{q(\cdot)}(\Omega)} + [u]^{s,p(\cdot,\cdot)}(\Omega).$$

The definition of the spaces $L^{q(\cdot)}(\mathbb{R}^n)$ and $W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$ is analogous to that of $L^{q(\cdot)}(\Omega)$ and $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$; one just changes every occurrence of Ω by $\mathbb{R}^n$ (for more detail, see [1], [2], [4], and [16]).

Fractional Sobolev spaces with variable exponents have major applications in variational problems related to a well-known fractional version of the $p(x)$-Laplace operator, given by $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, that is associated with the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$. We refer, for instance, to [6] and [13].

The denseness of regular functions such as $C^\infty(\Omega)$, functions which are continuously differentiable arbitrarily many times (smooth functions), or $C_0^\infty(\Omega)$, the subset of $C^\infty(\Omega)$ of functions which have compact support, was one of the central tools in the function spaces setting—it might have some additional ramifications in the variable exponent Sobolev spaces. Moreover, the density of regular functions was one of the questions considered early (from 1986) in the context of minimizers of variational integrals. It is well known that the regularity of the Dirichlet energy integral minimizer is related to the density of smooth functions in the corresponding function space (see [14], [23]–[26]).

The denseness of smooth functions in the variable exponent Sobolev spaces $W^{k,p(\cdot)}(\Omega)$ has proved to be a more difficult problem. Nowadays, denseness is a topic of intense focus and study by many researchers (we refer the reader to [6], [8]–[10], [15], [21], [27] and the references therein).

We recall that the continuity of the shift operator $T_h : L^{q(\cdot)} \to L^{q(\cdot)}, T_h f(x) := f(x+h)$ may fail (see [5]). In connection with the density problem for smooth function, we introduce the most important condition on the exponent in the study of variable exponent spaces, the well-known log-Hölder continuity condition: a function $q : \Omega \to \mathbb{R}$ is called *log-Hölder continuous* on Ω if there exists $C > 0$ such that

$$\big|q(x) - q(y)\big| \leq \frac{C}{-\log|x-y|}, \quad |x-y| \leq \frac{1}{2}. \tag{1.1}$$

We say that a function $p : \Omega \times \Omega \to \mathbb{R}$ satisfies condition (B-B) on $\Omega \times \Omega$ if there exists $C > 0$ such that

$$\big|p(x,y) - p(x^{'},y^{'})\big| \leq \frac{C}{-\log(|x-x^{'}| + |y-y^{'}|)}, \tag{1.2}$$

for all (x,y) and $(x^{'},y^{'}) \in \Omega \times \Omega$ such that $|x-x^{'}| + |y-y^{'}| \leq \frac{1}{2}$.

Notice that conditions (1.1) and (1.2) ensure that the typical mollifier function approximate function in Lebesgue Spaces with variable exponents (we refer the reader to [5], [21], [2] and [7]).

We define the following class of variable exponents

$$\mathcal{P}^{\log}(\Omega) := \{q : \Omega \to \mathbb{R} : q \text{ is measurable and log-Hölder continuous}\}$$

and

$$\mathcal{P}^{\log}(\Omega \times \Omega) := \big\{p : \Omega \times \Omega \to \mathbb{R} : p \text{ is measurable and satisfies condition (B–B)}\big\}.$$

We set $p^- := \operatorname{essinf}_{(x,y)\in\Omega\times\Omega} p(x,y)$, $p^+ := \operatorname{esssup}_{(x,y)\in\Omega\times\Omega} p(x,y)$, $q^- := \operatorname{essinf}_{x\in\Omega} q(x)$, and $q^+ := \operatorname{esssup}_{x\in\Omega} q(x)$.

We assume that

$$1 < p^- \leq p(x,y) \leq p^+ < \infty, \tag{1.3}$$

$$1 < q^- \leq q(x) \leq q^+ < \infty, \tag{1.4}$$

$$p\big((x,y) - (z,z)\big) = p(x,y), \quad \forall (x,y),(z,z) \in \Omega \times \Omega. \tag{1.5}$$

The aim of this paper is to prove a number of density properties of smooth and compactly supported functions in fractional Sobolev spaces with variable exponent. The first result is an approximation with a continuous and compactly supported function.

Theorem 1.1. *Let $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$. Then for any fixed $\delta > 0$, there exists a continuous and compactly supported function u_δ such that*

$$\|u - u_\delta\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0.$$

The second result is an approximation with smooth and compactly supported functions. More precisely, we have the following.

Theorem 1.2. *Let $q \in \mathcal{P}^{\log}(\Omega)$ and $p \in \mathcal{P}^{\log}(\Omega \times \Omega)$. Then the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$.*

We consider $\Omega \subset \mathbb{R}^n$ to be a $W^{s,q(\cdot),p(\cdot,\cdot)}$-extension domain if there exists a continuous linear extension operator

$$\mathcal{E} : W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega) \longrightarrow W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$$

such that $\mathcal{E}u|_\Omega = u$ for each $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$. From Theorem 1.2, we derive the density of $C^\infty(\overline{\Omega})$ also in $W^{s,q(\cdot),p(\cdot,\cdot)}$-extension domains, which include in particular domains with Lipschitz boundary (see [1]).

Theorem 1.3. *Let $q \in \mathcal{P}^{\log}(\Omega)$, $p \in \mathcal{P}^{\log}(\Omega \times \Omega)$ and suppose that Ω is a $W^{s,q(\cdot),p(\cdot,\cdot)}$-extension domain. Then the space $C^\infty(\overline{\Omega})$ is dense in $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$.*

This paper is organized as follows. In Section 2, we describe some properties of fractional Sobolev spaces with variable exponents that will be useful in our exposition. In Section 3, we prove the density results for fractional Sobolev spaces with variable exponents.

2. Notation and preliminary results

In this section we give some notation and prove several useful properties of fractional Sobolev spaces with variable exponents.

We use the following notation throughout this article. $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $n \in \mathbb{N}$ always stands for the dimension of the space. A domain $\Omega \subset \mathbb{R}^n$ is a connected open set equipped with the n-dimensional Lebesgue measure. For constants, we use the letter C whose value may change even within a string of estimates. A ball with radius R and center 0 will be denoted by B_R. The closure of a set A is denoted by $\overline{A}$. We use the usual convention of identifying two μ-measurable functions on A (almost everywhere (a.e.) in A, for short) if they agree almost everywhere—that is, if they agree up to a set of μ-measure zero. The Lebesgue integral of a Lebesgue measurable function $f : \Omega \longrightarrow \mathbb{R}$ is defined in the standard way and denoted by $\int_\Omega f(x)\,dx$. By ω_{n-1}, we denote the $(n-1)$-dimensional measure of the unit sphere S^{n-1}. By $\operatorname{supp} f$, we denote the support of f, which is the complement of the biggest open set on which it vanishes; in other words, $\operatorname{supp} f$ is the closure of the set $\{x; f(x) \neq 0\}$. We denote by $C(\overline{\Omega})$ the space of uniformly continuous functions equipped with the supremum norm $\|f\|_\infty = \sup_{x\in\overline{\Omega}} |f(x)|$. By $C^k(\overline{\Omega})$, $k \in \mathbb{N}$, we denote the space of all functions f such that $\partial_\alpha f := \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \in C(\overline{\Omega})$ for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq k$. The space is equipped with the norm $\sup_{|\alpha|\leq k} \|\partial_\alpha f\|_\infty$, $C^\infty(\overline{\Omega}) = \bigcap_k C^k(\overline{\Omega})$. The set of smooth functions in Ω is denoted by $C^\infty(\Omega)$—it consists of functions in Ω which are continuously differentiable arbitrarily many times. The set $C_0^\infty(\Omega)$ is the subset of $C^\infty(\Omega)$ of functions which have compact support.

The important role of manipulating Lebesgue–Sobolev spaces with variable exponents is played by the modular of the $L^{q(\cdot)}(\Omega)$ space, which is the mapping $\rho : L^{q(\cdot)}(\Omega) \longrightarrow \mathbb{R}^+$ defined by

$$\rho(u) = \int_\Omega |u(x)|^{q(x)}\,dx.$$

From [10, Theorems 1.4, 1.8], we obtain the following proposition.

Proposition 2.1. *Let* Ω *be an open subset of* $\mathbb{R}^n$, *and let* $u_n, u \in L^{q(\cdot)}(\Omega)$ *be such that* $n \in \mathbb{N}$. *Then*

(1) $C_0^\infty(\Omega)$ *is dense in the space* $(L^{q(\cdot)}(\Omega), \|\cdot\|_{L^{q(\cdot)}(\Omega)})$,
(2) $\lim_{n\longrightarrow+\infty} \rho(u_n - u) \longrightarrow 0 \Longleftrightarrow \lim_{n\longrightarrow+\infty}\|u_n - u\|_{L^{q(\cdot)}(\Omega)} = 0$.

Next, we prove several properties of $W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$ that are needed to obtain our main results.

Lemma 2.2. *Let* $\varphi \in C_0^\infty(\mathbb{R}^n)$. *Then there exists a positive constant* C *such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy \le C.$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subseteq B_R$. Therefore, if $x, y \in \mathbb{R}^n \setminus B_R$, then $\varphi(x) = \varphi(y) = 0$. Thus

$$\begin{aligned} I &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy \\ &= \int_{B_R}\int_{\mathbb{R}^n} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 := \int_{B_R}\int_{B_{2R}} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy$$

and

$$I_2 := \int_{B_R}\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy.$$

We first estimate I_1. We have

$$\begin{aligned} I_1 &= \int_{B_R}\int_{B_{2R}} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy \\ &= \int_{B_R}\int_{B_{2R}} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{p(x,y)}} \times \frac{|x-y|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy. \end{aligned}$$

Hence

$$I_1 \le C\int_{B_{2R}}\int_{B_{2R}} |x-y|^{-n+(1-s)p(x,y)}\,dx\,dy,$$

where C depends on the C^1-norm of φ, p^+, and p^-.

We set

$$d := \sup\big\{|x-y| : (x,y) \in B_{2R} \times B_{2R}\big\}.$$

Observe that

$$|x-y|^{-n+(1-s)p(x,y)} \leq \max\{^{-n+(1-s)p^+}, d^{-n+(1-s)p^-}\}.$$

Therefore,

$$I_1 \leq C \int_{B_{2R}} \int_{B_{2R}} dx\, dy \leq C.$$

Now we estimate I_2. For this, we observe that if $x \in B_R$ and $y \in \mathbb{R}^n \setminus B_{2R}$, then

$$|x-y| \geq |y| - |x| \geq \frac{|y|}{2}.$$

Thus

$$I_2 \leq C\big(n, s, p^+, p^-, \|\varphi\|_{L^\infty(\mathbb{R}^n)}\big) \int_{B_R} \Big(\int_{\mathbb{R}^n \setminus B_{2R}} \frac{dy}{|y|^{n+sp(x,y)}} \Big)\, dx.$$

Now, if $|y| > 1$, then

$$\int_{\mathbb{R}^n \setminus B_{2R}} \frac{dy}{|y|^{n+sp(x,y)}} \leq \int_{\mathbb{R}^n \setminus B_{2R}} \frac{dy}{|y|^{n+sp^-}} = \omega_{n-1} C(R, s, p^-) \leq C,$$

where ω_{n-1} denotes the $(n-1)$-dimensional measure of the unit sphere S^{n-1}.

Now, we assume that $|y| < 1$. We claim that

$$\int_{\mathbb{R}^n \setminus B_{2R}} \frac{dy}{|y|^{n+sp(x,y)}} \leq \int_{\mathbb{R}^n \setminus B_{2R}} \frac{dy}{|y|^{n+sp^+}} = \omega_{n-1} C(R, s, p^+) \leq C.$$

Hence, I is bounded. □

As an obvious consequence of Lemma 2.2 and Proposition 2.1, we have $C_0^\infty(\mathbb{R}^n) \subseteq W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$. Now we give two approximation results.

Lemma 2.3. *Let $u \in L^{q(\cdot)}(\mathbb{R}^n)$. Then there exists a sequence of functions $u_m \in L^{q(\cdot)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that*

$$\|u - u_m\|_{L^{q(\cdot)}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } m \longrightarrow +\infty.$$

Proof. We set

$$u_m(x) := \begin{cases} m & \text{if } u(x) \geq m, \\ u(x) & \text{if } u(x) \in (-m, m), \\ -m & \text{if } u(x) \leq -m. \end{cases}$$

We have

$$u_m \longrightarrow u \quad \text{a.e. in } \mathbb{R}^n$$

and

$$|u_m(x)|^{q(x)} \leq |u(x)|^{q(x)} \in L^1(\mathbb{R}^n).$$

By the dominated convergence theorem and Proposition 2.1, the claim follows. □

Lemma 2.4. *Let $u \in L^{p(\cdot,\cdot)}(\mathbb{R}^n \times \mathbb{R}^n)$. Then there exists a sequence of functions $u_M \in L^{p(\cdot,\cdot)}(\mathbb{R}^n \times \mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$\|u - u_M\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^n \times \mathbb{R}^n)} \longrightarrow 0 \quad \textit{as } M \longrightarrow +\infty.$$

Proof. We set

$$u_M(x,y) := \begin{cases} M & \text{if } u(x,y) \geq M, \\ u(x,y) & \text{if } u(x,y) \in (-M, M), \\ -M & \text{if } u(x,y) \leq -M. \end{cases}$$

We have

$$u_M \longrightarrow u \quad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}^n$$

and

$$|u_M(x,y)|^{p(x,y)} \leq |u(x,y)|^{p(x,y)} \in L^1(\mathbb{R}^n \times \mathbb{R}^n).$$

The claim follows from the dominated convergence theorem and Proposition 2.1. □

The proofs of density properties of smooth and compactly supported functions are mainly based on a basic technique of convolution (which makes functions C^{∞}), joined with a cutoff (which makes their support compact). In the remainder of this article, we describe properties of these operations with respect to the norm in fractional Sobolev spaces with variable exponents.

Let $\eta \in C_0^{\infty}(\mathbb{R}^n)$ be such that $\eta \geq 0$ in $\mathbb{R}^n$ and $\operatorname{supp} \eta \subseteq B_1$. We also assume that

$$\int_{B_1} \eta(x)\, dx = 1.$$

Let $\varepsilon > 0$, and let η_ε be the mollifier defined as

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\Big(\frac{x}{\varepsilon}\Big), \quad x \in \mathbb{R}^n.$$

For any $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$, we will denote by u_ε the function defined as the convolution between u and η_ε; that is,

$$u_\varepsilon(x) = (u * \eta_\varepsilon)(x) = \int_{\mathbb{R}^n} u(x-z)\eta_\varepsilon(z)\, dz, \quad x \in \mathbb{R}^n.$$

Note that, by construction, $u_\varepsilon \in C^{\infty}(\mathbb{R}^n)$. For small ε, the convolution does not change the norm too much, according to the following result (see, e.g., [2, Lemma 3.2]), which will be useful in the rest of this article.

Lemma 2.5. *Let $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$, $q \in \mathcal{P}^{\log}(\Omega)$ and $p \in \mathcal{P}^{\log}(\Omega \times \Omega)$. We assume that (1.3), (1.4) and (1.5) hold. Then $\|u - u_\varepsilon\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$.*

Proof. Let $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$. Since $u \in L^{q(\cdot)}(\Omega)$ and $q \in \mathcal{P}^{\log}(\Omega)$, by [21, Corollary], we know that

$$\|u - u_\varepsilon\|_{L^{q(\cdot)}(\mathbb{R}^n)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence, from Proposition 2.1, it suffices to prove that

$$\int_{\Omega\times\Omega}\left|u_\varepsilon(x)-u(x)-u_\varepsilon(y)+u(y)\right|^{p(x,y)}K(x,y)\,dx\,dy\to 0\quad\text{as }\varepsilon\to 0,$$

where $K(x,y)=|x-y|^{-n-sp(x,y)}$. By the same way in [2, Lemma 3.2], we can prove that

$$\begin{aligned}&\int_{\Omega\times\Omega}\left|u_\varepsilon(x)-u(x)-u_\varepsilon(y)+u(y)\right|^{p(x,y)}K(x,y)\,dx\,dy\\&\qquad\le|B_1|^{p^-+p^+-1}\int_{\Omega\times\Omega\times B_1}\left|u(x-\varepsilon z)-u(y-\varepsilon z)-u(x)+u(y)\right|^{p(x,y)}\\&\qquad\quad\times K(x,y)\big(\eta(z)^{p^+}+\eta(z)^{p^-}\big)\,dx\,dy\,dz.\end{aligned}$$

Fix $z\in B_1$ and put $w=(z,z)\in\Omega\times\Omega$. We define the function $v:\Omega\times\Omega\longrightarrow\mathbb{R}$ by

$$v(x,y):=\big(u(x)-u(y)\big)\big(K(x,y)\big)^{\frac{1}{p(x,y)}},\quad\forall(x,y)\in\Omega\times\Omega.$$

Then $v\in L^{p(\cdot,\cdot)}(\Omega\times\Omega)$. Let $\varepsilon^{'}>0$. Since $p\in\mathcal{P}^{\log}(\Omega\times\Omega)$, by [21, Corollary], there exists $g\in C_0^\infty(\Omega\times\Omega)$ with $\|v-g\|_{L^{p(\cdot,\cdot)}(\Omega\times\Omega)}\leqslant\frac{\varepsilon^{'}}{3}$, hence

$$\begin{aligned}&\big\|v(\cdot-\varepsilon w)-v\big\|_{L^{p(\cdot,\cdot)}(\Omega\times\Omega)}\\&\quad\leqslant\big\|v(\cdot-\varepsilon w)-g(\cdot-\varepsilon w)\big\|_{L^{p(\cdot,\cdot)}(\Omega\times\Omega)}\\&\qquad+\big\|g(\cdot-\varepsilon w)-g\big\|_{L^{p(\cdot,\cdot)}(\Omega\times\Omega)}+\|v-g\|_{L^{p(\cdot,\cdot)}(\Omega\times\Omega)}\\&\quad\leqslant\frac{\varepsilon^{'}}{3}+\frac{\varepsilon^{'}}{3}+\frac{\varepsilon^{'}}{3}=\varepsilon^{'},\end{aligned}$$

with ε is sufficiently small. Hence

$$\lim_{\varepsilon\to 0}\int_{\Omega\times\Omega}\left|u(x-\varepsilon z)-u(y-\varepsilon z)-u(x)+u(y)\right|^{p(x,y)}\times K(x,y)\,dx\,dy=0.$$

Moreover, for a.e. $z\in B_1$, there exists a positive constant C such that

$$\begin{aligned}&\big(\eta(z)^{p^+}+\eta(z)^{p^-}\big)\int_{\Omega\times\Omega}\left|u(x-\varepsilon z)-u(y-\varepsilon z)-u(x)+u(y)\right|^{p(x,y)}\\&\qquad\times K(x,y)\,dx\,dy\\&\quad\leqslant 2C\big(\eta(z)^{p^+}+\eta(z)^{p^-}\big)\int_{\Omega\times\Omega}\left|u(x)-u(y)\right|^{p(x,y)}\times K(x,y)\,dx\,dy\in L^\infty(B_1),\end{aligned}$$

for any $\varepsilon>0$. Hence, using the dominated convergence theorem, we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{B_1}\eta(z)^{p(x,y)}\int_{\Omega\times\Omega}&\left|u(x-\varepsilon z)-u(y-\varepsilon z)-u(x)+u(y)\right|^{p(x,y)}\\&\times K(x,y)\,dx\,dy\,dz=0.\end{aligned}$$

Consequently

$$\int_{\Omega\times\Omega}\left|u_\varepsilon(x)-u(x)-u_\varepsilon(y)+u(y)\right|^{p(x,y)}K(x,y)\,dx\,dy\to 0\quad\text{as }\varepsilon\to 0,$$

which concludes the proof. □

Now, we will discuss the cutoff technique needed for the density argument. For any $j \in \mathbb{N}$, let $\tau_j \in C^\infty(\mathbb{R}^n)$ be such that

$$0 \le \tau_j(x) \le 1, \quad \forall x \in \mathbb{R}^n,$$
$$\tau_j(x) = \begin{cases} 1 & \text{if } x \in B_j, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_{j+1}, \end{cases} \tag{2.1}$$

where B_j denotes the ball centered at zero with radius j. We have the following result.

Lemma 2.6. *Let $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$. Then $\tau_j u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$.*

Proof. Let $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$. It is clear that $\tau_j u \in L^{q(\cdot)}(\mathbb{R}^n)$ since $|\tau_j| \le 1$. Furthermore,

$$\begin{aligned} I &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\tau_j(x)u(x) - \tau_j(y)u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dx\,dy \\ &\le 2^{p^+-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\tau_j(x)(u(x) - u(y))|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dx\,dy \\ &\quad + 2^{p^+-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)(\tau_j(x) - \tau_j(y))|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dx\,dy \\ &\le 2^{p^+-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dx\,dy \\ &\quad + 2^{p^+-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)(\tau_j(x) - \tau_j(y))|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dx\,dy, \end{aligned}$$

where the integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dx\,dy$$

is finite since $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$.

According to Lemma 2.3, we can assume that $u \in L^\infty(\mathbb{R}^n)$. Hence,

$$|u(y)|^{p(x,y)} \le \|u\|^{p^+}_{L^\infty(\mathbb{R}^n)} + \|u\|^{p^-}_{L^\infty(\mathbb{R}^n)}.$$

Therefore,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)(\tau_j(x) - \tau_j(y))|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dx\,dy \le \frac{C}{\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|(\tau_j(x) - \tau_j(y))|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy,$$

where $\alpha = \min\{\lambda^{p^+}, \lambda^{p^-}\}$ and the constant C depends on p^+, p^-, and $\|u\|_{L^\infty(\mathbb{R}^n)}$. Finally, using Lemma 2.2, we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)(\tau_j(x) - \tau_j(y))|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dx\,dy < \infty.$$

This concludes the proof. ∎

Now, we will prove that we can approximate a function in $W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$ with another function with compact support, by keeping the error small. More precisely, we have the following result.

Lemma 2.7. *Let* $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$*. Then* $\operatorname{supp}(\tau_j u) \subseteq \overline{B}_{j+1} \cap \operatorname{supp} u$*, and*

$$\|\tau_j u - u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \longrightarrow 0 \quad \textit{as } j \longrightarrow +\infty.$$

Proof. By (2.1) and [11, Lemma 9], we get

$$\operatorname{supp}(\tau_j u) \subseteq \overline{B}_{j+1} \cap \operatorname{supp} u.$$

Now, let us show that

$$\|\tau_j u - u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } j \longrightarrow +\infty.$$

From Proposition 2.1, it suffices to prove that

$$\int_{\mathbb{R}^n} \big|\tau_j(x)u(x) - u(x)\big|^{q(x)}\, dx \longrightarrow 0 \quad \text{as } j \longrightarrow +\infty$$

and

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\tau_j(x)u(x) - u(x) - \tau_j(y)u(y) + u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\, dx\, dy \longrightarrow 0 \quad \text{as } j \longrightarrow +\infty.$$

We observe that

$$\begin{aligned}\big|\tau_j(x)u(x) - u(x)\big|^{q(x)} &= \big|\tau_j(x) - 1\big|^{q(x)}\big|u(x)\big|^{q(x)} \\ &\le 2^{q^+-1}\big(\big|\tau_j(x)\big|^{q(x)} + 1\big)\big|u(x)\big|^{q(x)} \\ &\le 2^{q^+}\big|u(x)\big|^{q(x)} \in L^1(\mathbb{R}^n).\end{aligned}$$

Moreover, by (2.1)

$$\big|\tau_j(x)u(x) - u(x)\big|^{q(x)} \longrightarrow 0 \quad \text{as } j \longrightarrow +\infty \text{ a.e. in } \mathbb{R}^n.$$

Then, by the dominated convergence theorem, we get

$$\int_{\mathbb{R}^n} \big|\tau_j(x)u(x) - u(x)\big|^{q(x)} \longrightarrow 0 \quad \text{as } j \longrightarrow +\infty.$$

Now, let us show that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\tau_j(x)u(x) - u(x) - \tau_j(y)u(y) + u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\, dx\, dy \longrightarrow 0 \quad \text{as } j \longrightarrow +\infty.$$

We set $\eta_j = 1 - \tau_j$. Then $\eta_j u = u - \tau_j u$. Moreover,

$$\begin{aligned}&\big|\tau_j(x)u(x) - u(x) - \tau_j(y)u(y) + u(y)\big| \\ &\quad = \big|\eta_j(x)\big(u(x) - u(y)\big) - \big(\tau_j(y) - \tau_j(x)\big)u(y)\big|.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\frac{|\tau_j(x)u(x)-u(x)-\tau_j(y)u(y)+u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy\\ &\leq 2^{p^+-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\tau_j(x)-\tau_j(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\big|u(y)\big|^{p(x,y)}\,dx\,dy\\ &\quad+2^{p^+-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\eta_j^{p(x,y)}(x)\,dx\,dy.\end{aligned}$$

Note that by Lemma 2.3, we can assume that $u\in L^\infty(\mathbb{R}^n)$. Hence,

$$\frac{|\tau_j(x)-\tau_j(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\big|u(y)\big|^{p(x,y)}\leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)},p^+,p^-\big)\frac{|\tau_j(x)-\tau_j(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}.$$

By Lemma 2.2, we deduce that

$$\frac{|\tau_j(x)-\tau_j(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\in L^1(\mathbb{R}^n\times\mathbb{R}^n).$$

Furthermore,

$$\frac{|\tau_j(x)-\tau_j(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\big|u(y)\big|^{p(x,y)}\longrightarrow 0\quad\text{as } j\longrightarrow\infty \text{ a.e. in } \mathbb{R}^n\times\mathbb{R}^n.$$

Hence, by the dominated convergence theorem,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\tau_j(x)-\tau_j(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\big|u(y)\big|^{p(x,y)}\,dx\,dy\longrightarrow 0\quad\text{as } j\longrightarrow\infty.$$

Also,

$$\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\big(\eta_j(x)\big)^{p(x,y)}\leq\big(\big(\eta_j(x)\big)^{p^+}+\eta_j(x))^{p^-}\big)\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}$$

and

$$\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\in L^1(\mathbb{R}^n\times\mathbb{R}^n).$$

Again by (2.1),

$$\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\big(\eta_j(x)\big)^{p(x,y)}\longrightarrow 0\quad\text{as } j\longrightarrow\infty \text{ a.e. in } \mathbb{R}^n\times\mathbb{R}^n.$$

Hence, by the dominated convergence theorem,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\big(\eta_j(x)\big)^{p(x,y)}\,dx\,dy\longrightarrow 0\quad\text{as } j\longrightarrow\infty.$$

This concludes the proof. $\square$

3. Approximation by smooth and compactly supported functions

In this section we give some density properties of smooth and compactly supported functions in fractional Sobolev spaces with variable exponents. The additional difficulty in this nonlocal setting is caused by the fact that the variable exponent Lebesgue spaces are not translation-invariant (see [6, Section 3.6]). To overcome this difficulty, we exploit the techniques developed in Section 2.

Theorem 3.1. *Let $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$. Then for any fixed $\delta > 0$, there exists a continuous and compactly supported function u_δ such that*

$$\|u - u_\delta\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0.$$

Proof. Let $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$. According to Lemma 2.2, we can approximate u with a sequence of bounded functions. Consequently, we can also assume that $u \in L^\infty(\mathbb{R}^n)$.

Let $\tau_j \in C^\infty(\mathbb{R}^n)$ be as in Section 2, with $\tau_j(P) = 1$ if $|P| \leq j$ and $\tau_j(P) = 0$ if $|P| \geq j+1$. Let $u_j = \tau_j u$. Then

$$u_j \longrightarrow u \quad \text{a.e. in } \mathbb{R}^n \text{ as } j \longrightarrow \infty,$$

and

$$\left|u(x) - u_j(x)\right|^{q(x)} \leq 2^{q^+}\left|u(x)\right|^{q(x)} \in L^1(\mathbb{R}^n).$$

As a consequence, by the dominated convergence theorem,

$$\int_{\mathbb{R}^n} \left|u(x) - u_j(x)\right|^{q(x)} dx \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.$$

Hence, for any fixed $\delta > 0$, there exists $j_\delta \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n} \left|u(x) - u_{j_\delta}(x)\right|^{q(x)} dx \leq \delta. \tag{3.1}$$

Note that u_{j_δ} is supported in $\overline{B}_{j+1}$ and that $\mu(A) = \int_A dx$ is finite over compact sets. Hence, we can use Lusin's theorem (see [12, Theorem 7.10, p. 121] for the definition of the uniform norm). We obtain that there exist a closed set $E_\delta \subset \mathbb{R}^n$ and a continuous and compactly supported function $u_\delta : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$u_\delta = u_{j_\delta} \quad \text{in } \mathbb{R}^n \setminus E_\delta, \qquad \mu(E_\delta) \leq \delta \qquad \text{and} \qquad \|u_{j_\delta}\|_{L^\infty(\mathbb{R}^n)} \leq \|u\|_{L^\infty(\mathbb{R}^n)}.$$

In particular, since $0 \leq \tau_{j_\delta}(x) \leq 1$, we have

$$\|u_\delta\|_{L^\infty(\mathbb{R}^n)} \leq \|u\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} \left|u_{j_\delta}(x) - u_\delta(x)\right|^{q(x)} dx &= \int_{E_\delta} \left|u_{j_\delta(x)} - u_\delta(x)\right|^{q(x)} dx \\ &\leq 2^{q^+-1}\Big(\int_{E_\delta} \mid u_{j_\delta(x)}\Big)|^{q(x)} dx + \int_{E_\delta} \left|u_\delta(x)\right|^{q(x)} dx) \\ &\leq C(q^+, q^-, \|u\|_{L^\infty}(\mathbb{R}^n)\mu(E_\delta).\end{aligned}$$

Hence

$$\int_{\mathbb{R}^n} \big|u_{j_\delta}(x) - u_\delta(x)\big|^{q(x)}\, dx \le C\delta. \tag{3.2}$$

On the other hand,

$$\begin{aligned}&\big|u(x) - u_\delta(x)\big|^{q(x)}\\ &\qquad \le 2^{q^+-1}\Big(\big|u(x) - u_{j_\delta}(x)\big|^{q(x)} + \big|u_{j_\delta}(x) - u_\delta(x)\big|^{q(x)}\Big).\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^n} \big|u(x) - u_\delta(x)\big|^{q(x)}\, dx \le 2^{q^+-1} &\int_{\mathbb{R}^n} \big|u(x) - u_{j_\delta}(x)\big|^{q(x)}\, dx\\ &+ 2^{q^+-1}\int_{\mathbb{R}^n} \big|u_{j_\delta}(x) - u_\delta(x)\big|^{q(x)}\, dx.\end{aligned}$$

Now, from (3.1) and (3.2), we deduce that

$$\int_{\mathbb{R}^n} \big|u(x) - u_\delta(x)\big|^{q(x)}\, dx \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0.$$

Therefore, by Proposition 2.1, we obtain

$$\|u - u_\delta\|_{L^{q(\cdot)}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0.$$

Moreover,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|(u-u_j)(x) - (u-u_j)(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\, dx\, dy \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.$$

Hence, for any fixed $\delta > 0$, there exists $j_\delta \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|(u-u_{j_\delta})(x) - (u-u_{j_\delta})(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\, dx\, dy \le \delta.$$

Notice that

$$v_{j_\delta}(x,y) = \frac{|u_{j_\delta}(x) - u_{j_\delta}(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}$$

is supported in $\{P \in \mathbb{R}^n \times \mathbb{R}^n; |P| \le j_\delta + 1\}$ and $\mu(A) = \int_A \int_A dx\, dy$ is finite over compact sets. Hence, we can again use Lusin's theorem. We get that there exist a closed set $E_\delta \subset \mathbb{R}^n \times \mathbb{R}^n$ and a continuous and compactly supported function $u_\delta : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$u_\delta = v_{j_\delta} \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n \setminus E_\delta, \qquad \mu(E_\delta) \le \delta \qquad \text{and}$$
$$\|u_{j_\delta}\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} \le \|v_{j_\delta}\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}.$$

In particular, since $0 \le \tau_{j\delta}(x) \le 1$, we have

$$\|u_\delta\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} \le \left\| \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} \right\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}.$$

Now, by putting

$$v(x,y) := \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{\frac{n}{p(x,y)}+s}}$$

in Lemma 2.4, we can assume that

$$\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n).$$

Therefore

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|(u-u_\delta)(x)-(u-u_\delta)(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}}\,dx\,dy \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0,$$

which concludes the proof. □

Theorem 3.2. *Let $q \in \mathcal{P}^{\log}(\Omega)$ and $p \in \mathcal{P}^{\log}(\Omega \times \Omega)$. Then the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$.*

Proof. We show that for any $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$, there exists a sequence $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|\rho_\varepsilon - u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Let $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$, and let us fix $\delta > 0$. Let τ_j be as in Section 2. By Lemma 2.7, we know that

$$\|u - \tau_j u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \le \frac{\delta}{2},$$

for j large enough.

For any $\varepsilon > 0$, let us consider

$$\rho_\varepsilon := \tau_j u * \eta_\varepsilon,$$

where η_ε is the mollifier function defined in Section 2. By construction, $\rho_\varepsilon \in C^\infty(\mathbb{R}^n)$. Furthermore, by [3, Proposition IV.18],

$$\operatorname{supp} \rho_\varepsilon \subseteq \operatorname{supp}(\tau_j u) + \overline{B_\varepsilon}.$$

Also, by Lemma 2.7,

$$\operatorname{supp}(\tau_j u) \subseteq \overline{B}_{j+1} \cap \operatorname{supp} u.$$

Hence,

$$\operatorname{supp} \rho_\varepsilon \subseteq (\overline{B}_{j+1} \cap \operatorname{supp} u) + \overline{B_\varepsilon}.$$

As a consequence of this,

$$\rho_\varepsilon \in C_0^\infty(\mathbb{R}^n),$$

for ε small enough. Furthermore, Lemma 2.5 gives

$$\|\rho_\varepsilon - \tau_j u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \le \frac{\delta}{2}$$

for ε small enough. Therefore,

$$\begin{aligned}\|u-\rho_\varepsilon\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} &\le \|u-\tau_j u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} + \|\tau_j u - \rho_\varepsilon\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \\ &\le \frac{\delta}{2} + \frac{\delta}{2} \\ &\le \delta.\end{aligned}$$

Since δ can be taken arbitrarily small, this concludes the proof. □

Theorem 3.3. *Let $q \in \mathcal{P}^{\log}(\Omega)$, $p \in \mathcal{P}^{\log}(\Omega \times \Omega)$ and suppose that Ω is a $W^{s,q(\cdot),p(\cdot,\cdot)}$-extension domain. Then $C^\infty(\overline{\Omega})$ is dense in $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$.*

Proof. Let $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$. Since Ω is a $W^{s,q(\cdot),p(\cdot,\cdot)}$-extension domain, we find that $\tilde{u} \in W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$ and

$$\|\tilde{u}\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)}.$$

Due to Theorem 3.2, we can choose $\tilde{u}_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ with

$$\tilde{u}_\varepsilon \longrightarrow u \quad \text{in } W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n).$$

We set $u_\varepsilon := \tilde{u}_\varepsilon|_\Omega$. Then

$$\|u - u_\varepsilon\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)} \leq \|\tilde{u} - \tilde{u}_\varepsilon\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^n)} \longrightarrow 0.$$

Hence $u_\varepsilon \in C^\infty(\overline{\Omega})$ are the required approximating functions. □

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