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ORTHOGONAL COMPLEMENTING IN HILBERT C^*-MODULES

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ABSTRACT. We characterize orthogonally complemented submodules in Hilbert C^*-modules by their orthogonal closures. Applying Magajna's characterization of Hilbert C^*-modules over C^*-algebras of compact operators by the complementing property of submodules, we give an elementary proof of Schweizer's characterization of Hilbert C^*-modules over C^*-algebras of compact operators. Also, we prove analogous characterization theorems for C^*-algebras of compact operators related to topological properties of submodules of strict completions of Hilbert modules over a nonunital C^*-algebra.

Introduction

It is well known that in Hilbert spaces—as well as in Hilbert C^*-modules over C^*-algebras of compact operators on some Hilbert space—every closed subspace/submodule is orthogonally complemented. There is a natural isomorphism between the C^*-algebras of bounded operators on Hilbert C^*-modules over C^*-algebras of compact operators on some Hilbert space and the C^*-algebras of bounded operators on a certain Hilbert space contained in these modules which is simply a restriction. This allows us to transfer many results from Hilbert spaces to Hilbert C^*-modules over C^*-algebras of compact operators.

The first goal of this article is to describe orthogonally complemented submodules in general Hilbert C^*-modules in terms of their biorthogonal closures (see see Theorem 2.1 below). The argument is based on simple formulas for sums and orthogonal complements of submodules obtained by Gebhardt and Schmüdgen

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in [4]. This enables us to give an elementary proof of Schweizer's theorem on the characterization of Hilbert C^*-modules over C^*-algebras of compact operators using the biorthogonal closure property for closed submodules. Although Magajna's theorem (based on the complementing property of closed submodules) preceded Schweizer's theorem (based on the orthogonal closures property of closed submodules), the original proof in [7] is independent of Magajna's result (cf. [5]) and uses pure algebraic techniques. Here we re-prove Schweizer's theorem by applying our Theorem 2.1 and Magajna's theorem.

The second goal of this article, if $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module over a nonunital C^*-algebra $\mathcal{A}$, is to characterize the complementing property of submodules in the Hilbert $\mathbf{M}(\mathcal{A})$-module $\mathbf{M}(\mathcal{X})$, which is the completion of $\mathcal{X}$ in the strict topology defined by $\mathcal{X}$. Also, we will prove characterization theorems of C^*-algebras of compact operators related to topological and orthogonal closedness properties of submodules of strictly complete modules.

1. Preliminaries

A (right) Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module $\mathcal{X}$ equipped with an $\mathcal{A}$-valued inner product $\langle\cdot|\cdot\rangle$ which is $\mathcal{A}$-linear in the second variable and conjugate linear in the first variable such that $\mathcal{X}$ is a Banach space with the norm $\|v\| = \|\langle v|v\rangle\|^{\frac{1}{2}}$. We say that $\mathcal{X}$ is a full Hilbert $\mathcal{A}$-module if $\mathcal{A} = \langle\mathcal{X}|\mathcal{X}\rangle$, where $\langle\mathcal{X}|\mathcal{X}\rangle$ is the closed linear span of all elements in the underlying C^*-algebra $\mathcal{A}$ of the form $\langle x|y\rangle$, $x, y \in \mathcal{X}$. We denote by $\mathcal{F}^\perp = \{x \in \mathcal{X}; \forall y \in \mathcal{F}, \langle y|x\rangle = 0\}$ the orthogonal complement of $\mathcal{F}$ in $\mathcal{X}$, and we denote by $\mathcal{F}^{\perp_\mathcal{Y}} = \mathcal{F}^\perp \cap \mathcal{Y}$ the orthogonal complement of $\mathcal{F}$ in a submodule $\mathcal{Y} \subset \mathcal{X}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules, then we denote by $\mathbf{B}(\mathcal{X},\mathcal{Y})$ the Banach space of all bounded $\mathcal{A}$-linear operators from $\mathcal{X}$ into $\mathcal{Y}$. When $\mathcal{X} = \mathcal{Y}$, we write $\mathbf{B}(\mathcal{X})$ instead of $\mathbf{B}(\mathcal{X},\mathcal{X})$. The Banach space of all adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathbf{B}_a(\mathcal{X},\mathcal{Y})$. In general, bounded $\mathcal{A}$-linear operators may fail to possess an adjoint, so $\mathbf{B}_a(\mathcal{X},\mathcal{Y})$ may properly be contained in $\mathbf{B}(\mathcal{X},\mathcal{Y})$.

We use some basic results on extensions of Hilbert C^*-modules and adjointable operators on these modules from [1], [2], and [3]. The following is a brief summary of the basic notions and facts used in this article.

If $\mathcal{X}$ is a Hilbert C^*-module over a nonunital C^*-algebra $\mathcal{A}$, then we have the so-called strict topology on the Hilbert $\mathbf{M}(\mathcal{A})$-module $\mathbf{M}(\mathcal{X}) = \mathbf{B}_a(\mathcal{A},\mathcal{X})$. It is a Hausdorff topology defined by the family of seminorms $v \mapsto \|\langle v|x\rangle\|$ $(x \in \mathcal{X})$ and $v \mapsto \|vb\|$ $(b \in \mathcal{A})$. This means that a net (x_i) in $\mathbf{M}(\mathcal{X})$ converges strictly to $x \in \mathbf{M}(\mathcal{X})$, which is denoted by $x = \text{st-lim}_i\, x_i$, if and only if $\langle x|y\rangle = \lim_i\langle x_i|y\rangle$, $\forall y \in \mathcal{X}$ and $xb = \lim_j x_j b$, $\forall b \in \mathcal{A}$. If (e_i) is any approximate unit for $\mathcal{A}$, then each $x \in \mathbf{M}(\mathcal{X})$ satisfies $x = \text{st-lim}_i\, xe_i$. In particular, $\mathcal{X}$ is dense in $\mathbf{M}(\mathcal{X})$ with respect to the strict topology and $\mathbf{M}(\mathcal{X})$ is the strict completion of $\mathcal{X}$. Also, $\mathbf{M}(\mathcal{X})\mathcal{A} = \mathcal{X} = \mathcal{X}\mathcal{A}$. Note that the strict topology is weaker than the norm topology on $\mathbf{M}(\mathcal{X})$. Particularly useful is the following theorem on adjointable operators.

Theorem 1.1 ([3, Theorem 2.3])**.** *Let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module, and let $\mathbf{M}(\mathcal{X})$ be its strict completion. Then the map $\alpha : \mathbf{B}_a(\mathbf{M}(\mathcal{X})) \to \mathbf{B}_a(\mathcal{X})$, $\alpha(T) = T|_{\mathcal{X}}$ is an isomorphism of C^*-algebras.*

Definition 1.2. For $\mathcal{S} \subseteq \mathbf{M}(\mathcal{X})$ we denote by $c\ell(\mathcal{S})$ the norm closure of $\mathcal{S}$, and we denote by $c\ell^{st}(\mathcal{S})$ the strict closure of $\mathcal{S}$, that is, the closure in the strict topology.

If $\mathcal{F}, \mathcal{G} \subseteq \mathbf{M}(\mathcal{X})$, then (cf. [8, Lemma 15.3.4])

$$\mathcal{F} \subseteq \mathcal{F}^{\perp\perp} \tag{1.1}$$

and

$$\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{G}^{\perp} \subseteq \mathcal{F}^{\perp}. \tag{1.2}$$

As an immediate consequence, we get $\mathcal{F}^{\perp} = \mathcal{F}^{\perp\perp\perp}$. Namely, (1.1) applied to $\mathcal{F}^{\perp}$ gives us $\mathcal{F}^{\perp} \subseteq (\mathcal{F}^{\perp})^{\perp\perp}$, and then by applying (1.2) to $\mathcal{G} = \mathcal{F}^{\perp\perp}$ we conclude that $(\mathcal{F}^{\perp\perp})^{\perp} \subseteq \mathcal{F}^{\perp}$. The same is true in each Hilbert C^*-module; in particular, for $\mathcal{F}, \mathcal{G} \subseteq \mathcal{X}$ and orthogonal complementing in $\mathcal{X}$. For subsets $\mathcal{F}, \mathcal{G} \subseteq \mathcal{X}$ we write $\mathcal{F} + \mathcal{G} = \{x + y : x \in \mathcal{X}, y \in \mathcal{G}\}$ and $\mathcal{F} \oplus \mathcal{G} := \mathcal{F} + \mathcal{G}$ when $\mathcal{F} \subseteq \mathcal{G}^{\perp}$ (and hence $\mathcal{G} \subseteq \mathcal{G}^{\perp\perp} \subseteq \mathcal{F}^{\perp}$). It is well known that in Hilbert C^*-modules orthogonal complements and orthogonal sums of closed submodules are closed submodules, same as in Hilbert spaces.

Definition 1.3. A submodule $\mathcal{F} \subseteq \mathcal{X}$ is called *essential* (cf. [4]) if $\mathcal{F}^{\perp} = \{0\}$, *orthogonally complemented* if $\mathcal{F} \oplus \mathcal{F}^{\perp} = \mathcal{X}$, and *orthogonally closed* if $\mathcal{F} = \mathcal{F}^{\perp\perp}$.

The following lemma contains simple links between sums of submodules and their orthogonal complements found in [4, Lemma 1].

Lemma 1.4. *If $\mathcal{F}, \mathcal{G} \subseteq \mathcal{X}$ are submodules, then $(\mathcal{F} \cap \mathcal{G})^{\perp} \supseteq (\mathcal{F}^{\perp} + \mathcal{G}^{\perp})^{\perp\perp}$ and $(\mathcal{F} + \mathcal{G})^{\perp} = \mathcal{F}^{\perp} \cap \mathcal{G}^{\perp}$. In particular, $\mathcal{F} \oplus \mathcal{F}^{\perp}$ is always an essential submodule.*

Proof. If $x \in \mathcal{F} \cap \mathcal{G}$, then $\langle x|y\rangle = 0$, $\forall y \in \mathcal{F}^{\perp}$ and $\langle x|z\rangle = 0$, $\forall z \in \mathcal{G}^{\perp}$; hence, $\langle x|y + z\rangle = 0$, $\forall y + z \in \mathcal{F}^{\perp} + \mathcal{G}^{\perp}$, that is, $\mathcal{F} \cap \mathcal{G} \subseteq (\mathcal{F}^{\perp} + \mathcal{G}^{\perp})^{\perp}$. We now apply (1.2) to get the first assertion. Furthermore, $x \in \mathcal{F}^{\perp} \cap \mathcal{G}^{\perp} \Leftrightarrow \langle x|y\rangle = 0$, $\forall y \in \mathcal{F} \wedge \langle x|z\rangle = 0$, $\forall z \in \mathcal{G} \Leftrightarrow \langle x|y + z\rangle = 0$, $\forall y + z \in \mathcal{F} + \mathcal{G} \Leftrightarrow x \in (\mathcal{F} + \mathcal{G})^{\perp}$. This gives us the equality $(\mathcal{F} + \mathcal{G})^{\perp} = \mathcal{F}^{\perp} \cap \mathcal{G}^{\perp}$. Taking $\mathcal{G} = \mathcal{F}^{\perp}$, we now see that $\mathcal{F} \oplus \mathcal{F}^{\perp}$ is essential. □

2. General Hilbert C^*-modules

The following theorem is a characterization of orthogonally complemented submodules in Hilbert C^*-modules based on the orthogonal closedness property.

Theorem 2.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, and let $\mathcal{F} \subseteq \mathcal{X}$ be a submodule. Then $\mathcal{F}$ is orthogonally complemented if and only if $\mathcal{F} \oplus \mathcal{F}^{\perp}$ is orthogonally closed.*

Proof. If $\mathcal{F}$ is orthogonally complemented, then $\mathcal{F} \oplus \mathcal{F}^{\perp} = \mathcal{X}$; hence $(\mathcal{F} \oplus \mathcal{F}^{\perp})^{\perp\perp} = \mathcal{X}^{\perp\perp} = \mathcal{X} = \mathcal{F} \oplus \mathcal{F}^{\perp}$. Conversely, if $\mathcal{F} \oplus \mathcal{F}^{\perp}$ is orthogonally closed, then from Lemma 1.4 we have $\mathcal{F} \oplus \mathcal{F}^{\perp} = (\mathcal{F} \oplus \mathcal{F}^{\perp})^{\perp\perp} = \{0\}^{\perp} = \mathcal{X}$. □

We quote Magajna's theorem on the characterization of C^*-algebras of compact operators.

Theorem 2.2 ([5, Theorem 1]). *Let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. Every closed submodule $\mathcal{F} \subseteq \mathcal{X}$ is orthogonally complemented if and only if $\mathcal{A}$ is isomorphic to some algebra of compact operators.*

The next theorem is due to Schweizer [7, Theorem 1.]. We give a simple proof of this theorem by applying Theorem 2.1.

Theorem 2.3. *Let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. Every closed submodule $\mathcal{F} \subseteq \mathcal{X}$ is orthogonally closed if and only if $\mathcal{A}$ is isomorphic to some algebra of compact operators.*

Proof. Suppose that all closed submodules are orthogonally closed. If $\mathcal{F} \subseteq \mathcal{X}$ is any closed submodule, then $\mathcal{F} \oplus \mathcal{F}^{\perp}$ is also a closed submodule. By assumption, both submodules are orthogonally closed and now the conclusion follows by applying Theorems 2.1 and 2.2. The converse is clear. □

3. Strict completions of Hilbert C^*-modules

Throughout this section, $\mathcal{A}$ denotes a nonunital C^*-algebra. The following properties of strict closure are of interest.

Lemma 3.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, and let $\mathcal{F}, \mathcal{G} \subseteq \mathbf{M}(\mathcal{X})$ be submodules. Then*

(i) $c\ell(\mathcal{F})\mathcal{A} = c\ell(\mathcal{F}\mathcal{A}) = c\ell^{st}(\mathcal{F})\mathcal{A}$,
(ii) $c\ell^{st}(c\ell(\mathcal{F})) = c\ell^{st}(\mathcal{F}) = c\ell^{st}(c\ell^{st}(\mathcal{F})\mathcal{A}) = c\ell^{st}(\mathcal{F}\mathcal{A})$,
(iii) $c\ell(\mathcal{F})\mathcal{A} = c\ell(\mathcal{G})\mathcal{A}$ *if and only if* $c\ell^{st}(\mathcal{F}) = c\ell^{st}(\mathcal{G})$,
(iv) $\mathcal{F} \subseteq \mathcal{X}$ *if and only if* $c\ell(\mathcal{F}) = c\ell(\mathcal{F})\mathcal{A}$,
(v) $c\ell(\mathcal{F}) \cap \mathcal{X} = c\ell(\mathcal{F})\mathcal{A}$.

Proof. In order to prove (i), observe that the inclusions $\mathcal{F} \subseteq c\ell(\mathcal{F}) \subseteq c\ell^{st}(\mathcal{F})$ imply that

$$\begin{aligned}\mathcal{F}\mathcal{A} \subseteq c\ell(\mathcal{F})\mathcal{A} &\subseteq c\ell^{st}(\mathcal{F})\mathcal{A} \\ &= \left\{xb; x = \text{st-}\lim_i x_i, x_i \in \mathcal{F}, b \in \mathcal{A}\right\} \\ &= \left\{xb; xb = \lim_i x_i b, x_i b \in \mathcal{F}\mathcal{A}\right\} \subseteq c\ell(\mathcal{F}\mathcal{A}).\end{aligned}$$

By applying the Cohen–Hewitt factorization theorem (see [6, Proposition 2.31]) the submodule $c\ell(\mathcal{F})\mathcal{A}$ of $c\ell(\mathcal{F})$ is closed; hence $c\ell(\mathcal{F})\mathcal{A} = c\ell(\mathcal{F}\mathcal{A}) = c\ell^{st}(\mathcal{F})\mathcal{A}$. Also, the inclusions $\mathcal{F} \subseteq c\ell(\mathcal{F}) \subseteq c\ell^{st}(\mathcal{F})$ imply that $c\ell^{st}(\mathcal{F}) \subseteq c\ell^{st}(c\ell(\mathcal{F})) \subseteq c\ell^{st}(c\ell^{st}(\mathcal{F})) = c\ell^{st}(\mathcal{F})$, and this is the first equality in (ii).

Furthermore, we show that $c\ell^{st}(\mathcal{F}) = c\ell^{st}(c\ell^{st}(\mathcal{F})\mathcal{A})$. Let $(e_\lambda)_\lambda$ be any approximate unit in $\mathcal{A}$. Then for each $x \in c\ell^{st}(\mathcal{F})$, we have $x = \text{st-}\lim_\lambda xe_\lambda$, that is, $c\ell^{st}(\mathcal{F}) \subseteq c\ell^{st}(c\ell^{st}(\mathcal{F})\mathcal{A})$. For the opposite inclusion, $c\ell^{st}(\mathcal{F})\mathcal{A} \subseteq c\ell^{st}(\mathcal{F})$ implies that $c\ell^{st}(c\ell^{st}(\mathcal{F})\mathcal{A}) \subseteq c\ell^{st}(c\ell^{st}(\mathcal{F})) = c\ell^{st}(\mathcal{F})$ and this gives the second equality in (ii).

For the proof of the last equality in (ii), observe that $\mathcal{F}\mathcal{A} \subseteq c\ell^{st}(\mathcal{F})\mathcal{A}$ implies that $c\ell^{st}(\mathcal{F}\mathcal{A}) \subseteq c\ell^{st}(c\ell^{st}(\mathcal{F})\mathcal{A})$. Also $c\ell(\mathcal{F}\mathcal{A}) \subseteq c\ell^{st}(\mathcal{F}\mathcal{A})$ by (i) implies in this case that $c\ell^{st}(c\ell^{st}(\mathcal{F})\mathcal{A}) = c\ell^{st}(c\ell(\mathcal{F}\mathcal{A})) \subseteq c\ell^{st}(c\ell^{st}(\mathcal{F}\mathcal{A})) = c\ell^{st}(\mathcal{F}\mathcal{A})$. Statement (iii) is a direct consequence of (i) and (ii).

For the proof of the necessity in (iv), because by (i) $c\ell(\mathcal{F})\mathcal{A}$ is a closed submodule, observe that if $x \in c\ell(\mathcal{F}) \subseteq \mathcal{X}$, then for any approximate unit $(e_\lambda)_\lambda$ in $\mathcal{A}$ we have $x = \lim_\lambda xe_\lambda \in c\ell(c\ell(\mathcal{F})\mathcal{A}) = c\ell(\mathcal{F})\mathcal{A}$; hence $c\ell(\mathcal{F}) = c\ell(\mathcal{F})\mathcal{A}$. Conversely, if $c\ell(\mathcal{F}) = c\ell(\mathcal{F})\mathcal{A}$, then $c\ell(\mathcal{F}) \subseteq \mathbf{M}(\mathcal{X})$ implies $c\ell(\mathcal{F}) = c\ell(\mathcal{F})\mathcal{A} \subseteq \mathbf{M}(\mathcal{X})\mathcal{A} = \mathcal{X}$. Claim (v) follows directly from (iv): namely, for $\mathcal{F} \subseteq \mathbf{M}(\mathcal{X})$ we have $c\ell(\mathcal{F})\mathcal{A} \subseteq c\ell(\mathcal{F}) \cap \mathcal{X} = (c\ell(\mathcal{F}) \cap \mathcal{X})\mathcal{A} \subseteq c\ell(\mathcal{F})\mathcal{A}$. □

Lemma 3.2. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module. If $\mathcal{F} \subseteq \mathbf{M}(\mathcal{X})$ is a submodule, then $\mathcal{F}^\perp$ is strictly closed, and hence $c\ell^{st}(\mathcal{F}) \subseteq \mathcal{F}^{\perp\perp}$. Also, $(\mathcal{F}\mathcal{A})^\perp = (c\ell(\mathcal{F})\mathcal{A})^\perp = (c\ell^{st}(\mathcal{F})\mathcal{A})^\perp = \mathcal{F}^\perp = c\ell(\mathcal{F})^\perp = c\ell^{st}(\mathcal{F})^\perp$ and $(\mathcal{F}\mathcal{A})^{\perp_\mathcal{X}} = \mathcal{F}^\perp\mathcal{A}$.*

Proof. The fact that $\mathcal{F}^\perp$ is strictly closed is a consequence of the strict continuity of the inner product in both variables. Namely, for any $x \in \mathcal{F}$ and $y \in c\ell^{st}(\mathcal{F}^\perp)$ such that $y = \text{st-lim}_\lambda y_\lambda$ for some net $(y_\lambda)_\lambda$ in $\mathcal{F}^\perp$, we have $\langle x|y\rangle = \text{st-lim}_\lambda \langle x|y_\lambda\rangle = 0$, and hence $c\ell^{st}(\mathcal{F}^\perp) \subseteq \mathcal{F}^\perp$; that is, $\mathcal{F}^\perp$ is strictly closed.

Similarly, for any $x \in \mathcal{F}^\perp$ and all $y \in c\ell^{st}(\mathcal{F})$, $y = \text{st-lim}_\lambda y_\lambda$ for some net $(y_\lambda)_\lambda$ in $\mathcal{F}$, we have $\langle x|y\rangle = \text{st-lim}_\lambda\langle x|y_\lambda\rangle = 0$, and hence $x \in c\ell^{st}(\mathcal{F})^\perp$. Because the opposite inclusion is clear, we have $\mathcal{F}^\perp = c\ell^{st}(\mathcal{F})^\perp$. Analogously, norm continuity of the inner product implies $\mathcal{F}^\perp = c\ell(\mathcal{F})^\perp$.

Now, we prove $(\mathcal{F}\mathcal{A})^\perp = \mathcal{F}^\perp$. Let $x \in (\mathcal{F}\mathcal{A})^\perp$. Then $\forall y \in \mathcal{F}\mathcal{A}$, $\langle x|y\rangle = 0$. This implies that $\forall z \in \mathcal{F}$, $\forall b \in \mathcal{A}$, $0 = \langle x|zb\rangle = \langle x|z\rangle b$; that is, for any $z \in \mathcal{F}$, $\langle x|z\rangle \in \mathcal{A}^\perp = \{0\}$ ($\mathcal{A}$ is an essential ideal in $\mathbf{M}(\mathcal{A})$), and hence $\langle x|z\rangle = 0$ or $x \in \mathcal{F}^\perp$. The opposite inclusion is obvious. Other equalities follow from Lemma 3.1. Finally, by Lemma 3.1(v), we have $(\mathcal{F}\mathcal{A})^{\perp_\mathcal{X}} = (\mathcal{F}\mathcal{A})^\perp \cap \mathcal{X} = \mathcal{F}^\perp \cap \mathcal{X} = \mathcal{F}^\perp\mathcal{A}$. □

Lemma 3.3. *If $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{F}, \mathcal{G} \subseteq \mathbf{M}(\mathcal{X})$ are closed submodules with $\mathcal{F} \perp \mathcal{G}$, then $(\mathcal{F} \oplus \mathcal{G})\mathcal{A} = \mathcal{F}\mathcal{A} \oplus \mathcal{G}\mathcal{A}$. In particular, we have $(c\ell^{st}(\mathcal{F}) \oplus \mathcal{F}^\perp)\mathcal{A} = \mathcal{F}\mathcal{A} \oplus (\mathcal{F}\mathcal{A})^{\perp_\mathcal{X}}$. Then $\mathcal{F}\mathcal{A}$ is orthogonally complemented in $\mathcal{X}$ exactly when $c\ell^{st}(\mathcal{F})$ is orthogonally complemented in $\mathbf{M}(\mathcal{X})$.*

Proof. The closed submodules $\mathcal{F}$, $\mathcal{G}$ are Hilbert $\mathcal{A}$-modules. After applying the Cohen–Hewitt factorization theorem, we obtain that $(\mathcal{F} \oplus \mathcal{G})\mathcal{A}$ is a closed submodule in $\mathcal{F} \oplus \mathcal{G}$. Clearly, $(\mathcal{F} \oplus \mathcal{G})\mathcal{A} \subseteq \mathcal{F}\mathcal{A} \oplus \mathcal{G}\mathcal{A}$. On the other hand, $(\mathcal{F} \oplus \{0\})\mathcal{A} = \mathcal{F}\mathcal{A} \oplus \{0\}$ and $(\{0\} \oplus \mathcal{G})\mathcal{A} = \{0\} \oplus \mathcal{G}\mathcal{A}$ are orthogonal submodules in $(\mathcal{F} \oplus \mathcal{G})\mathcal{A}$, and hence $\mathcal{F}\mathcal{A} \oplus \mathcal{G}\mathcal{A} = (\mathcal{F}\mathcal{A} \oplus \{0\}) \oplus (\{0\} \oplus \mathcal{G}\mathcal{A}) \subseteq (\mathcal{F} \oplus \mathcal{G})\mathcal{A}$. Now, by Lemmas 3.1(i) and 3.2, $(c\ell^{st}(\mathcal{F}) \oplus \mathcal{F}^\perp)\mathcal{A} = c\ell^{st}(\mathcal{F})\mathcal{A} \oplus \mathcal{F}^\perp\mathcal{A} = \mathcal{F}\mathcal{A} \oplus (\mathcal{F}\mathcal{A})^{\perp_\mathcal{X}}$.

If $c\ell^{st}(\mathcal{F})$ is orthogonally complemented in $\mathbf{M}(\mathcal{X})$, then $\mathcal{X} = \mathbf{M}(\mathcal{X})\mathcal{A} = (c\ell^{st}(\mathcal{F}) \oplus c\ell^{st}(\mathcal{F})^\perp)\mathcal{A} = c\ell^{st}(\mathcal{F})\mathcal{A} \oplus c\ell^{st}(\mathcal{F})^\perp\mathcal{A} = \mathcal{F}\mathcal{A} \oplus (\mathcal{F}\mathcal{A})^{\perp_\mathcal{X}}$. If $\mathcal{F}\mathcal{A}$ is orthogonally complemented in $\mathcal{X}$, then we have an orthogonal projector $P \in \mathbf{B}(\mathcal{X})$ such that $\mathcal{R}(P) = \mathcal{F}\mathcal{A}$ and $\mathcal{N}(P) = (\mathcal{F}\mathcal{A})^{\perp_\mathcal{X}}$. By Theorem 1.1 there exists a unique orthogonal projector $\widehat{P} \in \mathbf{B}(\mathbf{M}(\mathcal{X}))$, a strict extension of P (see [3, Remark 2.4.]), such that $P = \widehat{P}|_\mathcal{X}$. Consequently, $\mathcal{R}(\widehat{P}) = c\ell^{st}(\mathcal{R}(P)) = c\ell^{st}(\mathcal{F}\mathcal{A}) = c\ell^{st}(\mathcal{F})$ and by Lemmas 3.1(ii) and 3.2, $\mathcal{N}(\widehat{P}) = \mathcal{R}(\widehat{P})^\perp = c\ell^{st}(\mathcal{F})^\perp = (\mathcal{F}\mathcal{A})^\perp = c\ell^{st}((\mathcal{F}\mathcal{A})^\perp\mathcal{A}) = c\ell^{st}((\mathcal{F}\mathcal{A})^{\perp_\mathcal{X}}) = c\ell^{st}(\mathcal{N}(P))$. Now, $\mathbf{M}(\mathcal{X}) = \mathcal{R}(\widehat{P}) \oplus \mathcal{N}(\widehat{P}) = c\ell^{st}(\mathcal{F}) \oplus c\ell^{st}(\mathcal{F})^\perp$. □

The following result is analogous to Theorem 2.2.

Theorem 3.4. *Let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module, and let $\mathbf{M}(\mathcal{X})$ be its strict completion. Every strictly closed submodule in $\mathbf{M}(\mathcal{X})$ is orthogonally complemented if and only if $\mathcal{A}$ is isomorphic to some algebra of compact operators.*

Proof. First, let us note that by the Cohen–Hewitt factorization theorem every closed submodule $\mathcal{F} \subseteq \mathcal{X}$ is of the form $\mathcal{F} = \mathcal{F}\mathcal{A}$. Let us suppose that every strictly closed submodule in $\mathbf{M}(\mathcal{X})$ is orthogonally complemented. Take an arbitrary closed submodule $\mathcal{F} \subseteq \mathcal{X}$. Because $c\ell^{st}(\mathcal{F}) = c\ell^{st}(\mathcal{F}\mathcal{A})$ is strictly closed submodule in $\mathbf{M}(\mathcal{X})$ it is orthogonally complemented in $\mathbf{M}(\mathcal{X})$; hence, by Lemma 3.3, $\mathcal{F} = \mathcal{F}\mathcal{A}$ is orthogonally complemented in $\mathcal{X}$. By Theorem 2.2 this implies that $\mathcal{A}$ is isomorphic to some algebra of compact operators.

Conversely, let $\mathcal{A}$ be isomorphic to some algebra of compact operators. Then by Theorem 2.2 all closed submodules in $\mathcal{X}$ are orthogonally complemented. Take any strictly closed submodule $\mathcal{F} = c\ell^{st}(\mathcal{F}) = c\ell^{st}(\mathcal{F}\mathcal{A})$ in $\mathbf{M}(\mathcal{X})$. By Lemma 3.3 it is orthogonally complemented because $\mathcal{F}\mathcal{A}$ is orthogonally complemented in $\mathcal{X}$. □

The following result is analogous to Theorem 2.3.

Theorem 3.5. *Let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module, and let $\mathbf{M}(\mathcal{X})$ be its strict completion. Every strictly closed submodule in $\mathbf{M}(\mathcal{X})$ is orthogonally closed if and only if $\mathcal{A}$ is isomorphic to some algebra of compact operators.*

Proof. First, we prove that strictly closed submodule $\mathcal{F} \subseteq \mathbf{M}(\mathcal{X})$ is orthogonally closed exactly when $\mathcal{F}\mathcal{A}$ is orthogonally closed in $\mathcal{X}$. If strictly closed submodule $\mathcal{F} \subseteq \mathbf{M}(\mathcal{X})$ is orthogonally closed, then by Lemmas 3.1(i) and 3.2, $\mathcal{F}\mathcal{A} = c\ell^{st}(\mathcal{F})\mathcal{A} = c\ell^{st}(\mathcal{F})^{\perp\perp}\mathcal{A} = (c\ell^{st}(\mathcal{F})\mathcal{A})^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}} = (\mathcal{F}\mathcal{A})^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}$ is orthogonally closed in $\mathcal{X}$.

Conversely, if for strictly closed submodule $\mathcal{F} \subseteq \mathbf{M}(\mathcal{X})$ submodule $\mathcal{F}\mathcal{A}$ is orthogonally closed in $\mathcal{X}$, then $\mathcal{F}^{\perp\perp}\mathcal{A} = (\mathcal{F}\mathcal{A})^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}} = \mathcal{F}\mathcal{A}$ and by Lemma 3.1 we have $c\ell^{st}(\mathcal{F}^{\perp\perp}) = c\ell^{st}(\mathcal{F})$. But $\mathcal{F}^{\perp\perp}$ and $\mathcal{F}$ are strictly closed; hence $\mathcal{F} = \mathcal{F}^{\perp\perp}$.

Now, let us suppose that every strictly closed submodule in $\mathbf{M}(\mathcal{X})$ is orthogonally closed. Take any closed submodule $\mathcal{F} \subseteq \mathcal{X}$. Then $c\ell^{st}(\mathcal{F})$ is orthogonally closed in $\mathbf{M}(\mathcal{X})$. Then, as proved in Lemma 3.3, $\mathcal{F} = \mathcal{F}\mathcal{A} = c\ell^{st}(\mathcal{F})\mathcal{A}$ is orthogonally closed in $\mathcal{X}$ and Theorem 2.3 implies that $\mathcal{A}$ is isomorphic to some algebra of compact operators.

If $\mathcal{A}$ is isomorphic to some algebra of compact operators, then by Theorem 3.4 every strictly closed submodule in $\mathbf{M}(\mathcal{X})$ is orthogonally complemented; hence it is orthogonally closed. □

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